Axions
and
Generalized Symmetries

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Axions in Seoul 2026

Opening Remarks

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There is one more reason why AXION is special (and great).

In 4d, axion is special in that it can probe rich topology of gauge fields.

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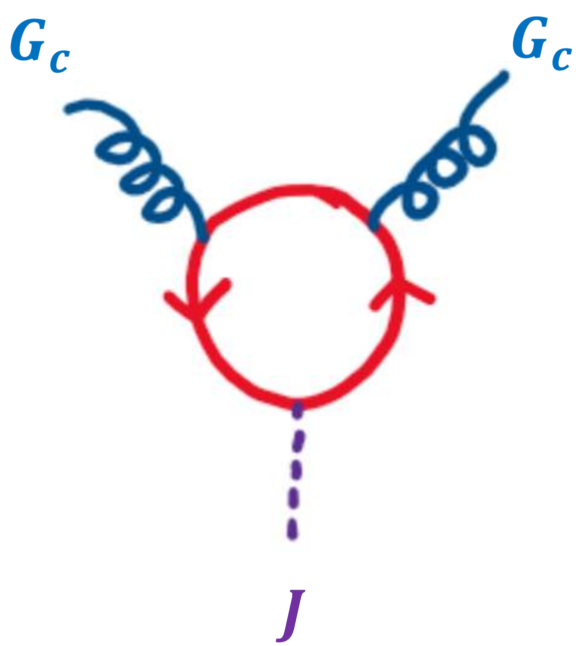
<u>Mixed (ABJ) anomaly</u>



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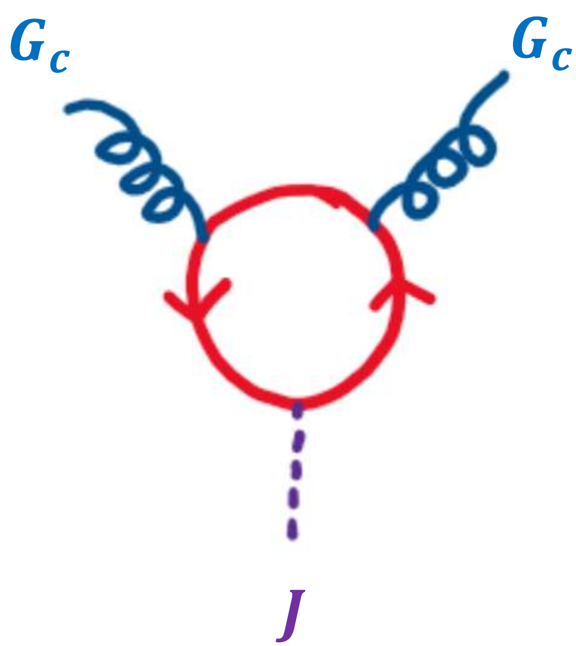

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(i) locally, one may have $G \wedge G(x) \neq 0$

(ii) globally, $\int \frac{1}{8\pi^2} tr(G \wedge G) \neq 0$

only for topologically non-trivial gauge fields

[Instantons]

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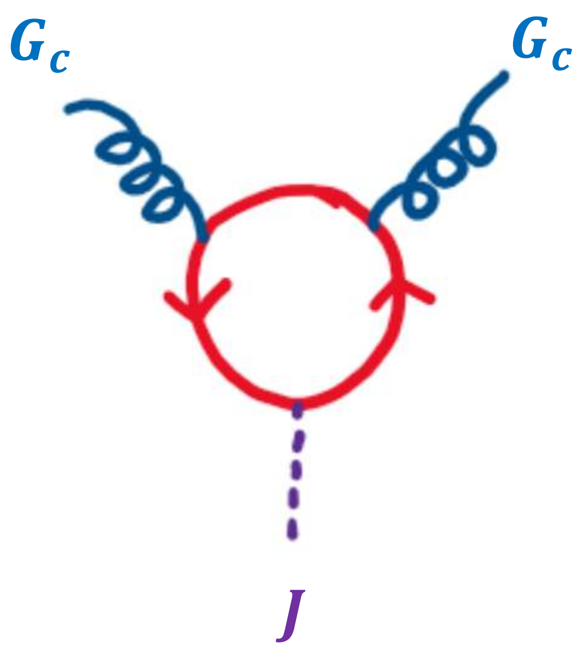

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(i) locally, one may have $G \wedge G(x) \neq 0$

(ii) globally, $\int \frac{1}{8\pi^2} tr(G \wedge G) \in \mathbb{Z}\,, \mathbb{Z}_N$

topological classification

[global form/topology of G_C]

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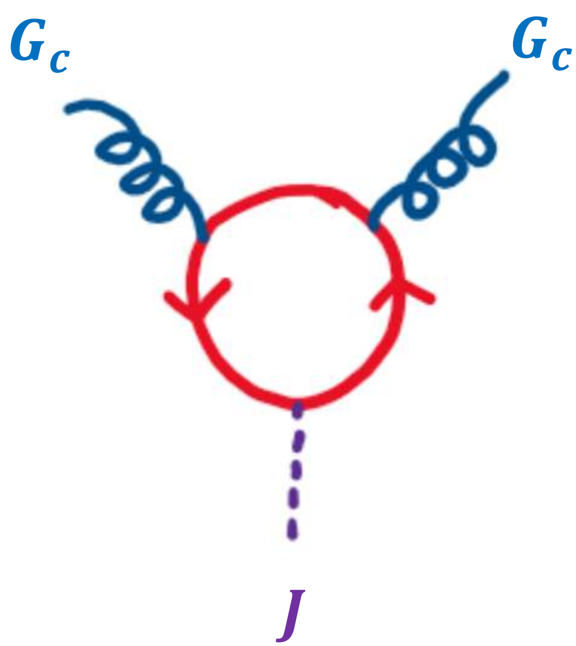

$$\boldsymbol{d * J = \frac{K}{8\pi^2}\, tr(G \wedge G)}$$

(iii) In 4d, only abelian J can have ABJ anomaly

anomaly $\propto tr(Q_i\, T_a\, T_b) = tr(Q_i) tr(T_a\, T_b)$

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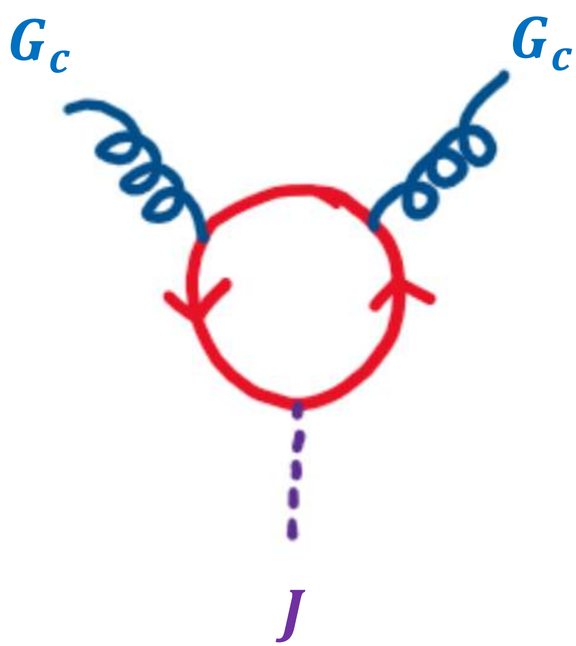

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anomaly $\propto tr(Q_i\, T_a\, T_b) = tr(Q_i) tr(T_a\, T_b)$

(iv) SSB of abelian J $\rightarrow$ NGB = AXION (ALP)

asymptotic state $\Rightarrow$ S-matrix

topology of Higgs manifold $\Rightarrow$ defects

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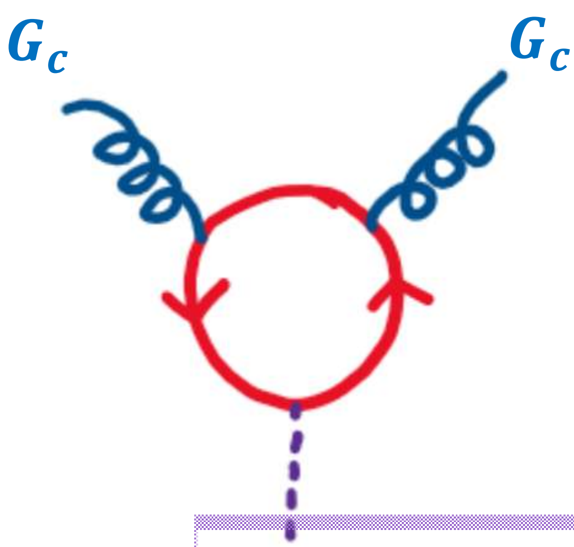

$J \quad \Rightarrow \quad \mathbf{SSB:} \quad J \sim d\theta$

(i) 0-form axion shift

(ii) 2-form axion winding

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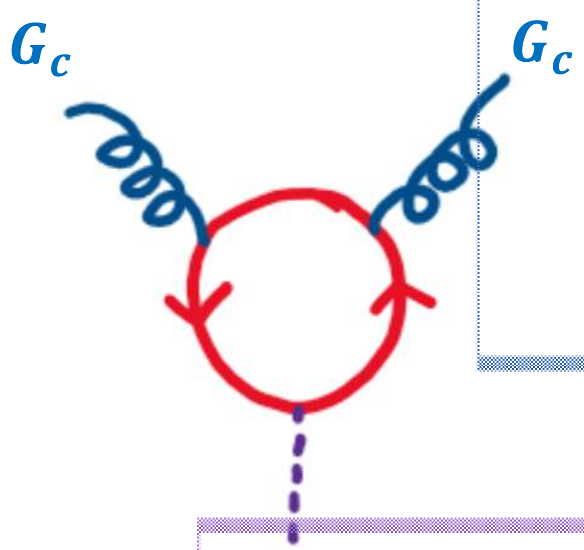

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(ii) 1-form magnetic : $dG = 0$

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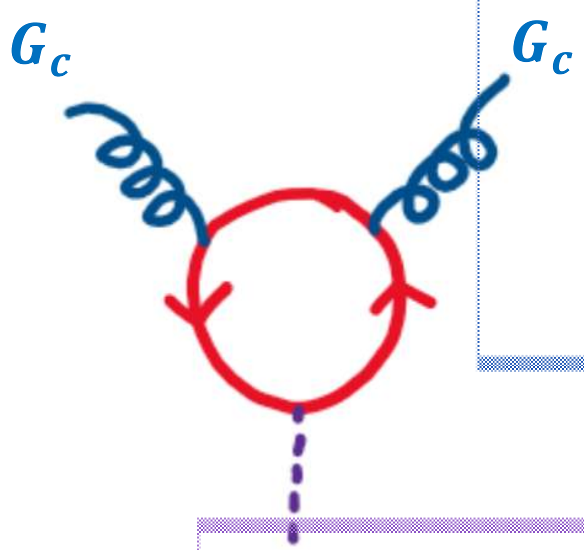

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ABJ anom = CS term

$\theta \, tr(G \wedge G)$

(i) higher-group

(ii) non-invertible

Outline

I. Generalized Global Symmetries

I-1. Higher-form symmetry
I-2. Non-invertible symmetry
I-3. Higher-group symmetry

II. Global Structure and Axion Defects

II-1. Global Structure of G_{SM} and Axion Domain Wall Problem
II-2. Global Structure and Lazarides-Shafi Mechanism

III. Extradimensional Axions and GGS

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- Axion-Maxwell Theory

$$S = \int \frac{1}{2} \left(\partial_\mu a\right)^2 + \int \frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} - \frac{iK}{16\pi^2} \int \frac{a}{f} F_{\mu\nu}\tilde{F}^{\mu\nu} \quad \text{(cf)} \; F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}\text{)}$$

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(i) axion is a periodic scalar: $\frac{a}{f} \sim \frac{a}{f} + 2\pi$ (gauge invariance)

(ii) This gauge invariance implies $K \in \mathbb{Z}$

$$\delta S = -2\pi K i \int \frac{F \wedge F}{8\pi^2}, \quad \int \frac{F \wedge F}{8\pi^2} \in \mathbb{Z}$$

(iii) Maxwell = $U(1)$ gauge theory: $F = dA, \quad A = A_\mu dx^\mu$

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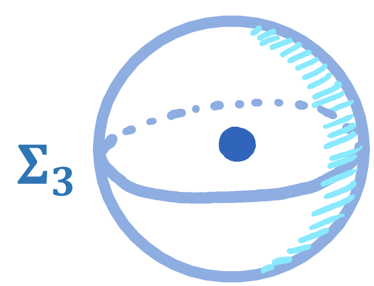
$$U_{aE}(\alpha, \Sigma_3)\!: \;\; I(m, x) = e^{im\frac{a(x)}{f}} \;\; \to \;\; e^{im\alpha} I(m, x)$$



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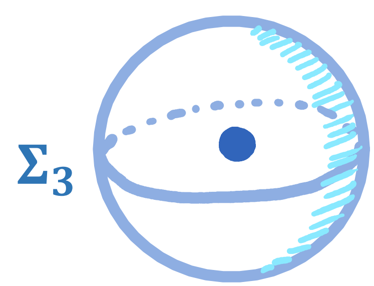

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$U_{aE}(\alpha, \Sigma_3)$ is invariant under $A \to A + d\lambda$ only for $\alpha = \frac{2\pi}{K}$
(equivalently, from the invariance of CS term)

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Note (a) Physical implications

*exact/unbroken $U(1)_a^{(0)} \;\; \Rightarrow$ massless axion

*unbroken ${\mathbb{Z}_K}^{(0)} \;\; \Rightarrow {\mathbb{Z}_K}^{(0)}$ symmetric axion potential $V(a)$

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(b) full symmetry: [invertible ${\mathbb{Z}_K}^{(0)} \;\subset$ **non-invertible** axion shift]

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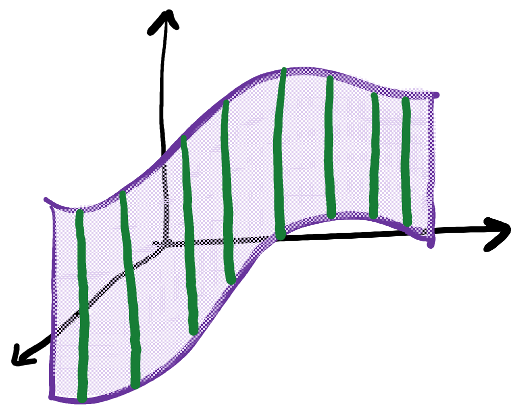

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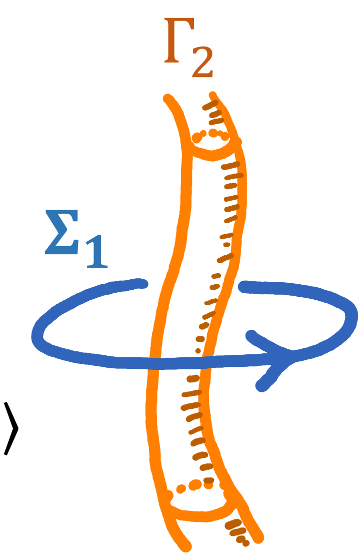

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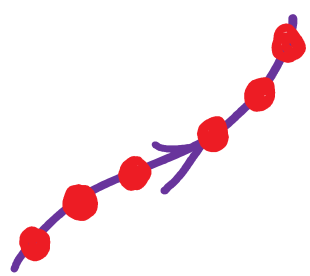
$$\text{EoM}(A)|_{K\neq 0}: \;\; d * \frac{1}{g^2} F = \frac{iK}{4\pi^2 f} d(aF) = J_{GW} \qquad \left[\mathbb{Z}_K{}^{(1)}\right]$$

<u>Note</u> charged object = Wilson line operators $W_1(m, \Gamma_1)$



Generalized Global Symmetries

I. Higher-form Symmetries of Axion-Maxwell

- Symmetries of Axion-Maxwell Theory

$$S = \int \frac{1}{2}\, da \wedge * \, da \; + \int \frac{1}{2g^2}\, F \wedge * \, F \; - \frac{iK}{8\pi^2} \int \frac{a}{f} F \wedge F$$

(iv) 1-form magnetic symmetry (dual to 1-form electric): $\left[U(1)_M^{(1)}\right]$

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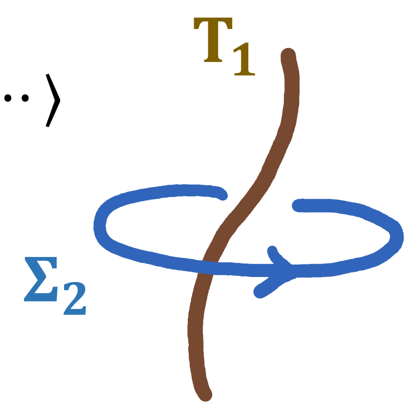
<u>Note</u> $U_{AM}(\alpha, \Sigma_2) = e^{i\alpha \oint_{\Sigma_2} * J_{2m}}, \;\; \alpha \in \mathbb{S}^1$

charged object = 't Hooft line operators $T_1(m, \Gamma_1)$

$$\langle U_{AE}(\alpha, \Sigma_2) T_1(m, \Gamma_1) \cdots \rangle = e^{i\alpha\ell m \mathrm{Link}(\Sigma_2, \Gamma_1)} \langle T_1(m, \Gamma_1) \cdots \rangle$$



Generalized Global Symmetries

I*. More on Higher-form Symmetries

1. $SU(N)$ YM (either pure YM or with only adj matter)

$\exists\ Z_N^{(1)}(e)$: under 0-form center $\Psi \to e^{\frac{2\pi i}{N} * N}\ \Psi$

$\to$ Wilson line with charge $= 0, 1, \cdots, (N-1)$ not screened

$\nexists$ mag 1-form : $\Pi_1\left(SU(N)\right) = \emptyset$

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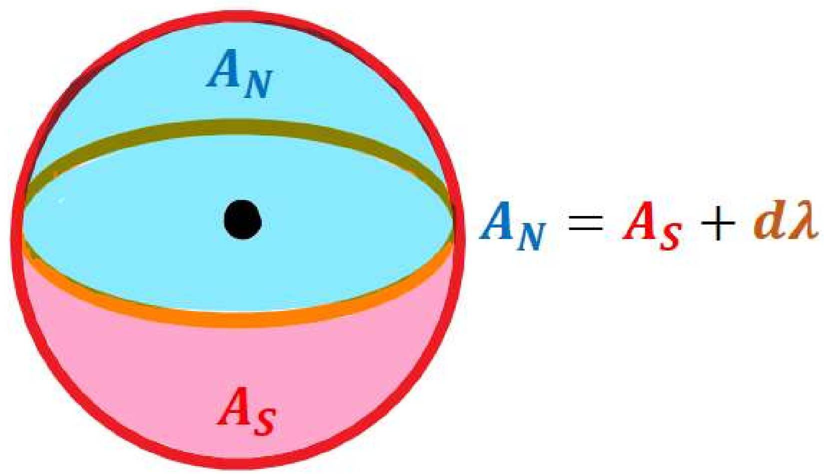

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2. $PSU(N) = \frac{SU(N)}{Z_N}$: $Z_N^{(1)}(e)$ is gauged (electric states projected out)

$\nexists$ electric 1-form

$\exists\, Z_N^{(1)}(m)$: $\Pi_1\left(PSU(N)\right) = Z_N$ or "$N * \frac{1}{N} = 1$"

$\Rightarrow \oint G_2 = 2\pi/N\,, \quad \int \mathrm{tr}(G_2 \wedge G_2) \supset \frac{N-1}{2N} \int w_2 \wedge w_2$

Fractional Instanton

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$$dF = 0 \;\; \rightarrow \;\; d(F \wedge F) = 0\,,\; J_4 \sim * J_0 = F \wedge F \qquad \left[U(1)^{(-1)}\right]$$

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(b) Axion as gauge field for (-1)-form symmetry

$$S \supset i \int B_{p+1} \wedge * J_{p+1} = i \int B_0 \wedge J_4 \sim i \int \theta F \wedge F$$

axion 0-form symmetry as "quantum" symmetry

cf) "No global symmetry" in QG

Generalized Global Symmetries

II. Non-invertible Symmetry (NIS)

$$S = \int \frac{1}{2}\, da \wedge\!* \, da \; + \int \frac{1}{2g^2}\, F \wedge\!* \, F \;\; - \frac{iK}{8\pi^2} \int \frac{a}{f} F \wedge F$$

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Note Chern-Simons term = anomaly-matching Wess-Zumino-Witten term

$\frac{a}{f} \leftrightarrow U(1)_{PQ}, \quad F \leftrightarrow U(1)_{EM} \;\; \Rightarrow \;\; \frac{a}{f}\, F \wedge F \;\leftrightarrow\; U(1)_{PQ}[U(1)_{EM}]^2$ anomaly

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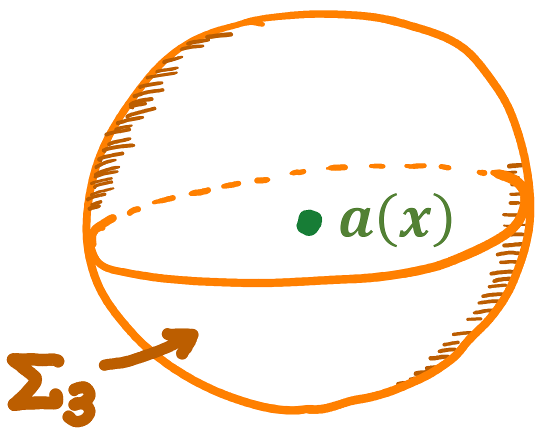
under $\frac{a}{f} \to \frac{a}{f} + \alpha, \qquad S \;\to S - iK\alpha \int \frac{F\wedge F}{8\pi^2}$ "anomalous transformation"



$\rightarrow \quad a(x) + \alpha f$

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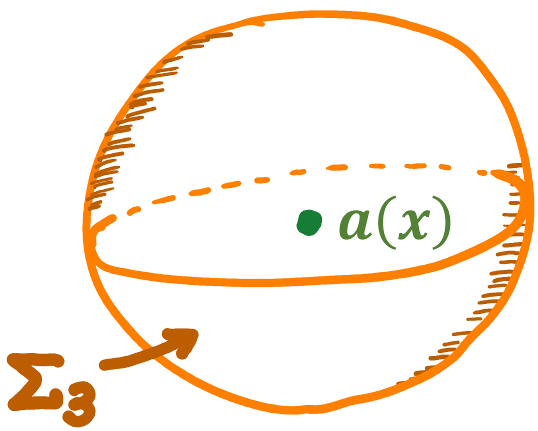
under $\frac{a}{f} \to \frac{a}{f} + \frac{2\pi}{z}, \qquad S \;\to S - \frac{2\pi i K}{z} \int \frac{F \wedge F}{8\pi^2}$ "anomalous transformation"



$$\to \quad a(x) + \frac{2\pi f}{z}$$

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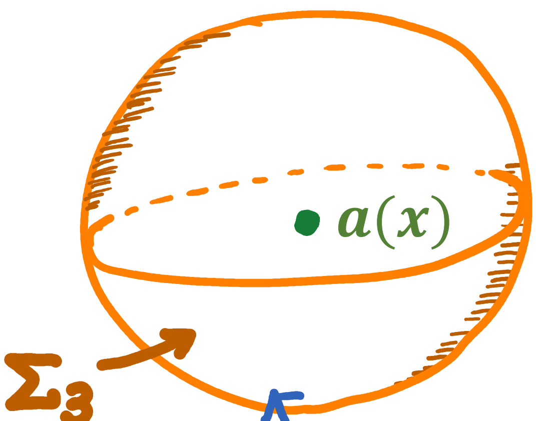

$$S_{\text{defect}} = \frac{iNp}{4\pi} \int_{\Sigma_3} C \wedge dC \ + \frac{ip}{2\pi} \int_{\Sigma_3} C \wedge dA$$

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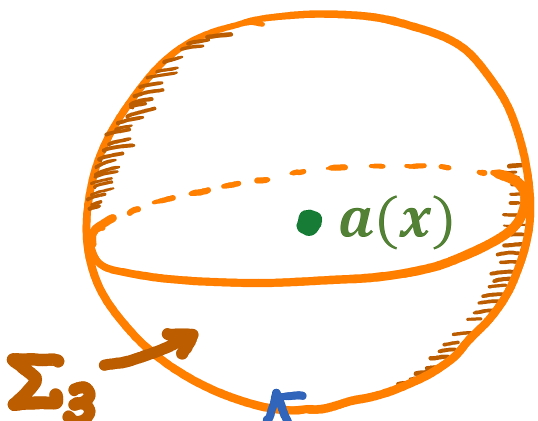

$Z_{Np}^{(1)}$ 1-form global symmetry : $C \rightarrow C + \lambda_1$

with $Z_N^{(1)} \subset Z_{Np}^{(1)}$ "partial gauging"

$$S_{\text{defect}} = \frac{iNp}{4\pi} \int_{\Sigma_3} C \wedge dC \; + \frac{ip}{2\pi} \int_{\Sigma_3} C \wedge dA$$

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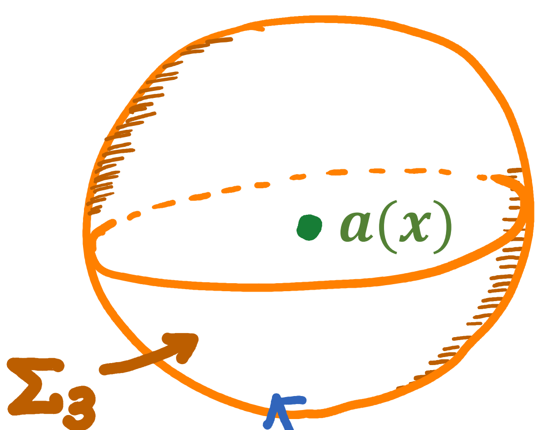

t 'Hooft anomaly of $Z_{Np}^{(1)}$ 1-form global symmetry

$$S_{\text{inflow}} = -\frac{2\pi i p}{N} \int_{M_4} \frac{\mathcal{P}(B_2)}{2} \;\rightarrow\; -\frac{2\pi i p}{N} \int_{M_4} \frac{F \wedge F}{8\pi^2}$$

$$S_{\text{defect}} = \frac{iNp}{4\pi} \int_{\Sigma_3} C \wedge dC \;+ \frac{ip}{2\pi} \int_{\Sigma_3} C \wedge dA$$

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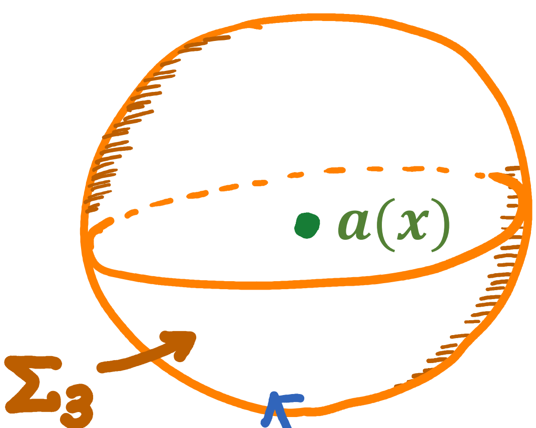

t 'Hooft anomaly of $Z_N^{(1)}$ 1-form global symmetry

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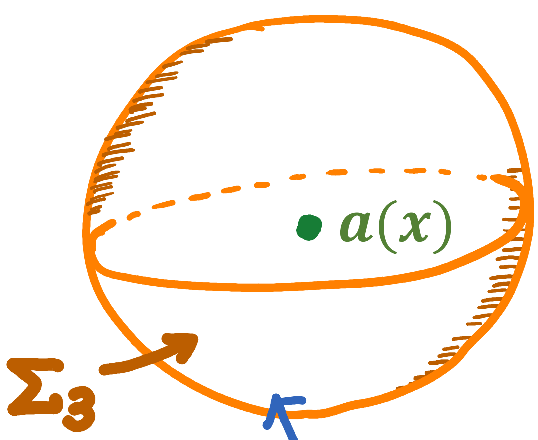


$$U\left(\frac{2\pi}{k}, \Sigma_3\right) \to D_k = U\left(\frac{2\pi}{k}, \Sigma_3\right) \times \mathcal{A}^{N,p}\left(\frac{F}{2\pi}\right)$$

$$D_k \times \overline{D}_k \sim \sum_S \xi(S) \exp\left(\frac{i}{2\pi N} \int_S F\right) \neq 1 \;\; \text{(Non-Invertible)}$$

$$S_{\text{defect}} = \frac{iNp}{4\pi} \int_{\Sigma_3} C \wedge dC \; + \frac{ip}{2\pi} \int_{\Sigma_3} C \wedge dA$$

More on Non-Invertible Symmetry

[Half-Space Gauging]

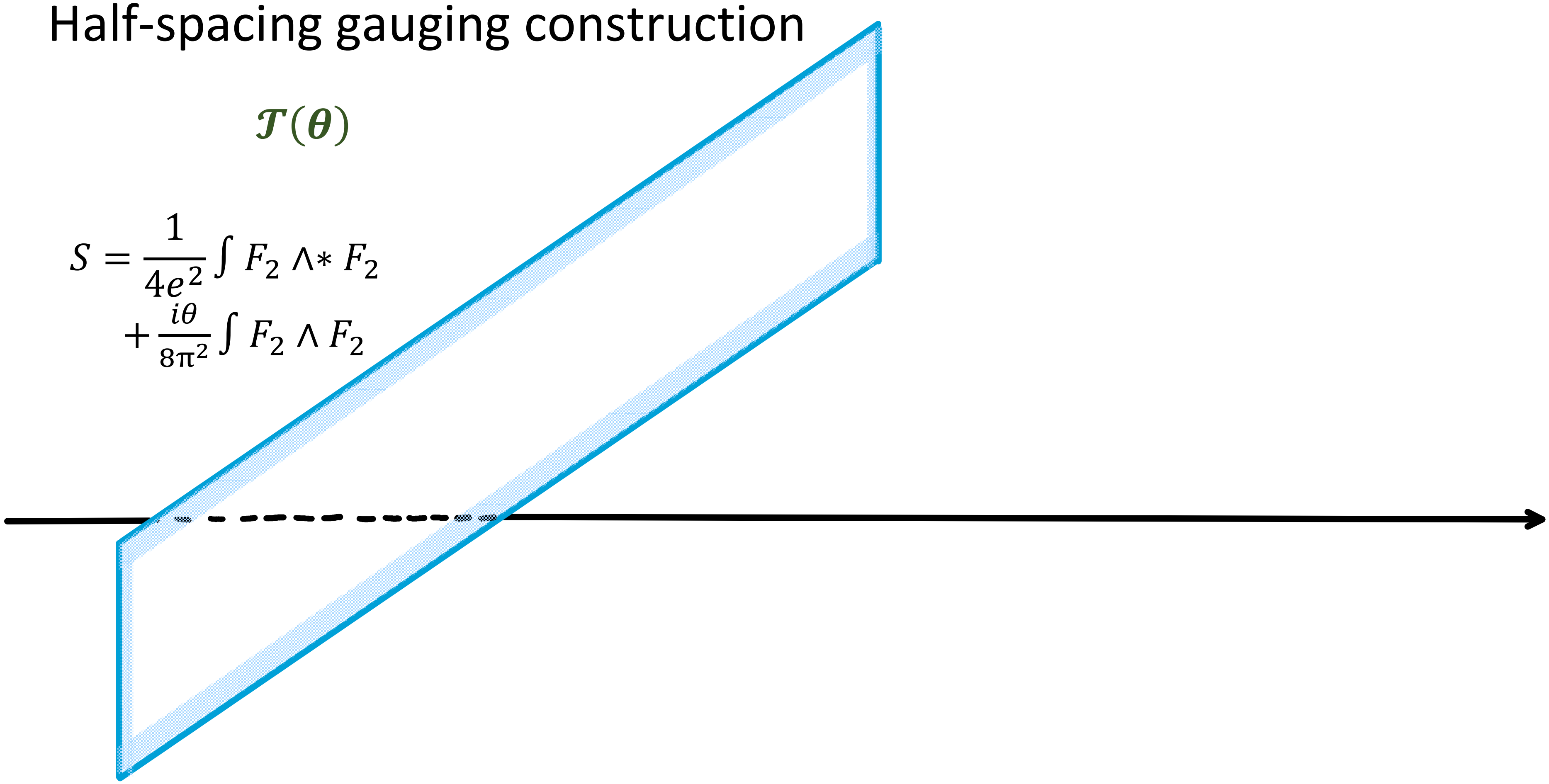
Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction

$\boldsymbol{\mathcal{T}(\theta)}$

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2$$



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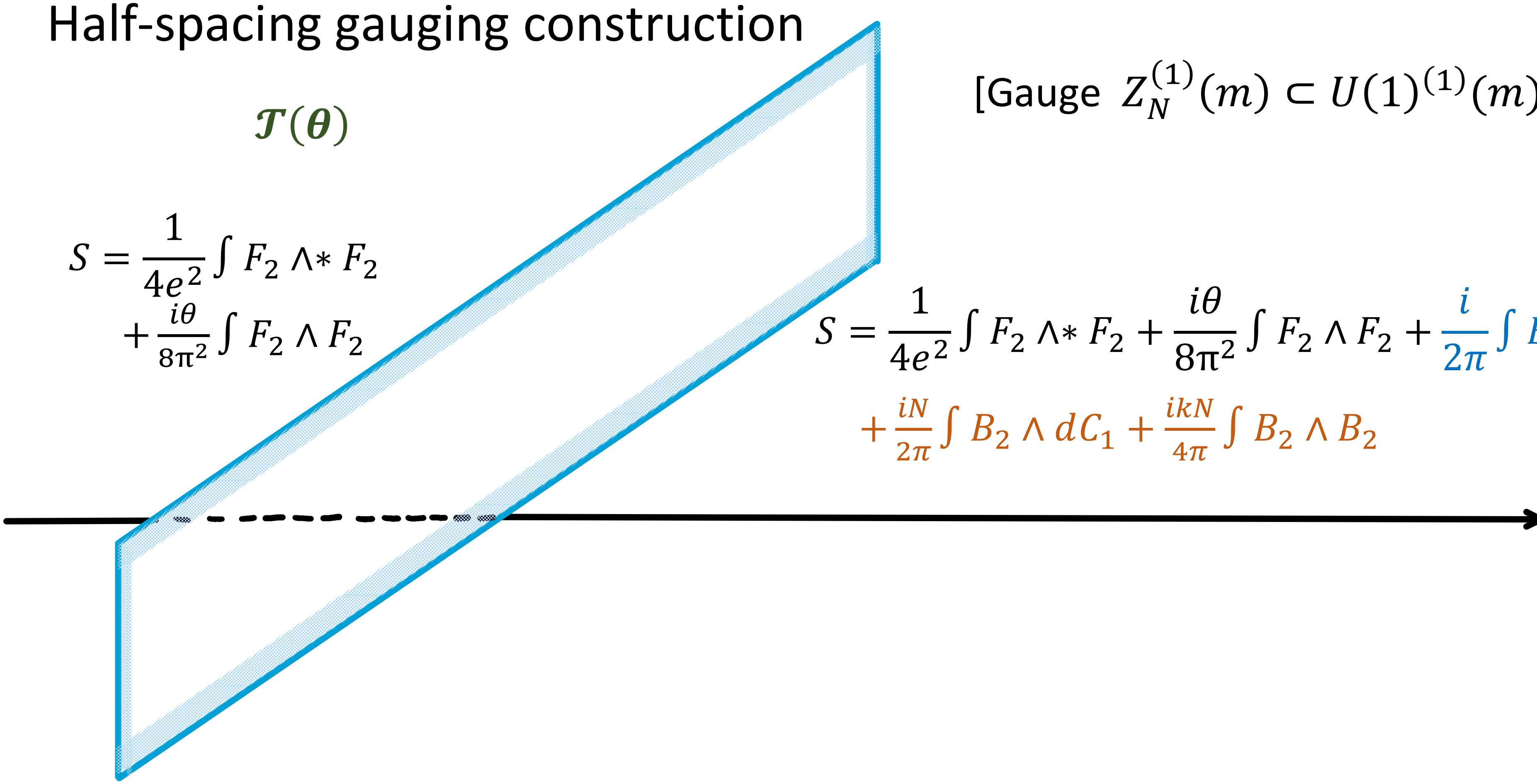
[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]



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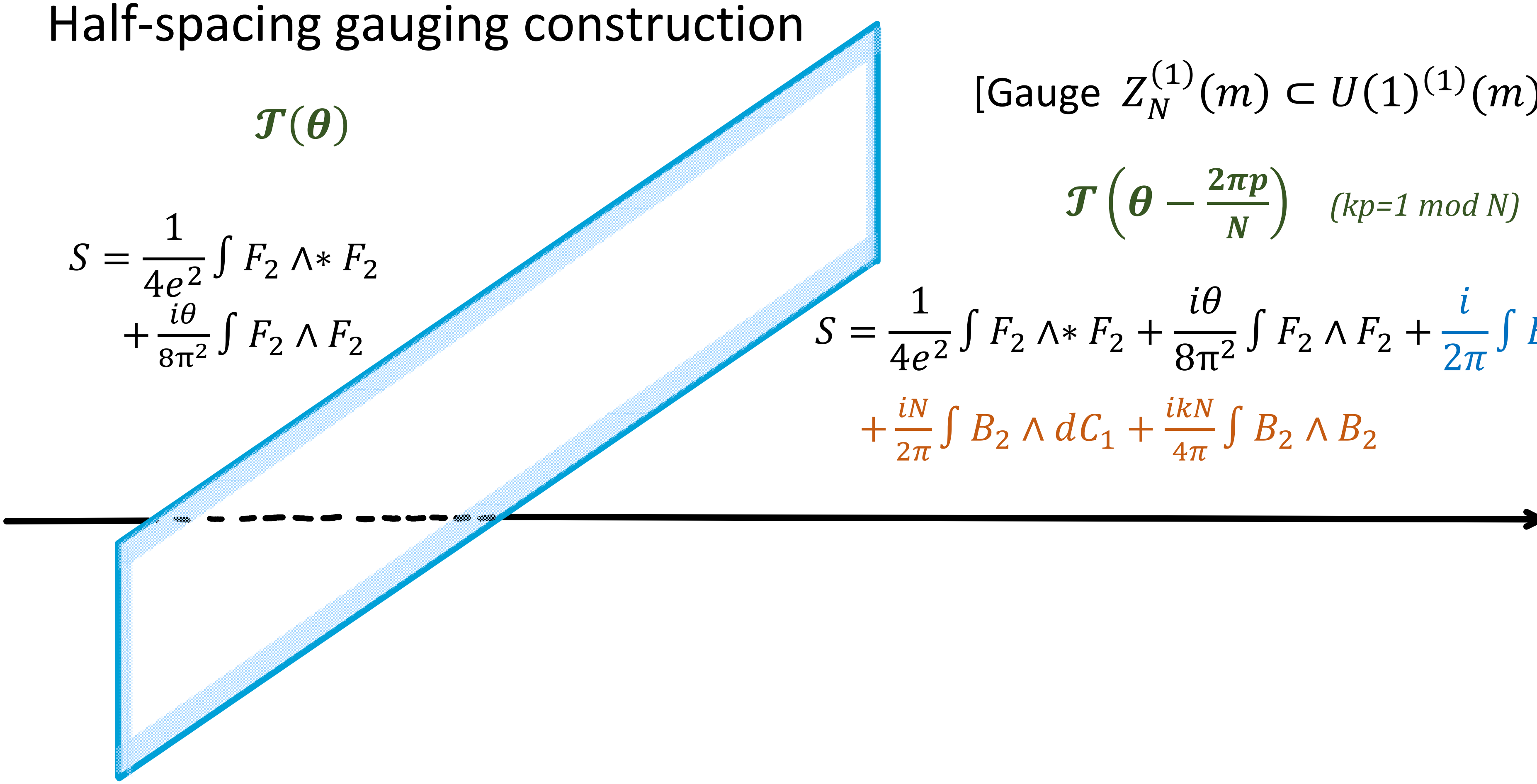

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[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]

$\boldsymbol{\mathcal{T}\left(\theta - \frac{2\pi p}{N}\right)}$ *(kp=1 mod N)*

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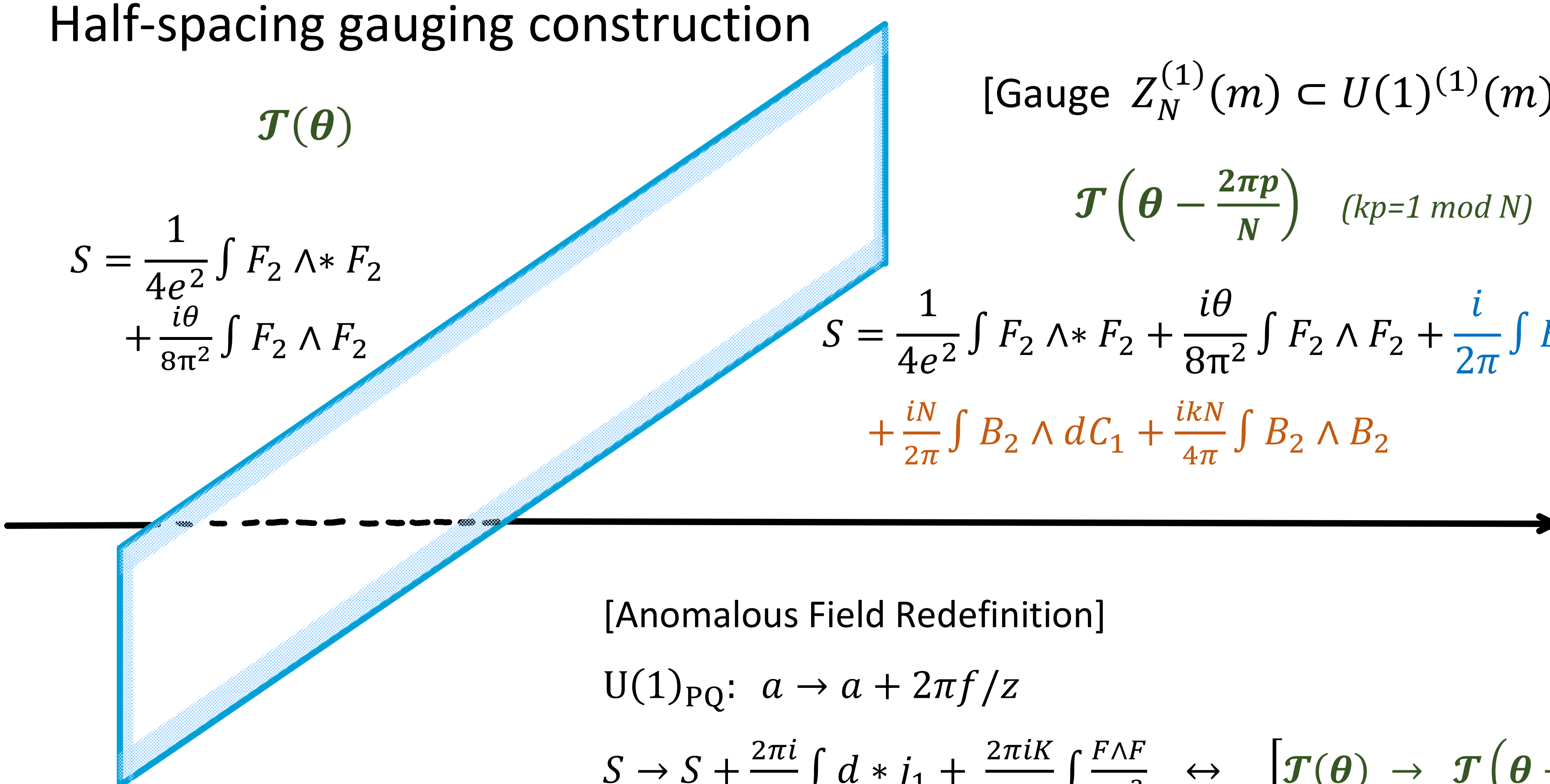

$\mathcal{T}(\boldsymbol{\theta})$

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[Anomalous Field Redefinition]

$\mathrm{U(1)_{PQ}}$: $a \to a + 2\pi f/z$

$$S \to S + \frac{2\pi i}{z}\int d * j_1 + \frac{2\pi i K}{z}\int \frac{F \wedge F}{8\pi^2} \leftrightarrow \left[\mathcal{T}(\boldsymbol{\theta}) \to \mathcal{T}\left(\boldsymbol{\theta} + \frac{2\pi K}{z}\right)\right]$$

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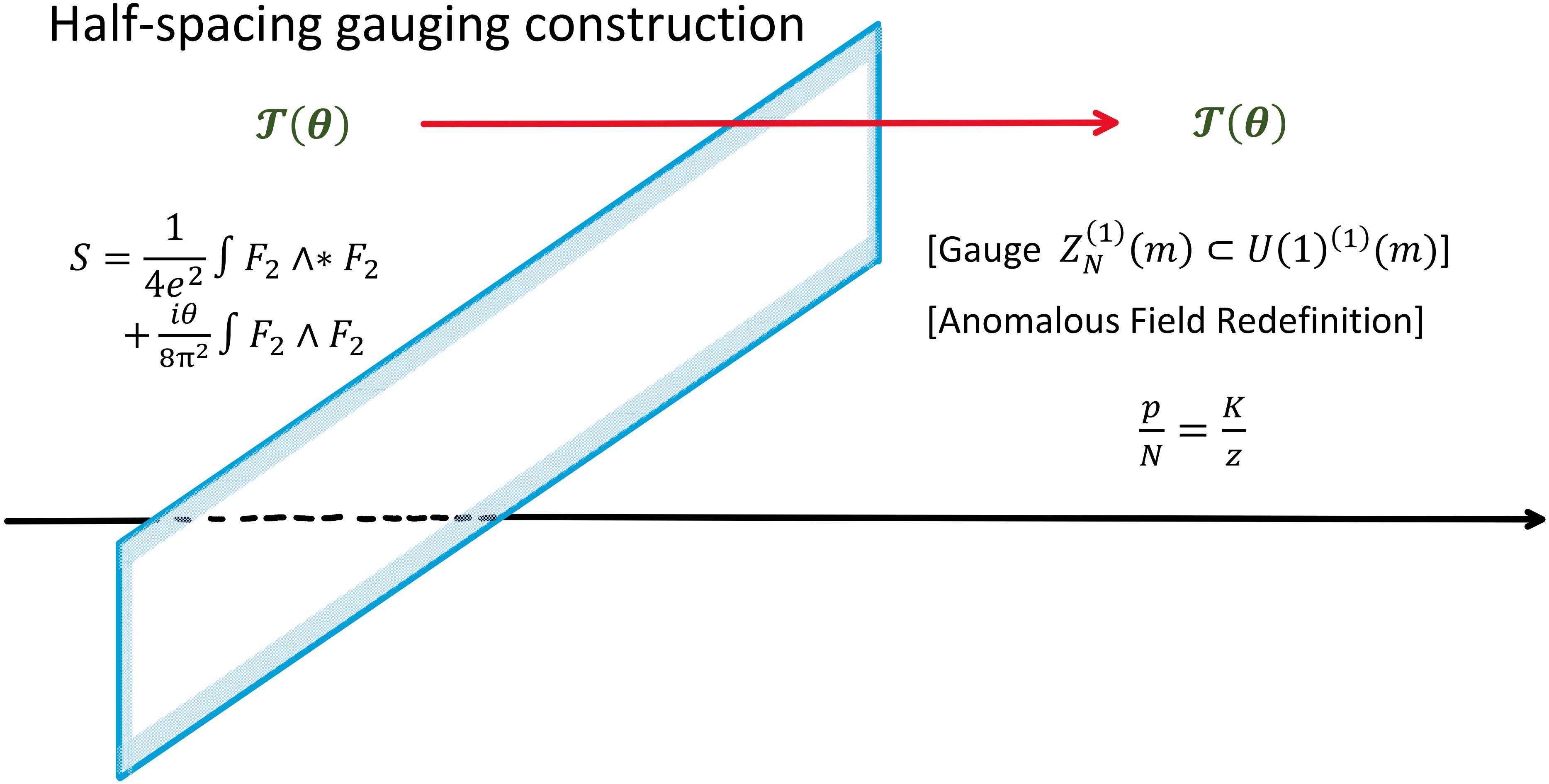

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[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]

[Anomalous Field Redefinition]

$$\frac{p}{N} = \frac{K}{z}$$

Non-Invertible Symmetry

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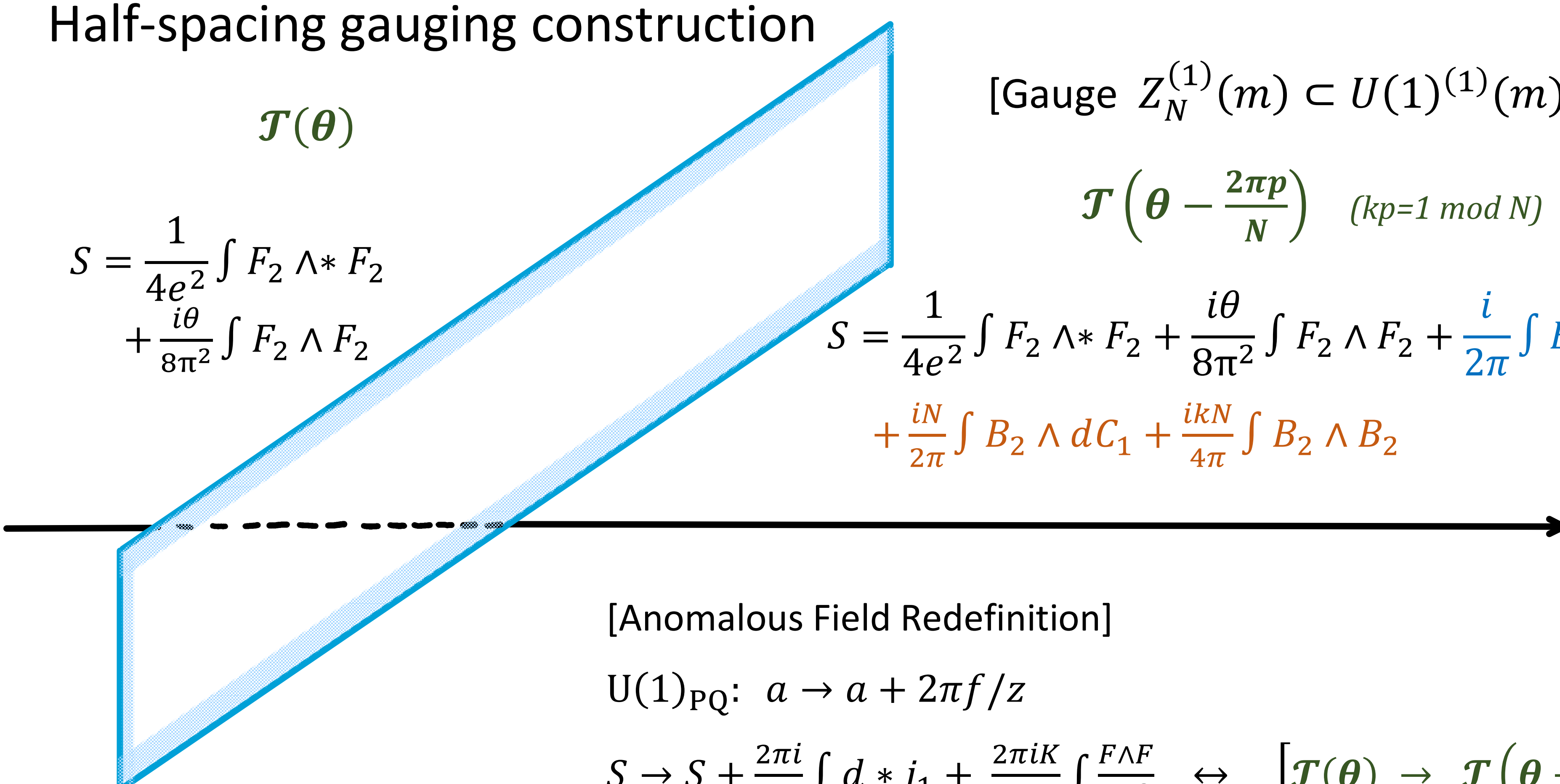

$\mathcal{T}(\theta)$

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Non-Invertible Symmetry

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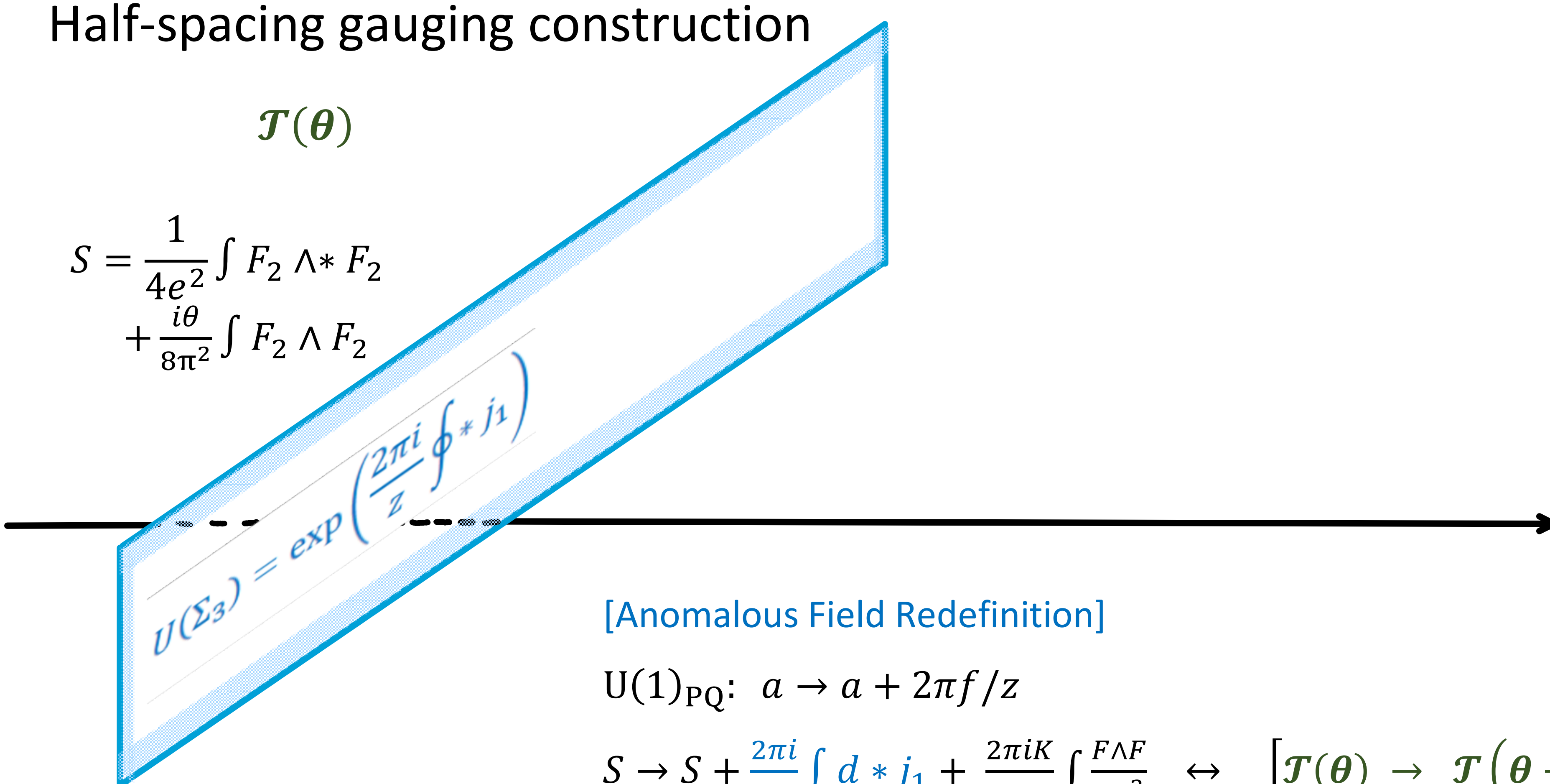
$\boldsymbol{\mathcal{T}(\theta)}$

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2$$



[Anomalous Field Redefinition]

$\mathrm{U}(1)_{\mathrm{PQ}}$: $a \to a + 2\pi f/z$

$$S \to S + \frac{2\pi i}{z}\int d * j_1 + \frac{2\pi i K}{z}\int \frac{F\wedge F}{8\pi^2} \quad \leftrightarrow \quad \left[\boldsymbol{\mathcal{T}(\theta)} \to \boldsymbol{\mathcal{T}\left(\theta + \frac{2\pi K}{z}\right)}\right]$$

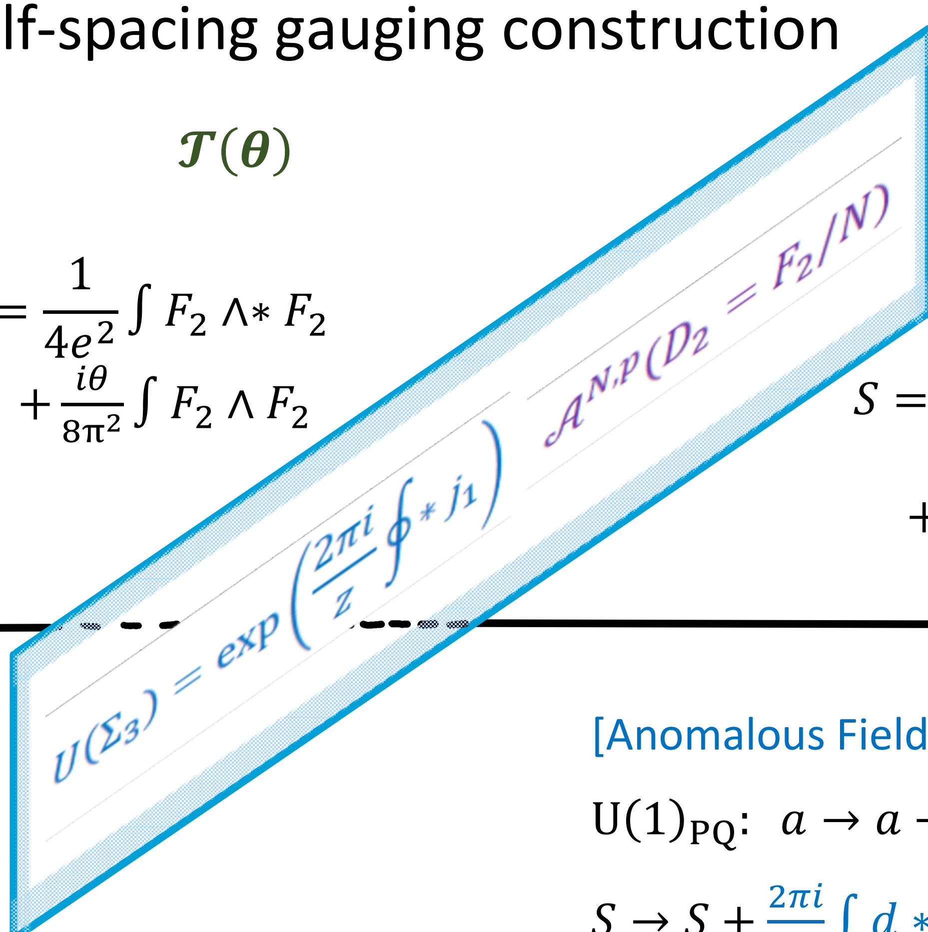
Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction

$\boldsymbol{\mathcal{T}(\theta)}$

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2$$



[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2 + \frac{i}{2\pi}\int B_2 \wedge F_2 + \frac{iN}{2\pi}\int B_2 \wedge dC_1 + \frac{ikN}{4\pi}\int B_2 \wedge B_2$$

[Anomalous Field Redefinition]

$\mathrm{U(1)_{PQ}}$: $a \to a + 2\pi f/z$

$$S \to S + \frac{2\pi i}{z}\int d * j_1 + \frac{2\pi i K}{z}\int \frac{F\wedge F}{8\pi^2} \quad \leftrightarrow \quad \left[\boldsymbol{\mathcal{T}(\theta)} \;\to\; \boldsymbol{\mathcal{T}\left(\theta + \frac{2\pi K}{z}\right)}\right]$$

Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction

$\mathcal{T}(\theta)$

[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]

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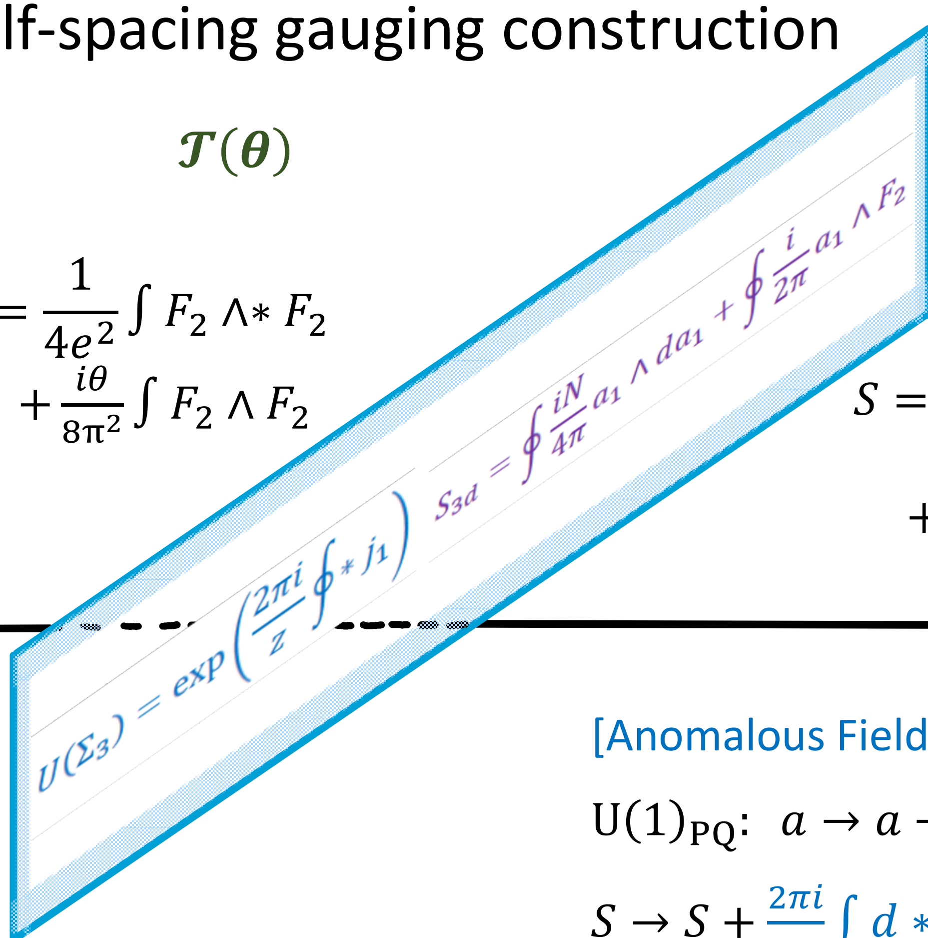

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2 + \frac{i}{2\pi}\int B_2 \wedge F_2 + \frac{iN}{2\pi}\int B_2 \wedge dC_1 + \frac{ikN}{4\pi}\int B_2 \wedge B_2$$

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From $U(1)$ Instanton

Half-spacing gauging construction

$\mathcal{T}(\theta)$

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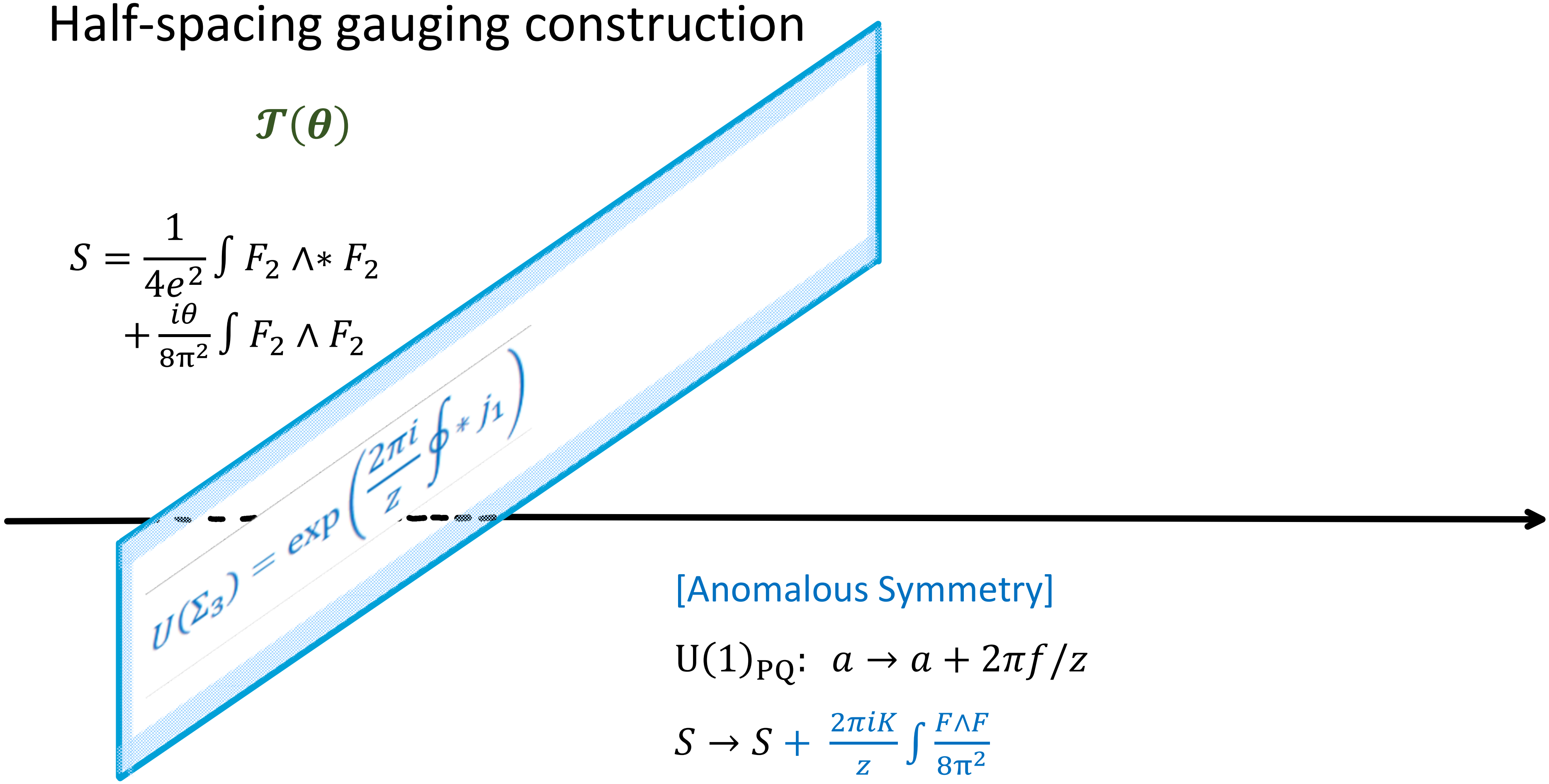

[Anomalous Symmetry]

$\mathrm{U}(1)_{\mathrm{PQ}}$: $a \to a + 2\pi f/z$

$$S \to S + \frac{2\pi i K}{z}\int \frac{F\wedge F}{8\pi^2}$$

Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction

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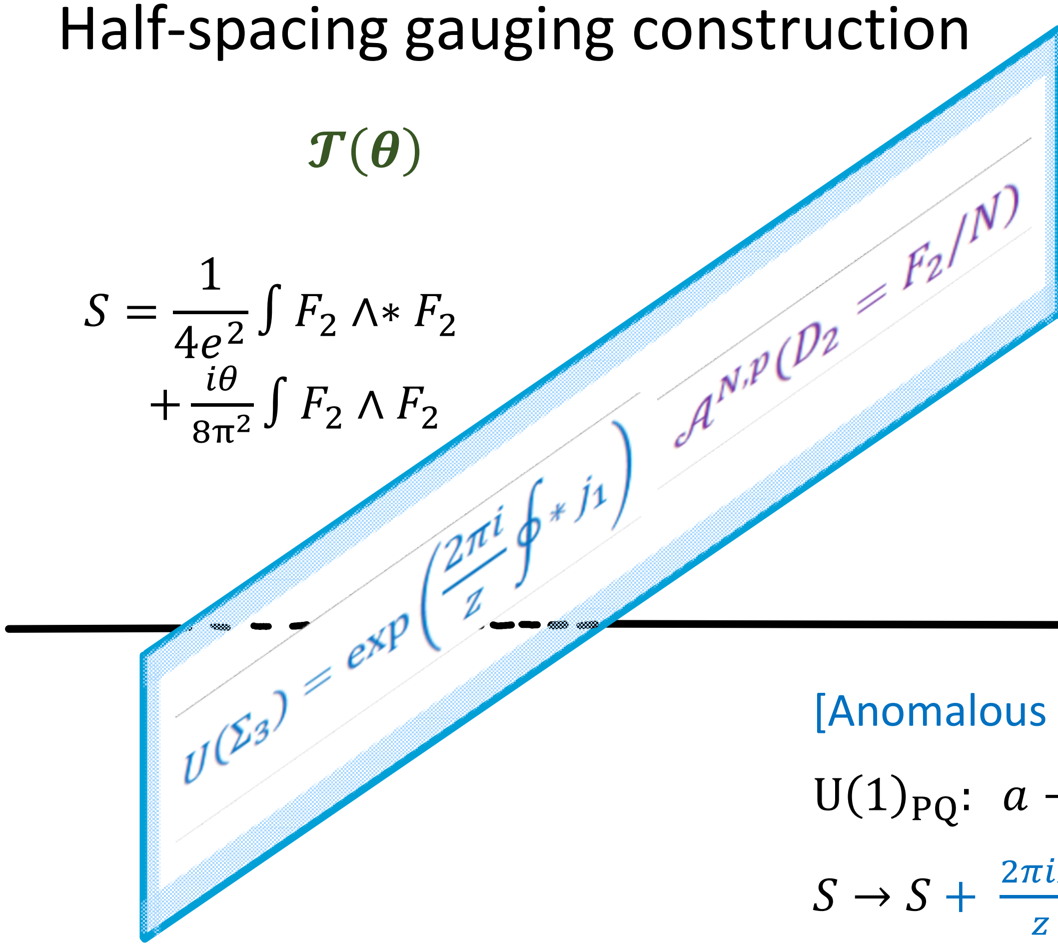
$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2$$



[Anomalous Symmetry] $\times$ [$\mathcal{A}^{N,p}(F_2/N)$]

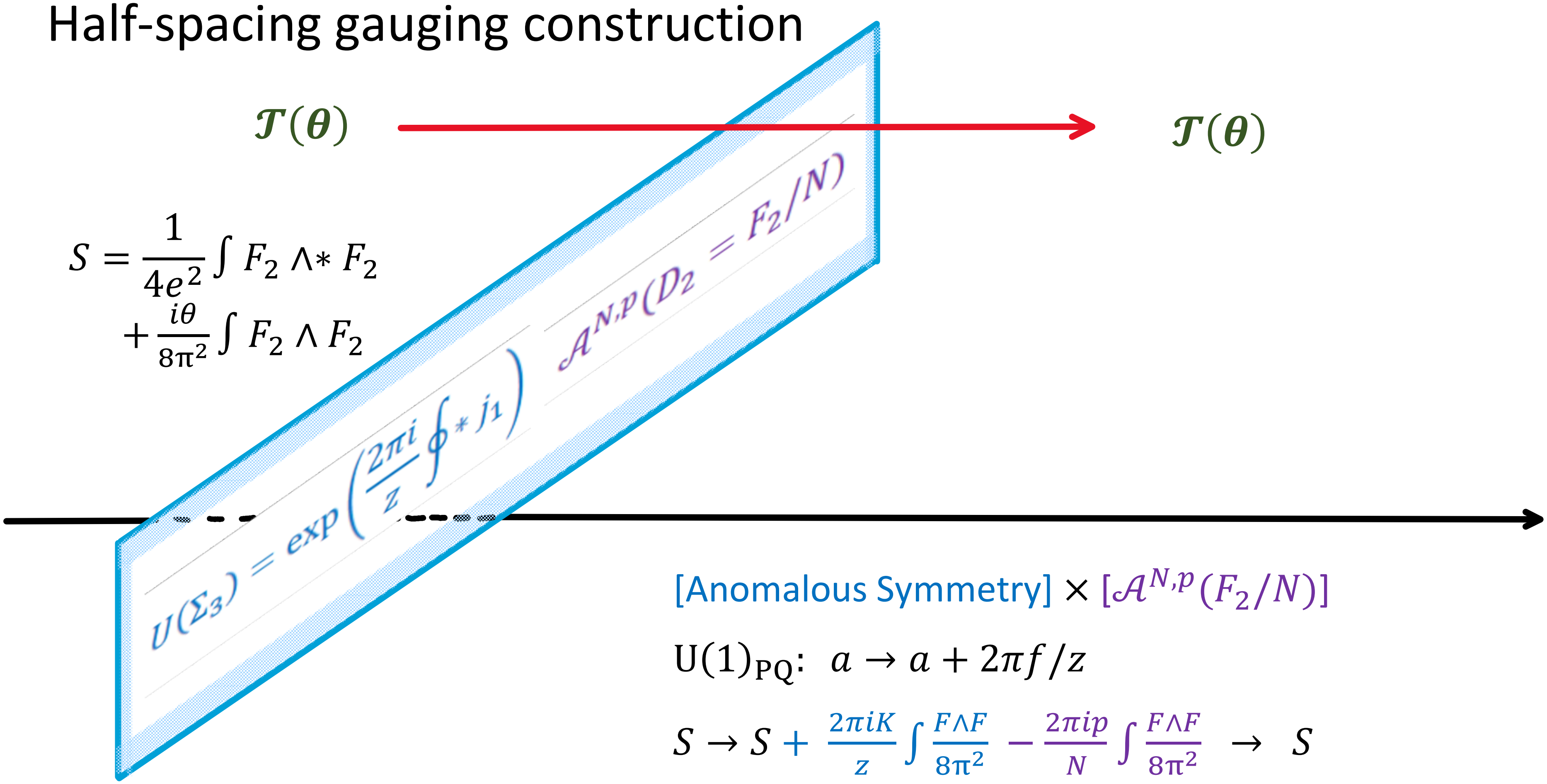
$\mathrm{U(1)_{PQ}}$: $a \to a + 2\pi f/z$

$$S \to S + \frac{2\pi i K}{z}\int \frac{F \wedge F}{8\pi^2} - \frac{2\pi i p}{N}\int \frac{F \wedge F}{8\pi^2} \;\to\; S$$

Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction



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$$S \to S + \frac{2\pi i K}{z}\int \frac{F\wedge F}{8\pi^2} - \frac{2\pi i p}{N}\int \frac{F\wedge F}{8\pi^2} \to S$$

Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction

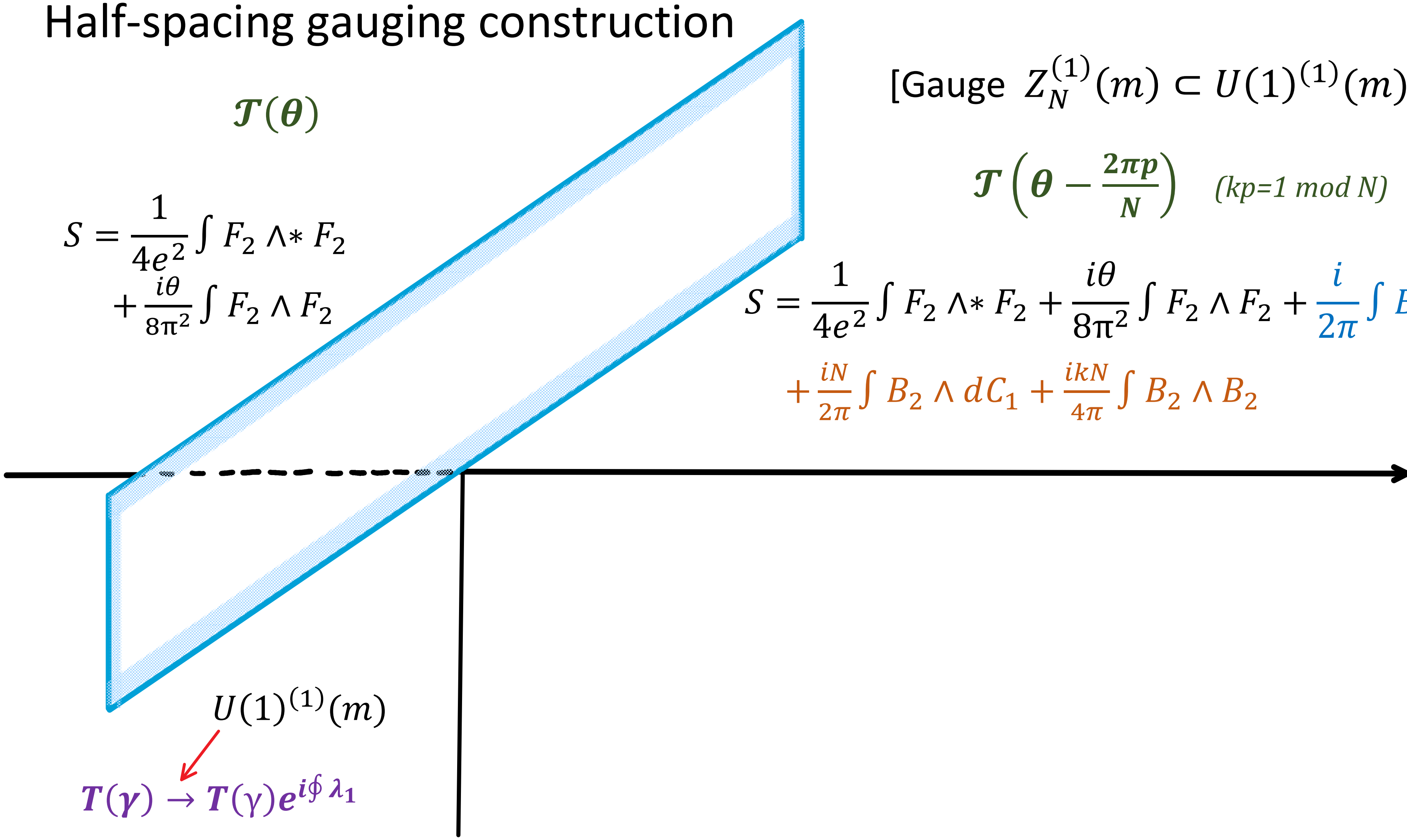
$\boldsymbol{\mathcal{T}(\theta)}$

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2$$

[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]

$\boldsymbol{\mathcal{T}\left(\theta - \frac{2\pi p}{N}\right)}$ *(kp=1 mod N)*

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2 + \frac{i}{2\pi}\int B_2 \wedge F_2 + \frac{iN}{2\pi}\int B_2 \wedge dC_1 + \frac{ikN}{4\pi}\int B_2 \wedge B_2$$



$U(1)^{(1)}(m)$

$\boldsymbol{T(\gamma) \to T(\gamma) e^{i\oint \lambda_1}}$

Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction

$\mathcal{T}(\theta)$

$$S = \frac{1}{4e^2}\int F_2 \wedge * F_2 + \frac{i\theta}{8\pi^2}\int F_2 \wedge F_2$$

[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]

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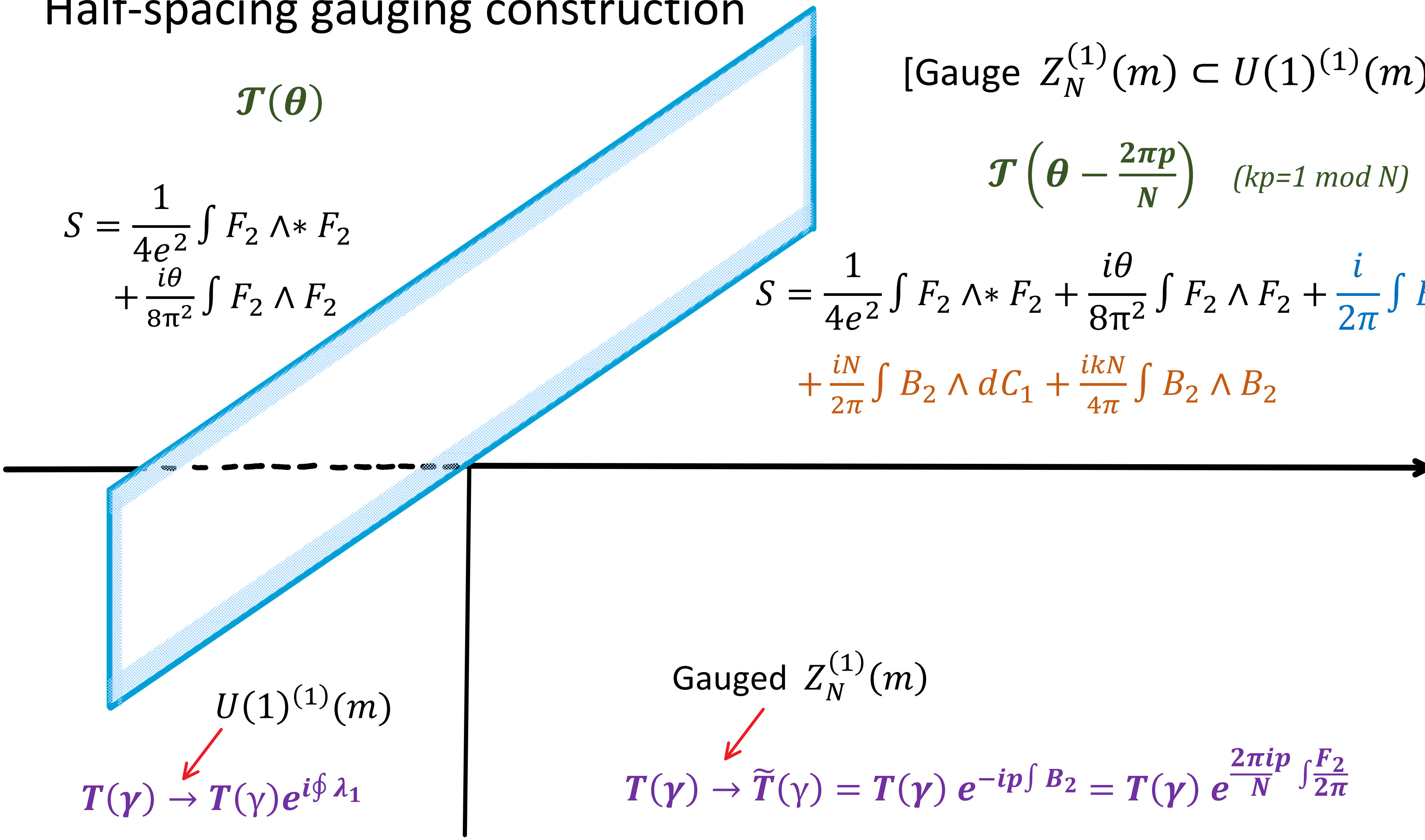


$U(1)^{(1)}(m)$

$T(\gamma) \to T(\gamma) e^{i\oint \lambda_1}$

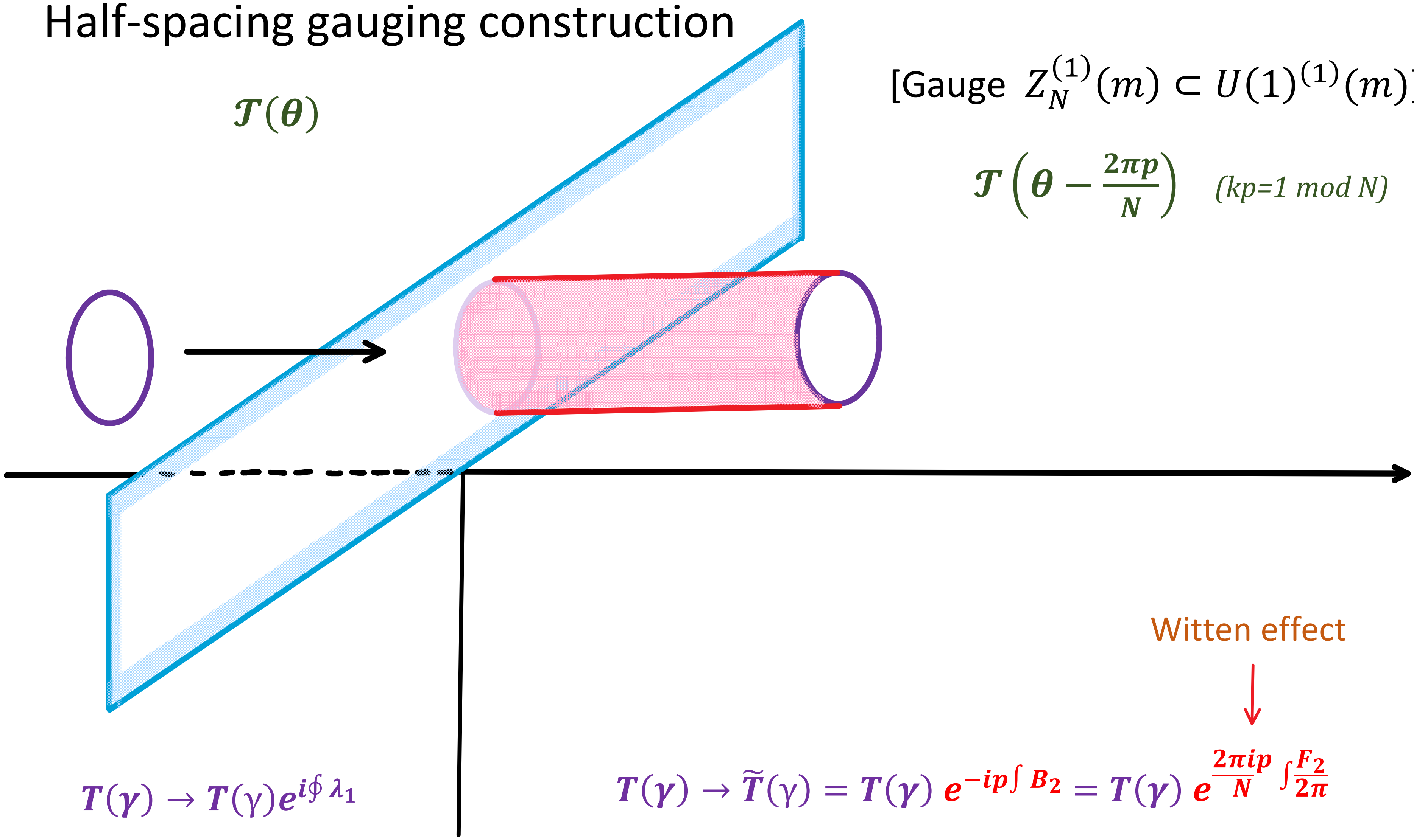
Gauged $Z_N^{(1)}(m)$

$T(\gamma) \to \widetilde{T}(\gamma) = T(\gamma)\, e^{-ip\int B_2} = T(\gamma)\, e^{\frac{2\pi i p}{N}\int \frac{F_2}{2\pi}}$

Non-Invertible Symmetry

From $U(1)$ Instanton

Half-spacing gauging construction



$\mathcal{T}(\theta)$

[Gauge $Z_N^{(1)}(m) \subset U(1)^{(1)}(m)$]

$\mathcal{T}\left(\theta - \frac{2\pi p}{N}\right)$ *(kp=1 mod N)*

Witten effect

$T(\gamma) \to T(\gamma) e^{i\oint \lambda_1}$

$T(\gamma) \to \widetilde{T}(\gamma) = T(\gamma)\, e^{-ip\int B_2} = T(\gamma)\, e^{\frac{2\pi i p}{N}\int \frac{F_2}{2\pi}}$

Generalized Global Symmetries

III. Higher-group symmetry: 3-group

$$S = \int \frac{1}{2}\, da \wedge * \, da \ + \int \frac{1}{2g^2}\ F \wedge * \, F \ \ - \int \frac{iK}{8\pi^2} \frac{a}{f} F \wedge F$$

Generalized Global Symmetries

III. Higher-group symmetry: 3-group

$$S = \int \frac{1}{2}\, da \wedge * \, da \; + \int \frac{1}{2g^2}\; F \wedge * \, F \; - \int_{N_5} \frac{iK}{8\pi^2 f} da \wedge F \wedge F$$

$$\partial N_5 = M_4$$

Generalized Global Symmetries

III. Higher-group symmetry: 3-group

$$S = \int \frac{1}{2}\,(da - f\mathcal{A}_1) \wedge * \,(da - f\mathcal{A}_1) \; + \frac{i}{2\pi f}\int a\mathcal{G}_4$$

$$+ \int \frac{1}{2g^2}\,(F - \mathcal{B}_2^e) \wedge * \,(F - \mathcal{B}_2^e) \; + \frac{i}{2\pi}\int A_1 \wedge \mathcal{H}_3$$

$$- \int_{N_5} \frac{iK}{8\pi^2 f}(da - f\mathcal{A}_1) \wedge (F - \mathcal{B}_2^e) \wedge (F - \mathcal{B}_2^e)$$

Cf) Global symmetry current $J^\mu \;\rightarrow\; S \supset -\int A_\mu J^\mu = -\int A \wedge * J$

Generalized Global Symmetries

III. Higher-group symmetry: 3-group

$$S = \int \frac{1}{2}\,(da - f\mathcal{A}_1) \wedge * \,(da - f\mathcal{A}_1) \;+\; \frac{i}{2\pi f}\int a\mathcal{G}_4$$

$$+ \int \frac{1}{2g^2}\,(F - \mathcal{B}_2^e) \wedge * \,(F - \mathcal{B}_2^e) \;+\; \frac{i}{2\pi}\int A_1 \wedge \mathcal{H}_3$$

$$- \int_{N_5} \frac{iK}{8\pi^2 f}(da - f\mathcal{A}_1) \wedge (F - \mathcal{B}_2^e) \wedge (F - \mathcal{B}_2^e)$$

$\mathcal{G}_4 = d\mathcal{A}_3 + \cdots$, $\mathcal{H}_3 = d\mathcal{B}_2^m + \cdots$

Generalized Global Symmetries

III. Higher-group symmetry: 3-group

$$S = \int \frac{1}{2}\,(da - f\mathcal{A}_1) \wedge * \,(da - f\mathcal{A}_1) \;+\; \frac{i}{2\pi f}\int a\mathcal{G}_4$$

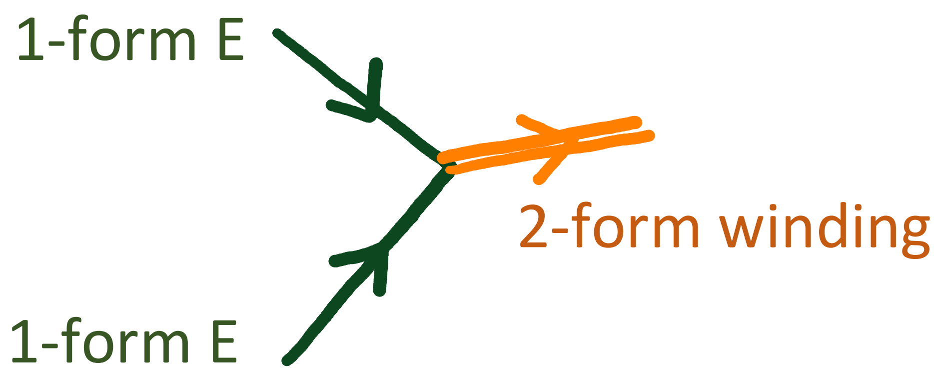
$$+ \int \frac{1}{2g^2}\;(F - \mathcal{B}_2^e) \wedge * \,(F - \mathcal{B}_2^e) \;+\; \frac{i}{2\pi}\int A_1 \wedge \mathcal{H}_3$$

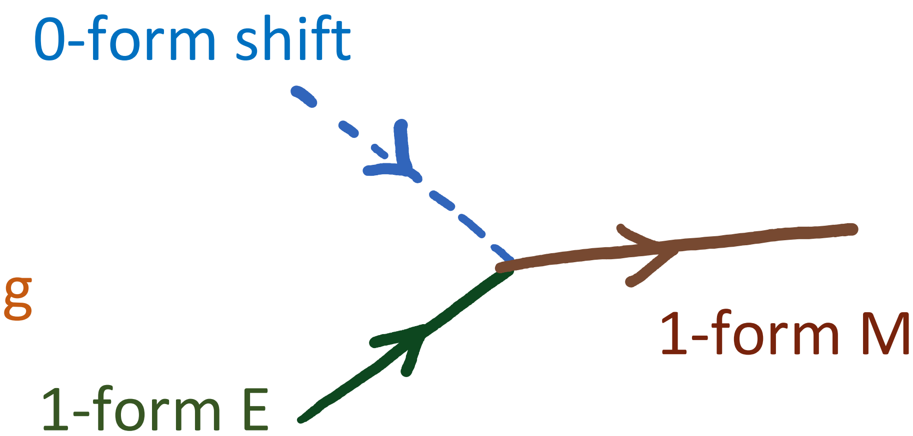
$$- \int_{N_5} \frac{iK}{8\pi^2 f}(da - f\mathcal{A}_1) \wedge (F - \mathcal{B}_2^e) \wedge (F - \mathcal{B}_2^e)$$

$$\mathcal{G}_4 = d\mathcal{A}_3 + \frac{K}{4\pi}\mathcal{B}_2^e \wedge \mathcal{B}_2^e \qquad , \mathcal{H}_3 = d\mathcal{B}_2^m + \frac{K}{2\pi}\mathcal{A}_1 \wedge \mathcal{B}_2^e$$

Generalized Global Symmetries

III. Higher-group symmetry: 3-group

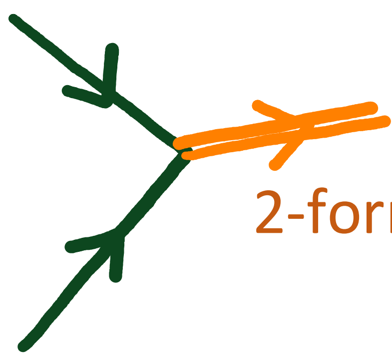


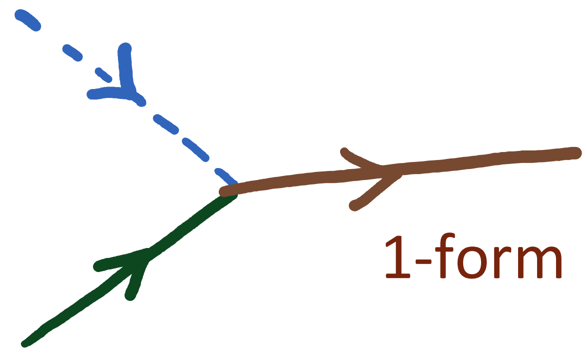


$$\mathcal{G}_4 = d\mathcal{A}_3 + \frac{K}{4\pi}\mathcal{B}_2^e \wedge \mathcal{B}_2^e \quad , \mathcal{H}_3 = d\mathcal{B}_2^m + \frac{K}{2\pi}\mathcal{A}_1 \wedge \mathcal{B}_2^e$$

Generalized Global Symmetries

III. Higher-group symmetry: 3-group





$$\mathcal{G}_4 = d\mathcal{A}_3 + \frac{K}{4\pi}\mathcal{B}_2^e \wedge \mathcal{B}_2^e \qquad , \mathcal{H}_3 = d\mathcal{B}_2^m + \frac{K}{2\pi}\mathcal{A}_1 \wedge \mathcal{B}_2^e$$

Symmetry Emergence Theorem

$$E_{1e} \lesssim E_{2w} \qquad \min\{E_{0s}, E_{1e}\} \lesssim E_{1m}$$

II. Global Structure and Axion Defects

II. Global Structure and Axion Defects

II-1. Global Structure of G_{SM} and Axion Domain Wall Problem

Axion-Standard Model with $G_{SM} = \widetilde{\mathrm{G}}_{\mathrm{SM}}/\Gamma$

Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr(G \wedge G) + \frac{i\ell_2}{8\pi^2}\int \frac{a}{f_a}\, Tr(W \wedge W) + \frac{i\ell_1}{8\pi^2}\int \frac{a}{f_a}\, B \wedge B$$

1. $a \sim a + 2\pi f_a$: axion (periodic scalar)

2. G, W, B = field strength of $SU(3)_C,\ SU(2)_L, U(1)_Y$, respectively.

Axion-Standard Model with $G_{SM} = \widetilde{\mathrm{G}}_{\mathrm{SM}}/\Gamma$

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1. $a \sim a + 2\pi f_a$: axion (periodic scalar)

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3. $G_{SM} = \frac{SU(3)_C \times SU(2)_L \times U(1)_Y}{\Gamma}$, $\Gamma = 1, Z_2, Z_3, Z_6$

Axion-Standard Model with $G_{SM} = \widetilde{G}_{SM}/\Gamma$

Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr(G\wedge G) + \frac{i\ell_2}{8\pi^2}\int \frac{a}{f_a}\, Tr(W\wedge W) + \frac{i\ell_1}{8\pi^2}\int \frac{a}{f_a}\, B\wedge B$$

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3. $G_{SM} = \frac{SU(3)_C\times SU(2)_L\times U(1)_Y}{\Gamma}$, $\Gamma = 1, Z_2, Z_3, Z_6$

   if we consider $\Gamma = 1$, instanton spectrum is

   $\int \frac{Tr(G\wedge G)}{8\pi^2} \in Z,\ \ \int \frac{Tr(W\wedge W)}{8\pi^2} \in Z$ and using $\oint \frac{B}{2\pi} \in Z$

   then the invariance of S under $a \sim a + 2\pi f_a \ \Rightarrow\ \ \ell_{12,3} \in Z$

Axion-Standard Model with $G_{SM} = \widetilde{\mathrm{G}}_{\mathrm{SM}}/\Gamma$

Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr(G \wedge G) + \frac{i\ell_2}{8\pi^2}\int \frac{a}{f_a}\, Tr(W \wedge W) + \frac{i\ell_1}{8\pi^2}\int \frac{a}{f_a}\, B \wedge B$$

$$G_{SM} = \frac{SU(3)_C \times SU(2)_L \times U(1)_Y}{\Gamma}, \quad \Gamma = 1, Z_2, Z_3, Z_6$$

Physics sensitive to $\mathbf{\Gamma}$

(i) Quantization conditions on ℓ_1, ℓ_2, ℓ_3 (IR parameters) depending on $\mathbf{\Gamma}$

(ii) Axion **non-invertible shift symmetry** (IR selection rules) depending on $\mathbf{\Gamma}$

(iii) Axion **domain walls** properties, and their **fates** (potential IR probe) from **non-invertible shift symmetry**, hence $\mathbf{\Gamma}$?

Axion-Gauge Coupling Quantization

Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr(G\wedge G) + \frac{i\ell_2}{8\pi^2}\int \frac{a}{f_a}\, Tr(W\wedge W) + \frac{i\ell_1}{8\pi^2}\int \frac{a}{f_a}\, B\wedge B$$

1. G, W, B = field strength of $SU(3)_C,\ SU(2)_L, U(1)_Y$, respectively.

2. $G_{SM} = \frac{SU(3)_C \times SU(2)_L \times U(1)_Y}{\Gamma}$, $\Gamma = 1, Z_2, Z_3, Z_6$

3. Quantization of Axion-Gauge Couplings from non-trivial global form

$\Gamma = 1$: $\ell_{1,2,3} \in Z$

$\Gamma = Z_2$: $\ell_1 \in 2Z$, $\ell_{2,3} \in Z$, and $\ell_1 + 2\ell_2 \in 4Z$

$\Gamma = Z_3$: $\ell_1 \in 3Z$, $\ell_{2,3} \in Z$, and $\ell_1 + 6\ell_3 \in 9Z$

$\Gamma = Z_6$: $\ell_1 \in 6Z$, $\ell_{2,3} \in Z$, $\ell_1 + 2\ell_2 \in 4Z$, and $\ell_1 + 6\ell_3 \in 9Z$

[Y. Choi, M. Forslund, H. T. Lam, S-H. Shao, 2309.03937]
[C. Cordova, S. Hong, L. T. Wang, 2309.05636]
[M. Reece, 2309.03939]

Axion-Gauge Coupling Quantization

Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr(G\wedge G) + \frac{i\ell_2}{8\pi^2}\int \frac{a}{f_a}\, Tr(W\wedge W) + \frac{i\ell_1}{8\pi^2}\int \frac{a}{f_a}\, B\wedge B$$

1. G, W, B = field strength of $SU(3)_C$, $SU(2)_L, U(1)_Y$, respectively.

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$$\begin{aligned}
\Gamma = 1 \ &: \quad \ell_{1,2,3} \in Z \\
\Gamma = Z_2 &: \quad \ell_1 \in 2Z\,,\ \ell_{2,3} \in Z\,,\ \text{ and } \ell_1 + 2\ell_2 \in 4Z \\
\Gamma = Z_3 &: \quad \ell_1 \in 3Z\,,\ \ell_{2,3} \in Z\,,\ \text{ and } \ell_1 + 6\ell_3 \in 9Z \\
\Gamma = Z_6 &: \quad \ell_1 \in 6Z\,,\ \ell_{2,3} \in Z\,,\ \ell_1 + 2\ell_2 \in 4Z,\ \text{ and } \ell_1 + 6\ell_3 \in 9Z
\end{aligned}$$

$\Gamma = Z_6$: $SU(5)$, $SO(10)$, E_6
$\Gamma = Z_3$: $SU(4)_C \times SU(2)_L \times SU(2)_R$ [Pati-Salam]
$\Gamma = Z_2$: $SU(3)_C \times SU(3)_L \times SU(3)_R$ [Trinification]

Axion-SM

$$
\begin{aligned}
&\Gamma = 1 \;\;: && \ell_{1,2,3} \in Z \\
&\Gamma = Z_2 : && \ell_1 \in 2Z\,,\; \ell_{2,3} \in Z\,,\;\; \text{and}\;\; \ell_1 + 2\ell_2 \in 4Z \\
&\Gamma = Z_3 : && \ell_1 \in 3Z\,,\; \ell_{2,3} \in Z\,,\;\; \text{and}\;\; \ell_1 + 6\ell_3 \in 9Z \\
&\Gamma = Z_6 : && \ell_1 \in 6Z\,,\; \ell_{2,3} \in Z\,,\;\; \ell_1 + 2\ell_2 \in 4Z,\;\; \text{and}\;\; \ell_1 + 6\ell_3 \in 9Z
\end{aligned}
$$

<u>Physics sensitive to $\boldsymbol{\Gamma}$</u>

(i) Quantization conditions on ℓ_1, ℓ_2, ℓ_3 (IR parameters) depending on $\boldsymbol{\Gamma}$

(ii) Axion **non-invertible shift symmetry** (IR selection rules) depending on $\boldsymbol{\Gamma}$

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Axion Non-Invertible Shift Symmetry

Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr(G\wedge G) + \frac{i\ell_2}{8\pi^2}\int \frac{a}{f_a}\, Tr(W\wedge W) + \frac{i\ell_1}{8\pi^2}\int \frac{a}{f_a}\, B\wedge B$$

Non-invertible Axion Shift Symmetry:

$$Z_K^I \subset Z^{NI}_{\mathrm{gcd}(\ell_2,\ell_3)} \approx Z^{NI}_{\ell_3}$$

$$\begin{aligned}
\Gamma = 1 \ &: \quad K = \mathrm{gcd}(\ell_1,\ell_2,\ell_2) \\
\Gamma = Z_2 &: \quad K = \mathrm{gcd}\left(\frac{\ell_1}{2},\ell_2,\ell_3,\frac{\ell_1+2\ell_2}{4}\right) \\
\Gamma = Z_3 &: \quad K = \mathrm{gcd}\left(\frac{\ell_1}{3},\ell_2,\ell_3,\frac{\ell_1+6\ell_3}{9}\right) \\
\Gamma = Z_6 &: \quad K = \mathrm{gcd}\left(\frac{\ell_1}{6},\ell_2,\ell_3,\frac{\ell_1+2\ell_2}{4},\frac{\ell_1+6\ell_3}{9}\right)
\end{aligned}$$

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Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr \ldots B \wedge B$$

$$G_{SM} = \frac{SU(3)_C \times SU(\ldots}{\Gamma}$$

Non-invertible Axion Shift Symmetry:

$$Z_K^I \subset Z^{NI}_{\gcd(\ell_2,\ell_3)} \approx Z^{NI}_{\ell_3}$$

$$\Gamma = 1 \;:\quad K = \gcd(\ell_1, \ell_2, \ell_2)$$

$$\Gamma = Z_2 :\quad K = \gcd\left(\frac{\ell_1}{2}, \ell_2, \ell_3, \frac{\ell_1+2\ell_2}{4}\right)$$

$$\Gamma = Z_3 :\quad K = \gcd\left(\frac{\ell_1}{3}, \ell_2, \ell_3, \frac{\ell_1+6\ell_3}{9}\right)$$

$$\Gamma = Z_6 :\quad K = \gcd\left(\frac{\ell_1}{6}, \ell_2, \ell_3, \frac{\ell_1+2\ell_2}{4}, \frac{\ell_1+6\ell_3}{9}\right)$$

Physics sensitive to $\boldsymbol{\Gamma}$

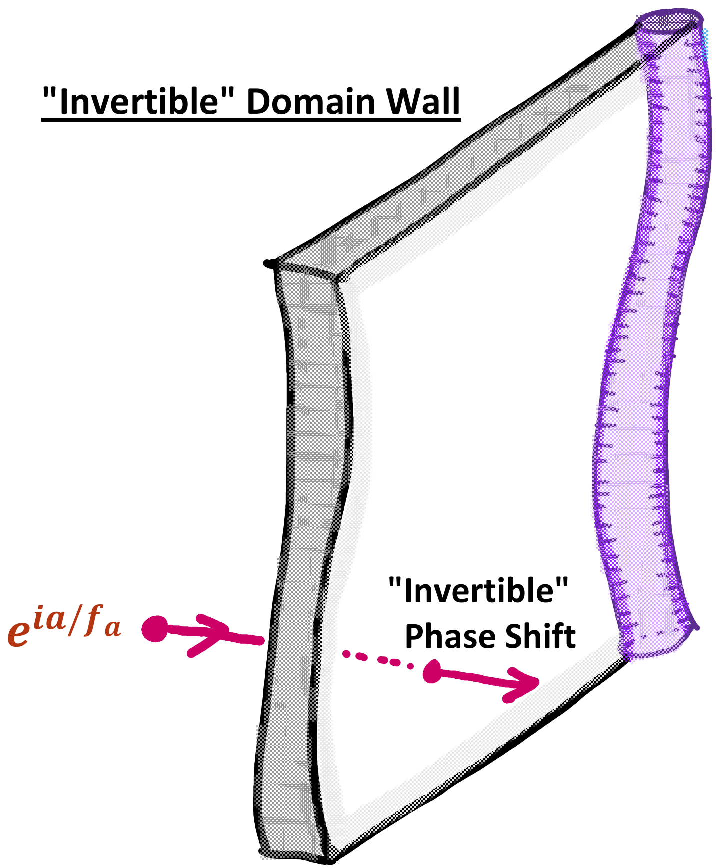
(i) Quantization conditions on ℓ_1, ℓ_2, ℓ_3 (IR parameters) depending on $\boldsymbol{\Gamma}$

(ii) Axion **non-invertible shift symmetry** (IR selection rules) depending on $\boldsymbol{\Gamma}$

(iii) Axion **domain walls** properties, and their **fates** (potential IR probe) from **non-invertible shift symmetry**, hence $\boldsymbol{\Gamma}$?

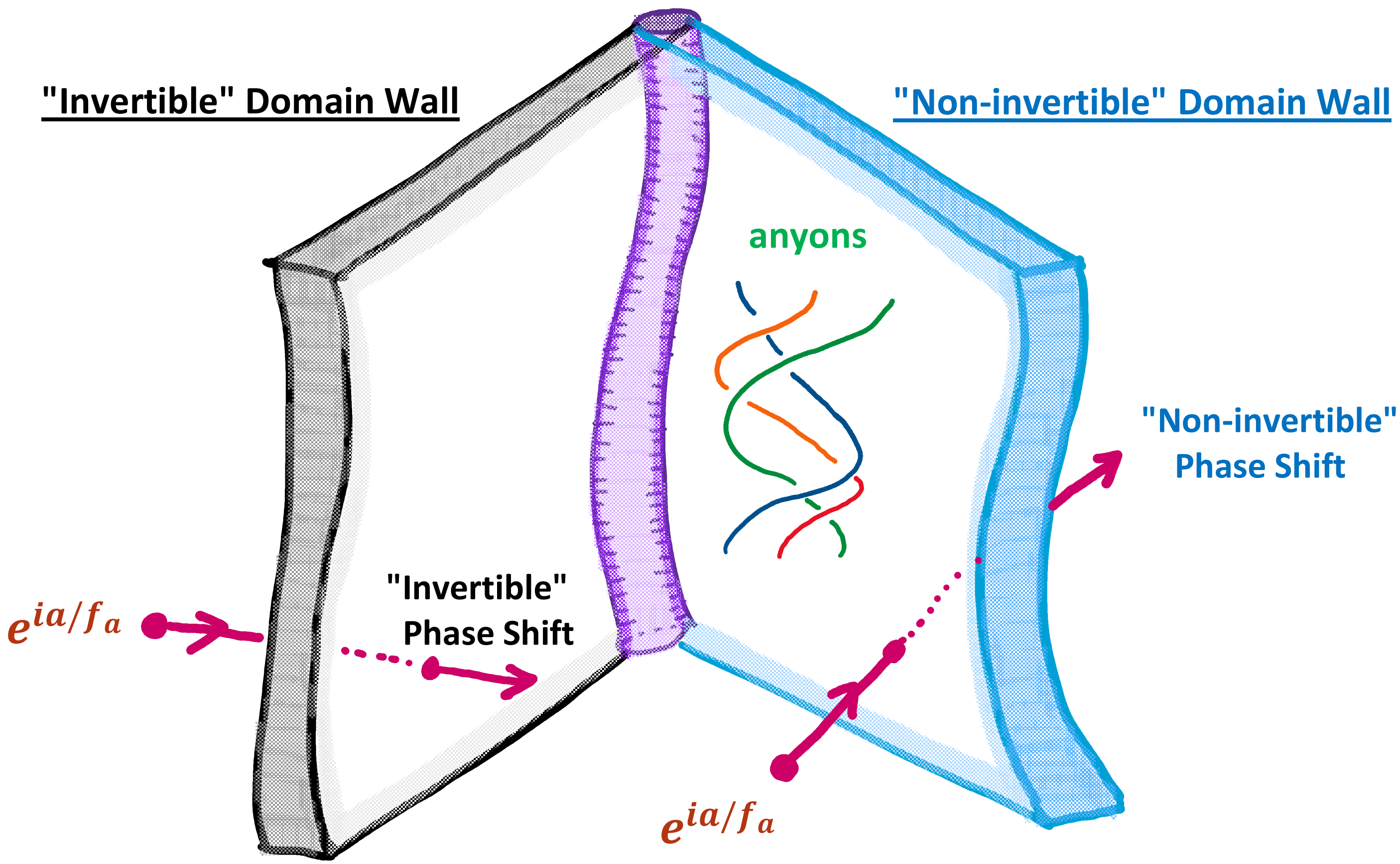
Axion Non-Invertible Shift Symmetry

Invertible DW vs Non-Invertible DW



Axion Non-Invertible Shift Symmetry

Invertible DW vs Non-Invertible DW



Axion Non-Invertible Shift Symmetry

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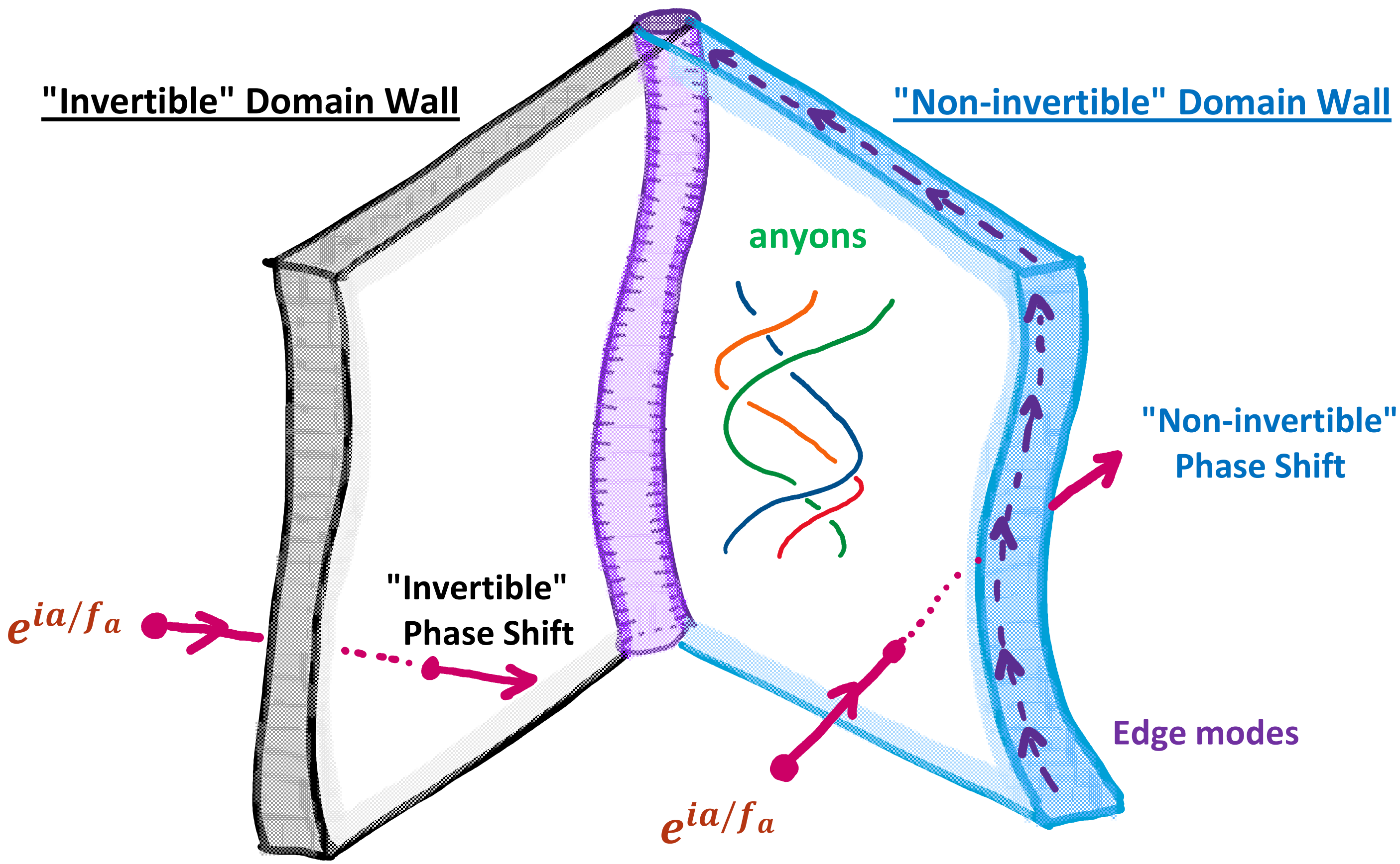
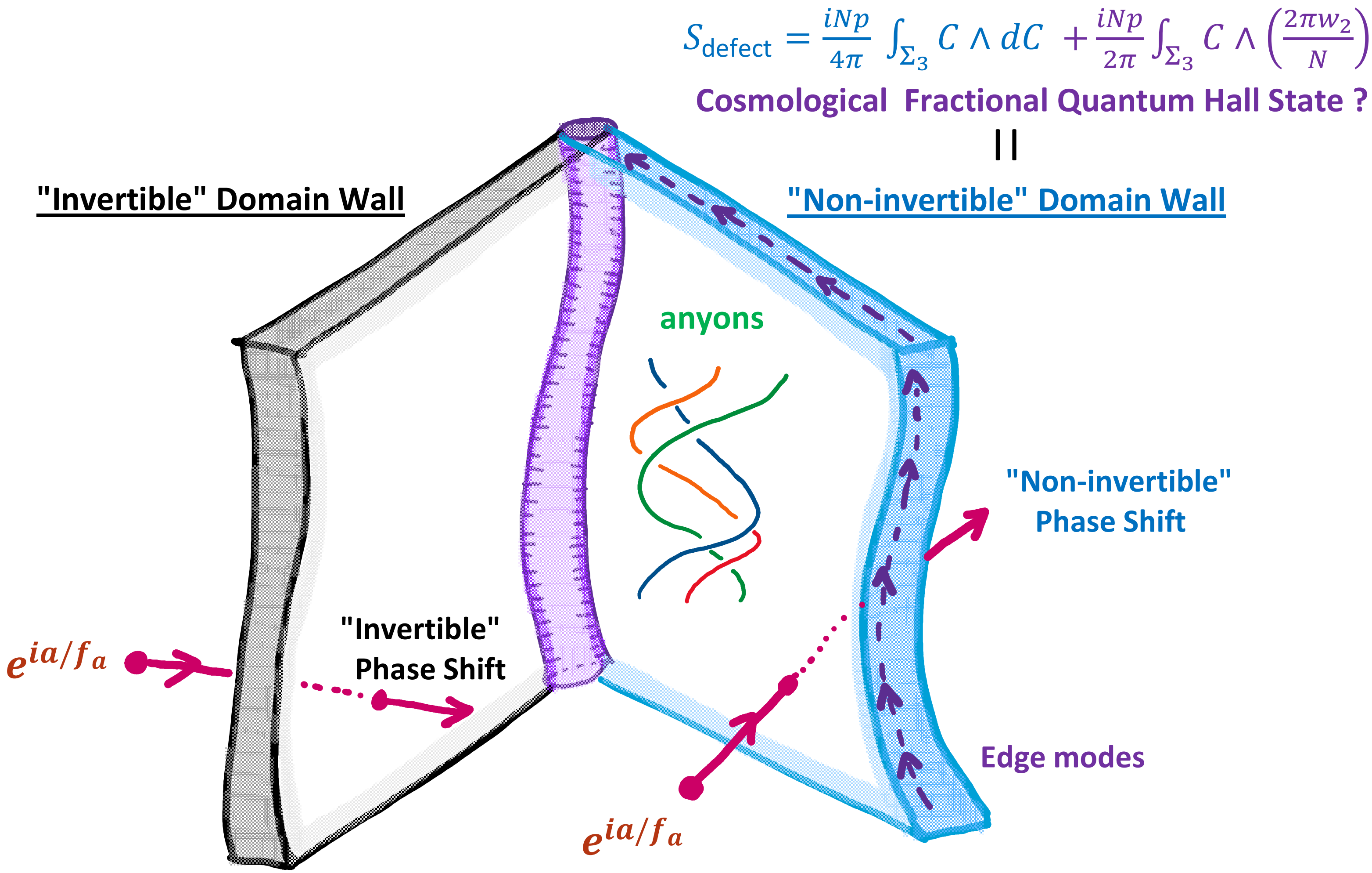


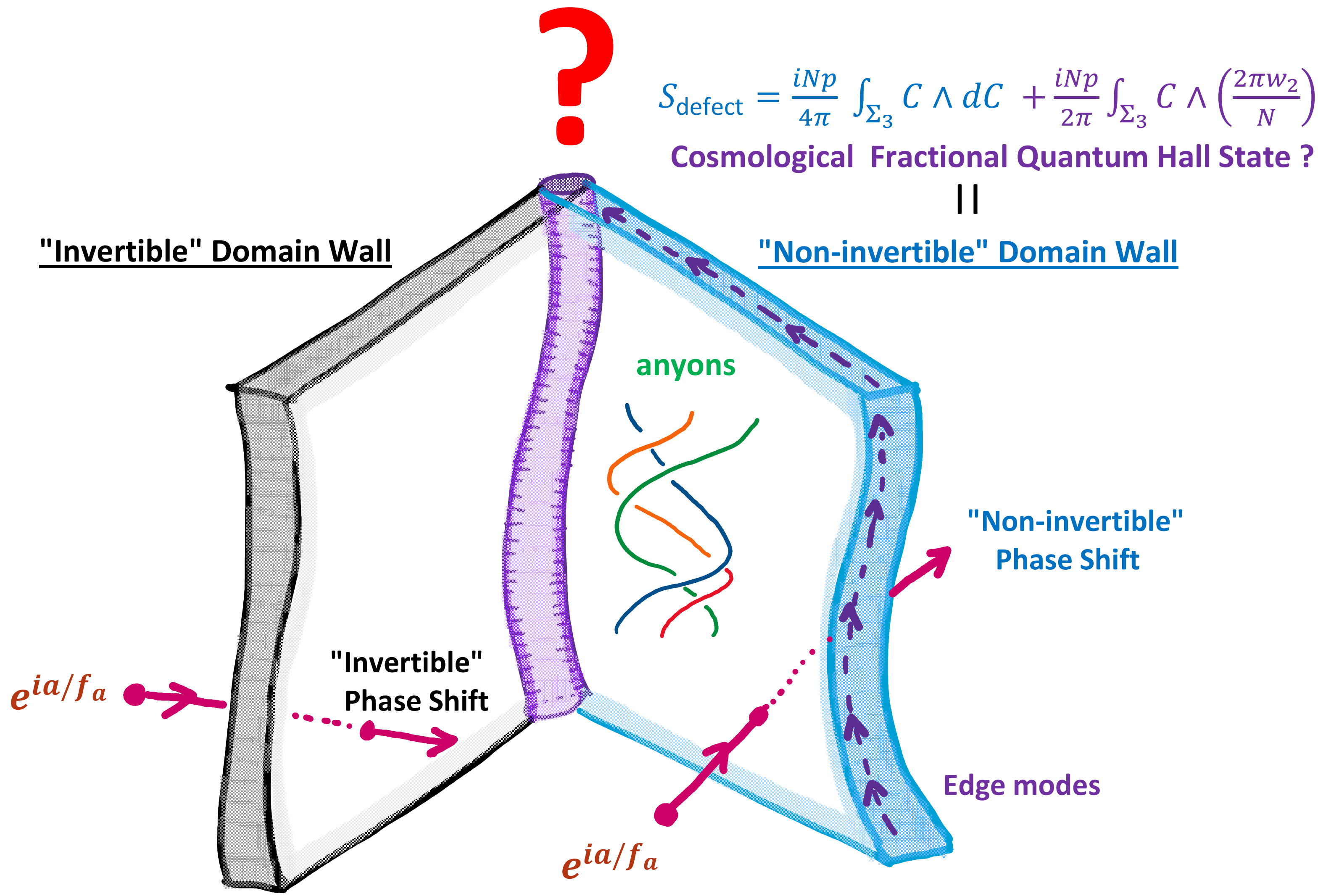
Axion Non-Invertible Shift Symmetry

Invertible DW vs Non-Invertible DW



Axion Non-Invertible Shift Symmetry

Invertible DW vs Non-Invertible DW



Axion-SM

$$S \supset \frac{i\ell_3}{8\pi^2}\int \frac{a}{f_a}\, Tr \cdots B \wedge B$$

$$G_{SM} = \frac{SU(3)_C \times SU(\cdots}{\Gamma}$$

Non-invertible Axion Shift Symmetry:

$$Z_K^I \subset Z^{NI}_{\gcd(\ell_2,\ell_3)} \approx Z^{NI}_{\ell_3}$$

$$\Gamma = 1 \;:\quad K = \gcd(\ell_1, \ell_2, \ell_2)$$

$$\Gamma = Z_2 \;:\quad K = \gcd\left(\frac{\ell_1}{2}, \ell_2, \ell_3, \frac{\ell_1+2\ell_2}{4}\right)$$

$$\Gamma = Z_3 \;:\quad K = \gcd\left(\frac{\ell_1}{3}, \ell_2, \ell_3, \frac{\ell_1+6\ell_3}{9}\right)$$

$$\Gamma = Z_6 \;:\quad K = \gcd\left(\frac{\ell_1}{6}, \ell_2, \ell_3, \frac{\ell_1+2\ell_2}{4}, \frac{\ell_1+6\ell_3}{9}\right)$$

Physics sensitive to $\mathbf{\Gamma}$

(i) Quantization conditions on ℓ_1, ℓ_2, ℓ_3 (IR parameters) depending on $\mathbf{\Gamma}$

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II. Global Structure and Axion Defects

II-2. Global Structure and Lazarides-Shafi Mechanism

Correct Domain Wall Spectrum from $\frac{G_c \times G_g}{\Gamma}$

(1) Lazarides and Shafi (1982):

embed $Z_{N_{DW}}^{PQ} \subset U(1)_{PQ}$ into continuous gauge group

$$\frac{\boldsymbol{Spin(10)} \times \boldsymbol{U(1)_{PQ}}}{\mathbb{Z}_{\mathbf{4}}} \quad \text{with} \quad A = 4N_g$$

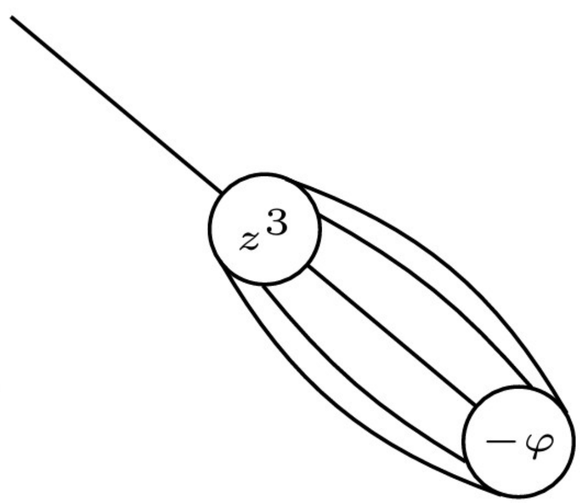
Correct Domain Wall Spectrum from $\frac{G_c \times G_g}{\Gamma}$

(2) Lazarides-Shafi mechanism does not always work:
precise string-wall network is UV-sensitive [Q. Lu, M. Reece, Z. Sun '23]

E.g. Gauge anom-free $\mathbb{Z}_5 \subset U(1)_{PQ}$ with $\left[\phi = v_\phi e^{i\theta_\phi}\right]_{PQ} = 2$

SSB of $\mathbb{Z}_5$ by $V(\theta)$: gauge-equivalent vacua $\theta_\phi \mapsto \theta_\phi + 2\pi/5$

$\mathbb{Z}_5$ string: $\oint_{z^k} \theta_\phi = 2\pi\left(\frac{2k}{5} + n\right) = 2\pi\left(\frac{2\cdot 3}{5} - 1\right)$

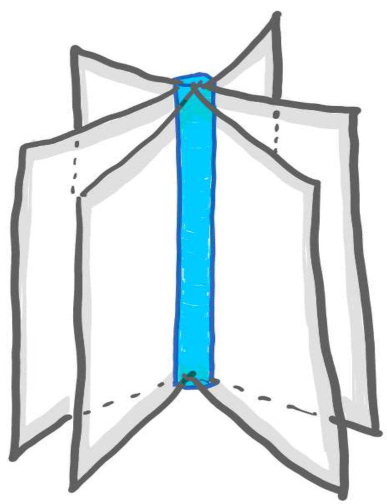


Correct Domain Wall Spectrum from $\frac{G_c \times G_g}{\Gamma}$

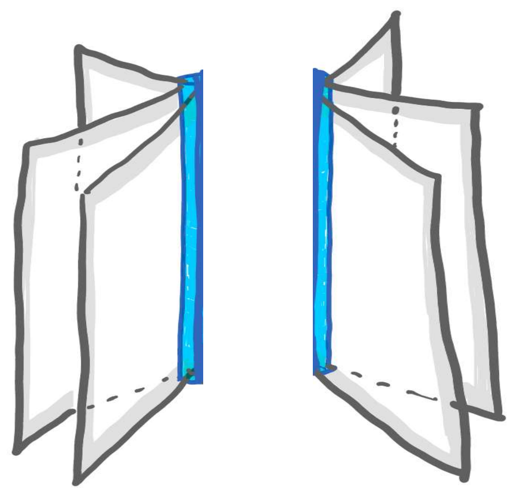
(3) Precise wall-string network from global structure
[G. Choi, S. Hong, S. Koren '26]

$$U(1)_Y \times U(1)_{PQ} \quad \text{vs} \quad \frac{U(1)_Y \times U(1)_{PQ}}{\mathbb{Z}_2}$$

$$\partial_\mu J^\mu_{PQ} = \frac{A}{16\pi^2}\mathrm{Tr}(G\tilde{G}),\ A = 2N_g$$



$\mathbb{Z}_{2N_g}$ Domain Wall Problem

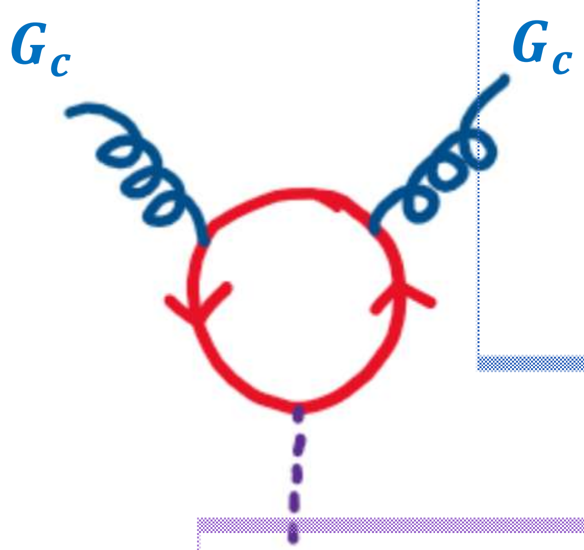


$\mathbb{Z}_{N_g}$ Domain Wall Problem

Closing Remarks

There is one more reason why AXION is special (and great).

In 4d, axion is special in that it can probe rich topology of gauge fields.



(i) 1-form electric: $d * G = 0$

(ii) 1-form magnetic : $dG = 0$

(iii) (-1)-form Chern-Weil: $d(G \wedge G) = 0$

ABJ anom = CS term

$$\boldsymbol{\theta\, tr(G \wedge G)}$$

(i) higher-group

(ii) non-invertible

$\boldsymbol{J}$ $\Rightarrow$ **SSB :** $\boldsymbol{J \sim d\theta}$

(i) 0-form axion shift

(ii) 2-form axion winding

Thank You
For
Your Attention!