General Relativity

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Some information about this course

- In three lectures, it is not possible to cover General Relativity in full.
- The goal of this course is to introduce some key aspects of GR. Whenever an asterisk (*) appears in the slides, it indicates additional material provided as **extra practice with fully solved exercises.**
- For further study, the following classic textbooks are highly recommended:
  1. R. M. Wald, *General Relativity*
  2. S. Weinberg, *Gravitation and Cosmology*
  3. R. d'Inverno, *Introducing Einstein's Relativity*
  4. C. G. Böhmer, *Introduction to General Relativity and Cosmology*

Outline

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Newton's Law of Universal Gravitation

In Newtonian gravity, the force exerted by a point mass m_1 on another point mass m_2 is (1687)

$$\vec{F}_{1\to 2} = -\,G\,\frac{m_1 m_2}{|\vec{r}|^2}\,\frac{\vec{r}}{|\vec{r}|},$$

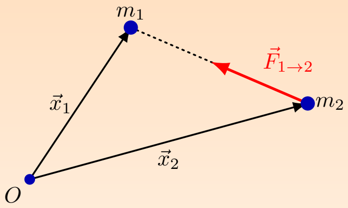
where

$$\vec{r} = \vec{x}_2 - \vec{x}_1$$

points from m_1 to m_2.

The gravitational constant (Cavendish) is

$$G = (6.67428 \pm 0.00067) \times 10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}.$$



Gravitational force.

The Gravitational Field (Point Mass)

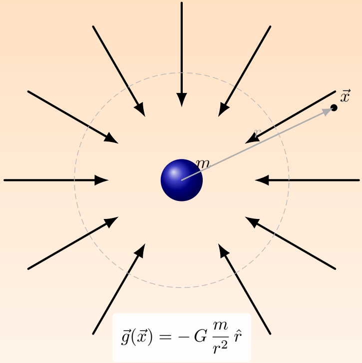
Definition. The gravitational field is force per unit test mass:

$$\vec{g}(\vec{x}) \equiv \frac{\vec{F}(\vec{x})}{m_{\text{test}}}.$$

For a point mass m located at $\vec{x}_m$, let $\vec{r} = \vec{x} - \vec{x}_m$. Newton's law gives

$$\vec{g}(\vec{x}) = -G\,m\,\frac{\vec{r}}{|\vec{r}|^3} \;=\; -G\,\frac{m}{r^2}\,\hat{r}.$$

Geometric picture: $\vec{g}$ always points toward the source and falls off as $1/r^2$.



Field lines for a point mass.

Superposition and Potential

Superposition. For many sources, fields add linearly. For a continuous mass density $\rho(\vec{x}')$,

$$\vec{g}(\vec{x}) = -G \int_{\mathbb{R}^3} \rho(\vec{x}')\, \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\, d^3x'.$$

Conservative field (Newtonian gravity). In regions without time-dependent effects, the field is irrotational:

$$\nabla \times \vec{g} = \vec{0}.$$

Therefore one can introduce a *gravitational potential* Φ such that

$$\vec{g} = -\nabla\Phi.$$

Gravitational Potential and Field Equations

If $\vec{g} = -\nabla\Phi$, then (up to an additive constant) the potential can be written as

$$\Phi(\vec{x}) = -G \int_{\mathbb{R}^3} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x' \; + \; \text{const.}$$

Poisson equation. The potential is sourced by the mass density:

$$\nabla^2\Phi = 4\pi G\,\rho.$$

Equivalent form (Gauss law for gravity). Using $\vec{g} = -\nabla\Phi$,

$$\nabla \cdot \vec{g} = -4\pi G\,\rho.$$

Instantaneous Interaction (Action at a Distance)

Instantaneous Interaction

In Newtonian gravity, changes in the mass distribution are transmitted instantaneously to all points in space. If the source of the field disappears ($\rho \to 0$), the gravitational field vanishes everywhere at the same time. This notion of instantaneous interaction conflicts with the relativistic principle that no physical influence can propagate faster than the speed of light.

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A question to think about

- If the Sun were suddenly removed, would Earth “notice” the change *immediately*?
- What does Special Relativity force us to conclude about how gravity should propagate?

Inertial vs. Gravitational Mass

Two notions of mass

- **Inertial mass** m_i: resistance to acceleration (appears in $\mathbf{F} = m_i\mathbf{a}$).

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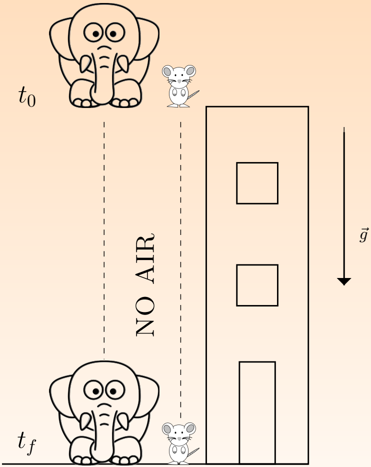
For a test body in a given gravitational field $\mathbf{g}$,

$$\mathbf{F} = m_i\,\mathbf{a} = m_g\,\mathbf{g} \quad \Rightarrow \quad \mathbf{a} = \left(\frac{m_g}{m_i}\right)\mathbf{g}.$$

Experimentally, the equality $m_g = m_i$ holds to extremely high precision ($\sim 10^{-17}$ of order in magnitude), implying the universality of free fall.



Weak Equivalence Principle (WEP)

Statement (Universality of Free Fall)

All test bodies fall the same way in a given gravitational field: their motion is independent of mass and internal composition.

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Newtonian viewpoint

If $m_g = m_i$, then

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so the trajectory depends only on the gravitational field, not on properties of the test body.

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Takeaway

WEP is the empirical clue that gravity is *geometric* (or at least universal).
This universality strongly suggests a geometric description.

Accelerated Frames and Gravity

This shows how an accelerated frame can reproduce the effect of a uniform gravitational field.

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Consider two reference frames:

- An inertial frame O.
- A frame O' accelerating with constant acceleration $\vec{a}$ relative to O.

Their coordinates are related by

$$t' = t, \qquad \vec{x}' = \vec{x} - \tfrac{1}{2}\,\vec{a}\,t^2.$$

Taking two time derivatives,

$$\frac{d^2\vec{x}'}{dt'^2} = \frac{d^2\vec{x}}{dt^2} - \vec{a}.$$

If the particle in O experiences a gravitational field $\vec{g}$,

$$\frac{d^2\vec{x}'}{dt'^2} = \vec{g} - \vec{a}.$$

Free-Falling Frames

From

$$\frac{d^2\vec{x}'}{dt'^2} = \vec{g} - \vec{a},$$

we observe a key fact:

Free-falling frame

If the accelerated frame satisfies

$$\vec{a} = \vec{g},$$

then

$$\frac{d^2\vec{x}'}{dt'^2} = 0.$$

- In a free-falling reference frame, particles move along straight lines with constant velocity.
- Gravitational effects can be locally eliminated (over sufficiently small regions of spacetime).

Einstein Elevator: Physical Equivalence

Thought experiment

- A person inside an elevator falling freely toward Earth.
- A person inside a rocket in deep space, far from all masses.

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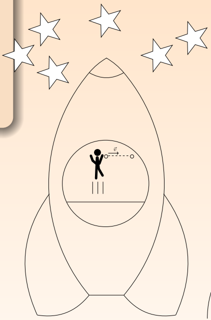
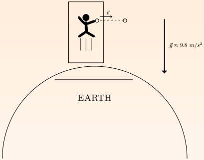
- A person inside an elevator falling freely toward Earth.
- A person inside a rocket in deep space, far from all masses.

- In both cases, objects float freely.
- A thrown ball follows a straight-line trajectory.
- No local experiment can distinguish the two situations.



(c) Person in a rocket.



(d) Person in a lift falling towards the Earth.

Acceleration Mimics Gravity (The Other Direction)

Key idea

Gravity can be *simulated* by going to a uniformly accelerating frame.
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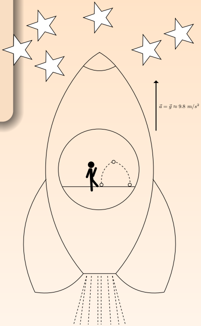
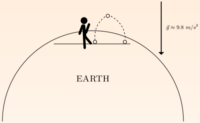
Consider a rocket far from all masses, so $\vec{g} = \vec{0}$. If the rocket accelerates with constant acceleration $\vec{a}$, the observed motion in the rocket frame satisfies

$$\frac{d^2\vec{x}'}{dt'^2} = \vec{g} - \vec{a} = -\vec{a}.$$



(g) Person in an accelerating rocket.



(h) Person on the Earth.

- If $\vec{a} = \vec{g}_{\oplus}$, objects fall inside the rocket exactly as on Earth.

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- In local inertial frames, the laws of physics reduce to those of **Special Relativity** including all non-gravitational law of physics.
- A test particle, neglecting other forces, is at rest or moves along a straight line with constant velocity.

A Physical Consequence of the Equivalence Principle

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Key consequence (qualitative)

Light must be deflected by a gravitational field.

This argument is purely local and qualitative. A quantitative description requires the spacetime geometry.

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Conceptual tension

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Key issue

Newtonian gravity cannot incorporate the equivalence principle in a fully frame-independent way.

From the Equivalence Principle to Geometry

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Key implication

Gravity cannot be described by a single global force field. Instead, it must be encoded in the geometry of spacetime.

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- Introduce geometric objects that transform covariantly.
- Encode gravitational effects in the geometry itself.

Outline

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2. the mapping is **one-to-one**;
3. if two such mappings overlap, they are related by **differentiable coordinate transformations**.

Example: The Circle S^1

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Consequence

The coordinate ϕ is not one-to-one globally. At least two coordinate patches are needed to cover the circle.

Example: The Sphere S^2

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Conclusion

No single coordinate system covers the entire sphere. Multiple overlapping charts are required.

From geometry to notation

Key transition

To describe physics on manifolds, we need a precise language to represent geometric objects and their components.
This language is provided by tensor calculus.

Indices and Components

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- This distinction will become essential once we introduce the metric and covariant derivatives.

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Given objects

$$A^a = (A^1, \ldots, A^n), \qquad B_a = (B_1, \ldots, B_n),$$

one writes

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- Free indices must match on both sides of an equation.

Free and Dummy Indices

- A **dummy index** is summed over and can be renamed:

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$$w^a = T^a{}_b \, v^b$$

Scalar Fields (no indices)

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- Scalars do **not** transform under coordinate changes (invariant).
- Their value at a point is coordinate-independent.

Vectors (Contravariant Vectors) - one index up

Definition

A **contravariant vector** is an object with one upper index whose components transform as

$$V'^a = \frac{\partial X'^a}{\partial X^b} V^b$$

under a coordinate transformation $X^a \to X'^a(X)$.

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- The transformation law defines the geometric nature of a vector.
- Vectors transform with the Jacobian of the coordinate map.

Covariant Vectors (1-Forms) - one index down

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A **covariant vector** (or **1-form**) is an object with one lower index that transforms as

$$W'_b = \frac{\partial X^c}{\partial X'^b} W_c.$$

- Covariant vectors transform with the inverse Jacobian.
- They are dual to contravariant vectors.

Tensors of Type (p, q)

Definition

A tensor of type (p, q) is an object with p upper and q lower indices. Its components transform as

$$T'^{a_1\cdots a_p}{}_{b_1\cdots b_q} = \frac{\partial X'^{a_1}}{\partial X^{c_1}} \cdots \frac{\partial X'^{a_p}}{\partial X^{c_p}} \frac{\partial X^{d_1}}{\partial X'^{b_1}} \cdots \frac{\partial X^{d_q}}{\partial X'^{b_q}} T^{c_1\cdots c_p}{}_{d_1\cdots d_q}.$$

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- Scalars: $(0, 0)$ Contravariant Vectors: $(1, 0)$ Covariant Vectors: $(0, 1)$
- The rank of a tensor is $p + q$.
- If a tensor is zero, then it will be zero in all coordinate systems as well.

Tensor Algebra: Basic Operations

Addition

Two tensors can be added only if they have the same type and index structure:

$$R^a{}_b{}^c + S^a{}_b{}^c = T^a{}_b{}^c, \qquad A^a + B_a \text{ is not defined.}$$

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Tensor Product

Given tensors of type (p, q) and (r, s), their tensor product is a tensor of type $(p + r, q + s)$:

$$M^a{}_b = V^a W_b.$$

Contraction and Trace

Contraction

Given a tensor of type (p, q), one may contract one upper and one lower index, producing a tensor of type $(p-1, q-1)$:

$$T^a{}_a \equiv \sum_a T^a{}_a.$$

Contraction and Trace

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Given a tensor of type (p, q), one may contract one upper and one lower index, producing a tensor of type $(p-1, q-1)$:

$$T^a{}_a \equiv \sum_a T^a{}_a.$$

Trace

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$$\operatorname{tr} M = M^a{}_a.$$

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Index check

In any tensor equation, the free indices on both sides must match exactly.

Total differential

Definition

Let f be a scalar function of local coordinates $(X^1, \ldots, X^n)$:

$$f = f(X^1, \ldots, X^n).$$

Its **total differential** is

$$\boxed{df := f_{,a}\, dX^a = \sum_{a=1}^{n} \frac{\partial f}{\partial X^a}\, dX^a.}$$

- $f_{,a} \equiv \partial f / \partial X^a$ (comma notation).
- Einstein convention: repeated upper/lower indices are summed.
- df is a covector (1-form): it acts linearly on displacements dX^a.

Measuring distances on a manifold

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- In flat Euclidean space, distance is well defined.
- On a general manifold, distance must be defined *locally*.
- This leads naturally to the concept of a **metric tensor** $g_{\mu\nu}$.

Euclidean distance in $\mathbb{R}^2$

In two-dimensional Euclidean space with Cartesian coordinates (x, y), the distance between two points is given by

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Taking the infinitesimal limit, we find the so-called **line-element**

$$\boxed{ds^2 = dx^2 + dy^2.}$$

Line-element

Distance is encoded in a quadratic form built from coordinate differentials.

From Euclidean space to manifolds

The line-element expression for Euclidean 2D (flat)

$$ds^2 = dx^2 + dy^2$$

can be written in index notation as

$$ds^2 = \delta_{ab}\, dX^a dX^b, \qquad a, b = 1, 2,$$

where $\delta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{diag}(1, 1)$ and $dX^a = (dx, dy)$.

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Let us verify this step by step with indices from $i = 1, 2$:

$$\begin{aligned} ds^2 &= \delta_{ab}\, dX^a dX^b \\ &= \delta_{1b}\, dX^1 dX^b + \delta_{2b}\, dX^2 dX^b \end{aligned}$$

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This suggests the generalization:

- Replace δ_{ab} by a position-dependent object $g_{ab}(X)$.
- Require ds^2 to be invariant under coordinate transformations.

Definition of a metric

Metric tensor

Let X^a and $X^a + dX^a$ be two infinitesimally separated points. A **metric** is a symmetric rank-2 tensor $g_{ab}(X)$ such that

$$\boxed{ds^2 = g_{ab}(X)\, dX^a dX^b.}$$

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- $g_{ab}(X)$ is an arbitrary function of the coordinates.
- It is assumed to be non-degenerate, so an inverse g^{ab} exists:

$$g_{ab}g^{bc} = \delta_a{}^c.$$

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The metric tensor is

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- Constant metric
- Flat geometry

Change of coordinates: polar coordinates

Introduce polar coordinates (r, φ) via

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Differentials:

$$dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi, \qquad dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi.$$

Line element in polar coordinates

Squaring and adding,

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Thus,

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The metric components are

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Why this matters

In gravity, we will interpret nontrivial metric components as encoding gravitational effects.

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- In a coordinate basis, the metric tensor in 4-dimensions can be represented as a symmetric 4×4 matrix:

$$g_{ab} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}.$$

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Physical relevance

In metric theories of gravity, all geometric and dynamical information ultimately derives from g_{ab}.

Signature of the Metric

Statement

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Definition

The signature of g_{ab} is the number of $(+)$ and $(-)$ signs in its diagonalised form.

Example: Minkowski spacetime

In flat spacetime, the line element is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

The corresponding metric has diagonal form

$$g_{ab} = \mathrm{diag}(-1, 1, 1, 1),$$

with one sign different from the others.

Lorentzian Metrics and Minkowski Spacetime

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Lorentzian metric

A metric with signature $(-, +, \ldots, +)$ or $(+, -, \ldots, -)$ is called **Lorentzian.** A manifold equipped with such a metric is called a **Lorentzian manifold.**

Metric: Inner Products, Norms and Angles

Given two vectors A^i and B^i at the same point of the manifold, the metric defines their inner product.

Inner product and norm

$$A \cdot B := g_{ij} A^i B^j, \qquad |A|^2 := g_{ij} A^i A^j.$$

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- These notions generalize familiar Euclidean concepts.
- The metric fully encodes the local geometry.

Inverse Metric and Index Manipulation

For a non-degenerate metric,

$$\det(g_{ij}) \neq 0,$$

there exists an inverse metric g^{ij} defined by

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Raising and lowering indices

Given a contravariant vector A^i, define

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Key point

The metric establishes a one-to-one correspondence between vectors and covectors. Only the *index position* matters.

Why A^i and A_i are different objects

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Why do we need a metric?

There is no canonical map between vectors and covectors. The metric g_{ij} provides an isomorphism:

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Key idea

The position of an index carries geometric meaning. Raising and lowering indices is only possible once a metric is specified.

Rank-2 tensors and index positions

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Lowering indices with the metric

Given a contravariant rank-2 tensor T^{kl}, define

$$T_{ij} := g_{ik}\, g_{jl}\, T^{kl}.$$

The metric maps $V \otimes V$ into $V^* \otimes V^*$.

Rank-2 tensors and index positions

Raising indices with the inverse metric

Conversely, given a covariant rank-2 tensor T_{kl}, define

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This requires the inverse metric g^{ij}.

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Raising indices with the inverse metric

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Note: Writing T_{ij} as a matrix is just a convenient way of listing components.
The tensor itself is defined by how its indices transform

Example: lowering indices in 2D polar coordinates

In 2D polar coordinates (r, φ),

$$ds^2 = dr^2 + r^2\, d\varphi^2, \qquad g_{rr} = 1, \quad g_{\varphi\varphi} = r^2, \quad g_{r\varphi} = g_{\varphi r} = 0.$$

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Consider a contravariant rank-2 tensor

$$T^{ij} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \iff \begin{cases} T^{rr} = 2, \\ T^{r\varphi} = 3, \\ T^{\varphi r} = 4, \\ T^{\varphi\varphi} = 5. \end{cases}$$

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Recall the definition

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$$T_{r\varphi} = g_{rk} g_{\varphi l} T^{kl} = g_{rr} g_{\varphi\varphi} T^{r\varphi}$$

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Metric: Length of a Curve

Given a metric tensor g_{ij} on a manifold M, the length of a curve $C : x^i(\lambda)$, with $\lambda \in [\lambda_i, \lambda_f]$, is defined as

(Recall that $ds^2 = g_{ij} dx^i dx^j$)

Definition

$$L = \int_{\lambda_i}^{\lambda_f} ds = \int_{\lambda_i}^{\lambda_f} \sqrt{g_{ij}(x(\lambda)) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}\, d\lambda.$$

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- This definition is coordinate invariant.
- It reduces to the usual arc length in Euclidean space.
- The metric determines how distances are measured locally.

Geodesics: Extremal Length

Consider a manifold (M, g) endowed with a metric tensor g_{ij}. Let $P, Q \in M$.

Definition

A **geodesic** is a curve $\gamma : \lambda \mapsto x^i(\lambda)$ joining P and Q whose length

$$L[\gamma] = \int_{\lambda_i}^{\lambda_f} \sqrt{g_{ij}(x(\lambda))\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\, d\lambda$$

is extremal (typically minimal).

Euler–Lagrange result

Applying the Euler–Lagrange equations to the length functional yields*

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = f(\lambda)\,\frac{dx^i}{d\lambda}.$$

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- The function $f(\lambda)$ reflects reparametrisation freedom.
- This equation determines free motion once the metric is specified.
- Particles in GR will follow this equation! (the shortest path)

Christoffel Symbols and Canonical Form

Christoffel symbols

The coefficients $\Gamma^i{}_{jk}$ are defined by

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right).$$

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Choosing an affine parameter (e.g. proper length ℓ), the geodesic equation takes the canonical form

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- In flat Euclidean space, ${\Gamma^i}_{jk} = 0$ and geodesics are straight lines.
- In curved spaces, ${\Gamma^i}_{jk}$ encodes the geometry.

A Glance Forward: From Geometry to Gravity

Let us continue with the *geodesic equation*, which governs free motion in a curved spacetime:

$$\frac{d^2X^a}{d\lambda^2} = -\Gamma^a{}_{bc}\frac{dX^b}{d\lambda}\frac{dX^c}{d\lambda}.$$

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Since the Christoffel symbols $\Gamma^a{}_{bc}$ contain *first derivatives of the metric*, it is natural to expect that the gravitational potential Φ is encoded in the spacetime metric itself.

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- Gravity is no longer a force, but a manifestation of spacetime geometry.
- Newtonian gravity should emerge as an appropriate limit of geodesic

Why Partial Derivatives Are Not Enough

- Scalars: $\partial_i f$ transforms covariantly.
- Tensors: $\partial_i T^a{}_b$ does not transform as a tensor*.

Key issue

Partial derivatives compare tensor components at *different points*. There is no canonical way to subtract tensors at distinct points on a manifold.

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Consequence

Without extra structure, there is no coordinate-independent notion of "constant tensor field".

Connection

Definition

A **connection** $\Gamma^i{}_{jk}$ is a set of n^3 functions such that under coordinate transformations

$$\bar{\Gamma}^i{}_{jk} = \frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^l{}_{pq} + \frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^k}.$$

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- $\Gamma^i{}_{jk}$ is not a tensor.
- The inhomogeneous term compensates the failure of ∂_i*.

Covariant Derivative

Definition (covariant vector)

For a covariant vector field A_j,

$$\nabla_i A_j \;:=\; \partial_i A_j - \Gamma^k{}_{ij} A_k .$$

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Contravariant vector

$$\nabla_i B^k = \partial_i B^k + \Gamma^k{}_{ij} B^j.$$

Covariant Derivative of a General Tensor

General rule

For a tensor $T^{a_1\cdots a_p}{}_{b_1\cdots b_q}$,

$$\nabla_c T^{a_1\ldots a_p}{}_{b_1\ldots b_q} = \partial_c T^{a_1\ldots a_p}{}_{b_1\ldots b_q} + \Gamma^{a_1}{}_{ck}\, T^{k\ldots a_p}{}_{b_1\ldots b_q} + \cdots + \Gamma^{a_p}{}_{ck}\, T^{a_1\ldots k}{}_{b_1\ldots b_q}$$
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For example:

$$\nabla_c T^{ab} = \partial_c T^{ab} + \Gamma^{a}{}_{ck}\, T^{kb} + \Gamma^{b}{}_{ck}\, T^{ak},$$

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- Different connections $\Rightarrow$ different notions of parallelism.

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- Covariant derivatives depend on the choice of connection.
- Different connections $\Rightarrow$ different notions of parallelism.
- The difference of two connections *is* a tensor.

Curvature tensor

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Key idea

Curvature arises when comparing the result of transporting vectors along *different infinitesimal paths*.

Curvature from covariant derivatives

Scalars

For a scalar field f,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) f = 0.$$

Curvature from covariant derivatives

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Vectors

For a vector field V^c,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c{}_{dab} V^d.$$

Riemann curvature tensor

The above tensor is defined as:

$$\boxed{R^c{}_{dab} = \partial_a \Gamma^c{}_{bd} - \partial_b \Gamma^c{}_{ad} + \Gamma^c{}_{ae} \Gamma^e{}_{bd} - \Gamma^c{}_{be} \Gamma^e{}_{ad}}$$

Geometric meaning of curvature

Parallel transport

A vector V^a is parallel transported along a curve with tangent T^a if

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Curvature

Transporting a vector around an infinitesimal closed loop does not return the same vector:

$$\Delta V^a \propto R^a{}_{bcd}.$$

Geometric content of a connection

Up to this point we have implicitly assumed the Levi–Civita connection.
More generally, an affine connection may have three independent features:

Curvature

Failure of vectors to return unchanged after transport around a loop.

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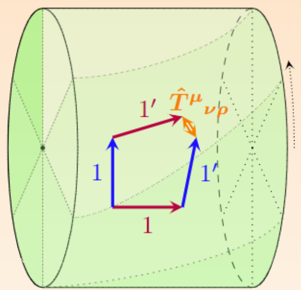
Non-metricity

Failure of lengths and angles to be preserved under transport.

$$\nabla_a g_{bc} \neq 0$$

Geometric meaning: torsion, non-metricity, curvature



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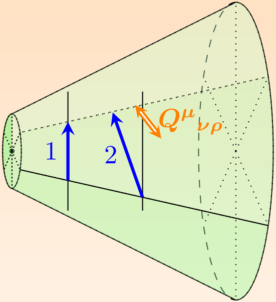
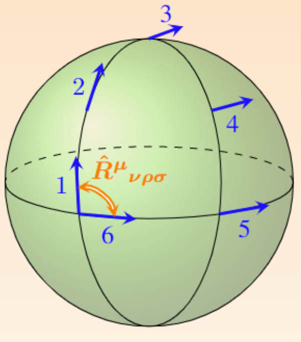


Non-metricity: lengths and angles not preserved



Curvature: vector changes after parallel transport around a loop

In General Relativity, we work with a unique connection satisfying:

$$\begin{cases} T^a{}_{bc} = 0 & \text{(torsion-free)} \\ \nabla_a g_{bc} = 0* & \text{(metric-compatible)} \end{cases}$$

Conclusion

All gravitational effects are encoded purely in the metric and then, only curvature is non-vanishing.

Structure of the Riemann Curvature Tensor

Independent components

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Important symmetries

- Antisymmetric in index pairs:

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc}$$

- Algebraic (cyclic) and differential Bianchi identities:

$$R_{abcd} + R_{cabd} + R_{bcad} = 0\,,$$
$$\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} = 0\,.$$

Contractions of the Riemann Tensor

The Riemann tensor $R^a{}_{bcd}$ encodes the full local curvature associated with a connection.

First contraction: Ricci tensor

Contracting one contravariant and one covariant index,

$$\boxed{R_{ab} := R^c{}_{acb},}$$

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Second contraction: scalar curvature

Taking the trace of the Ricci tensor,

$$\boxed{R := g^{ab} R_{ab},}$$

defines the **scalar curvature** that is a scalar invariant under coordinate transformations.

Outline

Towards a geometric theory of gravity

Guiding principles

We seek a theory of gravity that:

- is formulated in a geometric and covariant way,

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Strategy

These requirements suggest that gravity should be encoded in spacetime geometry, through tensors constructed from the metric and its derivatives.

Energy–momentum tensor

Motivation

In General Relativity, the gravitational field must be sourced by a *local, covariant object* encoding energy and momentum.

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$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} = \begin{pmatrix} T^{00} & T^{0j} \\ T^{i0} & T^{ij} \end{pmatrix} .$$

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- T_{00}: energy density
- T_{0i}: momentum density (energy flux)
- T_{ij}: stresses (pressure and shear)

Energy–momentum tensor: Maxwell field

Electromagnetic field

Let F_{ab} be the electromagnetic field strength tensor.

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- Traceless: $T^a{}_a = 0$
- Encodes energy density, Poynting flux and stresses
- Fully covariant and symmetric

Energy–momentum tensor: Perfect fluid

Perfect fluid

A perfect fluid is characterised by:

- Energy density ρ
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$$T_{ab} = (\rho + p) u_a u_b + p\, g_{ab}.$$

- In the rest frame: $u^a = (1, 0, 0, 0)$
- Non–relativistic matter: $p \ll \rho$

Non–relativistic limit

Assumptions

- Weak gravitational field
- Velocities $v \ll c$
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This limit will be used to fix the coupling constant and identify the correct gravitational source.

Newtonian limit of gravity

Weak field expansion

We write the metric as

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$$\frac{d^2\vec{x}}{dt^2} \simeq -\frac{1}{2}\vec{\nabla} h_{00}.$$

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Identifying $\vec{g} = -\vec{\nabla}\Phi$ gives

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Poisson equation and gravity

Newtonian gravity

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This identifies the energy density as the gravitational source.

From energy density to a covariant source

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Energy–momentum tensor

The natural covariant generalisation of the energy density is

$$T_{\mu\nu}.$$

General form of the field equations

Guided by the Newtonian limit, we postulate field equations of the form

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu},$$

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- At this stage, the precise form of $G_{\mu\nu}$ is still unknown.

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- Must recover Newtonian limit for weak grav fields.
- The most general rank-2 tensor (without a constant) satisfying those conditions is:

$$G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R,$$

  where C_1 and C_2 are constants that are related $C_1 = -2C_2$ to ensure covariant conservation.

Einstein tensor

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This tensor is called the **Einstein tensor.**

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The Einstein field equations without setting $c = 1$ read

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- In the weak-field (Newtonian) limit, it reproduces Poisson's equation.
- In four spacetime dimensions, $g_{\mu\nu}$ has 10 independent components. Therefore, the Einstein equations form a system of

10 coupled, non-linear, second-order partial differential equations.

Trace and vacuum Einstein equations

Taking the trace

In 4-dimensions, by contracting the Einstein equations with $g^{\mu\nu}$, we find

$$g^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=\frac{8\pi G}{c^4}\,g^{\mu\nu}T_{\mu\nu},$$

we obtain

$$R-\frac{1}{2}(4)R=-R=\frac{8\pi G}{c^4}\,T,$$

where $T=g^{\mu\nu}T_{\mu\nu}$ is the trace of the energy–momentum tensor.

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Equivalent form

Substituting back, the Einstein equations in 4D can be written as

$$R_{\mu\nu}=\frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{2}Tg_{\mu\nu}\right).$$

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- Vacuum solutions can still possess non-trivial curvature.

Geometrical interpretation of gravity

Wheeler's interpretation

"Spacetime tells matter how to move; matter tells spacetime how to curve."

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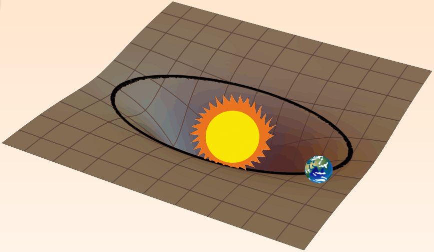
This contrasts with Newtonian gravity, where gravity acts as a force in a fixed, flat background.



Einstein equations from an action principle

Motivation

So far, the Einstein field equations were introduced from:

- geometric identities,
- physical requirements (symmetry, conservation),
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Alternative viewpoint

There exists an equivalent formulation:

- the Einstein equations can be obtained from a variational principle,
- by extremising an action with respect to the metric.

This approach places General Relativity within the standard framework of classical field theory.

Einstein–Hilbert action

Gravitational action

The dynamics of the gravitational field can be obtained from the action

$$S_{\mathsf{GR}} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,R + S_m,$$

where $g = \det(g_{\mu\nu})$ and S_m is the matter action.

- The fundamental variable is the metric $g_{\mu\nu}$.
- Field equations follow from the stationarity condition

$$\delta S_{\mathsf{GR}} = 0$$

under arbitrary variations $\delta g^{\mu\nu}$.

Variation of the gravitational action

Variation of the integrand

The variation of the gravitational term gives

$$\delta(\sqrt{-g}R) = R\,\delta\sqrt{-g} + \sqrt{-g}\,\delta R.$$

Metric determinant

Using

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\delta g_{\mu\nu}) = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu},$$

where we used $g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$ that can be found by using $\delta g^{\mu\nu} = -g^{\mu\alpha}\,(\delta g_{\alpha\beta})\,g^{\beta\nu}$.

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Thus, the first term becomes

$$-\frac{1}{2}\sqrt{-g}\,R\,g_{\mu\nu}\,\delta g^{\mu\nu}.$$

Variation of the scalar curvature

- Since $R = g^{\mu\nu} R_{\mu\nu}$,

$$\sqrt{-g}\delta R = \sqrt{-g}\delta(R_{\mu\nu}\delta g^{\mu\nu}) = \sqrt{-g}R_{\mu\nu}\,\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\,.$$

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- Now, by replacing the Ricci tensor in terms of Levi-Civita:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda}$$

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- Important identity: $\sqrt{-g}\,\nabla_\mu A^\mu = \nabla_\mu\left(\sqrt{-g}\,A^\mu\right) = \partial_\mu\left(\sqrt{-g}\,A^\mu\right)$
- Therefore, the second term does not contribute to the field equations (it is a boundary term!)

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$$\sqrt{-g}\delta R = \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_\rho\left(g^{\sigma\nu}\delta\Gamma^{\rho}{}_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^{\mu}{}_{\mu\sigma}\right).$$

- Important identity: $\sqrt{-g}\,\nabla_\mu A^\mu = \nabla_\mu\left(\sqrt{-g}\,A^\mu\right) = \partial_\mu\left(\sqrt{-g}\,A^\mu\right)$
- Therefore, the second term does not contribute to the field equations (it is a boundary term!)
- Hence, the variation of this term is

Variation of the scalar curvature

- Since $R = g^{\mu\nu} R_{\mu\nu}$,

$$\sqrt{-g}\delta R = \sqrt{-g}\delta(R_{\mu\nu}\delta g^{\mu\nu}) = \sqrt{-g}R_{\mu\nu}\,\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\,.$$

- Now, by replacing the Ricci tensor in terms of Levi-Civita:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda}$$

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Variation second term

$$\sqrt{-g}\delta R = \sqrt{-g}R_{\mu\nu}\,\delta g^{\mu\nu}$$

+boundary terms

Result of the gravitational variation

Gravitational contribution

After neglecting the boundary term, the variation yields

$$\delta S_{\text{GR}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu}.$$

The Einstein tensor emerges naturally from the variational principle.

Matter sector and field equations

Energy–momentum tensor

The energy–momentum tensor is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}.$$

Einstein field equations

Requiring $\delta S_{\text{GR}} = 0$ for arbitrary $\delta g^{\mu\nu}$ leads to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu}.$$

Cosmological constant

Generalised gravitational action

The most general gravitational action with at most second derivatives allows the addition of a constant term,

$$S_\Lambda = -\frac{1}{\kappa^2}\int d^4x\,\sqrt{-g}\,\Lambda.$$

Einstein equations with Λ

Varying the total action leads to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}.$$

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- Λ acts like a uniform vacuum energy density (present even when $T_{\mu\nu} = 0$).
- It is compatible with covariance and conservation laws.

Outline

Isometries and Killing vectors (minimal toolkit)

Isometry

A transformation is an **isometry** if it leaves the line element invariant:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \quad \text{is unchanged.}$$

Equivalently, the metric is constant along the symmetry flow.

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A vector field ξ^μ generates an isometry iff

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \quad \Longleftrightarrow \quad \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0.$$

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- These Killing vectors imply conserved quantities along geodesics (energy and angular momentum).

Schwarzschild problem: assumptions

Goal

Find the gravitational field **outside** a static, spherically symmetric mass.

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Input

- **Vacuum exterior**: $T_{ab} = 0 \Rightarrow R_{ab} = 0$.
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Find the gravitational field **outside** a static, spherically symmetric mass.

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How to model this?

Take the full metric and assume spherical symmetry (Killing) and solve the vacuum Einstein's field equations.

Most general static, spherically symmetric metric

Static spherically symmetric Metric $(-+++)$

The most general static and spherically symmetric line element can be written as

$$ds^2 = -e^{\nu(r)}dt^2 + e^{a(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

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- Time independence: $\partial_t g_{\mu\nu} = 0$
- Spherical symmetry fixes the angular sector
- Two unknown functions: $\nu(r)$ and $a(r)$

Connection coefficients

Non-vanishing Christoffel symbols

The independent non-zero Christoffel symbols are:*

$$\Gamma^t{}_{tr} = \frac{1}{2}\nu'(r), \qquad \Gamma^r{}_{tt} = \frac{1}{2}\nu'(r)e^{\nu-a},$$

$$\Gamma^r{}_{rr} = \frac{1}{2}a'(r), \qquad \Gamma^r{}_{\theta\theta} = -re^{-a},$$

$$\Gamma^r{}_{\phi\phi} = -r\sin^2\theta\, e^{-a},$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}, \qquad \Gamma^\phi{}_{\theta\phi} = \cot\theta.$$

(A prime denotes derivative with respect to r. All other components follow by symmetry.)

Ricci tensor components

Non-vanishing components

The Ricci tensor components are:*

$$R_{tt} = \frac{1}{2}e^{\nu - a}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}a'\nu' + \frac{2}{r}\nu'\right),$$

$$R_{rr} = -\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}a'\nu' + \frac{1}{r}a',$$

$$R_{\theta\theta} = 1 - e^{-a}\left(1 - \frac{r}{2}a' + \frac{r}{2}\nu'\right), \qquad R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}.$$

(All other components follow by symmetry.)

Vacuum Einstein equations

In vacuum, Einstein's equations reduce to

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For the metric ansatz above, there are three non-trivial equations, but only two of them are independent due to Bianchi identities.
We first consider the (tt) and (rr) components.

The (tt) and (rr) equations

The (tt) component yields

$$\frac{1}{2}\nu'' + \frac{1}{4}(\nu')^2 + \frac{1}{r}\nu' - \frac{1}{4}\nu' a' = 0.$$

The (rr) component gives

$$-\frac{1}{2}\nu'' - \frac{1}{4}(\nu')^2 + \frac{1}{4}\nu' a' + \frac{1}{r}a' = 0.$$

Here primes denote derivatives with respect to r.

Relation between metric functions

Adding the (tt) and (rr) equations, we obtain

$$\frac{1}{r}\left(\nu' + a'\right) = 0.$$

This integrates immediately to

$$\nu(r) + a(r) = \text{const.}$$

By a constant rescaling of the time coordinate t, the constant can be set to zero, so that

$$\nu(r) = -a(r).$$

The $(\theta\theta)$ equation

Using $a = -\nu$, the $(\theta\theta)$ component of $R_{ab} = 0$ reduces to

$$1 - e^{\nu} - r\nu' e^{\nu} = 0.$$

Noting that

$$\frac{d}{dr}(re^{\nu}) = e^{\nu} + r\nu' e^{\nu},$$

the equation can be rewritten as

$$\frac{d}{dr}(r - re^{\nu}) = 0.$$

Schwarzschild solution

Integrating, we find

$$r - re^{\nu} = C,$$

which implies

$$e^{\nu(r)} = 1 - \frac{C}{r}.$$

Since $a = -\nu$, the metric becomes

$$ds^2 = -\left(1 - \frac{C}{r}\right) dt^2 + \left(1 - \frac{C}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$.

Newtonian limit and identification of the constant

To identify the integration constant C, we restore physical units:

$$ds^2 = -\left(1 - \frac{GC}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{GC}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

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Comparing with Newtonian gravity,

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2},$$

we identify

$$\boxed{C = 2M.}$$

Final form of the Schwarzschild metric

Restoring G and c, the Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

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- Apparent singularities (metric diverges at some points!) occur at:*
    - $r = r_s := 2GM/c^2$ (coordinate singularity, event horizon),
    - $r = 0$ (true curvature singularity).
- Spherical coordinates are not good, one can introduce other ones (Eddington-Finkelstein) and then $r = r_s$ is non-singular.
- One way to check singularities is by looking into scalars constructed from curvature: For example Kretschmann invariant $K = R^{\lambda\mu\nu\rho} R_{\lambda\mu\nu\rho}$

Birkhoff’s theorem (statement)

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- Therefore, the exterior vacuum field of any spherically symmetric body is Schwarzschild.
- Even if the source changes in time (e.g. pulsations), the *exterior* vacuum metric remains static.

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- Therefore, the exterior vacuum field of any spherically symmetric body is Schwarzschild.
- Even if the source changes in time (e.g. pulsations), the *exterior* vacuum metric remains static. Schwarzschild is the unique spherically symmetric solution of GR in vacuum.

Schwarzschild radius: Sun and Earth

Characteristic length scale

$$r_s := \frac{2GM}{c^2}.$$

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- The hypersurface $r = r_s$ is the **event horizon**: once inside, future-directed causal curves cannot reach infinity.

Outline

Geodesics in Schwarzschild: setup

Goal: obtain the equations of motion by inserting the Schwarzschild connection ${\Gamma^\mu}_{\alpha\beta}$ into the geodesic equation

$$\frac{d^2X^\mu}{d\lambda^2} + {\Gamma^\mu}_{\alpha\beta}\frac{dX^\alpha}{d\lambda}\frac{dX^\beta}{d\lambda} = 0.$$

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Spherical symmetry: without loss of generality take motion in the equatorial plane

$$\theta = \frac{\pi}{2}, \qquad \dot{\theta} = 0,$$

so the dynamics reduces to $(t(\lambda), r(\lambda), \phi(\lambda))$.

Geodesic Lagrangian in the equatorial plane

Using the Schwarzschild line element and setting $\theta = \pi/2$, $\dot{\theta} = 0$, the geodesic Lagrangian

$$\mathcal{L} = \frac{1}{2}\, g_{\mu\nu} \dot{X}^{\mu} \dot{X}^{\nu}$$

becomes

$$\boxed{\mathcal{L} = \frac{1}{2}\left[-\Big(1 - \frac{2M}{r}\Big)\dot{t}^{\,2} + \Big(1 - \frac{2M}{r}\Big)^{-1}\dot{r}^{\,2} + r^2\dot{\phi}^{\,2}\right].}$$

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Interpretation: ϵ is fixed by the causal character of the worldline.

First integrals from cyclic coordinates (t and ϕ)

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Meaning: E is the energy per unit mass and ℓ the (specific) angular momentum.

Reducing the problem to a 1D radial equation

Using the conserved quantities

$$\dot{t} = \frac{E}{1 - \frac{2M}{r}}, \qquad \dot{\phi} = \frac{\ell}{r^2},$$

and the normalization of the four–velocity,

$$g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} = -\epsilon, \qquad \epsilon = \begin{cases} 1 & \text{timelike geodesics,} \\ 0 & \text{null geodesics,} \end{cases}$$

we obtain the radial equation

$$\boxed{\dot{r}^{\,2} = E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{\ell^2}{r^2} + \epsilon\right).}$$

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$$g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = -\epsilon, \qquad \epsilon = \begin{cases} 1 & \text{timelike geodesics,} \\ 0 & \text{null geodesics,} \end{cases}$$

we obtain the radial equation

$$\boxed{\dot{r}^{\,2} = E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{\ell^2}{r^2} + \epsilon\right).}$$

Physical interpretation

This is equivalent to a one–dimensional energy equation

$$\frac{1}{2}\,\dot{r}^{\,2} + V_{\text{eff}}(r) = \frac{1}{2}\,E^2, \quad V_{\text{eff}}(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{\ell^2}{r^2} + \epsilon\right).$$

Orbit equation $\phi(r)$

To compute observables (perihelion shift, light bending), we want $\phi = \phi(r)$.

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- Timelike (perihelion): $\epsilon = 1$
- Null (light bending): $\epsilon = 0$

Perihelion precession: relativistic orbit equation

For timelike geodesics in Schwarzschild spacetime, the orbital equation for $u(\phi) = 1/r$ reads

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{\ell^2} + 3Mu^2.$$

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In Newtonian gravity:

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{\ell^2},$$

which admits closed elliptical orbits.

The relativistic correction implies that bound orbits are not closed. The perihelion advances by

$$\Delta\phi = \frac{6\pi M}{a(1-e^2)}$$

per revolution, where a is the semi-major axis and e the eccentricity.

Perihelion precession: prediction and observation

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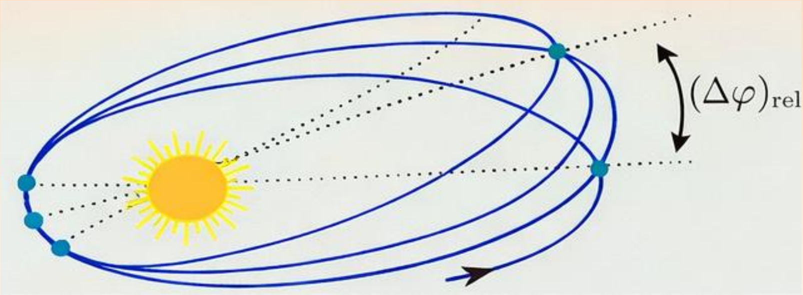
- Newtonian gravity predicts **no** perihelion shift.
- The observed excess matches the GR prediction precisely.

Perihelion precession (schematic)



- Before GR, the anomaly was attributed to a hypothetical planet (“Vulcan”).
- The orbit is not closed: the perihelion advances by $\Delta\varphi$ each revolution.
- In GR this comes from geodesic motion in Schwarzschild spacetime (extra relativistic correction).

Light rays follow null geodesics ($2\mathcal{L} = 0$). The orbital equation becomes

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Deflection of light: null geodesics

Light rays follow null geodesics ($2\mathcal{L} = 0$). The orbital equation becomes

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2.$$

Solving perturbatively for a light ray passing at impact parameter b, one finds a total deflection angle

$$\Delta\phi = \frac{4M}{b}.$$

where $b := \ell/E$ is the impact parameter.

Deflection of light: Newton vs GR

General Relativity

$$\Delta\phi_{\mathsf{GR}} = \frac{4GM}{c^2 b}.$$

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Deflection of light: Newton vs GR

General Relativity

$$\Delta\phi_{\text{GR}} = \frac{4GM}{c^2 b}.$$

Newtonian theory (naive)

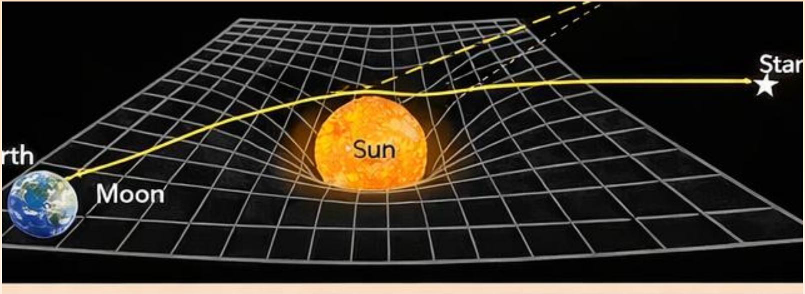
$$\Delta\phi_{\text{Newton}} = \frac{2GM}{c^2 b}.$$

- GR predicts **twice** the Newtonian deflection.
- For light grazing the Sun:

$$\Delta\phi_{\text{GR}} \simeq 1.75''.$$

- Confirmed during the 1919 solar eclipse.

Deflection of light by gravity (schematic)



- Light follows null geodesics: the trajectory bends when passing near a massive body.
- The apparent position of a background star is shifted due to spacetime curvature.

Gravitational redshift: origin

The proper time measured by a static observer is

$$d\tau = \sqrt{-g_{tt}}\, dt.$$

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The ratio of observed frequencies between two radii r_1 and r_2 is

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}}}.$$

Gravitational redshift: physical meaning

In a static gravitational field, clocks at different radii do not measure the same proper time:

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For two observers at r_1 and r_2,

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Key point

Gravity affects the *rate of time itself*, not only the motion of particles.

Gravitational time dilation and GPS

Satellites orbiting the Earth experience a weaker gravitational field than clocks on the Earth's surface.

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Physical consequence

Without relativistic corrections from General Relativity, GPS positioning errors would grow by several kilometers per day.

Gravitational redshift and GPS (schematic)

- Clock rates depend on gravitational potential: time at different altitudes runs differently.
- GPS needs relativistic corrections (GR + SR) to keep timing/position accurate.

Friedmann–Robertson–Walker spacetime

Cosmological assumptions

- **Homogeneity**: all spatial points are equivalent.
- **Isotropy**: no preferred spatial direction.
- Matter content described by a perfect fluid:

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p).$$

Most general metric compatible with these symmetries

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d^2\phi) \right],$$

where

- $a(t)$ is the scale factor,
- $k = 0, \pm 1$ determines the spatial curvature.

Friedmann equations and cosmological dynamics

Inserting the FRW metric into Einstein's equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu},$$

yields the **Friedmann equations***:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}}$$

Key physical consequences

- Cosmic expansion or contraction ($\dot{a} \neq 0$).
- Acceleration or deceleration determined by $\rho + 3p$.
- Λ can drive accelerated expansion.
- Cosmological redshift: $1 + z = a(t_0)/a(t_{\rm em})$.

Gravitational waves: linearised gravity

In the weak-field regime, the spacetime metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1,$$

where:

- $\eta_{\mu\nu}$ is the Minkowski metric,
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Physical content

In four dimensions, gravitational waves have **two independent polarizations**:

$$h_{+} \qquad \text{and} \qquad h_{\times},$$

called the *"plus"* and *"cross"* modes.

Gravitational waves: linearised gravity

- Gravitational waves are ripples of spacetime itself.
- They stretch and squeeze distances transverse to their direction of propagation.
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Why they matter

Gravitational waves provide a direct observational probe of strong-field General Relativity and were first detected by LIGO in 2015.

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- **Open questions:** There are many open problems in our understanding of gravity, from fundamental theory to observations, making it an excellent area to start research.
- **Next step:** read the material again and try to solve the exercises independently — this is where real understanding develops.