Dual holography from a non-perturbative generalization of the Wilsonian RG framework

Ki-Seok Kim

POSTECH

- What am I doing?
- Nonperturbative generalization of the Wilsonian RG theoretical framework:

  Given an exact functional renormalization group differential equation,

  I reformulate its solution in a path integral representation.
- Why?
- How?

Why ➔ To solve unsolved problems

cliché

List of unsolved problems in physics

Contents hide

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This is a dynamic list and may never be able to satisfy particular standards for completeness. You can help by adding missing items with reliable sources.

The following is a list of notable unsolved problems grouped into broad areas of physics.[1]

Some of the major unsolved problems in physics are theoretical, meaning that existing theories are currently unable to explain certain observed phenomena or experimental results. Others are experimental, involving challenges in creating experiments to test proposed theories or to investigate specific phenomena in greater detail.

A number of important questions remain open in the area of Physics beyond the Standard Model, such as the strong CP problem, determining the absolute mass of neutrinos, understanding matter–antimatter asymmetry, and identifying the nature of dark matter and dark energy.[2][3]

Another significant problem lies within the mathematical framework of the Standard Model itself, which remains inconsistent with general relativity. This incompatibility causes both theories to break down under extreme conditions, such as within known spacetime gravitational singularities like those at the Big Bang and at the centers of black holes beyond their event horizons.[4]

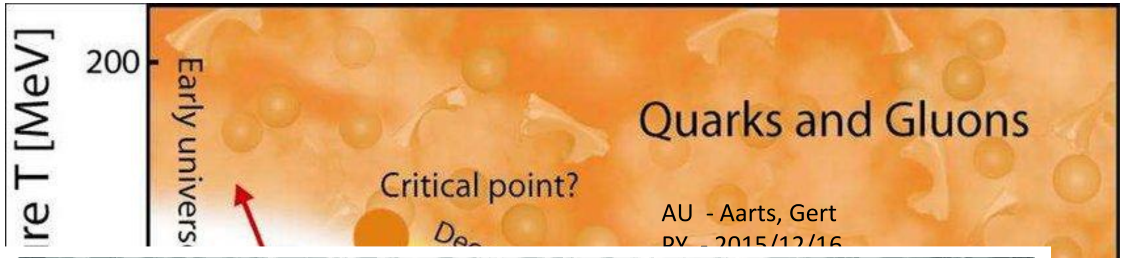
A weakly coupled UV fixed point:

RG flow to

A strongly correlated IR fixed point:

cliché

Strongly correlated syste



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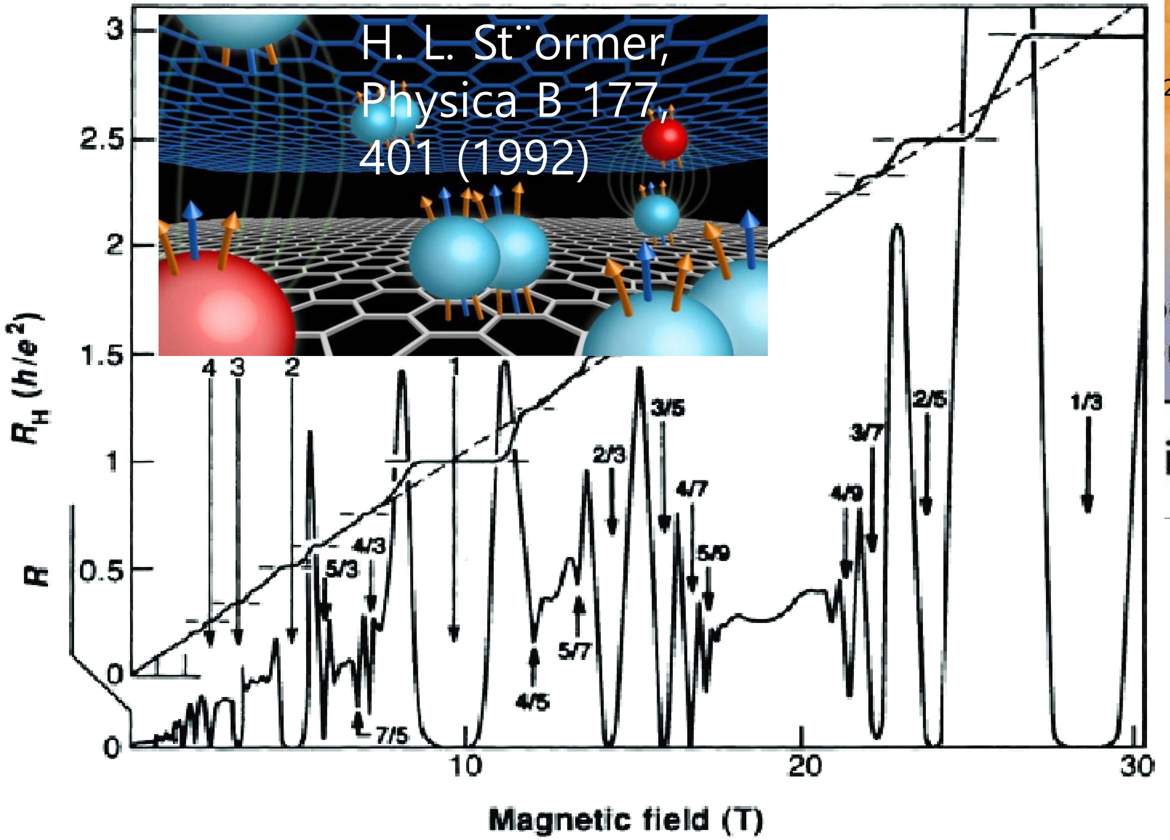

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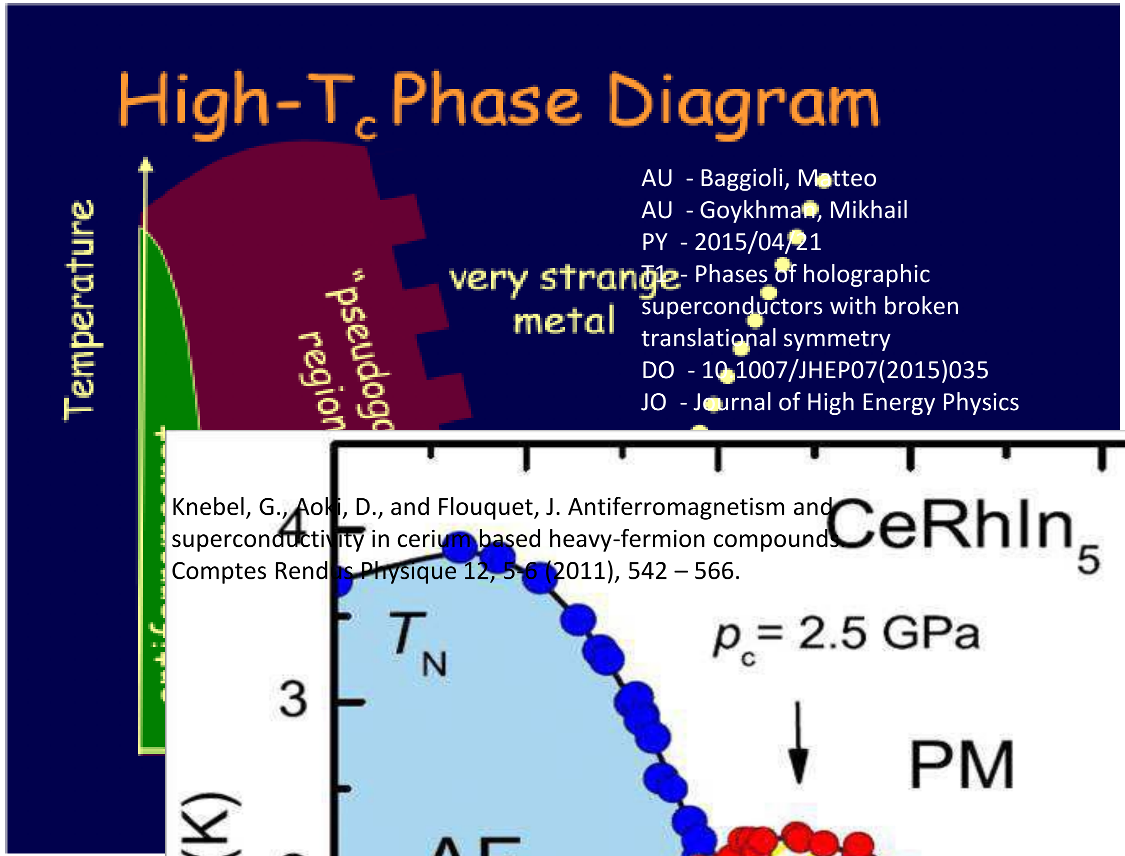
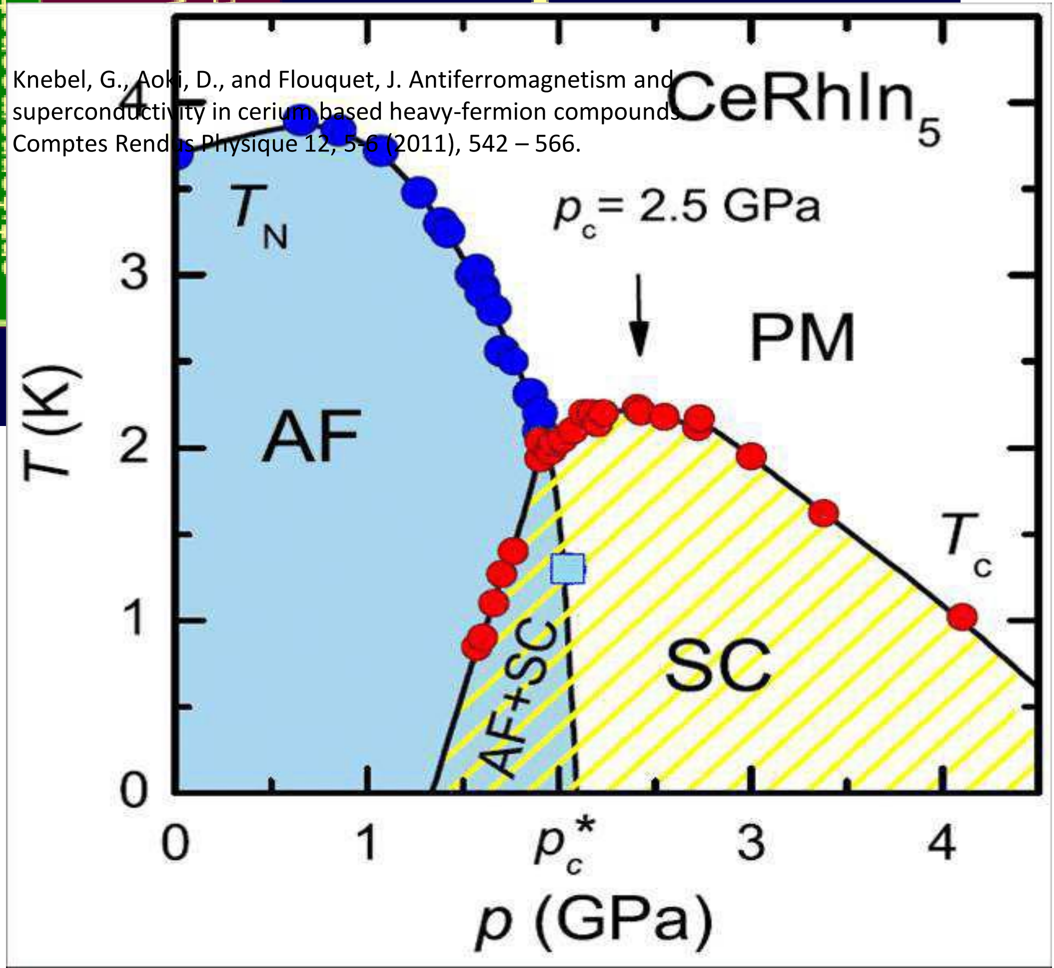


AU - Baggioli, Matteo
AU - Goykhman, Mikhail
PY - 2015/04/21
T1 - Phases of holographic superconductors with broken translational symmetry
DO - 10.1007/JHEP07(2015)035
JO - Journal of High Energy Physics



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A weakly coupled UV fixed point: Quasiparticles + perturbations

non-perturbatively RG flow to

A strongly correlated IR fixed point: Absence of quasiparticles or fractionalized (novel) quasiparticles + novel multiparticle spectra

Dynamics of quasiparticles are correlated with various multiparticle spectra,
the description of which needs higher-order quantum corrections.

How ➔ Renormalization group transformation

1950 ~ 1970: Era of symmetries, their spontaneous breaking, and resulting dynamical effects

cliché

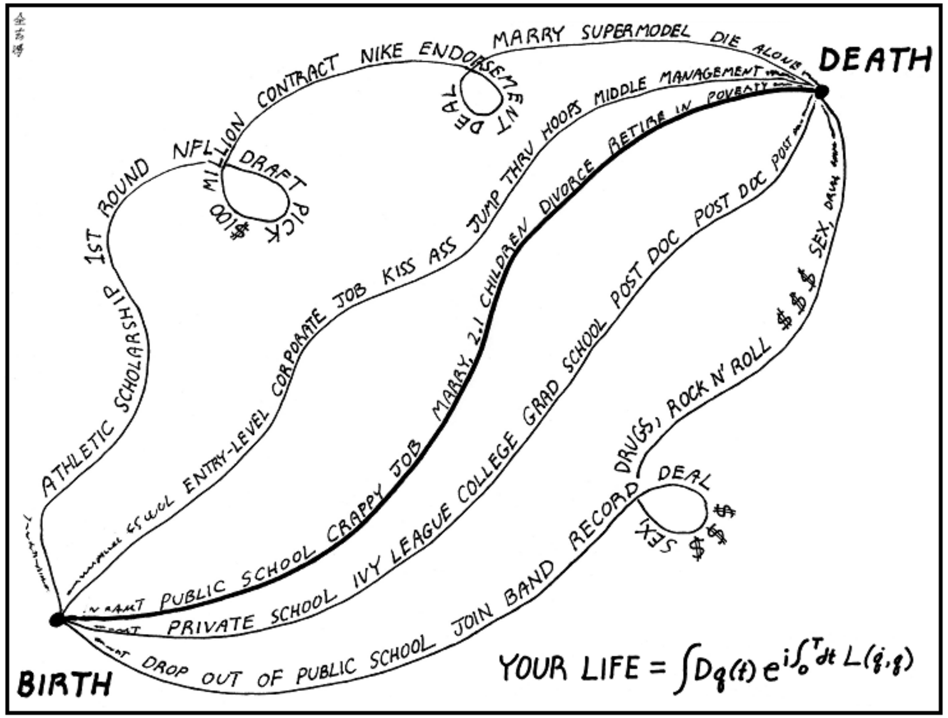
Era of symmetries, their spontaneous breaking, and resulting dynamical effects

High energy physics

- 1947 Lamb shift
- 1948 Renormalization of QED (Feynman, Tomonaga, Schwinger)
- 1954 Yang-Mills theory (nonabelian gauge theory)
- 1961 ~ 1962 Electroweak theory (Glashow-Weinberg-Salam)
- 1964 Anderson-Higgs mechanism
- 1971 ~ 1972 Renormalization of nonabelian gauge theories ('t Hooft & Veltman)
- 1973 Asymptotic freedom

Condensed matter physics

- 1950 Ginzburg-Landau theory
- 1957 Landau's Fermi liquid theory
- 1957 BCS theory
- 1958 Anderson localization
- 1962 Anderson-Higgs mechanism (Nambu, Goldstone, …)
- 1964 Kondo effect
- 1971 ~ 1972 Wilson's renormalization group procedure
- Theory of magnetism, Mott insulators, Heavy fermion systems, and etc.

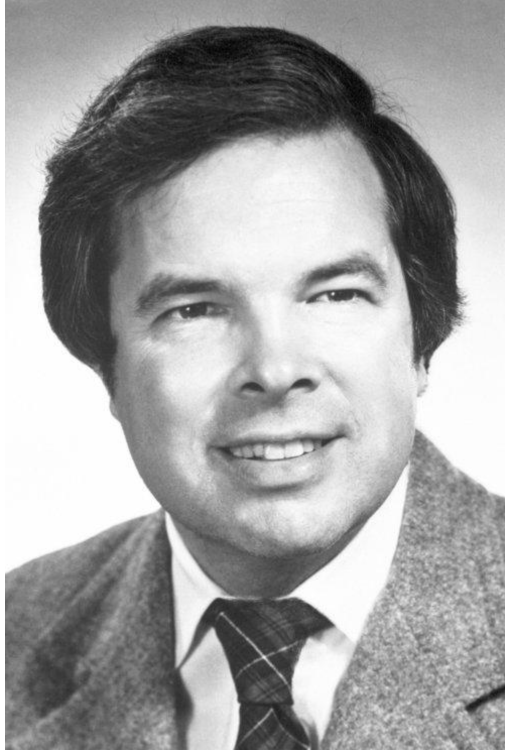


The Path Integral Formulation of Your Life

https://www.cyber-tec.org/2010/08/25/the-path-integral-formulation-of-your-life/

Kenneth G. Wilson

- RG = A careful way of the path integral for low energy physics = A way of re-summation of Feynman diagrams
- RG is to define an effective field theory (EFT) at a given scale.
- Wilson's RG allows us to be free from the concept of UV completion.
- Wilson's RG serves as a nonperturbative way of RG transformation. ➔ Numerical RG (NRG) & Functional RG
- Hamiltonian formulation for the RG transformation is possible. ➔ NRG & dual holography (QEC)
- Applications in condensed matter physics, statistical mechanics, and etc. in addition to particle physics



https://en.wikipedia.org/wiki/Kenneth_G._Wilson

quantum mechanics — spacetime — gravity

$$W = \int_{k<\Lambda} [Dg][DA][D\psi][D\Phi] \exp\left\{ i \int d^4x \sqrt{-g} \left[\frac{m_p^2}{2} R - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\bar{\psi}^i \gamma^\mu D_\mu \psi^i + \left(\bar{\psi}^i_L V_{ij} \Phi \psi^j_R + \text{h.c.} \right) - |D_\mu \Phi|^2 - V(\Phi) \right] \right\}$$

other forces — matter — Higgs

1982 Nobel Prize in Physics for his work on phase transitions—illuminating the subtle essence of phenomena like melting ice and emerging magnetism. It was embodied in his fundamental work on the renormalization group.

Wilson's renormalizatio

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Unité Mixte de Recherche 85
Scientifique et de
brez

A personal and subjective recolle
delivered over the spring of 1972 a
at Cornell University on Novemb
Kenneth G. Wilson conference).

1 Why was it so d viewpoint?

In 1971-72 I was on leave from Sa
physics department of Princeton U
mainly field theory. Just before th
the use of classical approximatio
pair creation by an oscillating el
the understanding of the intrigui
near a critical point was a difficu
was invited in the fall of 1971 to g

"Renormalization group and Kadanoff scaling picture [1] , Phase space cell analysis of critical behavior[2]" but Ken said that he could not explain his ideas in one talk. The University (thanks to David Gross I think) had the good idea of inviting him to talk as much as he needed and Ken ended up giving 15 lectures in the spring term of 1972, which resulted in the well-known 1974 Physics Reports " The renormalization group and the ϵ-expansion " by Wilson and Kogut (who had been taking notes throughout the lectures), one of the most influential articles of the last decades [3].

Wilson's ideas were so different from the standard views at the time, that they were not easy to accept. His style was also completely different : for him field theory was not a set of abstract axioms but a tool to perform practical calculations even at the expense of oversimplified models whose relationship with reality was tenuous. Let us try to understand why Ken's ideas, nowadays so inbred in the training of any young physicist that they seem nearly obvious, were so difficult to accept. Let us first review the conventional wisdom at the time.

- Renormalization was a technique to remove perturbatively the ultraviolet singularities for those theories which were known to be renormalizable, i.e. theories for which this removal could be done at the expense of a finite number of parameters.

- Why should one consider such theories? Presumably because they were the only ones for which practical calculations are conceivable. For instance in QED, the minimal replacement $p \to p - eA$ provides such a theory but gauge invariance alone would allow for more couplings, such as a spin current coupled to the field strength, which would ruin the renormalizability.

- As a result one obtained with renormalized QED a theory which, a priori, was potentially valid at all distances from astronomical scales down to vanishingly small ones.

- The renormalization group in its conventional formulation was applicable only to renormalizable theories ; it allowed one to understand leading logarithms, at a given order in perturbation theory, from lower orders.

So what is it which was surprising in Wilson's approach?

- Critical points such as liquid-vapor, Curie points in magnets, Ising models, etc., are a priori classical problems in statistical mechanics. Why would a quantum field theory be relevant?

- Why renormalization in a theory in which there is a momentum cut-off $\Lambda = a^{-1}$, in which a is a physical short distance such as a lattice spacing, or a typical interatomic distance, and there is no reason to let a go to zero?

- The momentum space reduction of the number of degrees of freedom that Wilson used introduced singularities in the Hamiltonian generated by the flow. It was not quite clear that the procedure could be systematized.

- Irrelevance of most couplings was of course a fundamental piece of the theory but it looked a priori purely dimensional. For instance if one adds to a $g_4\varphi^4$ theory in four dimensions a coupling $(g_6/\Lambda^2)\varphi^6$ one obtains a flow equation for the dimensionless g_6 which concludes at its irrelevance as if the $1/\Lambda^2$ sufficed :

$$\Lambda\frac{\partial}{\partial\Lambda}g_6 - 2g_6 = \beta_6(g_4, g_6)$$

  However the inclusion of the g_6/Λ^2 into a Feynman diagram with only g_4 vertices produces immediately an extra Λ^2 which cancels the previous one. So why is g_6 irrelevant?

- Although it is now physically very clear, it was not easy to understand that the critical surface had codimension two for an ordinary critical point, in others words that there are two and only two relevant operators in the large space of allowed coupling parameters.

- Even more difficult to take was the statement that four-dimensional renormalizable theories such as φ^4 or QED, i.e. non asymptotically free theories (an anachronistic name), if extended to all energy scales could only be free fields. How could one believe this, given the extraordinary agreement of QED with experiment?

I must admit that my initial difficulties at grasping the views exposed in Ken's lectures led me, in a friendly collaboration with David Wallace, to check their consistency. Amazed to see that the theory defeated our skepticism, we ended up working out with Ken the critical equation of state [4].

ORIGINAL PAPER

On the Wilsonian Meaning of Quantum Error Co

César Gómez

We sketch a recipe to define renormalizatio
Kadanoff-Wilson block packing using a quar
such a case the RG transformations of the c
error matrix of the QEC code. In order to de
couplings we use Weinberg's sum rule for a

PHYSICAL REVIEW D

Renormalization group and ap

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We show that in renormalization group (RG) flow th
approximately protected against the local short-dista
connection is, we consider three examples: the classica
scalar quantum field theory (QFT) in two spacetime dime
of strongly coupled systems. As a concrete example of r
multiscale renormalization ansatz for massive free fields
approximately protected from the quantum errors c
operators. In holographic RG flows, we study the phase
region and argue that one needs to define the price a
reconstructable wedge.

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PTEP

Quantum Error Correction I
Renormalization Group in Sc

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We demonstrate that quantum error correc
scalar field theories. We con
By acting on them the invers
group flow of the ground st
situations in which the Knil
herent states. We verify this t
a general relationship betw
and should give insights int
correspondence.

Subject Index B30, B32

PHYSICAL REVIEW X 14, 031044 (2024)

Mixed-State Quantum Phases: Renormalization and Quantum Error Correction

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Open system quantum dynamics can generate a variety of long-range entangled mixed states, yet it has been unclear in what sense they constitute phases of matter. To establish that two mixed states are in the same phase, as defined by their two-way connectivity via local quantum channels, we use the renormalization group (RG) and decoders of quantum error correcting codes. We introduce a real-space RG scheme for mixed states based on local channels which ideally preserve correlations with the complementary system, and we prove this is equivalent to the reversibility of the channel's action. As an application, we demonstrate an exact RG flow of finite temperature toric code in two dimensions to infinite temperature, thus proving it is in the trivial phase. In contrast, for toric code subject to local dephasing, we establish a mixed-state toric code phase using local channels obtained by truncating an RG-type decoder and the minimum weight perfect matching decoder. We also discover a precise relation between mixed-state phase and decodability, by proving that local noise acting on toric code cannot destroy logical information without bringing the state out of the toric code phase.

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Quantum Error Correction For Dummies

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Nonperturbative generalization of the Wilsonian RG theoretical framework → Path integral reformulation of the exact functional RG differential equation

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PAPER

Bayesian renormalization

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Keywords: renormalization, Bayesian inference, diffusion learning, information geometry, data compression, Fisher metric

Abstract

In this note we present a fully information theoretic approach to renormalization inspired Bayesian statistical inference, which we refer to as Bayesian renormalization. The main ins Bayesian renormalization is that the Fisher metric defines a correlation length that plays th an emergent renormalization group (RG) scale quantifying the distinguishability between points in the space of probability distributions. This RG scale can be interpreted as a proxy maximum number of unique observations that can be made about a given system during a statistical inference experiment. The role of the Bayesian renormalization scheme is subsec to prepare an effective model for a given system up to a precision which is bounded by the aforementioned scale. In applications of Bayesian renormalization to physical systems, the emergent information theoretic scale is naturally identified with the maximum energy tha probed by current experimental apparatus, and thus Bayesian renormalization coincides w ordinary renormalization. However, Bayesian renormalization is sufficiently general to app in circumstances in which an immediate physical scale is absent, and thus provides an idea approach to renormalization in data science contexts. To this end, we provide insight into Bayesian renormalization scheme relates to existing methods for data compression and dat generation such as the information bottleneck and the diffusion learning paradigm. We co by designing an explicit form of Bayesian renormalization inspired by Wilson's momentun renormalization scheme in quantum field theory. We apply this Bayesian renormalization to a simple neural network and verify the sense in which it organizes the parameters of the according to a hierarchy of information theoretic importance.

entropy MDPI

Article

The Inverse of Exact Renormalization Group Flows as Statistical Inference

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Abstract: We build on the view of the Exact Renormalization Group (ERG) as an instantiation of Optimal Transport described by a functional convection–diffusion equation. We provide a new information-theoretic perspective for understanding the ERG through the intermediary of Bayesian Statistical Inference. This connection is facilitated by the Dynamical Bayesian Inference scheme, which encodes Bayesian inference in the form of a one-parameter family of probability distributions solving an integro-differential equation derived from Bayes' law. In this note, we demonstrate how the Dynamical Bayesian Inference equation is, itself, equivalent to a diffusion equation, which we dub *Bayesian Diffusion*. By identifying the features that define Bayesian Diffusion and mapping them onto the features that define the ERG, we obtain a dictionary outlining how renormalization can be understood as the inverse of statistical inference.

Keywords: Bayesian Inference; Exact Renormalization; Renormalization Group; diffusion; diffusion learning; Stochastic Differential Equations; Fisher Information; Information Geometry; entropy; relative entropy; gradient flow; error correction; channels

In Polchinski's picture, $K_\Lambda(p^2)$ has a prescribed dependence on Λ, thus Polchinski's ERG equation arises by determining the equation which must be obeyed by $S_{\text{int},\Lambda}[\phi]$ in order to satisfy the principle (2). By a straightforward computation, one can show that the resulting equation can be put into the form:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}P_\Lambda[\phi] = \int_{M\times M}\mathrm{d}^d x\mathrm{d}^d y\left\{C_\Lambda^{\text{Pol.}}(x,y)\frac{\delta^2 P_\Lambda[\phi]}{\delta\phi(x)\delta\phi(y)} + \frac{\delta}{\delta\phi(x)}\left(P_\Lambda[\phi]C_\Lambda^{\text{Pol.}}(x,y)\frac{\delta V_\Lambda^{\text{Pol.}}[\phi]}{\delta\phi(y)}\right)\right\} \tag{4}$$

$$\equiv \Delta P_\Lambda[\phi] + \mathrm{div}\left(P_\Lambda[\phi]\,\mathrm{grad}_{C_\Lambda^{\text{Pol.}}}V_\Lambda^{\text{Pol.}}[\phi]\right), \tag{5}$$

where

$$C_\Lambda^{\text{Pol.}}(p^2) = (2\pi)^d G(p^2)^{-1}\frac{\partial K_\Lambda(p^2)}{\partial\ln\Lambda}; \quad V_\Lambda^{\text{Pol.}}[\phi] = \int\frac{\mathrm{d}^d p}{(2\pi)^d}\phi(p)G(p^2)K_\Lambda^{-1}(p^2)\phi(-p). \tag{6}$$

One might reco

governed by the

renormalization

the functional (i

functional version of Fokker–Plank.

RG flow is Markovian.

Specializing to Fokker–Planck ERG schemes, we can expand on this discussion. As was introduced in detail in [18], a (functional) Fokker–Planck equation of the form (4) is associated with a (functional) stochastic differential equation (SDE):

$$\mathrm{d}\phi(x) = -\mathrm{grad}_{C_\Lambda}V_\Lambda[\phi](d\ln\Lambda) + \sqrt{2}\int_M \mathrm{d}^d y\,\sigma_\Lambda(x,y)\,\mathrm{d}W_\Lambda(y). \tag{13}$$

Here, $W_\Lambda(x)$ is a function valued Weiner process, and σ_Λ is the diffusivity kernel defined by the property that it 'squares' to the covariance C_Λ:

$$\int_M \mathrm{d}^d z\,\sigma_\Lambda(x,z)\,\sigma_\Lambda(z,y) = C_\Lambda(x,y). \tag{14}$$

The Fokker–Planck equation corresponds to a bonafide ERG because it satisfies the ERG principle (2). To see that this is the case, let us now show that we can rewrite (4) in the form

Conservation law

$$-\frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}P_\Lambda[\phi] = \int_M \mathrm{d}^d x\, \frac{\delta}{\delta\phi(x)}\left(\Psi_\Lambda[\phi;x]\, P_\Lambda[\phi]\right), \tag{7}$$

where M is the spacetime manifold on which the theory is defined [12]. Hopefully it is clear that any one parameter family $P_\Lambda[\phi]$ satisfying (7) also satisfies (2). This is because (7) specifies a *divergence flow*, that is the right hand side of (7) is a divergence in the space of field configurations. We can therefore employ the divergence theorem to observe that

$$\frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}\int_{\mathcal{F}} \mathcal{D}\phi\; P_\Lambda[\phi] = -\int_{\mathcal{F}} \mathcal{D}\phi \int_M \mathrm{d}^d x\, \frac{\delta}{\delta\phi(x)}\left(\Psi_\Lambda[\phi;x]\, P_\Lambda[\phi]\right) = 0. \tag{8}$$

Conserved current for the RG transformation

In order to write (4) in the form (7) we take

Gradient flow

$$\Psi_\Lambda[\phi;x] = \int_M \mathrm{d}^d y\, C_\Lambda(x,y)\, \frac{\delta\Sigma_\Lambda[\phi;P_\Lambda]}{\delta\phi(y)}, \tag{9}$$

as has appeared previously in [12, 16, 17, 41, 45]. Here $C_\Lambda(x,y)$ is the *ERG kernel* appearing in the Fokker–Planck equation associated to the ERG, and $\Sigma_\Lambda[\phi;P_\Lambda]$ is called the *scheme functional* which is determined through the *ERG potential* V_Λ via the equation

$$\Sigma_\Lambda[\phi;P_\Lambda] = -\ln\left(\frac{P_\Lambda[\phi]}{\mathrm{e}^{-V_\Lambda[\phi]}}\right) = S_\Lambda[\phi] - V_\Lambda[\phi].$$

Relative entropy functional
= Kullback – Leibler divergence
= Fisher information

Plugging (9) back into (7), we reconcile (4) with the diffusion and drift aspects given by (C_Λ, V_Λ), as desired. Together (C_Λ, V_Λ) therefore specify a consistent scheme for regulating the high energy degrees of freedom of the field theory, in analogy with the regulating function $K_\Lambda^{-1}(p^2)$ appearing in (3).

PRL 95, 040602 (2005) PHYSICAL REVIEW LETTERS week ending 22 JULY 2005

Entropy Production along a Stochastic Trajectory and an Integral Fluctuation Theorem

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(Received 29 March 2005; published 20 July 2005)

For stochastic nonequilibrium dynamics like a Langevin equation for a colloidal particle or a master equation for discrete states, entropy production along a single trajectory is studied. It involves both genuine particle entropy and entropy production in the surrounding medium. The integrated sum of both

Reinterpretation & generalization of c − & a − theorem
for the monotonicity of the RG flow
in the nonequilibrium thermodynamics perspectives:
The gradient flow gives rise to the monotonicity of entropy.

- The path integral formulation of the functional RG equation gives rise to a non-perturbative theoretical framework, which can be identified with emergent dual holography.
- The fundamental property of the RG flow is governed by unitarity & KMS (Kubo-Martin-Schwinger) symmetry, which can be translated into N = 2 BRST symmetries. In other words, the Ward identities from these symmetries allow us to introduce a "c-function" or "relative entropy", which shows monotonicity.
- We discuss the monotonicity of the RG flow (c theorem in 2d & a theorem in 4d) in the nonequilibrium thermodynamics perspectives, i.e., the Fokker-Planck form of the FRG equation.

Brute force derivation

JHEP

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Entanglement transfer from quantum matter to classical geometry in an emergent holographic dual description of a scalar field theory

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ABSTRACT: Applying recursive renormalization group transformations to a scalar field theory, we obtain an effective quantum gravity theory with an emergent extra dimension, described by a dual holographic Einstein-Klein-Gordon type action. Here, the dynamics of both the dual order-parameter field and the metric tensor field originate from density-density and energy-momentum tensor-tensor effective interactions, respectively, in the recursive renormalization group transformation, performed approximately in the Gaussian level. This linear approximation in the recursive renormalization group transformation for the gravity sector gives rise to a linearized quantum Einstein-scalar theory along the z-directional emergent space. In the large N limit, where N is the flavor number of the original scalar fields, quantum fluctuations of both dynamical metric and dual scalar fields are suppressed, leading to a classical field theory of the Einstein-scalar type in $(D+1)$-spacetime dimensions. We show that this emergent background gravity describes the renormalization group flows of coupling functions in the UV quantum field theory through the extra dimension. More precisely, the IR boundary conditions of the gravity equations correspond to the renormalization group β-functions of the quantum field theory, where the infinitesimal distance in the extra-dimensional space is identified with an energy scale for the renormalization group transformation. Finally, we also show that this dual holographic formulation describes quantum entanglement in a geometrical way, encoding the transfer of quantum entanglement from quantum matter to classical gravity in the large N limit. We claim that this entanglement transfer serves as a microscopic foundation for the emergent holographic duality description.

KEYWORDS: AdS-CFT Correspondence, Renormalization Group, Holography and condensed matter physics (AdS/CMT), Resummation

ARXIV EPRINT: 2003.00165

1. Introduction of RG flows

2. UV & IR boundary conditions

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$$Z = \int D\phi_\alpha(x) \exp\Big\{ - S_{\rm UV}[\phi_\alpha(x); g^B_{\mu\nu}(x)]\Big\} \tag{2.1}$$

and the effective action is

$$S_{\rm UV}[\phi_\alpha(x); g^B_{\mu\nu}(x)] = \int d^D x \sqrt{g_B} \Big\{ g_B^{\mu\nu}(\partial_\mu \phi_\alpha)(\partial_\nu \phi_\alpha) + m^2 \phi_\alpha^2 + \xi R_B \phi_\alpha^2 + \boxed{\frac{u}{2N}\phi_\alpha^2 \phi_\beta^2} + \boxed{\frac{\lambda}{2N} T_{\mu\nu} T^{\mu\nu}} \Big\}. \tag{2.2}$$

$$Z = Z_\Lambda \int D\varphi(x,z) D g_{\mu\nu}(x,z) \exp\Big\{ - S_{\rm UV}[g_{\mu\nu}(x,0), \varphi(x,0)] - S_{\rm IR}[g_{\mu\nu}(x,z_f), \varphi(x,z_f)] - S_{\rm Bulk}[g_{\mu\nu}(x,z), \varphi(x,z)]\Big\}. \tag{2.4}$$

$$\varphi \leftrightarrow \phi_\alpha^2 \quad \& \quad g_{\mu\nu} \leftrightarrow T^{\mu\nu}$$

$$e^{-\int d^D x \sqrt{g_B}\, \frac{u}{2N}\phi_\alpha^2 \phi_\beta^2} = \int D\varphi\, e^{-\int d^D x \sqrt{g_B} \left(\frac{N}{2u}\varphi^2 - i\varphi \phi_\alpha^2\right)}$$

$$\varphi \to \boxed{\varphi^{(0)}} + \delta\varphi, \qquad g_{\mu\nu} \to \boxed{g_{\mu\nu}^{(0)}} + \delta g_{\mu\nu}, \qquad \phi_\alpha \to \phi_\alpha + \delta\phi_\alpha$$

$$\int D\delta\varphi \to \phi_\alpha^2\phi_\beta^2 \to \varphi^{(1)}, \qquad \int D\delta g_{\mu\nu} \to T_{\mu\nu}^2 \to g_{\mu\nu}^{(1)}, \qquad \int D\delta\phi_\alpha \to V_{eff}[\varphi^{(0)}, g_{\mu\nu}^{(0)}]$$

Heat kernel calculation

$$\varphi^{(1)} \to \boxed{\varphi^{(1)}} + \delta\varphi, \qquad g_{\mu\nu}^{(1)} \to \boxed{g_{\mu\nu}^{(1)}} + \delta g_{\mu\nu}, \qquad \phi_\alpha \to \phi_\alpha + \delta\phi_\alpha$$

$$\int D\delta\varphi \to \phi_\alpha^2\phi_\beta^2 \to \varphi^{(2)}, \qquad \int D\delta g_{\mu\nu} \to T_{\mu\nu}^2 \to g_{\mu\nu}^{(2)}, \qquad \int D\delta\phi_\alpha \to V_{eff}[\varphi^{(1)}, g_{\mu\nu}^{(1)}]$$

Heat kernel calculation

Just doing Gaussian integral
in the presence of general background fields
$\varphi^{(k-1)}$ & $g_{\mu\nu}^{(k-1)}$ *for every RG step* (k)

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Entanglement transfer from quantum matter to classical geometry in an emergent holographic dual description of a scalar field theory

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Abstract: Applying recursive renormalization group transformations to a scalar field theory, we obtain an effective quantum gravity theory with an emergent extra dimension, described by a dual holographic Einstein-Klein-Gordon type action. Here, the dynamics of both the dual order-parameter field and the metric tensor field originate from density-density and energy-momentum tensor-tensor effective interactions, respectively, in the recursive renormalization group transformation, performed approximately in the Gaussian level. This linear approximation in the recursive renormalization group transformation

$$dz \sum_{k=1}^{f} \frac{g_{\mu\nu}^{(k)}(x) - g_{\mu\nu}^{(k-1)}(x)}{dz[k-(k-1)]} \to \int_0^{z_f} dz\, \partial_z g_{\mu\nu}(x,z)$$

group flows of coupling functions in the UV quantum field theory through the extra dimension. More precisely, the IR boundary conditions of the gravity equations correspond to the renormalization group β-functions of the quantum field theory, where the infinitesimal distance in the extra-dimensional space is identified with an energy scale for the renormalization group transformation. Finally, we also show that this dual holographic formulation describes quantum entanglement in a geometrical way, encoding the transfer of quantum entanglement from quantum matter to classical gravity in the large N limit. We claim that this entanglement transfer serves as a microscopic foundation for the emergent holographic duality description.

Keywords: AdS-CFT Correspondence, Renormalization Group, Holography and condensed matter physics (AdS/CMT), Resummation

ArXiv ePrint: 2003.00165

1. Introduction of RG flows

2. UV & IR boundary conditions

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$$Z = \int D\phi_\alpha(x) \exp\left\{ - S_{\mathrm{UV}}[\phi_\alpha(x); g_{\mu\nu}^B(x)]\right\} \tag{2.1}$$

and the effective action is

$$S_{\mathrm{UV}}[\phi_\alpha(x); g_{\mu\nu}^B(x)] = \int d^D x \sqrt{g_B}\left\{ g_B^{\mu\nu}(\partial_\mu\phi_\alpha)(\partial_\nu\phi_\alpha) + m^2\phi_\alpha^2 + \xi R_B \phi_\alpha^2 + \frac{u}{2N}\phi_\alpha^2\phi_\beta^2 + \frac{\lambda}{2N}T_{\mu\nu}T^{\mu\nu}\right\}. \tag{2.2}$$

$$Z = Z_\Lambda \int D\varphi(x,z) Dg_{\mu\nu}(x,z) \exp\left\{ - S_{\mathrm{UV}}[g_{\mu\nu}(x,0), \varphi(x,0)] - S_{\mathrm{IR}}[g_{\mu\nu}(x,z_f), \varphi(x,z_f)] - S_{\mathrm{Bulk}}[g_{\mu\nu}(x,z), \varphi(x,z)]\right\}. \tag{2.4}$$

$$\varphi \leftrightarrow \phi_\alpha^2 \quad \& \quad g_{\mu\nu} \leftrightarrow T^{\mu\nu}$$

Heat kernel calculation

$$\begin{aligned} S_{\mathrm{Bulk}}[g_{\mu\nu}(x,z), \varphi(x,z)] = N \int_0^{z_f} dz \int d^D x \sqrt{g(x,z)}\Big\{ & + \frac{1}{2u}[\partial_z\varphi(x,z)]^2 + \frac{\mathcal{C}_\varphi}{2} g^{\mu\nu}(x,z)[\partial_\mu\varphi(x,z)][\partial_\nu\varphi(x,z)] + \mathcal{C}_\xi R(x,z)[\varphi(x,z)]^2 \\ & - \frac{1}{2\lambda}\Big(\partial_z g^{\mu\nu}(x,z) - g^{\mu\nu'}(x,z)(\partial_{\nu'}\partial_{\mu'} G_{xx'}[g_{\mu\nu}(x,z),\varphi(x,z)])_{x'\to x} g^{\mu'\nu}(x,z)\Big)^2 \\ & + \frac{1}{2\kappa}\Big(R(x,z) - 2\Lambda\Big)\Big\}. \end{aligned} \tag{2.5}$$

RG − improved Ginzburg − Landau theory on emergent curved spacetime with an extradimension

$$S_{\mathrm{IR}}[g_{\mu\nu}(x,z_f), \varphi(x,z_f)] = N \int d^D x \sqrt{g(x,z_f)}\left\{ \frac{\mathcal{C}_\varphi^f}{2} g^{\mu\nu}(x,z_f)[\partial_\mu\varphi(x,z_f)][\partial_\nu\varphi(x,z_f)] + \mathcal{C}_\xi^f R(x,z_f)[\varphi(x,z_f)]^2 + \frac{1}{2\kappa_f}\Big(R(x,z_f) - 2\Lambda_f\Big)\right\}, \tag{2.8}$$

$$Z=\int D\phi_\alpha(\tau,r)e^{-\int_0^\beta d\tau\int d^dr\left\{\phi_\alpha(\tau,r)\left(-\partial_\tau^2-v^2\partial^2+m^2\right)\phi_\alpha(\tau,r)+\frac{u}{2N}[\phi_\alpha(\tau,r)]^2[\phi_\beta(\tau,r)]^2\right\}}$$

After the RG transformation

$$Z=\int D\phi_\alpha(\tau,r)e^{-\int_0^\beta d\tau\int d^dr\left\{\phi_\alpha(\tau,r)\left(-Z_\phi\partial_\tau^2-Z_{v^2}v^2\partial^2+Z_{m^2}m^2\right)\phi_\alpha(\tau,r)+Z_u\frac{u}{2N}[\phi_\alpha(\tau,r)]^2[\phi_\beta(\tau,r)]^2\right\}}$$

These RG coefficients are determined in a nonperturbative way as follows:

$$Z[g^{(d+1)}(\tau,r,0),\varphi(\tau,r,0)]=$$

$$\int Dg_{AB}^{(d+2)}(\tau,r,z)D\varphi(\tau,r,z)\;e^{-N\int_0^{z_f}dz\int_0^\beta d\tau\int d^dr\sqrt{g^{(d+2)}(\tau,r,z)}\left\{\frac{1}{2\kappa}\left(R^{(d+2)}(\tau,r,z)-2\Lambda\right)\right\}}$$

$$\times\, e^{-N\int_0^{z_f}dz\int_0^\beta d\tau\int d^dr\sqrt{g^{(d+2)}(\tau,r,z)}\left\{\begin{matrix}g^{(d+2)AB}(\tau,r,z)[\partial_A\varphi(\tau,r,z)][\partial_B\varphi(\tau,r,z)]\\+M^2[\varphi(\tau,r,z)]^2+\xi R^{(d+2)}(\tau,r,z)[\varphi(\tau,r,z)]^2\end{matrix}\right\}}$$

$$\times\int Dg_{\mu\nu}^{(d+1)}(\tau,r,z_f)D\varphi(\tau,r,z_f)D\phi_\alpha(\tau,r)\;e^{-\int_0^\beta d\tau\int d^dr\sqrt{g^{(d+1)}(\tau,r,z_f)}\left\{\begin{matrix}g^{(d+1)\mu\nu}(\tau,r,z_f)[\partial_\mu\phi_\alpha(\tau,r)][\partial_\nu\phi_\alpha(\tau,r)]\\+(m^2-i\varphi(\tau,r,z_f))[\phi_\alpha(\tau,r)]^2+\xi_fR^{(d+1)}(\tau,r,z_f)[\phi_\alpha(\tau,r)]^2\end{matrix}\right\}}$$

As a result, the first-order RG flow differential equations at UV are promoted to be the second-order ones with UV & IR boundary conditions in the extremized RG path, self-consistently determined by the theoretical framework itself.

So what?

Claim: Field theoretic O(N), O(1), O(1/N), O(1/N^2), … quantum corrections are resumed and reorganized to form a holographic dual effective field theory in the large N limit.

$$\mathit{NOT\ PROVEN\ RIGOROUSLY\ YET}$$
$$=$$
$$\mathit{Corresponding\ Feynman\ diagrams\ are\ not\ identified.}$$

Emergent holographic description for the Kondo effect: Comparison with Bethe ansatz

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Implementing Wilsonian renormalization group transformations in an iterative way, we develop a nonperturbative field theoretical framework for strongly coupled quantum theories, which takes into account all-loop quantum corrections organized in the $1/N$ expansion. Here, N represents the flavor number of strongly correlated quantum fields. The resulting classical field theory is given by an effective Landau-Ginzburg theory for a local order parameter field, which appears in one-dimensional higher spacetime. We confirm the nonperturbative nature of this field theoretical framework for the Kondo effect. Intriguingly, we show that the recursive Wilsonian renormalization group method can explain nonperturbative thermodynamic properties of an impurity, consistent with Bethe ansatz for the whole temperature region.

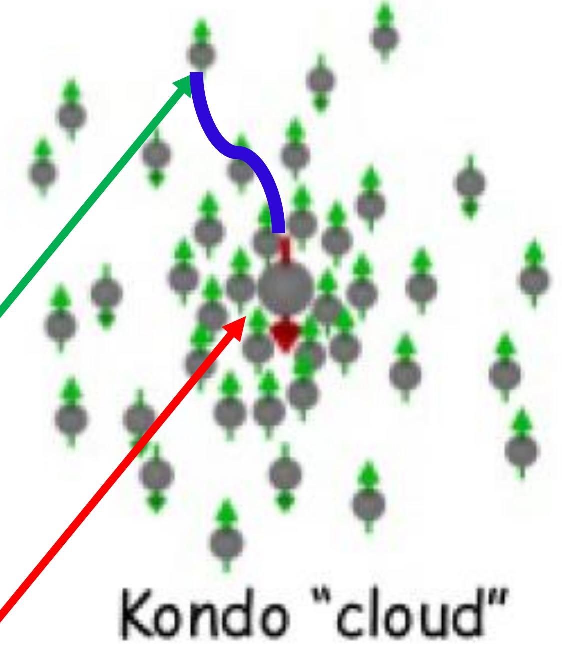
DOI: 10.1103/PhysRevD.99.105012

An effective Hamiltonian at UV

$$Z = \int Dc_\sigma(\boldsymbol{k},\tau) D\boldsymbol{S}(\tau) \exp\Big[- S_B[\boldsymbol{S}(\tau)]$$

$$- \int_0^\beta d\tau \Big\{ \int \frac{d^d\boldsymbol{k}}{(2\pi)^d} c_\sigma^\dagger(\boldsymbol{k},\tau) \Big(\partial_\tau - \mu + \frac{\boldsymbol{k}^2}{2m} \Big) c_\sigma(\boldsymbol{k},\tau)$$

$$+ \frac{J_K}{N} \int \frac{d^d\boldsymbol{k}}{(2\pi)^d} \int \frac{d^d\boldsymbol{k}'}{(2\pi)^d} c_\alpha^\dagger(\boldsymbol{k},\tau) \boldsymbol{\sigma}_{\alpha\beta} c_\beta(\boldsymbol{k}',\tau) \cdot \boldsymbol{S}(\tau) \Big\} \Big]$$



$$\boldsymbol{S}(\tau) = \frac{1}{2} f_\alpha^\dagger(\tau) \boldsymbol{\sigma}_{\alpha\beta} f_\beta(\tau) \qquad f_\sigma^\dagger(\tau) f_\sigma(\tau) = NS$$

$$Z = \int Dc_\sigma(\boldsymbol{k},\tau) Df_\sigma(\tau) Db(\tau) D\lambda(\tau) e^{-S},$$

$$S = \int_0^\beta d\tau \Big\{ \int \frac{d^d\boldsymbol{k}}{(2\pi)^d} c_\sigma^\dagger(\boldsymbol{k},\tau)\left(\partial_\tau - \mu + \frac{\boldsymbol{k}^2}{2m}\right) c_\sigma(\boldsymbol{k},\tau) + f_\sigma^\dagger(\tau)(\partial_\tau - i\lambda(\tau)) f_\sigma(\tau) + iNS\lambda(\tau)$$
$$- b(\tau) f_\sigma^\dagger(\tau) c_\sigma(\tau) - b^\dagger(\tau) c_\sigma^\dagger(\tau) f_\sigma(\tau) + \frac{N}{J_K} b^\dagger(\tau) b(\tau) \Big\}. \qquad b(\tau) = \frac{J_K}{N}\left\langle c_\sigma^\dagger(\tau) f_\sigma(\tau) \right\rangle$$

$$Z = Z_c Z_h^f \int Df_\sigma(\tau) Db(\tau,z) \exp\Big[- \int_0^\beta d\tau \Big\{ \int_0^\beta d\tau' f_\sigma^\dagger(\tau) b(\tau,z_f) G_c(\tau-\tau') b^\dagger(\tau',z_f) f_\sigma(\tau') + f_\sigma^\dagger(\tau)(\partial_\tau - \lambda) f_\sigma(\tau)$$
$$+ NS\lambda + \frac{N}{J_K} b^\dagger(\tau,0) b(\tau,0) \Big\} - \qquad \Big\} \Big].$$

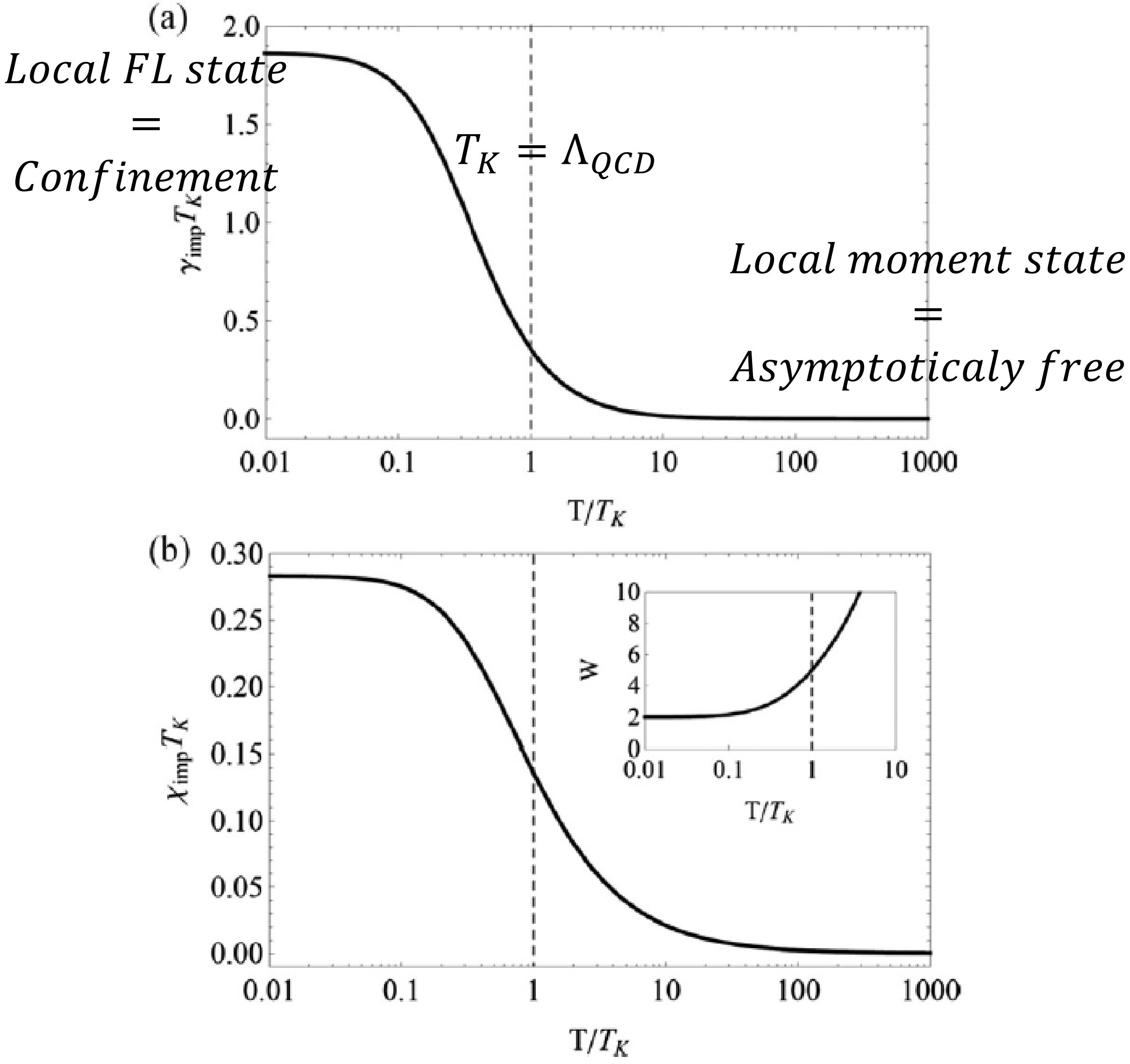


FIG. 1. (a) Linear-log plot of the impurity specific heat coefficient [Eq. (16)] as a function of temperature. (b) Linear-log plot of the impurity spin susceptibility [Eq. (17)] as a function of temperature. The Wilson ratio is plotted in the inset. The vertical dotted line denotes $T/T_K = 1$.

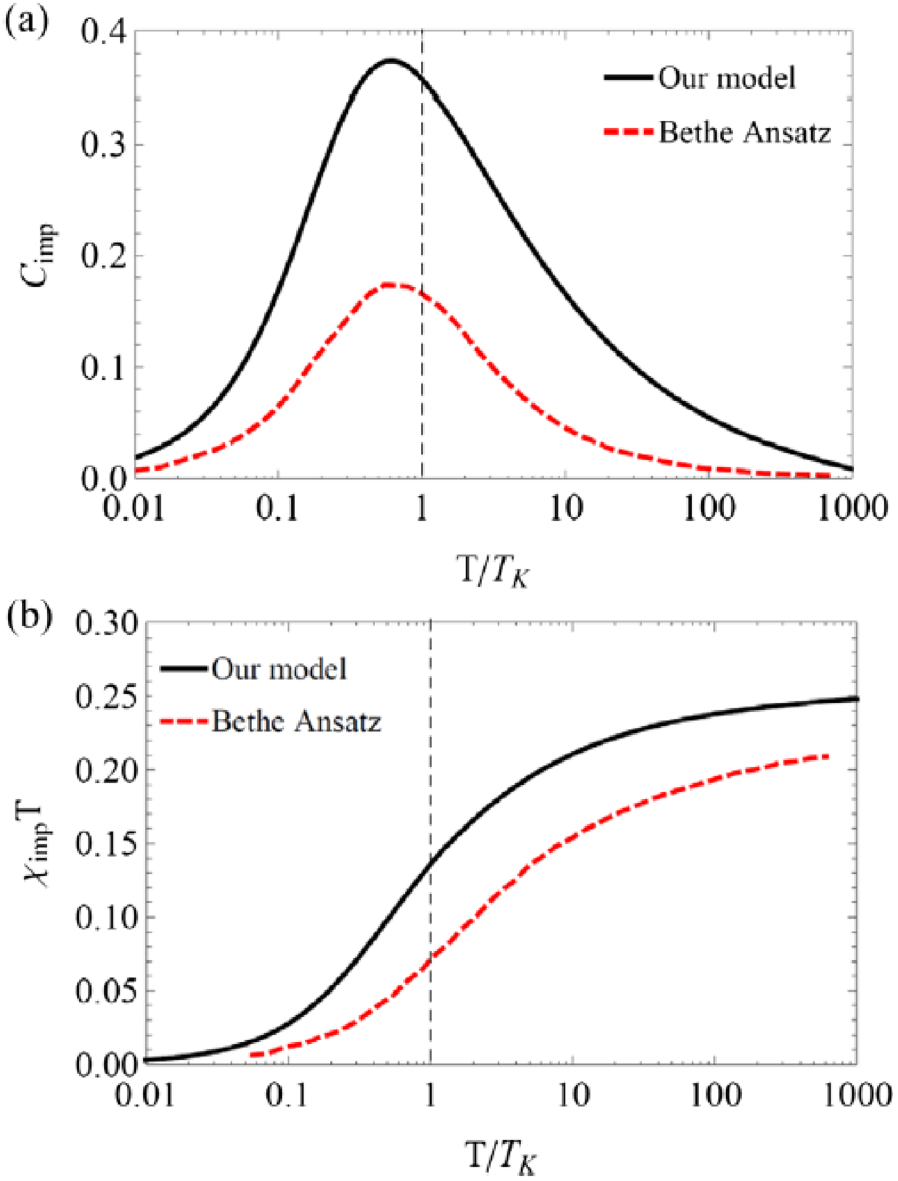


FIG. 2. (a) Linear-log plot of the impurity specific heat as a function of temperature. (b) Linear-log plot of the impurity spin susceptibility multiplied by temperature T. Solid black (red dotted) lines are our (Bethe ansatz) results.

Is there a general prescription beyond explicit and detailed implementations of the Wilsonian RG transformations? More rigorously, can I put the RG transformation into the framework of a topological quantum field theory to manifest the mathematical structure of RG? This construction would correspond to the path integral formulation of the Fokker-Planck type FRG equation.

Key aspects of the derivation or construction

- Local approximation for the Wilsonian effective action and the RG flows
- "One loop" level in every step of the RG transformations (cf. FRG looks like one-loop RG transformation. Of course, it's not.)
- RG flows as a gradient flow of an effective potential (cf. can be translated into the WZ consistency condition for the Weyl anomaly in the local RG equation)
- Non-perturbative in nature: Resumed in the all-loop order of the action level
- A cohomological type topological field theory construction

A cohomological-type topological field theory construction a la Witten

cliché

Physics Reports (Review Section of Physics Letters) 209, Nos. 4 & 5 (1991) 129–340. Nort

TOPOLOGICAL FIELD THEORY

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Contents:

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SUPERSYMMETRY AND MORSE THEORY

EDWARD WITTEN

Abstract

It is shown that the Morse inequalities can be obtained by consideration of a certain supersymmetric quantum mechanics Hamiltonian. Some of the implications of modern ideas in mathematics for supersymmetric theories are discussed.

Communications in Mathematical Physics
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Topological Quantum Field Theory

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Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

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Random Magnetic Fields, Supersymmetry, and Negative Dimensions

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(Received 26 June 1979)

We prove the equivalence, near the critical point, of a D-dimensional spin system in a random external magnetic field with a $(D-2)$-dimensional spin system in the absence of a magnetic field. This is due to the hidden supersymmetry of the associated stochastic
uting coordinate with a space

Z. Phys. B – Condensed Matter 57, 249–253 (1984)

cliché

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Zeitschrift für Physik B

Ward Takahashi Identities and Fluctuation-Dissipation Theorem in a Superspace Formulation of the Langevin Equation

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J. Phys. A: Math. Theor. **44** (2011) 095002 (19pp) doi:10.1088/1751-8113/44/9/095002

A field-theoretic approach to non-equilibrium work identities

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Abstract
We study non-equilibrium work relations for a space
stochastic dynamics (model A). Jarzynski's equality
symmetries of the dynamical action in the path-integra
derive a set of exact identities that generalize the
relations to non-stationary and far-from-equilibrium situa
are prone to experimental verification. Furthermore,
studied invariance of the Langevin equation under sup
known to be broken when the external potential is ti
partially restored by adding to the action a term which i
work. The work identities can then be retrieved as
associated Ward–Takahashi identities.

PHYSICAL REVIEW E **98**, 042140 (2018)

Symmetries in the path integral formulation of the Langevin d

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We study dissipative Langevin dynamics in the path integral formulation using the M
formalism. The effective action is supersymmetric and we identify the supercharges. In addi
transformations generated by superderivatives, which were recently included in the cohom
emerging in the dissipative systems. We find that these transformations do not generate Ward
are explicitly broken; however, they lead to universal sum-rule-type identities, which we derive
Dyson equations. We confirm that the above identities hold in an explicit example of the O
process.

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entropy MDPI

Review

Introduction to Supersymmetric Theory of Stochastics

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Abstract: Many natural and engineered dynamical systems, including all living objects, exhibit signatures of what can be called spontaneous dynamical long-range order (DLRO). This order's omnipresence has long been recognized by the scientific community, as evidenced by a myriad of related concepts, theoretical and phenomenological frameworks, and experimental phenomena such as turbulence, $1/f$ noise, dynamical complexity, chaos and the butterfly effect, the Richter scale for earthquakes and the scale-free statistics of other sudden processes, self-organization and pattern formation, self-organized criticality, *etc.* Although several successful approaches to various realizations of DLRO have been established, the universal theoretical understanding of this phenomenon remained elusive. The possibility of constructing a unified theory of DLRO has emerged recently within the approximation-free supersymmetric theory of stochastics (STS). There, DLRO is the spontaneous breakdown of the topological or de Rahm supersymmetry that all stochastic differential equations (SDEs) possess. This theory may be interesting to researchers with very different backgrounds because the ubiquitous DLRO is a truly interdisciplinary entity. The STS is also an interdisciplinary construction. This theory is based on dynamical systems theory, cohomological field theories, the theory of pseudo-Hermitian operators, and the conventional theory of SDEs. Reviewing the literature on all these mathematical disciplines can be time consuming. As such, a concise and self-contained introduction to the STS, the goal of this paper, may be useful.

Keywords: supersymmetry; stochastic differential equations; non-equilibrium dynamics; cohomological field theory; ergodicity; thermodynamic equilibrium; complexity; chaos; butterfly effect; turbulence; $1/f$ noise; self-organization; self-organized criticality

Nonequilibrium thermodynamics perspectives for the monotonicity of the renormalization group flow

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We investigate the monotonicity of the renormalization group (RG) flow from the perspectives of nonequilibrium thermodynamics. Applying the Martin-Siggia-Rose formalism to the Wilsonian RG transformation, we incorporate the RG flow equations manifestly in an effective action, where all coupling functions are dynamically promoted. As a result, we obtain an emergent holographic dual effective field theory, where an extra dimension appears from the Wilsonian RG transformation. We observe that Becchi-Rouet-Stora-Tyutin (BRST)-type transformations play an important role in the bulk effective action, which give rise to novel Ward identities for correlation functions between the renormalized coupling fields. As generalized fluctuation-dissipation theorems in the semiclassical nonequilibrium dynamics can be understood from the Ward identities of such BRST symmetries, we find essentially the same principle for the RG flow in the holographic dual effective field theory. Furthermore, we discuss how these "nonequilibrium work identities" can be related to the monotonicity of the RG flow, for example, the c-theorem. In particular, we introduce an entropy functional for the dynamical coupling field and show that the production rate of the total entropy functional is always positive, indicating the irreversibility of the RG flow.

DOI: 10.1103/PhysRevD.108.126022

Remarkably, it turns out that our cohomological-type topological field theory construction for the RG-based emergent dual holography is "essentially identical" to Daniel Friedan, *A tentative theory of large distance physics*, JHEP10(2003)063, where the Fokker-Planck-type functional RG equation has been introduced.

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cliché

A tentative theory of large distance physics

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Abstract: A theoretical mechanism is devised to determine the large distance physics of spacetime. It is a two dimensional nonlinear model, the lambda model, set to govern the string worldsurface in an attempt to remedy the failure of string theory, as it stands. The lambda model is formulated to cancel the infrared divergent effects of handles at short distance on the worldsurface. The target manifold is the manifold of background spacetimes. The coupling strength is the spacetime coupling constant. The lambda model operates at 2d distance Λ^{-1}, very much shorter than the 2d distance μ^{-1} where the worldsurface is seen. A large characteristic spacetime distance L is given by $L^2 = \ln(\Lambda/\mu)$. Spacetime fields of wave number up to $1/L$ are the local coordinates for the manifold of spacetimes. The distribution of fluctuations at 2d distances shorter than Λ^{-1} gives the *a priori* measure on the target manifold, the manifold of spacetimes. If this measure concentrates at a macroscopic spacetime, then, nearby, it is a measure on the spacetime fields. The lambda model thereby constructs a spacetime quantum field theory, cutoff at ultraviolet distance L, describing physics at distances larger than L. The lambda model also constructs an effective string theory with infrared cutoff L, describing physics at distances smaller than L. The lambda model evolves outward from zero 2d distance, $\Lambda^{-1} = 0$, building spacetime physics starting from $L = \infty$ and proceeding downward in L. L can be taken smaller than any distance practical for experiments, so the lambda model, if right, gives all actually observable physics. The harmonic surfaces in the manifold of spacetimes are expected to have novel nonperturbative effects at large distances.

Writing the *a priori* measure in the variables λ_e^i as $\mathrm{d}\rho_e(\Lambda, \lambda_e)$, the driven diffusion process is

$$-\Lambda \frac{\partial}{\partial \Lambda_{/\lambda_e}} \mathrm{d}\rho_e(\Lambda, \lambda_e) = \nabla_i^e \left(T\, g_e^{ij}(\lambda_e) \nabla_j^e + \beta_e^i(\lambda_e) \right) \mathrm{d}\rho_e(\Lambda, \lambda_e) \tag{2.33}$$

where ∇_i^e is the covariant derivative with respect to the effective metric $T^{-1} g_{ij}^e$. The coefficients, $T\, g_e^{ij}$ and β_e^i, of the diffusion process are stationary, independent of Λ^{-1}, because of the generalized scale invariance of the effective lambda model.

All the considerations that applied to the classical lambda model carry over to the effective lambda model. The effective *a priori* measure satisfies an effective diffusion equation, which takes the same form as the tree-level diffusion equation. The generalized scale invariance of the effective lambda model implies that the diffusion equation has stationary coefficients,

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda_{/\lambda_e}} \rho_e(\Lambda, \lambda_e) &= \nabla_i^e \left(T_e g_e^{ij}\, \partial_j + \beta_e^i \right) \rho_e \\ &= \nabla_i^e\, T_e g_e^{ij} \left(\partial_j + \partial_j (T_e^{-1} a_e) \right) \rho_e\,. \end{aligned} \tag{8.16}$$

The effective *a priori* measure is the equilibrium measure

$$\mathrm{dvol}_e(\lambda_e)\; \mathrm{e}^{-T_e^{-1} a_e(\lambda_e)} \tag{8.17}$$

which satisfies the equation of motion $\beta_e = 0$,

$$0 = \left(\partial_i + T_e^{-1} g_{ij}^e\, \beta_e^j \right)\, \mathrm{e}^{-T_e^{-1} a_e(\lambda_e)} \tag{8.18}$$

II. A REVIEW ON STOCHASTIC THERMODYNAMICS IN THE LANGEVIN SYSTEM

A procedure for the proof on the entropy production along a stochastic trajectory is as follows. One starts from the Fokker-Planck equation for the probability distribution function of a stochastic trajectory described by the Langevin equation. Seifert introduced a path-dependent entropy functional in terms of the probability distribution away from equilibrium and proved that the Gibbs or Shannon-type microscopic entropy shows its monotonic behavior for the Markovian process. Since we follow all these steps to show the monotonicity of the microscopic entropy functional of the holographic dual effective field theory, we review ref. [1] rather sincerely.

We introduce the Langevin equation,

$$\partial_t x(t) = \mu F(x(t), \lambda(t)) + \xi(t) \tag{1}$$

describing the overdamped dynamics of Brownian motion. Here, $F(x(t), \lambda(t)) = -\partial_x V(x(t), \lambda(t)) + f(x(t), \lambda(t))$ is the force, where $V(x(t), \lambda(t))$ is a conservative potential and $f(x(t), \lambda(t))$ is an external force. These force sources may be time-dependent through an external control parameter $\lambda(t)$ varied according to some prescribed experimental protocol from $\lambda(0) = \lambda_0$ to $\lambda(t_f) = \lambda_f$. μ is the mobility of the particle. $\xi(t)$ serves as stochastic increments modeled as Gaussian white noise,

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t - t') \tag{2}$$

where D is the diffusion constant, given by the Einstein relation $D = \beta^{-1}\mu$ at temperature $T = \beta^{-1}$ in equilibrium.

One can reformulate this Langevin equation in the path integral representation, introducing a generating functional analogous to the partition function in equilibrium. To translate the equation of motion into the generating functional, we consider the following identity

$$1 = \int_{x_i}^{x_f} Dx(t)\delta\Big(\partial_t x(t) - \mu F(x(t),\lambda(t)) - \xi(t)\Big)\det\Big(\partial_t - \mu\partial_x F(x(t),\lambda(t))\Big), \tag{3}$$

referred to as the Faddeev-Popov procedure. $\det\Big(\partial_t - \mu\partial_x F(x(t),\lambda(t))\Big)$ is a Jacobian factor to describe the change of an integration measure. This is nothing but $1 = \int_{x_i}^{x_f} dx\delta(f(x))f'(x)$, where $f'(x) = \frac{df(x)}{dx}$. Then, the generating functional is given by

$$\mathcal{W} = \mathcal{N}\int D\xi(t)\exp\Big(-\frac{1}{4D}\int_{t_i}^{t_f} dt\xi^2(t)\Big)\int_{x_i}^{x_f} Dx(t)\delta\Big(\partial_t x(t) - \mu F(x(t),\lambda(t)) - \xi(t)\Big)\det\Big(\partial_t - \mu\partial_x F(x(t),\lambda(t))\Big). \tag{4}$$

Here, $\mathcal{N}$ is a normalization constant to reproduce Eq. (2).

Introducing a Lagrange multiplier field $p(t)$ and a ghost field $c(t)$, one can exponentiate this expression as follows

The path integral localizes into the Langevin eq.

This is guaranteed by $N = 2$ BRST symmetries. (5)

Here, the Lagrange multiplier $p(t)$ may be regarded as the canonical momentum of the position $x(t)$. The ghost $c(t)$ is a fermion variable to take the Jacobian factor with its canonical conjugate partner $\bar{c}(t)$. Carrying out the Gaussian integral for random noise fluctuations, we obtain an effective 'partition function' as follows

$$\mathcal{W} = \mathcal{N}\int_{x_i}^{x_f} Dx(t)Dp(t)Dc(t)D\bar{c}(t)\exp\Big[-\int_{t_i}^{t_f} dt\Big\{ip(t)\Big(\partial_t x(t) - \mu F(x(t),\lambda(t))\Big) + Dp^2(t) + \bar{c}(t)\Big(\partial_t - \mu\partial_x F(x(t),\lambda(t))\Big)c(t)\Big\}\Big]. \tag{6}$$

Based on this path integral formulation, Refs. [15–20] investigated BRST symmetries and discussed Ward identities. In this study, we apply this framework to the RG flow and reveal symmetries of the RG flow.

One important ingredient involved with the monotonicity of the RG flow is entropy production in the Langevin system. Introducing the following probability distribution,

$$p(x,t) = \langle \delta(x - x(t)) \rangle$$
$$= \mathcal{N} \int D\xi(t') \exp\left(-\frac{1}{4D} \int_{t_i}^{t} dt' \xi^2(t') \right) \delta(x - x(t)), \tag{6}$$

where the average of random noise fluctuations is taken. Then, one obtains the Fokker-Planck equation for the probability distribution function to find the particle at x and at time t,

$$\partial_t p(x,t) = -\partial_x j(x,t) = -\partial_x [(\mu F(x,\lambda) - D\partial_x) p(x,t)]. \tag{7}$$

$j(x,t) = (\mu F(x,\lambda) - D\partial_x) p(x,t)$ is the conserved current. This partial differential equation must be augmented by a normalized initial distribution, $p(x,0) = p_0(x)$. In Appendix A, we show our intuitive derivation for this Fokker-Planck equation. It is straightforward to see the formal path integral expression for the probability distribution function as follows

$$p(x,t) = \frac{\mathcal{N}}{\mathcal{W}} \int D\xi(t') \exp\left(-\frac{1}{4D} \int_{t_i}^{t} dt' \xi^2(t') \right) \int_{x_i}^{x} Dx(t') Dp(t') D\bar{c}(t') Dc(t')$$
$$\times \exp\left[-\int_{t_i}^{t} dt' \{ ip(t')(\partial_{t'} x(t') - \mu F(x(t'), \lambda(t')) - \xi(t')) + \bar{c}(t')(\partial_{t'} - \mu \partial_x F(x(t'), \lambda(t'))) c(t') \} \right] \tag{8}$$

Wilsonian RG transformation

We consider a partition function as follows

$$Z(\Lambda_{uv}) = \int D\psi_\sigma(x;\Lambda_{uv}) \exp\Big\{ - \int d^D x \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(\Lambda_{uv})\}; \Lambda_{uv}] \Big\}. \tag{17}$$

Here, Λ_{uv} is a UV cutoff, where the corresponding effective Lagrangian $\mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(\Lambda_{uv})\}; \Lambda_{uv}]$ is defined. $\psi_\sigma(x;\Lambda_{uv})$ is a dynamical matter field at a given spacetime x, where σ denotes its flavor index $\sigma = 1, ..., N$. $\{\lambda_a(\Lambda_{uv})\}$ represents a set of coupling functions such as velocity, interaction coefficients, and etc., denoted by the subscript a.

Performing the Wilsonian RG transformation, we obtain the following expression for the partition function

$$Z(z_f) = \int D\psi_\sigma(x;z_f) \exp\Big\{ - \int d^D x \Big(\mathcal{L}[\psi_\sigma(x;z_f); \{\lambda_a(x,z_f)\}; z_f] + N \int_{\Lambda_{uv}}^{z_f} dz \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z] \Big) \Big\}, \tag{18}$$

where the UV cutoff Λ_{uv} is lowered to be z_f. In other words, all the dynamical fields $\psi_\sigma(x;z_f)$ and all the coupling functions $\lambda_a(x,z_f)$ are defined at a lower cutoff z_f, where the dynamical fields between z_f and Λ_{uv} are integrated over to introduce an effective potential $N \int_{\Lambda_{uv}}^{z_f} dz \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z]$ into the partition function.

Considering that the partition function is invariant under the RG transformation, regardless of the cutoff scale, we observe that the effective potential is

(1) *Locality approx.*

$$\mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z] = -\frac{1}{N} \ln \int_{\Lambda(z)} D\psi_\sigma(x;z) \exp\Big\{ - \int d^D x \mathcal{L}[\psi_\sigma(x;z); \{\lambda_a(x,z)\}; z] \Big\}, \tag{19}$$

at a given scale z. Accordingly, all the coupling functions are renormalized to be

(2) *One − loop effective action*

$$\frac{\partial \lambda_a(x,z)}{\partial z} = \beta_a[\{\lambda_a(x,z)\}; z] \tag{20}$$

Here, the RG β−function for a coupling function $\lambda_a(x,z)$ is given by the first order derivative of the effective potential with respect to $\lambda_a(x,z)$ as follows

(3) *Gradient RG flow*

$$\beta_a[\{\lambda_a(x,z)\}; z] = -\frac{\partial \mathcal{V}_{rg}[\{\lambda_a(x,z)\}; z]}{\partial \lambda_a(x,z)} \tag{21}$$

To manifest the renormalization group flow in the level of an effective action

To manifest the RG flow in the level of an effective action, we consider the following identity

$$1 = \int D\lambda_a(x,z)\delta\Big(\partial_z\lambda_a(x,z) - \beta_a[\{\lambda_a(x,z)\};z]\Big)\det\Big(\partial_z\delta_{ab} - \frac{\partial\beta_a[\{\lambda_a(x,z)\};z]}{\partial\lambda_b(x,z)}\Big). \quad (24)$$

Here, $\det\Big(\partial_z\delta_{ab} - \frac{\partial\beta_a[\{\lambda_a(x,z)\};z]}{\partial\lambda_b(x,z)}\Big)$ may be regarded as a Jacobian factor for the functional integral.

$$\mathbf{1 = \int dx\, \delta\big(f(x)\big)\, f'(x)}$$

Introducing this $\delta-$function identity into the partition function, we obtain

$$Z(z_f) = \int D\psi_\sigma(x,z_f)D\lambda_a(x,z)\delta\Big(\partial_z\lambda_a(x,z) - \beta_a[\{\lambda_a(x,z)\};z]\Big)\det\Big(\partial_z\delta_{ab} - \frac{\partial\beta_a[\{\lambda_a(x,z)\};z]}{\partial\lambda_b(x,z)}\Big)$$
$$\exp\Big\{-\int d^Dx\Big(\mathcal{L}[\psi_\sigma(x,z_f);\{\lambda_a(x,z_f)\};z_f] + N\int_{\Lambda_{uv}}^{z_f} dz\mathcal{V}_{rg}[\{\lambda_a(x,z)\};z]\Big)\Big\}. \quad (25)$$

Now, the coupling function is promoted to be a dynamical coupling field, which appears as the path integral formulation with the $\delta-$function constraint. This is essentially the same as the Faddeev-Popov procedure for the path integral quantization of gauge fields [47], also applied to the semiclassical nonequilibrium physics, for example, the path integral formulation of Langevin dynamics, and referred to as the MSR formalism [33–35] discussed before. Here, the RG flow corresponds to the Langevin equation.

Almost similar, but not complete

It is straightforward to exponentiate the $\delta-$function constraint as follows

$$Z(z_f) = \int D\psi_\sigma(x, z_f) D\lambda_a(x, z) D\pi_a(x, z) D\bar{c}_a(x, z) Dc_a(x, z) \exp\Big[- \int d^D x \mathcal{L}[\psi_\sigma(x, z_f); \{\lambda_a(x, z_f)\}; z_f] - N \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \Big\{ \pi_a(x, z) \Big(\partial_z \lambda_a(x, z) - \beta_a[\{\lambda_a(x, z)\}; z] \Big) + \bar{c}_a(x, z) \Big(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x, z)\}; z]}{\partial \lambda_b(x, z)} \Big) c_b(x, z) + \mathcal{V}_{rg}[\{\lambda_a(x, z)\}; z] \Big\} \Big]. \quad (26)$$

$\pi_a(x, z)$ is a Lagrange multiplier field to impose the RG flow constraint, which corresponds to the canonical momentum of the coupling field $\lambda_a(x, z)$. In the Schwinger-Keldysh formulation, it is identified with a quantum field denoted by the subscript a or qu in the standard notation. $c_a(x, z)$ ($\bar{c}_a(x, z)$) is an auxiliary fermion field to take the Jacobian factor, referred to as the Faddeev-Popov ghost. z is an RG scale, which serves as a cutoff scale for the Wilsonian RG transformation. Interestingly, this RG scale plays the role of an extradimension, which reminds us of the holographic duality conjecture [48–54], where $\mathcal{S}_{eff}[\{\pi_a(x, z), \lambda_a(x,)\}, \{\bar{c}_a(x, z), c_a(x, z)\}; z_f, \Lambda_{uv}] = N \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \Big\{ \pi_a(x, z) \Big(\partial_z \lambda_a(x, z) - \beta_a[\{\lambda_a(x, z)\}; z] \Big) + \bar{c}_a(x, z) \Big(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x,z)\};z]}{\partial \lambda_b(x,z)} \Big) c_b(x, z) + \mathcal{V}_{rg}[\{\lambda_a(x, z)\}; z] \Big\}$ corresponds to an effective bulk action, supported by an effective boundary action of $\int d^D x \mathcal{L}[\psi_\sigma(x, z_f); \{\lambda_a(x, z_f)\}; z_f]$. Here, 'dual' means that the bulk effective action is written in terms of the coupling fields $\{\lambda_a(x, z)\}$, regarded to be collective dual fields to the corresponding matter composites. In other words, $\lambda_a(x, z)$ is dual to $\frac{\partial \mathcal{L}[\psi_\sigma(x;z); \{\lambda_a(x,z)\}; z]}{\partial \lambda_a(x,z)}$.

To promote coupling functions to dynamical fields:

Irrelevant (noise or TT-bar type) deformations

Although the above reformulation for the Wilsonian RG transformation is rather analogous to the holographic dual effective field theory, there exists one important difference: The coupling field $\lambda_a(x, z)$ is not fully dynamical, whose dynamics is semiclassical, given by the RG flow equation. To promote the coupling field to be fully dynamical, we consider the following UV deformation,

$$Z(\Lambda_{uv}) = \int D\psi_\sigma(x;\Lambda_{uv}) D\lambda_a(x;\Lambda_{uv}) \exp\Big\{ - \int d^D x \Big(\mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}] + \boxed{\frac{1}{2\Gamma_a}[\lambda_a(x;\Lambda_{uv}) - \bar{\lambda}_a(\Lambda_{uv})]^2 \Big)} \Big\}. \tag{27}$$

Here, we introduced random fluctuations of the coupling fields at UV, where Γ_a denotes the variance around the mean value $\bar{\lambda}_a(\Lambda_{uv})$. Taking the $\Gamma_a \to 0$ limit, we recover the previous formulation, where $\lambda_a(\Lambda_{uv})$ is replaced with $\bar{\lambda}_a(\Lambda_{uv})$.

To understand the physical meaning of this UV deformation, we perform the Gaussian integral with respect to $\lambda_a(x;\Lambda_{uv})$. Then, we obtain

$$Z(\Lambda_{uv}) = \int D\psi_\sigma(x;\Lambda_{uv}) \exp\Big[- \int d^D x \Big\{ \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\bar{\lambda}_a(\Lambda_{uv})\}; \Lambda_{uv}] + \boxed{\frac{\Gamma_a}{2}\Big(\frac{\partial \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}]}{\partial \lambda_a(x;\Lambda_{uv})}\Big)^2} \Big\} \Big]. \tag{28}$$

Suppose the Gross-Neveu model for spontaneous chiral symmetry breaking as $\mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}] = \bar{\psi}_\sigma(x;\Lambda_{uv}) i\gamma^\mu \partial_\mu \psi_\sigma(x;\Lambda_{uv}) + \frac{\lambda_\chi(x;\Lambda_{uv})}{2} \bar{\psi}_\sigma(x;\Lambda_{uv})\psi_\sigma(x;\Lambda_{uv})\bar{\psi}_{\sigma'}(x;\Lambda_{uv})\psi_{\sigma'}(x;\Lambda_{uv})$. Then, the last term is $\Big(\frac{\partial \mathcal{L}[\psi_\sigma(x;\Lambda_{uv}); \{\lambda_a(x;\Lambda_{uv})\}; \Lambda_{uv}]}{\partial \lambda_a(x;\Lambda_{uv})}\Big)^2 \sim [\bar{\psi}_\sigma(x;\Lambda_{uv})\psi_\sigma(x;\Lambda_{uv})]^4$ [46], generally irrelevant at the Gaussian fixed point and expected not to change the RG flow as long as weak Γ_χ is concerned.

Considering these random fluctuations of the coupling functions at UV and performing the Wilsonian RG transformation, we obtain

$$Z(z_f) = \int D\psi_\sigma(x, z_f) D\lambda_a(x, z) D\pi_a(x, z) D\bar{c}_a(x, z) Dc_a(x, z)$$
$$\exp\Big[- \int d^D x \Big(\mathcal{L}[\psi_\sigma(x, z_f); \{\lambda_a(x, z_f)\}; z_f] + \frac{1}{2\Gamma_a} [\lambda_a(x, \Lambda_{uv}) - \bar{\lambda}_a(\Lambda_{uv})]^2 \Big)$$
$$-N \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \Big\{ \pi_a(x, z) \Big(\partial_z \lambda_a(x, z) - \beta_a[\{\lambda_a(x, z)\}; z] \Big) - \frac{\Gamma_a}{2} \pi_a^2(x, z)$$
$$+ \bar{c}_a(x, z) \Big(\partial_z \delta_{ab} - \frac{\partial \beta_a[\{\lambda_a(x, z)\}; z]}{\partial \lambda_b(x, z)} \Big) c_b(x, z) \text{[redacted]} \Big\} \Big], \qquad (29)$$

where $-\frac{\Gamma_a}{2} \pi_a^2(x, z)$ appeared to give the dynamics to $\lambda_a(x, z)$ in the extradimension. It is trivial to check out that the $\Gamma_a \to 0$ limit reproduces Eq. (26).

This construction is idential to the brute − force integral derivation discussed before.
You can check it out by discretizing the z − coordinate into the RG step (k),
which shows the recursive RG structure.
Although this cohomological construction is NOT EXACT, the resulting EFT
satisfies the necessary condition, that is, the monotonicity of the RG flow,
which can be translated into the WZ consistency.

Monotonicity of the RG flow in an emergent dual holography of a worldsheet nonlinear σ model

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Based on the renormalization group (RG) flow of worldsheet bosonic string theory, we construct an effective holographic dual description of the target space theory identifying the RG scale with the emergent extra dimension. This results in an effective dilaton-gravity-gauge theory, analogous to the low-energy description of bosonic M theory. We argue that this holographic dual effective field theory is nonperturbative in the α' expansion, where a class of string quantum fluctuations are resummed to all orders. To investigate the monotonicity of the RG flow of the target space metric in the emergent spacetime, we consider entropy production along the RG flow. We construct a microscopic entropy functional based on the probability distribution function of the holographic dual effective field theory, regarded as Gibbs- or Shannon-type entropy. Given that the Ricci flow represents the 1-loop RG flow equation of the target space metric for the 2D nonlinear sigma model, and motivated by Perelman's proof of the monotonicity of Ricci flow, we propose a Perelman's entropy functional for the holographic dual effective field theory. This entropy functional is also nonperturbative in the α' expansion, and thus, generalizes the 1-loop result to the all-loop order. Furthermore, utilizing the equivalence between the Hamilton-Jacobi equation and the local RG equation, we suggest that the RG flow of holographic Perelman's entropy functional is the Weyl anomaly. This eventually reaffirms the monotonicity of RG flow for the emergent target spacetime but in a nonperturbative way. Interestingly, we find that the microscopic entropy production rate can be determined by integrating the rate of change of the holographic Perelman's entropy functional over all possible metric configurations along the flow.

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$$Z = \int Dx^\mu(\sigma) Db_{ab}(\sigma) Dc^a(\sigma) \exp\left[-\frac{1}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g(\sigma)}\{(g^{ab}(\sigma)G_{\mu\nu}(x) + i\epsilon^{ab}B_{\mu\nu}(x))\partial_a x^\mu(\sigma)\partial_b x^\nu(\sigma) + \alpha' R^{(2)}(\sigma)\Phi(x)\} - \frac{1}{2\pi}\int_M d^2\sigma\sqrt{g(\sigma)}b_{ab}(\sigma)\nabla^a c^b(\sigma)\right]. \quad (1)$$

Here, $\frac{1}{2\pi}\int_M d^2\sigma\sqrt{g(\sigma)}b_{ab}(\sigma)\nabla^a c^b(\sigma)$ is the ghost action to implement the Jacobian factor involved with the gauge fixing of the worldsheet metric $g_{ab}(\sigma)$ in the Faddeev-Popov procedure [101]. The worldsheet covariant derivative for the ghost field is

$$\begin{aligned}\nabla^a c^b(\sigma) &= g^{ac}(\sigma)\nabla_c c^b(\sigma) \\ &= g^{ac}(\sigma)(\partial_c c^b(\sigma) + \Gamma^b_{cd}(\sigma)c^d(\sigma)), \qquad (2)\end{aligned}$$

where $\Gamma^b_{cd}(\sigma)$ is the worldsheet connection. ϵ^{ab} is the antisymmetric tensor in the worldsheet. $\Phi(x)$ is the dilaton field to manage the conformal anomaly, coupled to the worldsheet scalar curvature $R^{(2)}(\sigma)$.

Specifically, we consider the conformal gauge where [101]

$$\hat{g}^{ab}(\sigma) = e^{2\omega(\sigma)}\delta^{ab}. \qquad (3)$$

Accordingly, the worldsheet connection is given by

$$\begin{aligned}\hat{\Gamma}^b_{cd} &= \frac{1}{2}\hat{g}^{be}(\partial_c\hat{g}_{ed} + \partial_d\hat{g}_{ce} - \partial_e\hat{g}_{cd}) \\ &= -(\partial_c\omega(\sigma)\delta^b_d + \partial_d\omega(\sigma)\delta^b_c - \delta^{be}\partial_e\omega(\sigma)\delta_{cd}). \qquad (4)\end{aligned}$$

The worldsheet Riemann tensor is given by

$$\hat{R}^{(2)}_{abcd}(\sigma) = \frac{1}{2}(\hat{g}_{ac}(\sigma)\hat{g}_{bd}(\sigma) - \hat{g}_{ad}(\sigma)\hat{g}_{bc}(\sigma))\hat{R}^{(2)}(\sigma), \qquad (5)$$

where the Ricci scalar is

$$\hat{R}^{(2)}(\sigma) = -2e^{-2\omega(\sigma)}\delta^{ab}\partial_a\partial_b\omega(\sigma) \qquad (6)$$

in this conformal gauge.

It is straightforward to perform the RG transformation at the 1-loop level in α' [102,103]. One separates $x^\mu(\sigma)$ into its slow (classical or background) and fast (quantum) degrees of freedom, and expands the resulting worldsheet string action in terms of the fast degrees of freedom up to the second order in the α' expansion. One can do this task in the Riemann normal coordinate, where the connection coefficient of the target spacetime vanishes locally to allow for the expansion to be written conveniently. Carrying out the Gaussian integral for the fast degrees of freedom in the conformal gauge, one finds the RG flow equations for the target spacetime metric, the Kalb-Ramond gauge field, and the dilaton field, respectively, as follows [101]:

$$\begin{aligned}\partial_z G_{\mu\nu}(x) &= -\beta^G_{\mu\nu}\\ &= -\alpha' R_{\mu\nu}(x) - 2\alpha'\nabla_\mu\nabla_\nu\Phi(x)\\ &\quad + \frac{\alpha'}{4}H_{\mu\lambda\omega}(x)H_\nu^{\lambda\omega}(x),\end{aligned} \tag{7}$$

$$\partial_z B_{\mu\nu}(x) = -\beta^B_{\mu\nu} = \frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu}(x) - \alpha'\partial^\omega\Phi(x)H_{\omega\mu\nu}(x), \tag{8}$$

$$\begin{aligned}\partial_z\Phi(x) &= -\beta^\Phi\\ &= -\frac{D-26}{6} + \frac{\alpha'}{2}\nabla^2\Phi(x) - \alpha'\partial_\mu\Phi(x)\partial^\mu\Phi(x)\\ &\quad + \frac{\alpha'}{24}H_{\mu\nu\lambda}(x)H^{\mu\nu\lambda}(x).\end{aligned} \tag{9}$$

Here, z is an RG scale for the RG transformation, where $z = 0$ is identified with the UV fixed point. $H_{\omega\mu\nu}(x) = \partial_\omega B_{\mu\nu}(x) + \partial_\mu B_{\nu\omega}(x) + \partial_\nu B_{\omega\mu}(x)$ is

sor for the two-form gauge field $B_{\mu\nu}$

Considering the target spacetime metric, we modify the RG flow equation (7) as

$$\begin{aligned}\partial_z G_{\mu\nu}(x,z) &= -\alpha' R_{\mu\nu}(x,z) - 2\alpha'\nabla_\mu\nabla_\nu\Phi(x,z)\\ &\quad + \frac{\alpha'}{4}H_{\mu\lambda\omega}(x,z)H_\nu^{\lambda\omega}(x,z) + \xi_{\mu\nu}(x,z)\end{aligned} \tag{10}$$

Here, $\xi_{\mu\nu}(x,z)$ plays the role of the Gaussian

given by

$$\langle\xi_{\mu\nu}(x,z)\mathcal{G}^{\mu\nu\rho\lambda}(x,z)\xi_{\rho\lambda}(x',z')\rangle = \lambda\delta(x-x')\delta(z-z'), \tag{11}$$

where its average value vanishes, i.e., $\langle\xi_{\mu\nu}(x,z)\rangle = 0$.

$\mathcal{G}_{\mu\nu\rho\lambda}(x,z)$ is the Wheeler-DeWitt metric [104], given by

$$\begin{aligned}\mathcal{G}_{\mu\nu\rho\gamma}(x,z) &\equiv \frac{1}{2}G_{\mu\rho}(x,z)G_{\nu\gamma}(x,z) + \frac{1}{2}G_{\nu\rho}(x,z)G_{\mu\gamma}(x,z)\\ &\quad - \frac{1}{D-1}G_{\mu\nu}(x,z)G_{\rho\gamma}(x,z),\end{aligned} \tag{12}$$

$$\begin{aligned}\partial_z B_{\mu\nu}(x,z) &= \frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu}(x,z) - \alpha'\partial^\omega\Phi(x,z)H_{\omega\mu\nu}(x,z)\\ &\quad + \zeta_{\mu\nu}(x,z)\end{aligned} \tag{14}$$

$^{\mu\nu\rho\gamma}(x,z)$ is the inverse Wheeler-DeWitt metric, ing $\mathcal{G}_{\alpha\beta\gamma\delta}\mathcal{G}^{\gamma\delta\mu\nu} = \delta_{(\alpha}{}^\mu\delta_{\beta)}{}^\nu$ and given by

$$\langle\zeta_{\mu\nu}(x,z)\zeta_{\rho\lambda}(x',z')\rangle = q\delta_{\mu\rho}\delta_{\nu\lambda}\delta(x-x')\delta(z-z') \tag{15}$$

$$\begin{aligned}\mathcal{G}^{\mu\nu\rho\gamma}(x,z) &= \frac{1}{2}\left(G^{\mu\rho}(x,z)G^{\nu\gamma}(x,z) + G^{\nu\rho}(x,z)G^{\mu\gamma}(x,z)\right)\\ &\quad - G^{\mu\nu}(x,z)G^{\rho\gamma}(x,z).\end{aligned} \tag{13}$$

Next, we can construct a generating functional for this Langevin-type equation, following the strategy discussed in Appendix B. Recalling the Faddeev-Popov procedure [86], we introduce the following identity:

$$1 = \int_{G_{\mu\nu}(x,0)}^{G_{\mu\nu}(x,z_f)} DG_{\mu\nu}(x,z) \int_{B_{\mu\nu}(x,0)}^{B_{\mu\nu}(x,z_f)} DB_{\mu\nu}(x,z) \int_{\Phi(x,0)}^{\Phi(x,z_f)} D\Phi(x,z)\delta(\partial_z G_{\mu\nu}(x,z) + \beta^G_{\mu\nu} - \xi_{\mu\nu}(x,z))\delta(\partial_z B_{\mu\nu}(x,z) + \beta^B_{\mu\nu} - \zeta_{\mu\nu}(x,z))\delta(\partial_z \Phi(x,z) + \beta^\Phi)\mathcal{J}\left(\frac{\partial}{\partial G_{\mu\nu}(x,z)}, \frac{\partial}{\partial B_{\mu\nu}(x,z)}, \frac{\partial}{\partial \Phi(x,z)}\right). \tag{16}$$

Here, the Jacobian factor is given by

$$\mathcal{J}\left(\frac{\partial}{\partial G_{\mu\nu}(x,z)}, \frac{\partial}{\partial B_{\mu\nu}(x,z)}, \frac{\partial}{\partial \Phi(x,z)}\right) \equiv \frac{\partial(\partial_z G_{\omega\lambda} + \beta^G_{\omega\lambda}, \partial_z B_{\omega\lambda} + \beta^B_{\omega\lambda}, \partial_z \Phi + \beta^\Phi)}{\partial(G_{\mu\nu}, B_{\mu\nu}, \Phi)}$$
$$= \mathrm{Det}\begin{pmatrix} \frac{\partial[\partial_z G_{\omega\lambda}+\beta^G_{\omega\lambda}]}{\partial G_{\mu\nu}} & \frac{\partial[\partial_z G_{\omega\lambda}+\beta^G_{\omega\lambda}]}{\partial B_{\mu\nu}} & \frac{\partial[\partial_z G_{\omega\lambda}+\beta^G_{\omega\lambda}]}{\partial \Phi} \\ \frac{\partial[\partial_z B_{\omega\lambda}+\beta^B_{\omega\lambda}]}{\partial G_{\mu\nu}} & \frac{\partial[\partial_z B_{\omega\lambda}+\beta^B_{\omega\lambda}]}{\partial B_{\mu\nu}} & \frac{\partial[\partial_z B_{\omega\lambda}+\beta^B_{\omega\lambda}]}{\partial \Phi} \\ \frac{\partial[\partial_z \Phi+\beta^\Phi]}{\partial G_{\mu\nu}} & \frac{\partial[\partial_z \Phi+\beta^\Phi]}{\partial B_{\mu\nu}} & \frac{\partial[\partial_z \Phi+\beta^\Phi]}{\partial \Phi} \end{pmatrix}, \tag{17}$$

where all the 1-loop β functions were introduced before.

$$
\begin{aligned}
Z = \int & D\xi_{\mu\nu}(x,z) D\zeta_{\mu\nu}(x,z) DG_{\mu\nu}(x,z) DB_{\mu\nu}(x,z) D\Phi(x,z) \mathcal{J}\left(\frac{\partial}{\partial G_{\mu\nu}(x,z)}, \frac{\partial}{\partial B_{\mu\nu}(x,z)}, \frac{\partial}{\partial \Phi(x,z)}\right) \\
& \times \delta\left(\partial_z G_{\mu\nu}(x,z) + \beta^G_{\mu\nu} - \xi_{\mu\nu}(x,z)\right) \delta\left(\partial_z B_{\mu\nu}(x,z) + \beta^B_{\mu\nu} - \zeta_{\mu\nu}(x,z)\right) \delta\left(\partial_z \Phi(x,z) + \beta^\Phi\right) \\
& \times \exp\left[-\frac{1}{\alpha'}\int_0^{z_f} dz \int d^D x \sqrt{G(x,z)} e^{-2\Phi(x,z)} \left\{ -\frac{1}{2\lambda}\xi_{\mu\nu}(x,z)\mathcal{G}^{\mu\nu\rho\lambda}(x,z)\xi_{\rho\lambda}(x,z) - \frac{1}{2q}\zeta_{\mu\nu}(x,z)\zeta^{\mu\nu}(x,z) \right.\right. \\
& \left.\left. + \frac{\alpha'}{4} R(x,z) - \frac{D-26}{6} - \frac{\alpha'}{48} H_{\mu\nu\lambda}(x,z) H^{\mu\nu\lambda}(x,z) + \alpha' \partial_\mu \Phi(x,z) \partial^\mu \Phi(x,z) \right\}\right]. \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
Z = \int & DG_{\mu\nu}(x,z) D\Pi^{\mu\nu}_G(x,z) DB_{\mu\nu}(x,z) D\Pi^{\mu\nu}_B(x,z) D\Phi(x,z) D\Pi_\Phi(x,z) \mathcal{J}\left(\frac{\partial}{\partial G_{\mu\nu}(x,z)}, \frac{\partial}{\partial B_{\mu\nu}(x,z)}, \frac{\partial}{\partial \Phi(x,z)}\right) \\
& \times \exp\left[-\frac{1}{\alpha'}\int_0^{z_f} dz \int d^D x \sqrt{G(x,z)} e^{-2\Phi(x,z)} \left\{ \Pi^{\mu\nu}_G(x,z)(\partial_z G_{\mu\nu}(x,z) + \beta^G_{\mu\nu}) + \frac{\lambda}{2}\Pi^{\mu\nu}_G(x,z)\mathcal{G}_{\mu\nu\rho\lambda}(x,z)\Pi^{\rho\lambda}_G(x,z) \right.\right. \\
& + \Pi^{\mu\nu}_B(x,z)(\partial_z B_{\mu\nu}(x,z) + \beta^B_{\mu\nu}) + \frac{q}{2}\Pi_{B,\mu\nu}(x,z)\Pi^{\mu\nu}_B(x,z) + \Pi_\Phi(x,z)(\partial_z \Phi(x,z) + \beta^\Phi) \\
& \left.\left. + \frac{\alpha'}{4} R(x,z) - \frac{D-26}{6} - \frac{\alpha'}{48} H_{\mu\nu\lambda}(x,z) H^{\mu\nu\lambda}(x,z) + \alpha' \partial_\mu \Phi(x,z) \partial^\mu \Phi(x,z) \right\}\right]. \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
Z_{\mathrm{IR}} = \int Dx^{\mu}(\sigma)Db_{ab}(\sigma)Dc^{a}(\sigma)\exp\Bigg[-\frac{1}{4\pi\alpha'}\int_{M} d^{2}\sigma\sqrt{g(\sigma)}\{(g^{ab}(\sigma)G_{\mu\nu}(x,z_{f}) + i\epsilon^{ab}B_{\mu\nu}(x,z_{f}))\partial_{a}x^{\mu}(\sigma)\partial_{b}x^{\nu}(\sigma) \\
+ \alpha' R^{(2)}(\sigma)\Phi(x,z_{f})\} - \frac{1}{2\pi}\int_{M} d^{2}\sigma\sqrt{g(\sigma)}b_{ab}(\sigma)\nabla^{a}c^{b}(\sigma)\Bigg] \\
\approx \exp\left[-\frac{1}{\alpha'}\int d^{D}x\sqrt{G(x,z_{f})}e^{-2\Phi(x,z_{f})}\left\{\frac{\alpha'}{4}R(x,z_{f}) - \frac{D-26}{6} - \frac{\alpha'}{48}H_{\mu\nu\lambda}(x,z_{f})H^{\mu\nu\lambda}(x,z_{f}) + \alpha'\partial_{\mu}\Phi(x,z_{f})\partial^{\mu}\Phi(x,z_{f})\right\}\right],
\end{aligned}
\tag{24}
$$

Can we solve the tachyon problem in the bosonic string theory?

$$
\begin{aligned}
\mathcal{Z} = \int & DG_{\mu\nu}(x,z) D\Pi_G^{\mu\nu}(x,z) DB_{\mu\nu}(x,z) D\Pi_B^{\mu\nu}(x,z) D\Phi(x,z) D\Pi_\Phi(x,z) \mathcal{J}\left(\frac{\partial}{\partial G_{\mu\nu}(x,z)}, \frac{\partial}{\partial B_{\mu\nu}(x,z)}, \frac{\partial}{\partial \Phi(x,z)}\right) \\
& \times \exp\left[-\frac{1}{\alpha'}\int d^D x \sqrt{G(x,z_f)} e^{-2\Phi(x,z_f)} \left\{\frac{\alpha'}{4} R(x,z_f) - \frac{D-26}{6} - \frac{\alpha'}{48} H_{\mu\nu\lambda}(x,z_f) H^{\mu\nu\lambda}(x,z_f) + \alpha' \partial_\mu \Phi(x,z_f) \partial^\mu \Phi(x,z_f)\right\}\right] \\
& \times \exp\left[-\frac{1}{\alpha'}\int_0^{z_f} dz \int d^D x \sqrt{G(x,z)} e^{-2\Phi(x,z)} \left\{\Pi_G^{\mu\nu}(x,z)\left(\partial_z G_{\mu\nu}(x,z) + \alpha' R_{\mu\nu}(x,z) + 2\alpha' \nabla_\mu \nabla_\nu \Phi(x,z) \right.\right.\right. \\
& \left. - \frac{\alpha'}{4} H_{\mu\lambda\omega}(x,z) H_\nu^{\lambda\omega}(x,z)\right) + \frac{\lambda}{2} \Pi_G^{\mu\nu}(x,z) \mathcal{G}_{\mu\nu\rho\lambda}(x,z) \Pi_G^{\rho\lambda}(x,z) \\
& + \Pi_B^{\mu\nu}(x,z)\left(\partial_z B_{\mu\nu}(x,z) - \frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu}(x,z) + \alpha' \nabla^\omega \Phi(x,z) H_{\omega\mu\nu}(x,z)\right) + \frac{q}{2} \Pi_{B,\mu\nu}(x,z) \Pi_B^{\mu\nu}(x,z) \\
& + \Pi_\Phi(x,z)\left(\partial_z \Phi(x,z) + \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi(x,z) + \alpha' \partial_\mu \Phi(x,z) \partial^\mu \Phi(x,z) - \frac{\alpha'}{24} H_{\mu\nu\lambda}(x,z) H^{\mu\nu\lambda}(x,z)\right) \\
& \left.\left. + \frac{\alpha'}{4} R(x,z) - \frac{D-26}{6} - \frac{\alpha'}{48} H_{\mu\nu\lambda}(x,z) H^{\mu\nu\lambda}(x,z) + \alpha' \partial_\mu \Phi(x,z) \partial^\mu \Phi(x,z)\right\}\right]. \qquad (27)
\end{aligned}
$$

Physics of four types of BRST transformations

- Ward identities of these four types of BRST symmetries and their explicit breaking ➔ Generalized fluctuation-dissipation theorems

One may consider four types of BRST transformations in this holographic dual effective field theory as the case of the Langevin dynamics [15–20]. We recall that in the Schwinger-Keldysh formulation, the first two BRST symmetries with their charges Q and $\bar{Q}$ are topological in origin, related with the unitarity. These two BRST symmetries do not commute with the KMS ones [33]. Considering both the BRST and KMS symmetries, we have to introduce additional two fermion-type symmetries with their charges D and $\bar{D}$. Although D and $\bar{D}$ correspond to the superderivatives in the superspace formulation as discussed in Appendix B, we also call these additional fermionic symmetries BRST-type symmetries. The first two BRST transformations lead the bulk kinetic energy $\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)})c(x,z)$ to be invariant while the last two do not. The effective potential $\mathcal{V}_{rg}[\lambda(x,z);z]$ does transform under all these BRST transformations. As a result, there do not exist any BRST-type emergent symmetries in this RG flow, precisely speaking. However, such BRST noninvariant terms are expressed in a "universal" way. As a result, we can derive generalized Ward identities from these four types of BRST transformations and find some constraints for correlation functions of the coupling field.

$$\delta_Q \lambda(x,z) = \epsilon[Q, \lambda(x,z)] = \epsilon c(x,z), \tag{31}$$

$$\delta_Q \pi(x,z) = \epsilon[Q, \pi(x,z)] = 0, \tag{32}$$

$$\delta_Q \bar{c}(x,z) = \epsilon[Q, \bar{c}(x,z)] = -\epsilon\pi(x,z), \tag{33}$$

$$\delta_Q c(x,z) = \epsilon[Q, c(x,z)] = 0, \tag{34}$$

$$Q = c(x,z)\frac{\delta}{\delta\lambda(x,z)} - \pi(x,z)\frac{\delta}{\delta\bar{c}(x,z)}$$

$$\delta_Q\left\{\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\left(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z) + \mathcal{V}_{rg}[\lambda(x,z);z]\right\}$$
$$= \delta_Q\mathcal{V}_{rg}[\lambda(x,z);z] = -\epsilon c(x,z)\beta[\lambda(x,z);z], \tag{36}$$

$$\delta_{\bar{Q}} \lambda(x,z) = \bar{\epsilon}[\bar{Q}, \lambda(x,z)] = \bar{\epsilon}\,\bar{c}(x,z), \tag{37}$$

$$\delta_{\bar{Q}} \pi(x,z) = \bar{\epsilon}[\bar{Q}, \pi(x,z)] = \bar{\epsilon}\frac{2}{\Gamma}\partial_z\bar{c}(x,z), \tag{38}$$

$$\delta_{\bar{Q}} \bar{c}(x,z) = \bar{\epsilon}[\bar{Q}, \bar{c}(x,z)] = 0, \tag{39}$$

$$\delta_{\bar{Q}} c(x,z) = \bar{\epsilon}[\bar{Q}, c(x,z)] = \bar{\epsilon}\left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right), \tag{40}$$

$$\bar{Q} = \bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z\bar{c}(x,z)]\frac{\delta}{\delta\pi(x,z)} + \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta c(x,z)}.$$

$$\delta_{\bar{Q}}\left\{\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\left(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z) + \mathcal{V}_{rg}[\lambda(x,z);z]\right\}$$
$$= \bar{\epsilon}\frac{d}{dz}\left\{\frac{2}{\Gamma}\bar{c}(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \bar{c}(x,z)\pi(x,z)\right\} - \bar{\epsilon}\,\bar{c}(x,z)\beta[\lambda(x,z);z],$$

$$\delta_D\lambda(x,z) = \varepsilon[D,\lambda(x,z)] = \varepsilon\bar{c}(x,z),$$

$$\delta_D\pi(x,z) = \varepsilon[D,\pi(x,z)] = 0,$$

$$\delta_D\bar{c}(x,z) = \varepsilon[D,\bar{c}(x,z)] = 0,$$

$$\delta_D c(x,z) = \varepsilon[D,c(x,z)] = \varepsilon\pi(x,z),$$

$$\delta_{\bar{D}}\lambda(x,z) = \bar{\varepsilon}[\bar{D},\lambda(x,z)] = \bar{\varepsilon}c(x,z),$$

$$\delta_{\bar{D}}\pi(x,z) = \bar{\varepsilon}[\bar{D},\pi(x,z)] = \bar{\varepsilon}\frac{2}{\Gamma}\partial_z c(x,z),$$

$$\delta_{\bar{D}}\bar{c}(x,z) = \bar{\varepsilon}[\bar{D},\bar{c}(x,z)] = -\bar{\varepsilon}\left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right),$$

$$\delta_{\bar{D}}c(x,z) = \bar{\varepsilon}[\bar{D},c(x,z)] = 0,$$

$$D = \bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \pi(x,z)\frac{\delta}{\delta c(x,z)}$$

$$\bar{D} = c(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z c(x,z)]\frac{\delta}{\delta\pi(x,z)} - \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta\bar{c}(x,z)}.$$

$$\delta_D\left\{\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\left(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z) + \mathcal{V}_{rg}[\lambda(x,z);z]\right\}$$
$$= 2\varepsilon\pi(x,z)\partial_z\bar{c}(x,z) - \varepsilon\partial_z(\bar{c}(x,z)\pi(x,z)) - \varepsilon\bar{c}(x,z)\beta[\lambda(x,z);z],$$

$$\delta_{\bar{D}}\left\{\pi(x,z)(\partial_z\lambda(x,z) - \beta[\lambda(x,z);z]) - \frac{\Gamma}{2}\pi^2(x,z) + \bar{c}(x,z)\left(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)c(x,z) + \mathcal{V}_{rg}[\lambda(x,z);z]\right\}$$
$$= -2\bar{\varepsilon}\pi(x,z)[\partial_z c(x,z)] + \bar{\varepsilon}\frac{4}{\Gamma}[\partial_z c(x,z)][\partial_z\lambda(x,z)] - \bar{\varepsilon}\frac{2}{\Gamma}\partial_z(c(x,z)\beta[\lambda_a(x,z);z]) - \bar{\varepsilon}c(x,z)\beta[\lambda(x,z);z],$$

B. Generalized fluctuation-dissipation theorems for the RG flows of correlation functions of the coupling functions

To derive the Ward identities from these BRST transformations, we consider an action for sources as follows [18,49]

$$\begin{aligned}\mathcal{S}_{Source} &= N\int_{\Lambda_{uv}}^{z_f} dz \int d^D x(\bar{T}(x,z)\lambda(x,z) + \pi(x,z)T(x,z) + \bar{G}(x,z)c(x,z) + \bar{c}(x,z)G(x,z)) \\ &= N\int_{\Lambda_{uv}}^{z_f} dz \int d^D x\left(\bar{T}(x,z)\frac{\partial}{\partial \bar{T}(x,z)} + T(x,z)\frac{\partial}{\partial T(x,z)} + \bar{G}(x,z)\frac{\partial}{\partial \bar{G}(x,z)} - \frac{\partial}{\partial G(x,z)}G(x,z)\right) \\ &= N\int_{\Lambda_{uv}}^{z_f} dz \int d^D x\left(\frac{\partial}{\partial \lambda(x,z)}\lambda(x,z) + \frac{\partial}{\partial \pi(x,z)}\pi(x,z) - \frac{\partial}{\partial c(x,z)}c(x,z) + \bar{c}(x,z)\frac{\partial}{\partial \bar{c}(x,z)}\right). \end{aligned} \tag{57}$$

Here, $\bar{T}(x,z)$ ($T(x,z)$) is the bosonic source field for $\lambda(x,z)$ ($\pi(x,z)$), and $\bar{G}(x,z)$ ($G(x,z)$) is the fermionic source field for $c(x,z)$ ($\bar{c}(x,z)$). Accordingly, the four BRST charges are represented as follows

$$Q = c(x,z)\frac{\delta}{\delta\lambda(x,z)} - \pi(x,z)\frac{\delta}{\delta\bar{c}(x,z)} = \bar{T}(x,z)\frac{\partial}{\partial\bar{G}(x,z)} - G(x,z)\frac{\partial}{\partial T(x,z)}, \tag{58}$$

$$\begin{aligned}\bar{Q} &= \bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z\bar{c}(x,z)]\frac{\delta}{\delta\pi(x,z)} + \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta c(x,z)} \\ &= -\bar{T}(x,z)\frac{\partial}{\partial G(x,z)} - \frac{2}{\Gamma}T(x,z)\left(\partial_z\frac{\partial}{\partial G(x,z)}\right) - \bar{G}(x,z)\left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma}\partial_z\frac{\partial}{\partial\bar{T}(x,z)}\right)\end{aligned} \tag{59}$$

and

$$D = \bar{c}(x,z)\frac{\delta}{\delta\lambda(x,z)} + \pi(x,z)\frac{\delta}{\delta c(x,z)} = -\bar{T}(x,z)\frac{\partial}{\partial G(x,z)} - \bar{G}(x,z)\frac{\partial}{\partial T(x,z)}, \tag{60}$$

$$\begin{aligned}\bar{D} &= c(x,z)\frac{\delta}{\delta\lambda(x,z)} + \frac{2}{\Gamma}[\partial_z c(x,z)]\frac{\delta}{\delta\pi(x,z)} - \left(\pi(x,z) - \frac{2}{\Gamma}\partial_z\lambda(x,z)\right)\frac{\delta}{\delta\bar{c}(x,z)} \\ &= \bar{T}(x,z)\frac{\partial}{\partial\bar{G}(x,z)} + \frac{2}{\Gamma}T(x,z)\left(\partial_z\frac{\partial}{\partial\bar{G}(x,z)}\right) - G(x,z)\left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma}\partial_z\frac{\partial}{\partial\bar{T}(x,z)}\right),\end{aligned} \tag{61}$$

respectively.

Taking the first two BRST transformations to the partition function, we find

$$\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left(\bar{T}(x,z) \frac{\partial}{\partial \bar{G}(x,z)} - G(x,z) \frac{\partial}{\partial T(x,z)} \right) Z(z_f) = \frac{1}{2} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial \bar{G}(x,z)} Z(z_f), \tag{62}$$

$$\begin{aligned} &\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left\{ \bar{T}(x,z) \frac{\partial}{\partial G(x,z)} + \frac{2}{\Gamma} T(x,z) \left(\partial_z \frac{\partial}{\partial G(x,z)} \right) + \bar{G}(x,z) \left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma} \partial_z \frac{\partial}{\partial \bar{T}(x,z)} \right) \right\} Z(z_f) \\ &= \frac{1}{2} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left[-\partial_z \left\{ \frac{2}{\Gamma} \left(\partial_z \frac{\partial}{\partial \bar{T}(x,z)} - \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \right) \frac{\partial}{\partial G(x,z)} - \frac{\partial}{\partial T(x,z)} \frac{\partial}{\partial G(x,z)} \right\} \right. \\ &\left. + \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial G(x,z)} \right] Z(z_f). \end{aligned} \tag{63}$$

Considering the second two BRST transformations to the partition function, we obtain

$$\begin{aligned} &\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left(\bar{T}(x,z) \frac{\partial}{\partial G(x,z)} + \bar{G}(x,z) \frac{\partial}{\partial T(x,z)} \right) Z(z_f) \\ &= \frac{1}{2} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left[\partial_z \left(\frac{\partial}{\partial T(x,z)} \frac{\partial}{\partial G(x,z)} \right) - 2 \frac{\partial}{\partial T(x,z)} \partial_z \frac{\partial}{\partial G(x,z)} + \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial G(x,z)} \right] Z(z_f), \end{aligned} \tag{64}$$

$$\begin{aligned} &\int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left\{ \bar{T}(x,z) \frac{\partial}{\partial \bar{G}(x,z)} + \frac{2}{\Gamma} T(x,z) \left(\partial_z \frac{\partial}{\partial \bar{G}(x,z)} \right) - G(x,z) \left(\frac{\partial}{\partial T(x,z)} - \frac{2}{\Gamma} \partial_z \frac{\partial}{\partial \bar{T}(x,z)} \right) \right\} Z(z_f) \\ &= \frac{1}{2} \int_{\Lambda_{uv}}^{z_f} dz \int d^D x \left[\frac{2}{\Gamma} \partial_z \left\{ \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial \bar{G}(x,z)} \right\} + 2 \frac{\partial}{\partial T(x,z)} \partial_z \frac{\partial}{\partial \bar{G}(x,z)} - \frac{4}{\Gamma} \left(\partial_z \frac{\partial}{\partial \bar{T}(x,z)} \right) \partial_z \frac{\partial}{\partial \bar{G}(x,z)} \right. \\ &\left. + \beta \left(\frac{\partial}{\partial \bar{T}(x,z)}; z \right) \frac{\partial}{\partial \bar{G}(x,z)} \right] Z(z_f). \end{aligned} \tag{65}$$

Applying $\frac{\partial}{\partial G(x',z')}\frac{\partial}{\partial \bar{T}(x'',z'')}$ to Eq. (62) and $\frac{\partial}{\partial \bar{G}(x',z')}\frac{\partial}{\partial \bar{T}(x'',z'')}$ to Eq. (63), respectively, we obtain

$$\left(\frac{\partial}{\partial G(x',z')}\frac{\partial}{\partial \bar{G}(x'',z'')}-\frac{\partial}{\partial \bar{T}(x'',z'')}\frac{\partial}{\partial T(x',z')}\right)Z(z_f)=\frac{1}{2}\frac{\partial}{\partial G(x',z')}\frac{\partial}{\partial \bar{T}(x'',z'')}\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\beta\left(\frac{\partial}{\partial \bar{T}(x,z)};z\right)\frac{\partial}{\partial \bar{G}(x,z)}Z(z_f), \tag{66}$$

$$\left(\frac{\partial}{\partial \bar{G}(x',z')}\frac{\partial}{\partial G(x'',z'')}+\frac{\partial}{\partial \bar{T}(x'',z'')}\frac{\partial}{\partial T(x',z')}-\frac{2}{\Gamma}\frac{\partial}{\partial \bar{T}(x'',z'')}\partial_{z'}\frac{\partial}{\partial \bar{T}(x',z')}\right)Z(z_f)$$
$$=\frac{1}{2}\frac{\partial}{\partial \bar{G}(x',z')}\frac{\partial}{\partial \bar{T}(x'',z'')}\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\beta\left(\frac{\partial}{\partial \bar{T}(x,z)};z\right)\frac{\partial}{\partial G(x,z)}Z(z_f), \tag{67}$$

where all other source fields were set to zero. These two equations lead to

$$\langle \bar{c}(x',z')c(x,z)\rangle+\langle \lambda(x,z)\pi(x',z')\rangle=-\frac{1}{2}\langle \bar{c}(x',z')\lambda(x,z)\int_{\Lambda_{uv}}^{z_f}dw\int d^Dy\beta[\lambda(y,w);w]c(y,w)\rangle, \tag{68}$$

$$\langle c(x',z')\bar{c}(x,z)\rangle-\langle \lambda(x,z)\pi(x',z')\rangle+\frac{2}{\Gamma}\langle \lambda(x,z)\partial_{z'}\lambda(x',z')\rangle=\frac{1}{2}\langle c(x',z')\lambda(x,z)\int_{\Lambda_{uv}}^{z_f}dw\int d^Dy\beta[\lambda(y,w);w]\bar{c}(y,w)\rangle, \tag{69}$$

respectively.

Considering the ghost Green's function $(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)})\langle c(x,z)\bar{c}(x',z')\rangle = -\delta^{(D)}(x-x')\delta(z-z')$, we obtain

$$\langle\lambda(x',z')\pi(x,z)\rangle - \langle\lambda(x,z)\pi(x',z')\rangle = -\frac{2}{\Gamma}\langle\lambda(x,z)\partial_{z'}\lambda(x',z')\rangle - \langle\lambda(x',z')\left(\partial_z - \frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\right)^{-1}\beta[\lambda(x,z);z]\rangle \quad (70)$$

from Eqs. (68) and (69). If we consider a fixed point defined by $\beta[\lambda(x,z);z] = 0$, we obtain

$$\langle\lambda(x',z')\pi(x,z)\rangle - \langle\lambda(x,z)\pi(x',z')\rangle = -\frac{2}{\Gamma}\langle\lambda(x,z)\partial_{z'}\lambda(x',z')\rangle. \quad G_R - G_A = G_K$$

This is essentially the same as the fluctuation-dissipation theorem of the Langevin dynamics in equilibrium. Away from the fixed point, there is an RG flow given by the RG β-function, which plays the role of the nonequilibrium work in the dynamics, reflected in the last term of Eq. (70).

Generalized fluctuation − dissipation theorem to constrain multi − particle dynamics

Monotonicity of the RG flow

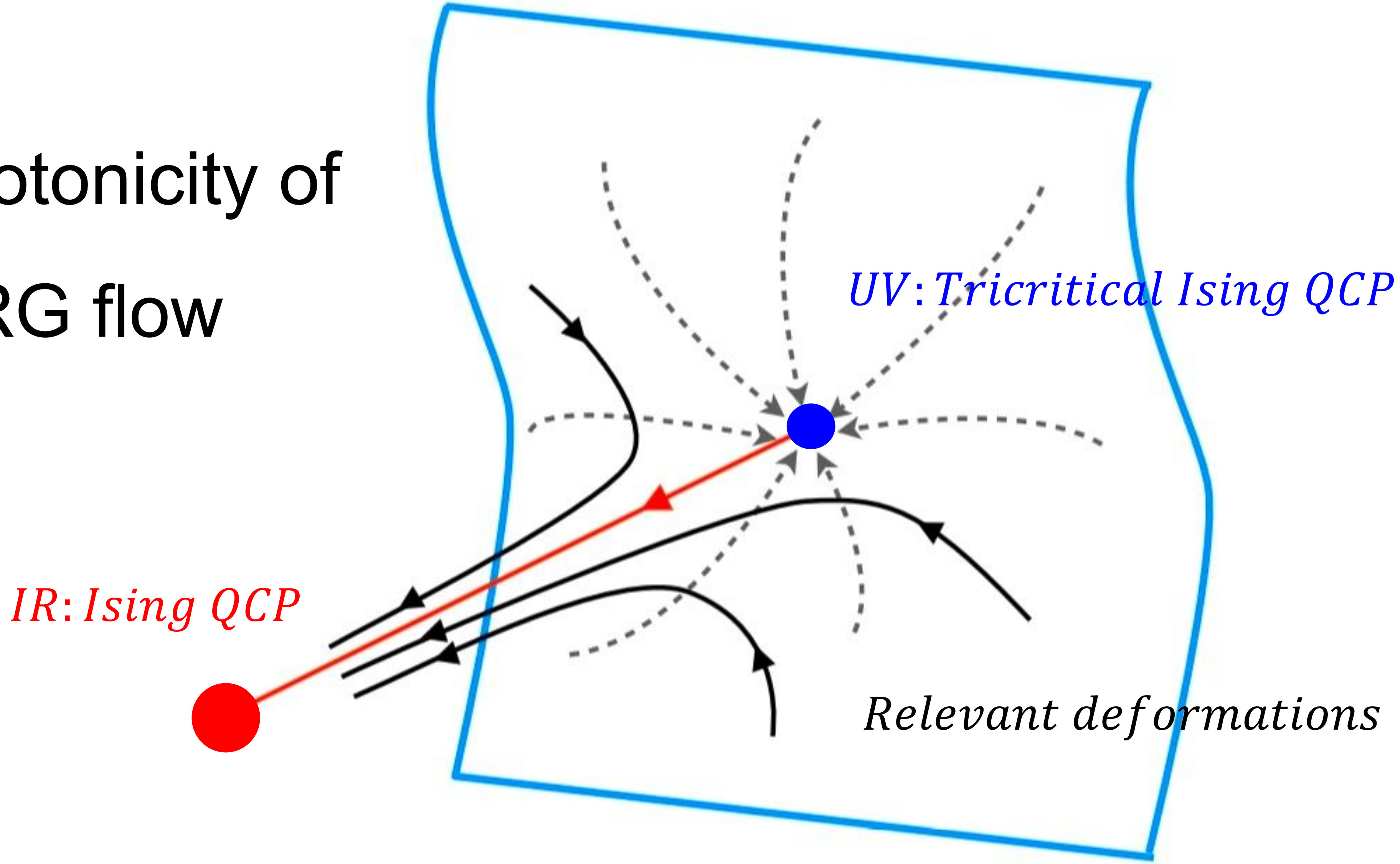
in the entropy production perspective

$\leftrightarrow$

WZ consistency condition for the Weyl anomaly

in the local RG equation

Monotonicity of RG flow



$c - theorem\ is\ a\ No - Go\ \ theorem\ in\ dynamics.$

$c - theorem\ may\ be\ the\ most\ general$

$No - Go\ \ theorem\ in\ physics.$

Is there a mechanism of

symmetry or topology protection

as a “conventional” No-Go theorem?

"Irreversibility" of the flux of the renormalization group in a 2D field theory

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There exists a function $c(g)$ of the coupling constant g in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where c is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

On Renormalization Group Flows in Four Dimensions

cliché

Zohar Komargodski ♠♡ *and Adam Schwimmer* ♠

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We discuss some general aspects of renormalization group flows in four dimensions. Every such flow can be reinterpreted in terms of a spontaneously broken conformal symmetry. We analyze in detail the consequences of trace anomalies for the effective action of the Nambu-Goldstone boson of broken conformal symmetry. While the c-anomaly is algebraically trivial, the a-anomaly is "non-Abelian," and leads to a positive-definite universal contribution to the S-matrix element of $2 \to 2$ dilaton scattering. Unitarity of the S-matrix r … Euler anomali…

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IS THERE A c-THEOREM IN FOUR DIMENSIONS?

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Received 27 September 1988

The difficulties of extending Zamolodchikov's c-theorem to dimensions $d \neq 2$ are discussed. It is shown that, for d even, the one-point function of the trace of the stress tensor on the sphere, S^d, when suitably regularized, defines a c-function, which, at least to one loop order, is decreasing along RG trajectories and is stationary at RG fixed points, where it is proportional to the usual conformal anomaly. It is shown that the existence of such a c-function, if it satisfies these properties to all orders, is consistent with the expected behavior of QCD in four dimensions.

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A finite entanglement entropy and the c-theorem

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Abstract

The trace over the degrees of freedom located in a subset of the space transforms the vacuum state into a mixed density matrix with nonzero entropy. This is usually called entanglement entropy, and it is known to be divergent in quantum field theory (QFT). However, it is possible to define a finite quantity $F(A, B)$ for two given different subsets A and B which measures the degree of entanglement between their respective degrees of freedom. We show that the function $F(A, B)$ is severely constrained by the Poincaré symmetry and the mathematical properties of the entropy. In particular, for one component sets in two-dimensional conformal field theories its general form is completely determined. Moreover, it allows to prove an alternative entropic version of the c-theorem for $(1 + 1)$-dimensional QFT. We propose this well-defined quantity as the meaningfull entanglement entropy and comment on possible applications in QFT and the black hole evaporation problem.

Relevant deformations in CFT2:
RG flows as gradient flows

$$Z_{CFT_2} = \int D\varphi_k(x) \exp\left\{ - S_{CFT_2}[\varphi_k(x)] - \sum_\alpha g_\alpha \int \frac{d^2x}{a^{2(1-\Delta_\alpha)}} \mathcal{O}^\alpha[\varphi_k(x)] \right\}$$

$$\frac{dg_\alpha}{d\ln\tau} = \beta_\alpha(\{g_\alpha\}) = 2(1-\Delta_\alpha)g_\alpha - \pi \sum_{\beta,\gamma} \mathcal{C}_{\alpha\beta\gamma} g_\beta g_\gamma.$$

$$O^\alpha O^\beta \sim \sum_\gamma C^{\alpha\beta\gamma} O_\gamma$$

$$\beta_\alpha(\{g_\alpha\}) = -\frac{\partial \mathcal{C}(\{g_\alpha\})}{\partial g_\alpha}, \qquad \mathcal{C}(\{g_\alpha\}) = -\sum_\alpha (1-\Delta_\alpha) g_\alpha^2 + \frac{\pi}{3} \sum_{\alpha,\beta,\gamma} \mathcal{C}_{\alpha\beta\gamma} g_\alpha g_\beta g_\gamma.$$

Zamolodchikov c−theorem in the RG-flow perspectives

$$\Big(\frac{1}{2}R\frac{\partial}{\partial R}-\sum_{\alpha}\beta_{\alpha}(R,\{g_{\alpha}\})\frac{\partial}{\partial g_{\alpha}}\Big)C(R,\{g_{\alpha}\})=0, \qquad R^2=z\bar{z}$$

$$\sum_{\alpha}\beta_{\alpha}(R,\{g_{\alpha}\})\frac{\partial}{\partial g_{\alpha}}C(R,\{g_{\alpha}\})=-\frac{3}{4}\sum_{\alpha,\beta}\beta_{\alpha}(R,\{g_{\alpha}\})G^{\alpha\beta}(R)\beta_{\beta}(R,\{g_{\alpha}\})$$

$$\Theta(z,\bar{z})=\sum_{\alpha}\beta_{\alpha}(R,\{g_{\alpha}\})\mathcal{O}^{\alpha}[\varphi_k(z,\bar{z})]$$

$$T_{\mu}^{\mu}+\sum_{\alpha}\beta_{\alpha}O^{\alpha}=0$$

$$G^{\alpha\beta}(R)=(2\pi)^2R^2\langle\mathcal{O}^{\alpha}[\varphi_k(z,\bar{z})]\mathcal{O}^{\beta}[\varphi_k(0,0)]\rangle$$

We not only reinterpret the c-theorem in the perspective of entropy production but also “generalize” it beyond the one loop order.

Previously, the monotonicity of the RG flow (c-theorem & a-theorem) has been rederived from the WZ consistency condition of the Weyl anomaly in the local RG equation.

Here, we discuss it in the entropy production perspective, which turns out to be consistent with the WZ consistency condition of the local RG equation.

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Nuclear Physics B286 (1987) 349–377
North-Holland, Amsterdam

A LOCAL RENORMALIZATION GROUP EQUATION, DIFFEOMORPHISMS AND CONFORMAL INVARIANCE IN SIGMA MODELS*

G.M. SHORE**

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The c and a-theorems and the Local Renormalisation Group

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Abstract: The Zamolodchikov c-theorem has led to important new insights in our understanding of the renormalisation group and the geometry of the space of QFTs. Here, we review the parallel developments of the search for a higher-dimensional generalisation of the c-theorem and of the Local Renormalisation Group.

The idea of renormalisation with position-dependent couplings, running under local Weyl scaling, is traced from its early realisations to the elegant modern formalism of the local renormalisation group. The key rôle of the associated Weyl consistency conditions in establishing RG flow equations for the coefficients of the trace anomaly in curved spacetime, and their relation to the c-theorem and four-dimensional a-theorem, is explained in detail.

A number of different derivations of the c-theorem in two dimensions are presented – using spectral functions, RG analysis of Green functions of the energy-momentum tensor $T_{\mu\nu}$, and dispersion relations – and are generalised to four dimensions. The obstruction to establishing monotonic C-functions related to the β_c and β_b trace anomaly coefficients in four dimensions is discussed. The possibility of deriving an a-theorem, involving the coefficient β_a of the Euler-Gauss-Bonnet density in the trace anomaly, is explored initially by formulating the QFT on maximally symmetric spaces. Then the formulation of the weak a-theorem using a dispersion relation for four-point functions of $T^{\mu}{}_{\mu}$ is presented.

Finally, we describe the application of the local renormalisation group to the issue of limit cycles in theories with a global symmetry and it is shown how this sheds new light on the geometry of the space of couplings in QFT.

Nuclear Physics B363 (1991) 486–526
North-Holland

WEYL CONSISTENCY CONDITIONS AND A LOCAL RENORMALISATION GROUP EQUATION FOR GENERAL RENORMALISABLE FIELD THEORIES

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Received 5 March 1991
(Revised 13 May 1991)

A local renormalisation group equation which realises infinitesimal Weyl rescalings of the metric and which is an extension of the usual Callan–Symanzik equation is described. In order to ensure that any local composite operators, with dimensions so that on addition to the basic lagrangian they preserve renormalisability, are well defined for arbitrarily many insertions into correlation fu
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Local Renormalisation Group Equations in Quantum Field Theory

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ABSTRACT

A local renormalisation group equation is formulated for renormalisable theories which describes the effect of local Weyl rescalings of the metric. In two dimensions the resulting equations are shown to correspond to Zamolodchikov's c-theorem and in four dimensions to give results which may have a similar significance.

$\delta\gamma^{\mu\nu} = 2\sigma\gamma^{\mu\nu}$

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$\Delta^{W}_{\sigma} W = \Delta^{\beta}_{\sigma} W - \frac{1}{l}\int \mathrm{d}v\, \sigma\left(\tfrac{1}{2}\beta^{\Phi} R - \tfrac{1}{2}\chi_{ij}\partial_\mu g^i \partial^\mu g^j\right) + \frac{1}{l}\int \mathrm{d}v\, \partial_\mu\sigma\, w_i \partial^\mu g^i ,$

$\Delta^{W}_{\sigma} = 2$ …

$$\left(\frac{1}{2}R\frac{\partial}{\partial R} - \sum_\alpha \beta_\alpha(R,\{g_\alpha\})\frac{\partial}{\partial g_\alpha}\right)C(R,\{g_\alpha\}) = 0, \qquad R^2 = z\bar{z}$$

l scale

$\Delta^{\beta}_{\sigma} =$ …

$$0 = \left[\Delta^{W}_{\sigma} - \Delta^{\beta}_{\sigma}, \Delta^{W}_{\sigma'} - \Delta^{\beta}_{\sigma'}\right] W = \frac{1}{l}\int \mathrm{d}v(\sigma'\partial_\mu\sigma - \sigma\partial_\mu\sigma')V^\mu ,$$

$$\sum_\alpha \beta_\alpha(R,\{g_\alpha\})\frac{\partial}{\partial g_\alpha}C(R,\{g_\alpha\}) = -\frac{3}{4}\sum_{\alpha,\beta}\beta_\alpha(R,\{g_\alpha\})G^{\alpha\beta}(R)\beta_\beta(R,\{g_\alpha\})$$

the couplings. If

$$\delta W = \frac{1}{l}\int \mathrm{d}v\left(\tfrac{1}{2}bR - \tfrac{1}{2}c_{ij}\partial_\mu g^i \partial^\mu g^j\right), \qquad (2.10)$$

then, for $b(g), c_{ij}(g)$ an arbitrary scalar, tensor respectively on the space of couplings,

$$\delta\beta^{\Phi} = \beta^i\partial_i b, \qquad \delta\chi_{ij} = \mathscr{L}_\beta c_{ij},$$

$$\delta w_i = -\partial_i b + c_{ij}\beta^j, \qquad \delta\tilde{\beta}^{\Phi} = c_{ij}\beta^i\beta^j. \qquad (2.11)$$

It is easy to see that eq. (2.7), or eq. (2.8), are invariant under the changes (2.11). In general it is not possible to set $w_i = 0$ under such a redefinition except when $w_i = \partial_i X$.

$$\partial_i\beta^{\Phi} = \chi_{ij}\beta^j - \mathscr{L}_\beta w_i, \qquad \mathscr{L}_\beta w_i = \beta^j\partial_j w_i + \partial_i\beta^j w_j, \qquad (2.7)$$

$\mathscr{L}_\beta$ denotes the Lie derivative defined by the vector field β^i. From eq. (2.7)

$$\partial_i\tilde{\beta}^{\Phi} = \chi_{ij}\beta^j + (\partial_i w_j - \partial_j w_i)\beta^j, \qquad \tilde{\beta}^{\Phi} = \beta^{\Phi} + w_i\beta^i, \qquad (2.8)$$

nce

$$\beta^i\partial_i\tilde{\beta}^{\Phi} = \chi_{ij}\beta^i\beta^j \qquad (2.9)$$

The entropy formula for the Ricci flow and its geometric applications

Grisha Perelman*

February 1, 2008

Introduction

1. The Ricci flow equation, introduced by Richard Hamilton [H 1], is the evolution equation $\frac{d}{dt}g_{ij}(t) = -2R_{ij}$ for a riemannian metric $g_{ij}(t)$. In his seminal paper, Hamilton proved that this equation has a unique solution for a short time for an arbitrary (smooth) metric on a closed manifold. The evolution equation for the metric tensor implies the evolution equation for the curvature tensor of the form $Rm_t = \triangle Rm + Q$, where Q is a certain quadratic expression of the curvatures. In particular, the scalar curvature R satisfies $R_t = \triangle R + 2|\mathrm{Ric}|^2$, so by the maximum principle its minimum is non-decreasing along the flow. By developing a maximum principle for tensors, Hamilton [H 1,H 2] proved that Ricci flow preserves the positivity of the Ricci tensor in dimension three and of the curvature operator in all dimensions; moreover, the eigenvalues of the Ricci tensor in dimension three and of the curvature operator in dimension four are getting pinched pointwisely as the curvature is getting large. This observation allowed him to prove the convergence results: the evolving metrics (on a closed manifold) of positive Ricci curvature in dimension three, or positive curvature operator

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cliché

Perelman's interpretation for the $Zamolodchikov$'s c-theorem

$$\partial_t g_{ij}(x,t) = -2R_{ij}(x,t)$$

$$RG\ flow \rightarrow Ricci\ flow$$

$$Zamolodchikov\ c - functional$$
$$for\ the\ gradient\ flow\ of\ the\ RG\ flow$$

⬇

$$Perelman's\ entropy\ functional$$
$$for\ the\ gradient\ flow\ of\ the\ Ricci\ flow$$

PHYSICAL REVIEW D **75**, 064024 (2007)

Sigma model renormalization group flow, "central charge" action, a

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(Received 25 January 2007; published 20 March 2007)

Zamolodchikov's c-theorem type argument (and also string theory effective action
that the RG flow in 2d sigma model should be a gradient one to all loop orders. Howe
of the flow of the target-space metric is not obvious since the metric on the spa
couplings is indefinite. To leading (one-loop) order when the RG flow is simply
monotonicity was proved by Perelman [G. Perelman, math.dg/0211159.] by constr
functional which is essentially the metric-dilaton action extremized with respect t
condition that the target-space volume is fixed. We discuss how to generalize the Per
to all loop orders (i.e. all orders in α'). The resulting entropy is equal to minus the
fixed points, in agreement with the general claim of the c-theorem.

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NUCLEAR PHYSICS B

A gradient flow for worldsheet nonlinear sigma models

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Abstract

We discuss certain recent mathematical advances, mainly due to Perelman, in the theory of Ricci flows and their relevance for renormalization group (RG) flows. We consider nonlinear sigma models with closed target manifolds supporting a Riemannian metric, dilaton, and 2-form B-field. By generalizing recent mathematical results to incorporate the B-field and by decoupling the dilaton, we are able to describe the 1-loop β-functions of the metric and B-field as the components of the gradient of a potential functional on the space of coupling constants. We emphasize a special choice of diffeomorphism gauge generated by the lowest eigenfunction of a certain Schrödinger operator whose potential and kinetic terms evolve along the flow. With this choice, the potential functional is the corresponding lowest eigenvalue, and gives the order α' correction to the Weyl anomaly at fixed points of $(g(t), B(t))$. The lowest eigenvalue is monotonic along the flow, and since it reproduces the Weyl anomaly at fixed points, it accords with the c-theorem for flows that remain always in the first-order regime. We compute the Hessian of the lowest eigenvalue functional and use it to discuss the linear stability of points where the 1-loop β-functions vanish, such as flat tori and K3 manifolds.

PRL **95,** 040602 (2005) PHYSICAL REVIEW LETTERS week ending 22 JULY 2005

Entropy Production along a Stochastic Trajectory and an Integral Fluctuation Theorem

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(Received 29 March 2005; published 20 July 2005)

For stochastic nonequilibrium dynamics like a Langevin equation for a colloidal particle or a master equation for discrete states, entropy production along a single trajectory is studied. It involves both genuine particle entropy and entropy production in the surrounding medium. The integrated sum of both Δs_{tot} is shown to obey a fluctuation theorem $\langle \exp[-\Delta s_{\text{tot}}] \rangle = 1$ for arbitrary initial conditions and arbitrary time-dependent driving over a finite time interval.

DOI: 10.1103/PhysRevLett.95.040602 PACS numbers: 05.40.−a, 05.70.−a

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Nonequilibrium Equality for Free Energy Differences

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(Received 7 June 1996)

An expression is derived for the equilibrium free energy difference between two configurations of a system, in terms of an ensemble of *finite-time* measurements of the work performed in parametrically switching from one configuration to the other. Two well-known identities emerge as limiting cases of this result. [S0031-9007(97)02845-7]

PHYSICAL REVIEW E VOLUME 56, NUMBER 5 NOVEMBER 1997

Equilibrium free-energy differences from nonequilibrium measurements: A master-equation approach

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(Received 18 June 1997)

In has recently been shown that the Helmholtz free-energy difference between two equilibrium configurations of a system may be obtained from an ensemble of *finite-time* (nonequilibrium) measurements of the work performed in switching an external parameter of the system. Here this result is established, as an identity, within the master equation formalism. Examples are discussed and numerical illustrations provided. [S1063-651X(97)10710-3]

Journal of Statistical Physics, Vol. 90, Nos. 5/6, 1998

cliché

Nonequilibrium Measurements of Free Energy Differences for Microscopically Reversible Markovian Systems

Gavin E. Crooks[1]

Received October 20, 1997; final December 16, 1997

An equality has recently been shown relating the free energy difference between two equilibrium ensembles of a system and an ensemble average of the work required to switch between these two configurations. In the present paper it is shown that this result can be derived under the assumption that the system's dynamics is Markovian and microscopically reversible.

KEY WORDS: Nonequilibrium statistical mechanics; free energy; work; thermodynamic integration; thermodynamic perturbation.

PHYSICAL REVIEW E VOLUME 60, NUMBER 3 SEPTEMBER 1999

Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences

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(Received 17 February 1999)

There are only a very few known relations in statistical dynamics that are valid for systems driven arbitrarily far-from-equilibrium. One of these is the fluctuation theorem, which places conditions on the entropy production probability distribution of nonequilibrium systems. Another recently discovered far from equilibrium expression relates nonequilibrium measurements of the work done on a system to equilibrium free energy differences. In this paper, we derive a generalized version of the fluctuation theorem for stochastic, microscopically reversible dynamics. Invoking this generalized theorem provides a succinct proof of the nonequilibrium work relation. [S1063-651X(99)10109-0]

PHYSICAL REVIEW E VOLUME 61, NUMBER 3 MARCH 2000

Path-ensemble averages in systems driven far from equilibrium

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(Received 30 August 1999)

The Kawasaki nonlinear response relation, the transient fluctuation theorem, and the Jarzynski nonequilibrium work relation are all expressions that describe the behavior of a system that has been driven from equilibrium by an external perturbation. In contrast to linear response theory, these expressions are exact no matter the strength of the perturbation, or how far the system has been driven away from equilibrium. In this paper, I show that these three relations (and several other closely related results) can all be considered special cases of a single theorem. This expression is explicitly derived for discrete time and space Markovian dynamics, with the additional assumptions that the unperturbed dynamics preserve the appropriate equilibrium ensemble, and that the energy of the system remains finite.

where the normalization constant or the generating functional is given by Eq. (5). One can verify

$$\int_{x_i}^{x_f} dx p(x,t) = 1. \tag{9}$$

To discuss the entropy production in the forced overdamped Langevin dynamics, Ref. [27] proposed a trajectory-dependent entropy for the particle or system as

$$s_{\mathrm{sys}}(x,t) = -\ln p(x,t). \tag{10}$$

This definition is consistent with the common definition of a nonequilibrium Gibbs entropy, given by

$$S_{\mathrm{sys}}(t) = \langle s_{\mathrm{sys}}(x,t)\rangle = -\int_{x_i}^{x_f} dx p(x,t) \ln p(x,t). \tag{11}$$

This microscopic entropy gives rise to the macroscopic thermodynamic entropy for an equilibrium Boltzmann distribution at fixed λ,

$$\partial_t S_{\mathrm{tot}}(t) = \langle \partial_t s_{\mathrm{tot}}(x,t)\rangle = \int_{x_i}^{x_f} dx \frac{j^2(x,t)}{Dp(x,t)} \geq 0, \tag{15}$$

where the equality holds in equilibrium only. The ensemble-averaged entropy production rate of the environment is given by

where the equilibrium free energy $F(\lambda)$ is $F(\lambda) = -\beta^{-1} \ln \int_{x_i}^{x_f} dx e^{-\beta V(x,\lambda)}$ with the conserved potential $V(x,\lambda)$ introduced before. Then, it is natural to consider the rate of heat dissipation in the environment as

$$\partial_t q(x,t) = F(x,\lambda)\partial_t x(t) = \beta^{-1}\partial_t s_{\mathrm{env}}(x,t) \tag{13}$$

Accordingly, one may identify the exchanged heat with an increase in the environment entropy $s_{\mathrm{env}}(x,t)$ at temperature $\beta^{-1} = D/\mu$.

Combining these two contributions, Ref. [27] found the trajectory-dependent total entropy production rate as follows

$$\begin{aligned}\partial_t s_{\mathrm{tot}}(x,t) &= \partial_t s_{\mathrm{env}}(x,t) + \partial_t s_{\mathrm{sys}}(x,t)\\ &= \frac{\partial_x j(x,t)}{p(x,t)} + \frac{j(x,t)}{Dp(x,t)}\partial_t x(t).\end{aligned} \tag{14}$$

Taking the ensemble average, Ref. [27] showed that the

$$\partial_t S_{\mathrm{env}}(x,t) = \langle \partial_t s_{\mathrm{env}}(x,t)\rangle = \beta \int_{x_i}^{x_f} dx F(x,t) j(x,t), \tag{16}$$

where the force $F(x,t)$ and the conserved current $j(x,t)$ have been introduced above. In this study, we discuss the entropy production of the RG flow, following this line of thought.

The "probability distribution" function for the coupling field is defined as follows

$$\rho(\lambda, z) = \langle \delta(\lambda - \lambda(x, z)) \rangle$$
$$= \mathcal{N} \int D\xi(x, z') \exp\left\{ -N \int_{\Lambda_{uv}}^{z} dz' \int d^D x \left(\frac{1}{2\Gamma} \xi^2(x, z') + \mathcal{V}_{rg}[\lambda(x, z'); z'] \right) \right\} \delta(\lambda - \lambda(x, z)) \tag{84}$$

Following the standard procedure to derive the Fokker-Planck equation from the Langevin equation, we obtain

$$(\partial_z - \mathcal{V}_{rg}(\lambda, z))\rho(\lambda, z) = -\partial_\lambda \left\{ \left(\beta(\lambda, z) - \frac{\Gamma}{2} \partial_\lambda \right) \rho(\lambda, z) \right\}. \tag{87}$$

where the RG effective potential $\mathcal{V}_{rg}(\lambda, z)$ serves as the "time" component of a background gauge field. The conserved current is given by

$$j(\lambda, z) = \left(\beta(\lambda, z) - \frac{\Gamma}{2} \partial_\lambda \right) \rho(\lambda, z), \tag{88}$$

which shares essentially the same structure as that of the overdamped Langevin dynamics, discussed before. In Appendix A, we show our intuitive derivation for this Fokker-Planck equation.

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Following Ref. [27], it is natural to introduce the entropy of a system, given by

$$s_{\rm sys}(\lambda, z) = -\ln \rho(\lambda, z). \tag{89}$$

Then, the ensemble average of the system or bulk entropy is

$$S_{\rm sys}(z) = \langle s_{\rm sys}(\lambda, z) \rangle = -\int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \rho(\lambda, z) \ln \rho(\lambda, z), \tag{90}$$

as expected.

The time evolution of the bulk entropy is given by

$$\partial_z s_{\rm sys}(z) = -\frac{\partial_z \rho(\lambda, z)}{\rho(\lambda, z)} - \frac{\partial_\lambda \rho(\lambda, z)}{\rho(\lambda, z)} \partial_z \lambda(x, z). \tag{91}$$

Resorting to the Fokker-Planck equation and considering the definition of the conserved current, we rewrite the above expression as follows

$$\partial_z s_{\rm sys}(z) = \frac{\partial_\lambda j(\lambda,z)}{\rho(\lambda,z)} - \mathcal{V}_{rg}(\lambda,z) + \frac{2}{\Gamma}\frac{j(\lambda,z)}{\rho(\lambda,z)}\partial_z\lambda(x,z) - \frac{2}{\Gamma}\beta(\lambda,z)\partial_z\lambda(x,z). \quad (92)$$

Here, we introduce the time evolution of the "environment" entropy in a similar way as Ref. [27],

$$\partial_z s_{\rm env}(\lambda,z) = \partial_z q(\lambda,z) = \frac{2}{\Gamma}\beta(\lambda,z)\partial_z\lambda(x,z) + \mathcal{V}_{rg}(\lambda,z). \quad (93)$$

$\partial_z q(\lambda,z)$ is the rate of heat dissipation in the medium, where we identify the exchanged heat with an increase in entropy of the medium.

Summing over these two contributions, we obtain

$$\begin{aligned}\partial_z s_{\rm tot}(\lambda,z) &= \partial_z s_{\rm env}(\lambda,z) + \partial_z s_{\rm sys}(\lambda,z)\\ &= \frac{\partial_\lambda j(\lambda,z)}{\rho(\lambda,z)} + \frac{2}{\Gamma}\frac{j(\lambda,z)}{\rho(\lambda,z)}\partial_z\lambda(x,z),\end{aligned} \quad (94)$$

fully consistent with that of the overdamped Langevin dynamics [27], although there exists a clear modification in the Fokker-Planck equation, Eq. (87). As a result, we find the irreversibility of the RG flow, given by the total entropy function,

$$\partial_z S_{\rm tot}(z) = \langle \partial_z s_{\rm tot}(\lambda,z)\rangle = \int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \frac{2}{\Gamma}\frac{j^2(\lambda,z)}{\rho(\lambda,z)} \geq 0, \quad (95)$$

where the ensemble average has been taken, and the following current conservation has been used,

$$\left\langle \frac{\partial_\lambda j(\lambda,z)}{\rho(\lambda,z)} \right\rangle = \int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \partial_\lambda j(\lambda,z) = 0. \quad (96)$$

More explicitly, we have

$$\langle \partial_z s_{\rm tot}(\lambda,z)\rangle = \int_{\lambda_{uv}}^{\lambda_{ir}} d\lambda \rho(\lambda,z)\left\{\frac{2}{\Gamma}[\beta(\lambda,z)]^2 + \frac{\Gamma}{2}(\partial_\lambda \ln\rho(\lambda,z))^2 - 2\beta(\lambda,z)\partial_\lambda \ln\rho(\lambda,z)\right\} \geq 0. \quad (97)$$

III. ENTROPY PRODUCTION RATE IN THE EMERGENT HOLOGRAPHIC DUAL EFFECTIVE FIELD THEORY

We now focus on the bulk part to derive the Fokker-Planck equation. The bulk effective Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} &= \Pi_G^{\mu\nu}(x,z)\left(\partial_z G_{\mu\nu}(x,z) + \alpha' R_{\mu\nu}(x,z) + 2\alpha'\nabla_\mu\nabla_\nu\Phi(x,z) - \frac{\alpha'}{4}H_{\mu\lambda\omega}(x,z)H_\nu^{\lambda\omega}(x,z)\right) + \frac{\lambda}{2}\Pi_G^{\mu\nu}(x,z)\mathcal{G}_{\mu\nu\rho\lambda}(x,z)\Pi_G^{\rho\lambda}(x,z) \\ &+ \Pi_B^{\mu\nu}(x,z)\left(\partial_z B_{\mu\nu}(x,z) - \frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu}(x,z) + \alpha'\nabla^\omega\Phi(x,z)H_{\omega\mu\nu}(x,z)\right) + \frac{q}{2}\Pi_{B,\mu\nu}(x,z)\Pi_B^{\mu\nu}(x,z) \\ &+ \frac{\alpha'}{4}R(x,z) - \frac{D-26}{6} - \frac{\alpha'}{48}H_{\mu\nu\lambda}(x,z)H^{\mu\nu\lambda}(x,z) + \alpha'\partial_\mu\Phi(x,z)\partial^\mu\Phi(x,z). \qquad (30)\end{aligned}$$

Considering the Legendre transformation,

$$\mathcal{L}_{\mathrm{eff}} = \Pi_G^{\mu\nu}(x,z)\partial_z G_{\mu\nu}(x,z) + \Pi_B^{\mu\nu}(x,z)\partial_z B_{\mu\nu}(x,z) + \mathcal{H}_{\mathrm{eff}} \qquad (31)$$

in Euclidean spacetime, we obtain the effective bulk Hamiltonian as

$$\begin{aligned}\mathcal{H}_{\mathrm{eff}} &= \Pi_G^{\mu\nu}(x,z)\left(\alpha' R_{\mu\nu}(x,z) + 2\alpha'\nabla_\mu\nabla_\nu\Phi(x,z) - \frac{\alpha'}{4}H_{\mu\lambda\omega}(x,z)H_\nu^{\lambda\omega}(x,z)\right) + \frac{\lambda}{2}\Pi_G^{\mu\nu}(x,z)\mathcal{G}_{\mu\nu\rho\lambda}(x,z)\Pi_G^{\rho\lambda}(x,z) \\ &+ \Pi_B^{\mu\nu}(x,z)\left(-\frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu}(x,z) + \alpha'\nabla^\omega\Phi(x,z)H_{\omega\mu\nu}(x,z)\right) + \frac{q}{2}\Pi_{B,\mu\nu}(x,z)\Pi_B^{\mu\nu}(x,z) \\ &+ \frac{\alpha'}{4}R(x,z) - \frac{D-26}{6} - \frac{\alpha'}{48}H_{\mu\nu\lambda}(x,z)H^{\mu\nu\lambda}(x,z) + \alpha'\partial_\mu\Phi(x,z)\partial^\mu\Phi(x,z). \qquad (32)\end{aligned}$$

To obtain an effective Fokker-Planck equation, we introduce the identification

$$\Pi_G^{\mu\nu}(x,z) \equiv -\frac{\partial}{\partial G_{\mu\nu}(x,z)}, \qquad \Pi_B^{\mu\nu}(x,z) \equiv -\frac{\partial}{\partial B_{\mu\nu}(x,z)} \tag{33}$$

into the effective Hamiltonian, in a similar manner discussed in the review discussed in Appendix B. Then, we construct the corresponding Fokker-Planck equation for the Langevin-type RG flow equation (modified by noise fluctuations) as follows:

$$\begin{aligned}
&\left(\partial_z - \frac{\alpha'}{4}R(x,z) + \frac{D-26}{6} + \frac{\alpha'}{48}H_{\mu\nu\lambda}(x,z)H^{\mu\nu\lambda}(x,z) - \alpha'\partial_\mu\Phi(x,z)\partial^\mu\Phi(x,z)\right)\mathcal{P}(G_{\mu\nu},B_{\mu\nu};z) \\
&\quad = -\frac{\partial}{\partial G_{\mu\nu}(x,z)}\left\{\left(\alpha' R_{\mu\nu}(x,z) + 2\alpha'\nabla_\mu\nabla_\nu\Phi(x,z) - \frac{\alpha'}{4}H_{\mu\lambda\omega}(x,z)H_\nu^{\lambda\omega}(x,z) - \frac{\lambda}{2}\mathcal{G}_{\mu\nu\rho\gamma}(x,z)\frac{\partial}{\partial G_{\rho\gamma}(x,z)}\right)\mathcal{P}(G_{\mu\nu},B_{\mu\nu};z)\right\} \\
&\qquad - \frac{\partial}{\partial B_{\mu\nu}(x,z)}\left\{\left(-\frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu}(x,z) + \alpha'\nabla^\omega\Phi(x,z)H_{\omega\mu\nu}(x,z) - \frac{q}{2}\frac{\partial}{\partial B_{\mu\nu}(x,z)}\right)\mathcal{P}(G_{\mu\nu},B_{\mu\nu};z)\right\},
\end{aligned} \tag{34}$$

where $\mathcal{P}(G_{\mu\nu},B_{\mu\nu};z)$ is the probability distribution function associated with both the RG flows of the metric and two-form gauge field along z direction.

$$J^G_{\mu\nu}(x,z) = \left(\alpha' R_{\mu\nu}(x,z) + 2\alpha'\nabla_\mu\nabla_\nu\Phi(x,z) - \frac{\alpha'}{4}H_{\mu\lambda\omega}(x,z)H^{\lambda\omega}_{\nu}(x,z) - \frac{\lambda}{2}\mathcal{G}_{\mu\nu\rho\gamma}(x,z)\frac{\partial}{\partial G_{\rho\gamma}(x,z)}\right)\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z), \quad (37)$$

$$J^B_{\mu\nu}(x,z) = \left(-\frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu}(x,z) + \alpha'\nabla^\omega\Phi(x,z)H_{\omega\mu\nu}(x,z) - \frac{q}{2}\frac{\partial}{\partial B^{\mu\nu}(x,z)}\right)\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z). \quad (38)$$

Based on these currents, the Fokker-Planck equation (34) is formally expressed as

$$\partial_z\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z) + \partial_{G_{\mu\nu}}J^G_{\mu\nu}(x,z) + \partial_{B_{\mu\nu}}J^B_{\mu\nu}(x,z) = \mathcal{V}_{\text{eff}}(x,z)\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z), \quad (41)$$

where $\mathcal{V}_{\text{eff}}(x,z)$ is the Wilsonian effective potential,

$$\mathcal{V}_{\text{eff}}(x,z) = \frac{\alpha'}{4}R(x,z) - \frac{D-26}{6} - \frac{\alpha'}{48}H_{\mu\nu\lambda}(x,z)H^{\mu\nu\lambda}(x,z) + \alpha'\partial_\mu\Phi(x,z)\partial^\mu\Phi(x,z). \quad (42)$$

Since there is an external "potential" term in the RG flow, the conservation law is modified.

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A tentative theory of large distance physics

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Abstract: A theoretical mechanism is devised to determine the large distance physics of spacetime. It is a two dimensional nonlinear model, the lambda model, set to govern the string worldsurface in an attempt to remedy the failure of string theory, as it stands. The lambda model is formulated to cancel the infrared divergent effects of handles at short distance on the worldsurface. The target manifold is the manifold of background spacetimes. The coupling strength is the spacetime coupling constant. The lambda model operates at 2d distance Λ^{-1}, very much shorter than the 2d distance μ^{-1} where the worldsurface is seen. A large characteristic spacetime distance L is given by $L^2 = \ln(\Lambda/\mu)$. Spacetime fields of wave number up to $1/L$ are the local coordinates for the manifold of spacetimes. The distribution of fluctuations at 2d distances shorter than Λ^{-1} gives the *a priori* measure on the target manifold, the manifold of spacetimes. If this measure concentrates at a macroscopic spacetime, then, nearby, it is a measure on the spacetime fields. The lambda model thereby constructs a spacetime quantum field theory, cutoff at ultraviolet distance L, describing physics at distances larger than L. The lambda model also constructs an effective string theory with infrared cutoff L, describing physics at distances smaller than L. The lambda model evolves outward from zero 2d distance, $\Lambda^{-1} = 0$, building spacetime physics starting from $L = \infty$ and proceeding downward in L. L can be taken smaller than any distance practical for experiments, so the lambda model, if right, gives all actually observable physics. The harmonic surfaces in the manifold of spacetimes are expected to have novel nonperturbative effects at large distances.

Writing the *a priori* measure in the variables λ_e^i as $\mathrm{d}\rho_e(\Lambda,\lambda_e)$, the driven diffusion process is

$$-\Lambda\frac{\partial}{\partial\Lambda_{/\lambda_e}}\,\mathrm{d}\rho_e(\Lambda,\lambda_e) = \nabla_i^e\left(T\,g_e^{ij}(\lambda_e)\nabla_j^e + \beta_e^i(\lambda_e)\right)\mathrm{d}\rho_e(\Lambda,\lambda_e) \tag{2.33}$$

where ∇_i^e is the covariant derivative with respect to the effective metric $T^{-1}g_{ij}^e$. The coefficients, $T\,g_e^{ij}$ and β_e^i, of the diffusion process are stationary, independent of Λ^{-1}, because of the generalized scale invariance of the effective lambda model.

All the considerations that applied to the classical lambda model carry over to the effective lambda model. The effective *a priori* measure satisfies an effective diffusion equation, which takes the same form as the tree-level diffusion equation. The generalized scale invariance of the effective lambda model implies that the diffusion equation has stationary coefficients,

$$\begin{aligned}-\Lambda\frac{\partial}{\partial\Lambda_{/\lambda_e}}\,\rho_e(\Lambda,\lambda_e) &= \nabla_i^e\left(T_e g_e^{ij}\,\partial_j + \beta_e^i\right)\rho_e \\ &= \nabla_i^e\,T_e g_e^{ij}\left(\partial_j + \partial_j(T_e^{-1}a_e)\right)\rho_e\,. \end{aligned} \tag{8.16}$$

The effective *a priori* measure is the equilibrium measure

$$\mathrm{dvol}_e(\lambda_e)\,\mathrm{e}^{-T_e^{-1}a_e(\lambda_e)} \tag{8.17}$$

which satisfies the equation of motion $\beta_e = 0$,

$$0 = \left(\partial_i + T_e^{-1}g_{ij}^e\,\beta_e^j\right)\,\mathrm{e}^{-T_e^{-1}a_e(\lambda_e)}\,. \tag{8.18}$$

Following Seifert [77], we introduce a path-dependent microscopic entropy in the target space as

$$s_{\rm sys}(G_{\mu\nu}, B_{\mu\nu}) = -\ln \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z). \tag{43}$$

Then, the observable "system" entropy is given by the average of the path-dependent entropy with respect to the probability distribution in the following way:

$$S_{\rm sys}(z) = \langle s_{\rm sys}(G_{\mu\nu}, B_{\mu\nu}; z)\rangle = -\int_{[G_{\mu\nu}(x,0),B_{\mu\nu}(x,0)]}^{[G_{\mu\nu}(x,z_f),B_{\mu\nu}(x,z_f)]} d[G_{\mu\nu}, B_{\mu\nu}]\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z) \ln \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z). \tag{44}$$

Here, $\int_{[G_{\mu\nu}(x,0),B_{\mu\nu}(x,0)]}^{[G_{\mu\nu}(x,z_f),B_{\mu\nu}(x,z_f)]} d[G_{\mu\nu}, B_{\mu\nu}]$ means the line integral, given the dilaton field as a functional of $G_{\mu\nu}(x, z)$ and $B_{\mu\nu}(x, z)$ in the δ-function constraint of Eq. (29). We suspect that this Gibbs entropy would reproduce the black hole entropy at high temperatures.

Benchmarking Ref. [77], we define the *environmental entropy* production rate as follows:

$$\begin{aligned}\partial_z s_{\rm env}(G_{\mu\nu}, B_{\mu\nu}; z) = &\frac{\alpha'}{4} R(x,z) - \frac{D-26}{6} - \frac{\alpha'}{48} H_{\mu\nu\lambda}(x,z) H^{\mu\nu\lambda}(x,z) + \alpha' \partial_\mu \Phi(x,z) \partial^\mu \Phi(x,z) \\ &+ \frac{2}{\lambda}\left(\alpha' R^{\mu\nu}(x,z) + 2\alpha' \nabla^\mu \nabla^\nu \Phi(x,z) - \frac{\alpha'}{4} H^{\mu\lambda\omega}(x,z) H^{\nu}_{\lambda\omega}(x,z)\right)[\partial_z G_{\mu\nu}(x,z)] \\ &+ \frac{2}{q}\left(-\frac{\alpha'}{2} \nabla_\omega H^{\omega\mu\nu}(x,z) + \alpha' \nabla_\omega \Phi(x,z) H^{\omega\mu\nu}(x,z)\right)[\partial_z B_{\mu\nu}(x,z)]. \end{aligned} \tag{47}$$

In fact, this identification is quite natural since the RG β function and the Wilsonian effective action play the roles of external force and potential, respectively. Thus, the production rate of the path-dependent *microscopic total entropy* ($s_{\rm tot} \equiv s_{\rm sys} + s_{\rm env}$) is given by

$$\begin{aligned}\partial_z s_{\rm tot}(G_{\mu\nu}, B_{\mu\nu}; z) &= \partial_z s_{\rm env}(G_{\mu\nu}, B_{\mu\nu}; z) + \partial_z s_{\rm sys}(G_{\mu\nu}, B_{\mu\nu}; z) \\ &= \frac{\partial_{G_{\mu\nu}} J^G_{\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)} + \frac{2}{\lambda}\frac{J^{G,\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)}[\partial_z G_{\mu\nu}(x,z)] + \frac{\partial_{B_{\mu\nu}} J^B_{\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)} + \frac{2}{q}\frac{J^{B,\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)}[\partial_z B_{\mu\nu}(x,z)]. \end{aligned} \tag{48}$$

We must emphasize at this point that $s_{\rm tot}$ is in fact not the thermodynamic entropy.

Taking the ensemble average with the probability distribution $\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)$, we obtain

$$\partial_z S_{\rm tot}(z) = \langle \partial_z s_{\rm tot}(G_{\mu\nu}, B_{\mu\nu}; z)\rangle$$
$$= \frac{2}{\lambda}\int_{G_{\mu\nu}(x,0)}^{G_{\mu\nu}(x,z_f)} dG_{\mu\nu} \frac{J^G_{\mu\nu}(x,z)\mathcal{G}^{\mu\nu\rho\gamma}(x,z)J^G_{\rho\gamma}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)} + \frac{2}{q}\int_{B_{\mu\nu}(x,0)}^{B_{\mu\nu}(x,z_f)} dB_{\mu\nu} \frac{J^B_{\mu\nu}(x,z)J^{B,\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)} \geq 0, \qquad (49)$$

where we have used the following conservation equation, namely,

$$\left\langle \frac{\partial_{G_{\mu\nu}} J^G_{\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)} + \frac{\partial_{B_{\mu\nu}} J^B_{\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)} \right\rangle = \int_{G_{\mu\nu}(x,0)}^{G_{\mu\nu}(x,z_f)} dG_{\mu\nu}\partial_{G_{\mu\nu}} J^G_{\mu\nu}(x,z) + \int_{B_{\mu\nu}(x,0)}^{B_{\mu\nu}(x,z_f)} dB_{\mu\nu}\partial_{B_{\mu\nu}} J^B_{\mu\nu}(x,z) = 0. \qquad (50)$$

In the line integral expression $\int_{G_{\mu\nu}(x,0)}^{G_{\mu\nu}(x,z_f)} dG_{\mu\nu}$ $(\int_{B_{\mu\nu}(x,0)}^{B_{\mu\nu}(x,z_f)} dB_{\mu\nu})$, $B_{\mu\nu}(x,z)$ and $\Phi(x,z)$ $[G_{\mu\nu}(x,z)$ and $\Phi($
the fixed path determined by the RG transformation. Equation (49) indicates that the microscopic
always increases during the RG flow. In particular, this expression is quite similar to Eq. (C17) basec

It seems to be consistent with the WZ consistency condition of the conformal anomaly in the local RG equation.

definition of the entropy functional except for the integration in the metric and Kalb-Ramond field during the RG flow. We recall $J^G_{\mu\nu}(x,z) \sim \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)\partial_z G_{\mu\nu}(x,z)$ and $J^B_{\mu\nu}(x,z) \sim \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)\partial_z B_{\mu\nu}(x,z)$, where the RG flows of the metric and the Kalb-Ramond gauge field are modified by the introduction of noise. Based on the IR boundary condition to be discussed in the next section, we obtain $J^G_{\mu\nu}(x,z) \sim \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)\beta^G_{\mu\nu}$ and $J^B_{\mu\nu}(x,z) \sim \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)\beta^B_{\mu\nu}$, where $\partial_z G_{\mu\nu}(x,z) \sim \beta^G_{\mu\nu}$ and $\partial_z B_{\mu\nu}(x,z) \sim \beta^B_{\mu\nu}$. As a result, we obtain

$$\frac{J^G_{\mu\nu}(x,z)\mathcal{G}^{\mu\nu\rho\gamma}(x,z)J^G_{\rho\gamma}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)} + \frac{J^B_{\mu\nu}(x,z)J^{B,\mu\nu}(x,z)}{\mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)}$$
$$\sim \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)\beta^G_{\mu\nu}\mathcal{G}^{\mu\nu\rho\gamma}(x,z)\beta^G_{\rho\gamma} + \mathcal{P}(G_{\mu\nu}, B_{\mu\nu}; z)\beta^B_{\mu\nu}\beta^{B,\mu\nu}, \qquad (51)$$

which is essentially the same as the entropy production rate *á la* Perelman.

Quantum effective action

To discuss the RG flow of an IR effective action, we take the large N limit and obtain equations of motion with boundary conditions. We recall the holographic dual field theory,

$$Z(z_f)=\int D\psi_\sigma(x,z_f)D\lambda(x,z)D\pi(x,z)D\bar{c}(x,z)Dc(x,z)\exp\Big[-\int d^Dx\Big(\mathcal{L}[\psi_\sigma(x,z_f);\lambda(x,z_f);z_f]+\frac{N}{2\Gamma}[\lambda(x,\Lambda_{uv})-\bar{\lambda}(\Lambda_{uv})]^2\Big)$$
$$-N\int_{\Lambda_{uv}}^{z_f}dz\int d^Dx\Big\{\pi(x,z)(\partial_z\lambda(x,z)-\beta[\lambda(x,z);z])-\frac{\Gamma}{2}\pi^2(x,z)+\bar{c}(x,z)\Big(\partial_z-\frac{\partial\beta[\lambda(x,z);z]}{\partial\lambda(x,z)}\Big)c(x,z)+\mathcal{V}_{rg}[\lambda(x,z);z]\Big\}\Big].$$

$$\beta[\lambda(x,z);z]=-\frac{\partial\mathcal{V}_{rg}[\lambda(x,z);z]}{\partial\lambda(x,z)}.$$

$$\mathcal{V}_{rg}[\lambda(x,z_f);z_f]=-\frac{1}{N}\ln\int_{\Lambda(z_f)}D\psi_\sigma(x;z_f)\exp\Big\{-\int d^Dx\mathcal{L}[\psi_\sigma(x,z_f);\lambda(x,z_f);z_f]\Big\},$$

Following the standard procedure to derive the Fokker-Planck equation from the Langevin equation, we obtain

$$(\partial_z - \mathcal{V}_{rg}(\lambda, z))\rho(\lambda, z) = -\partial_\lambda\left\{\left(\beta(\lambda, z) - \frac{\Gamma}{2}\partial_\lambda\right)\rho(\lambda, z)\right\}, \tag{87}$$

where the RG effective potential $\mathcal{V}_{rg}(\lambda, z)$ serves as the "time" component of a background gauge field. The conserved current is given by

$$j(\lambda, z) = \left(\beta(\lambda, z) - \frac{\Gamma}{2}\partial_\lambda\right)\rho(\lambda, z), \tag{88}$$

which shares essentially the same structure as that of the overdamped Langevin dynamics, discussed before. In Appendix A, we show our intuitive derivation for this Fokker-Planck equation.

$$\rho(\lambda, z) = \frac{1}{Z(z_f)}\int_{\lambda_{uv}}^{\lambda} D\lambda(x, z')D\pi(x, z')D\bar{c}(x, z')Dc(x, z') \times \int D\xi(x, z')\exp\left\{-N\int_{\Lambda_{uv}}^{z} dz' \int d^D x\left(\frac{1}{2\Gamma}\xi^2(x, z') + \mathcal{V}_{rg}[\lambda(x, z'); z']\right)\right\} \times \exp\left[-N\int_{\Lambda_{uv}}^{z} dz' \int d^D x\left\{\pi(x, z')(\partial_{z'}\lambda(x, z') - \beta[\lambda(x, z'); z'] - \xi(x, z')) + \bar{c}(x, z')\left(\partial_{z'} - \frac{\partial\beta[\lambda(x, z'); z']}{\partial\lambda(x, z')}\right)c(x, z')\right\}\right],$$

$$-\frac{d}{dz_f}\ln Z(z_f) = 0.$$

$$\mathcal{S}_{\text{eff}}(z_f) = N\int d^D x\left(\mathcal{V}_{rg}[\lambda(x, z_f); z_f] + \pi(x, z_f)\lambda(x, z_f) + \bar{c}(x, z_f)c(x, z_f) + \frac{N}{2\Gamma}[\lambda(x, \Lambda_{uv}) - \bar{\lambda}(\Lambda_{uv})]^2 - \pi(x, \Lambda_{uv})\lambda(x, \Lambda_{uv}) - \bar{c}(x, \Lambda_{uv})c(x, \Lambda_{uv})\right).$$

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PAPER

Bayesian renormalization

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Keywords: renormalization, Bayesian inference, diffusion learning, information geometry, data compression, Fisher metric

Abstract

In this note we present a fully information theoretic approach to renormalization inspired Bayesian statistical inference, which we refer to as Bayesian renormalization. The main ins Bayesian renormalization is that the Fisher metric defines a correlation length that plays th an emergent renormalization group (RG) scale quantifying the distinguishability between points in the space of probability distributions. This RG scale can be interpreted as a proxy maximum number of unique observations that can be made about a given system during a statistical inference experiment. The role of the Bayesian renormalization scheme is subsec to prepare an effective model for a given system up to a precision which is bounded by the aforementioned scale. In applications of Bayesian renormalization to physical systems, the emergent information theoretic scale is naturally identified with the maximum energy tha probed by current experimental apparatus, and thus Bayesian renormalization coincides w ordinary renormalization. However, Bayesian renormalization is sufficiently general to app in circumstances in which an immediate physical scale is absent, and thus provides an idea approach to renormalization in data science contexts. To this end, we provide insight into Bayesian renormalization scheme relates to existing methods for data compression and dat generation such as the information bottleneck and the diffusion learning paradigm. We co by designing an explicit form of Bayesian renormalization inspired by Wilson's momentun renormalization scheme in quantum field theory. We apply this Bayesian renormalization to a simple neural network and verify the sense in which it organizes the parameters of the according to a hierarchy of information theoretic importance.

entropy

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Article

The Inverse of Exact Renormalization Group Flows as Statistical Inference

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Abstract: We build on the view of the Exact Renormalization Group (ERG) as an instantiation of Optimal Transport described by a functional convection–diffusion equation. We provide a new information-theoretic perspective for understanding the ERG through the intermediary of Bayesian Statistical Inference. This connection is facilitated by the Dynamical Bayesian Inference scheme, which encodes Bayesian inference in the form of a one-parameter family of probability distributions solving an integro-differential equation derived from Bayes' law. In this note, we demonstrate how the Dynamical Bayesian Inference equation is, itself, equivalent to a diffusion equation, which we dub *Bayesian Diffusion*. By identifying the features that define Bayesian Diffusion and mapping them onto the features that define the ERG, we obtain a dictionary outlining how renormalization can be understood as the inverse of statistical inference.

Keywords: Bayesian Inference; Exact Renormalization; Renormalization Group; diffusion; diffusion learning; Stochastic Differential Equations; Fisher Information; Information Geometry; entropy; relative entropy; gradient flow; error correction; channels

In Polchinski's picture, $K_\Lambda(p^2)$ has a prescribed dependence on Λ, thus Polchinski's ERG equation arises by determining the equation which must be obeyed by $S_{\text{int},\Lambda}[\phi]$ in order to satisfy the principle (2). By a straightforward computation, one can show that the resulting equation can be put into the form:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}P_\Lambda[\phi] = \int_{M\times M}\mathrm{d}^d x\mathrm{d}^d y\left\{C^{\text{Pol.}}_\Lambda(x,y)\frac{\delta^2 P_\Lambda[\phi]}{\delta\phi(x)\delta\phi(y)} + \frac{\delta}{\delta\phi(x)}\left(P_\Lambda[\phi]C^{\text{Pol.}}_\Lambda(x,y)\frac{\delta V^{\text{Pol.}}_\Lambda[\phi]}{\delta\phi(y)}\right)\right\} \quad (4)$$

$$\equiv \Delta P_\Lambda[\phi] + \operatorname{div}\left(P_\Lambda[\phi]\operatorname{grad}_{C^{\text{Pol.}}_\Lambda}V^{\text{Pol.}}_\Lambda[\phi]\right), \quad (5)$$

where

Functional Schrodinger equation

$$C^{\text{Pol.}}_\Lambda(p^2) = (2\pi)^d G(p^2)^{-1}\frac{\partial K_\Lambda(p^2)}{\partial\ln\Lambda}; \quad V^{\text{Pol.}}_\Lambda[\phi] = \int\frac{\mathrm{d}^d p}{(2\pi)^d}\phi(p)G(p^2)K^{-1}_\Lambda(p^2)\phi(-p). \quad (6)$$

One might recognize (4) as the *Fokker–Planck* equation with diffusion governed by $C^{\text{Pol.}}_\Lambda(p^2)$ and drift governed by the potential $V^{\text{Pol.}}_\Lambda[\phi]$. This is the first indication of a deep relationship between exact renormalization and diffusion. Note that the equivalence between (4) and (5) is just a rewriting in terms of the functional (infinite dimensional) equivalent of vector operators. This is so one can identify (4) as a functional version of Fokker–Plank.

Specializing to Fokker–Planck ERG schemes, we can expand on this discussion. As was introduced in detail in [18], a (functional) Fokker–Planck equation of the form (4) is associated with a (functional) stochastic differential equation (SDE):

Heisenberg operator equation

$$\mathrm{d}\phi(x) = -\operatorname{grad}_{C_\Lambda}V_\Lambda[\phi](d\ln\Lambda) + \sqrt{2}\int_M \mathrm{d}^d y\,\sigma_\Lambda(x,y)\,\mathrm{d}W_\Lambda(y). \quad (13)$$

Here, $W_\Lambda(x)$ is a function valued Weiner process, and σ_Λ is the diffusivity kernel defined by the property that it 'squares' to the covariance C_Λ:

$$\int_M \mathrm{d}^d z\,\sigma_\Lambda(x,z)\,\sigma_\Lambda(z,y) = C_\Lambda(x,y). \quad (14)$$

The Fokker–Planck equation corresponds to a bonafide ERG because it satisfies the ERG principle (2). To see that this is the case, let us now show that we can rewrite (4) in the form

$$-\frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}P_\Lambda[\phi]=\int_M \mathrm{d}^d x\,\frac{\delta}{\delta\phi(x)}\left(\Psi_\Lambda[\phi;x]\,P_\Lambda[\phi]\right),\tag{7}$$

where M is the spacetime manifold on which the theory is defined [12]. Hopefully it is clear that any one parameter family $P_\Lambda[\phi]$ satisfying (7) also satisfies (2). This is because (7) specifies a *divergence flow*, that is the right hand side of (7) is a divergence in the space of field configurations. We can therefore employ the divergence theorem to observe that

$$\frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}\int_{\mathcal{F}}\mathcal{D}\phi\;P_\Lambda[\phi]=-\int_{\mathcal{F}}\mathcal{D}\phi\int_M \mathrm{d}^d x\,\frac{\delta}{\delta\phi(x)}\left(\Psi_\Lambda[\phi;x]\,P_\Lambda[\phi]\right)=0.\tag{8}$$

Conserved current for the RG transformation

In order to write (4) in the form (7) we take

Gradient flow

$$\Psi_\Lambda[\phi;x]=\int_M \mathrm{d}^d y\,C_\Lambda(x,y)\,\frac{\delta\Sigma_\Lambda[\phi;P_\Lambda]}{\delta\phi(y)},\tag{9}$$

as has appeared previously in [12, 16, 17, 41, 45]. Here $C_\Lambda(x,y)$ is
Fokker–Planck equation associated to the ERG, and $\Sigma_\Lambda[\phi;P_\Lambda]$ is c
determined through the *ERG potential* V_Λ via the equation

$$S_{rel.}=N\int d^D x\,\pi(x,z_f)\,\lambda(x,z_f)$$

Relative entropy functional

$$\Sigma_\Lambda[\phi;P_\Lambda]=-\ln\left(\frac{P_\Lambda[\phi]}{\mathrm{e}^{-V_\Lambda[\phi]}}\right)=S_\Lambda[\phi]-V_\Lambda[\phi].\tag{10}$$

Plugging (9) back into (7), w
Together (C_Λ, V_Λ) therefore

$$S_{eff}(z_f)=N\int d^D x\left(\pi(x,z_f)\,\lambda(x,z_f)+V_{rg}[\lambda(x,z_f)]\right)$$

the field theory, in analogy with the regulating function $K_\Lambda^{-1}(p^2)$ appearing in (3).

In general, RG flow is Markovian, which gives rise to thermalization at the fixed point.