Proposed exercises

1. **Exercise 1:**

   Consider a 2-dimensional space.

   (a) Let $A^i = (A^x, A^y) = (1, 2)$ and $B_i = (B_x, B_y) = (3, 4)$. Using the Einstein summation convention, compute

   $$A^i B_i. \tag{1}$$

   (b) Consider flat space written in Cartesian coordinates (x, y), with line element

   $$ds^2 = dx^2 + dy^2, \qquad \delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

   Lower the index of A^i using

   $$A_i = \delta_{ij} A^j, \tag{3}$$

   and compute

   $$A^i A_i. \tag{4}$$

   (c) Now describe the *same flat space* using polar coordinates (r, ϕ). The line element becomes

   $$ds^2 = dr^2 + r^2 d\phi^2. \tag{5}$$

   a) Write the metric components g_{ij} in matrix form.

   b) Consider a vector with contravariant components

   $$A^i = (A^r, A^\phi). \tag{6}$$

   Lower the index using $A_i = g_{ij} A^j$ and compute A_r and A_ϕ.

   c) Compare with part (b) and explain why the components of A_i are different, even though the space is still flat.

   (d) Finally, consider a genuinely curved geometry: a 2-dimensional torus parametrized by angles (θ, ϕ), with induced metric

   $$ds^2 = r^2\, d\theta^2 + (R + r\cos\theta)^2\, d\phi^2, \qquad R > r > 0, \tag{7}$$

   where R and r are constants (geometric parameters of the torus, not coordinates).

   a) Write the metric components g_{ij} in matrix form.

   b) Given a vector $V^i = (V^\theta, V^\phi)$, compute the covariant components

   $$V_i = g_{ij} V^j. \tag{8}$$

   c) Compute the scalar $V^i V_i$ and comment on its dependence on θ.

2. **Exercise 2**

Consider polar coordinates (r, ϕ) on the plane, with metric (flat)

$$ds^2 = dr^2 + r^2 d\phi^2. \tag{9}$$

(a) Write the metric components g_{ij} and the inverse metric g^{ij}.

(b) Compute the non-vanishing Christoffel symbols

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell} \left(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk} \right), \tag{10}$$

and check that the Riemann tensor is zero (meaning that the spacetime is flat).

(c) Consider the contravariant vector field

$$V^i = (1, 0). \tag{11}$$

Compute the partial derivatives $\partial_j V^i$.

(d) Define the covariant derivative of a contravariant vector by

$$\nabla_j V^i = \partial_j V^i + \Gamma^i{}_{jk} V^k, \tag{12}$$

and compute $\nabla_j V^i$ explicitly.

3. **Exercise 3**

Let A_{ij} be a covariant tensor field of type $(0, 2)$. Under a change of coordinates $x^i \mapsto \bar{x}^i(x)$, its components transform as

$$\bar{A}_{jk} = \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} A_{lm}. \tag{13}$$

(a) Differentiate the expression above with respect to $\bar{x}^i$ and use the chain rule to show that

$$\boxed{\bar{\partial}_i \bar{A}_{jk} = \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^i} \partial_n A_{lm} + \left(\frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} + \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^k} \right) A_{lm} \,,} \tag{14}$$

and explain why it does not transform as a tensor.

(b) Now consider the covariant derivative $\nabla_i A_{jk}$. Using its transformation properties, show that

$$\bar{\nabla}_i \bar{A}_{jk} = \frac{\partial x^n}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \nabla_n A_{lm}, \tag{15}$$

i.e. that the covariant derivative transforms as a tensor of type $(0, 3)$.

(c) Explain the difference between $\partial_i A_{jk}$ and $\nabla_i A_{jk}$, and why the introduction of the covariant derivative is necessary in a generally covariant theory.

4. **Exercise 4:**

   Consider a curve $x^\mu(\lambda)$ connecting two fixed points A and B in a spacetime with metric $g_{\mu\nu}(x)$. Define the (length) action

$$S[x] \;=\; \int_{\lambda_1}^{\lambda_2} d\lambda\, L, \qquad L \;=\; \sqrt{g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu}\,, \tag{16}$$

   where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$. By varying the curve $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$ with fixed endpoints

$$\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0, \tag{17}$$

   derive the geodesic equation

$$\ddot{x}^\rho + \Gamma^\rho{}_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = 0, \tag{18}$$

   where

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{19}$$

5. **Exercise 5:**

   Show that for the Levi-Civita connection (Christoffel symbols), the covariant derivative of the metric vanishes:

$$\nabla_\alpha g_{\mu\nu} = 0. \tag{20}$$

6. **Exercise 6:** Use the Killing equation

$$\nabla_{(\mu}\xi_{\nu)} = 0 \tag{21}$$

   to show that the most general **spherically symmetric** line element in spherical coordinates (t, r, θ, ϕ) can be written as

$$ds^2 = -A(t,r)\,dt^2 + 2B(t,r)\,dt\,dr + C(t,r)\,dr^2 + D(t,r)\,(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{22}$$

   i.e. (i) there are no $dt\,d\theta$, $dt\,d\phi$, $dr\,d\theta$, $dr\,d\phi$, $d\theta\,d\phi$ terms, and (ii) the angular block is proportional to the metric on the unit 2-sphere.

   (Optional 1) Show that by a redefinition of the time coordinate

$$t \;\to\; t'(t,r), \tag{23}$$

   one can always set $B(t,r) = 0$ locally.

   (Optional 2) Show that in the *static* case one can always choose the radial coordinate such that

$$D(r) = r^2. \tag{24}$$

7. **Exercise 7:**

   Consider the static and spherically symmetric line element

$$ds^2 = -e^{\nu(r)}dt^2 + e^{a(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{25}$$

   a) Find (step by step) the *independent non-vanishing* components of the Levi–Civita connection *b*) Using these results, compute the non-vanishing components of the Ricci tensor $R_{\mu\nu}$ and express them in terms of $\nu(r)$ and $a(r)$.

c) Finally, impose the Schwarzschild form

$$e^{\nu(r)} = 1 - \frac{2M}{r}, \qquad e^{a(r)} = \left(1 - \frac{2M}{r}\right)^{-1}, \tag{26}$$

and show explicitly that

$$R_{\mu\nu} = 0 \tag{27}$$

for $r > 2M$.

8. **Exercise 8:**

Consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{28}$$

A useful curvature invariant is the **Kretschmann scalar**

$$K \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \tag{29}$$

For Schwarzschild it is known that

$$\boxed{K = \frac{48M^2}{r^6}.} \tag{30}$$

Since K is a scalar (coordinate-invariant), if it diverges at some radius then the spacetime has a **true curvature singularity** there. Conversely, if the metric coefficients blow up but K stays finite, the "singularity" may be only a **coordinate singularity**.

(a) Using $K = \frac{48M^2}{r^6}$, check whether the spacetime is singular at $r = 2M$ and at $r = 0$.

(b) Introduce ingoing Eddington–Finkelstein coordinates by defining

$$r_* \equiv r + 2M \ln\left|\frac{r}{2M} - 1\right|, \qquad v \equiv t + r_*. \tag{31}$$

Rewrite the Schwarzschild metric in coordinates (v, r, θ, ϕ) and show that the metric is regular at $r = 2M$.

(c) Explain why this shows that $r = 2M$ is not a true singularity, while $r = 0$ is.

9. **Exercise 9:**

The spatially flat FRW metric ($k = 0$) is usually written in spherical coordinates as

$$ds^2 = -dt^2 + a(t)^2\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right). \tag{32}$$

(a) Show that by introducing Cartesian coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \tag{33}$$

the metric can be written as

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right). \tag{34}$$

(b) Using the Cartesian form of the metric, compute the independent non-vanishing Christoffel symbols.

(c) Compute explicitly the non-vanishing components of the Ricci tensor and the Ricci scalar.

(d) Assume the matter content is a perfect fluid with

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\, g_{\mu\nu}, \qquad u^\mu = (1, 0, 0, 0). \tag{35}$$

Use Einstein field equations with a cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{36}$$

to derive the Friedmann equations for $k = 0$:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{37}$$

(e) Show that energy–momentum conservation,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{38}$$

leads to the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0, \qquad H \equiv \frac{\dot{a}}{a}. \tag{39}$$