Solutions of exercises

1. **Solution to Exercise 1:**

(a)

Step 1: Expand the contraction.
By Einstein summation,

$$A^i B_i = A^x B_x + A^y B_y. \tag{1}$$

Step 2: Substitute the components.
With $A^i = (1, 2)$ and $B_i = (3, 4)$,

$$A^i B_i = (1)(3) + (2)(4) = 3 + 8 = 11. \tag{2}$$

(b)

We are given

$$ds^2 = dx^2 + dy^2, \qquad \delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Step 1: Lower the index.
By definition,

$$A_i = \delta_{ij} A^j. \tag{4}$$

Thus

$$A_x = \delta_{xx} A^x + \delta_{xy} A^y = 1 \cdot A^x + 0 \cdot A^y = A^x, \tag{5}$$

$$A_y = \delta_{yx} A^x + \delta_{yy} A^y = 0 \cdot A^x + 1 \cdot A^y = A^y. \tag{6}$$

So

$$A_i = (A_x, A_y) = (1, 2). \tag{7}$$

Step 2: Compute $A^i A_i$.

$$A^i A_i = A^x A_x + A^y A_y = (1)(1) + (2)(2) = 5. \tag{8}$$

(c)

We are given

$$ds^2 = dr^2 + r^2 d\phi^2. \tag{9}$$

Metric components.

Step 1: Identify $x^i = (r, \phi)$ and match $ds^2 = g_{ij} dx^i dx^j$.
Comparing terms,

$$g_{rr} = 1, \qquad g_{\phi\phi} = r^2, \qquad g_{r\phi} = g_{\phi r} = 0. \tag{10}$$

Step 2: Write g_{ij} as a matrix.

$$g_{ij}(r) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{11}$$

Lower the components $A^i = (A^r, A^\phi)$.

Step 1: Use $A_i = g_{ij}A^j$.
Then

$$A_r = g_{rr}A^r + g_{r\phi}A^\phi = 1 \cdot A^r + 0 \cdot A^\phi = A^r, \tag{12}$$

$$A_\phi = g_{\phi r}A^r + g_{\phi\phi}A^\phi = 0 \cdot A^r + r^2 A^\phi = r^2 A^\phi. \tag{13}$$

Hence

$$A_i = (A_r, A_\phi) = \big(A^r,\ r^2 A^\phi\big). \tag{14}$$

Why does this differ from part (b)?
The *space is still flat*, but the metric components in polar coordinates are not δ_{ij}; instead $g_{\phi\phi} = r^2$ depends on position. Therefore lowering an index changes the numerical components:

$$A_\phi \neq A^\phi \quad \text{for all } r. \tag{15}$$

This change is due to the coordinate choice, not curvature.

(d)

We are given

$$ds^2 = r^2\, d\theta^2 + (R + r\cos\theta)^2\, d\phi^2, \qquad R > r > 0. \tag{16}$$

Metric components and matrix form.

Step 1: Match $ds^2 = g_{ij}dx^i dx^j$ with $x^i = (\theta, \phi)$.
Thus

$$g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = (R + r\cos\theta)^2, \qquad g_{\theta\phi} = g_{\phi\theta} = 0. \tag{17}$$

Step 2: Matrix form.

$$g_{ij}(\theta) = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos\theta)^2 \end{pmatrix}. \tag{18}$$

Lower $V^i = (V^\theta, V^\phi)$.

Step 1: Use $V_i = g_{ij}V^j$.
Then

$$V_\theta = g_{\theta\theta}V^\theta + g_{\theta\phi}V^\phi = r^2 V^\theta, \tag{19}$$

$$V_\phi = g_{\phi\theta}V^\theta + g_{\phi\phi}V^\phi = (R + r\cos\theta)^2 V^\phi. \tag{20}$$

So

$$V_i = (V_\theta, V_\phi) = \big(r^2 V^\theta,\ (R + r\cos\theta)^2 V^\phi\big). \tag{21}$$

Compute V^iV_i and comment.

Step 1: Contract.

$$V^iV_i = V^\theta V_\theta + V^\phi V_\phi. \tag{22}$$

Step 2: Substitute the lowered components.

$$V^iV_i = V^\theta(r^2V^\theta) + V^\phi\big((R+r\cos\theta)^2V^\phi\big) = r^2(V^\theta)^2 + (R+r\cos\theta)^2(V^\phi)^2. \tag{23}$$

Comment.

Even if V^θ and V^ϕ are constant numbers, the scalar V^iV_i generally depends on position through θ, because the metric component

$$g_{\phi\phi}(\theta) = (R + r\cos\theta)^2 \tag{24}$$

varies along the torus. This illustrates that inner products and norms are determined by the metric.

2. **Solution to Exercise 2**

We work on the plane in polar coordinates (r, ϕ) with metric

$$ds^2 = dr^2 + r^2 d\phi^2. \tag{25}$$

We use index labels $(1, 2) = (r, \phi)$, i.e. $x^i = (r, \phi)$.

(a) Metric components g_{ij} and inverse metric g^{ij}.

Step 1: Read off g_{ij} from $ds^2 = g_{ij} dx^i dx^j$.
Comparing

$$ds^2 = 1 \cdot dr^2 + r^2 \cdot d\phi^2, \tag{26}$$

we obtain

$$g_{rr} = 1, \qquad g_{\phi\phi} = r^2, \qquad g_{r\phi} = g_{\phi r} = 0. \tag{27}$$

Step 2: Matrix form.

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{28}$$

Step 3: Invert the matrix to obtain g^{ij}.
Since g_{ij} is diagonal,

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{r^2} \end{pmatrix}, \tag{29}$$

that is,

$$g^{rr} = 1, \qquad g^{\phi\phi} = \frac{1}{r^2}, \qquad g^{r\phi} = g^{\phi r} = 0. \tag{30}$$

(b) Non-vanishing Christoffel symbols.

We use

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell} \left(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk} \right). \tag{31}$$

Step 1: Derivatives of the metric components.
The only coordinate-dependent component is

$$g_{\phi\phi} = r^2, \tag{32}$$

so

$$\partial_r g_{\phi\phi} = 2r, \qquad \partial_\phi g_{\phi\phi} = 0, \tag{33}$$

while all derivatives of $g_{rr} = 1$ and $g_{r\phi} = 0$ vanish.

Step 2: Compute $\Gamma^r{}_{\phi\phi}$.
Setting $i = r$, $j = \phi$, $k = \phi$,

$$\Gamma^r{}_{\phi\phi} = \frac{1}{2} g^{r\ell} \left(\partial_\phi g_{\phi\ell} + \partial_\phi g_{\phi\ell} - \partial_\ell g_{\phi\phi} \right). \tag{34}$$

Only $\ell = r$ contributes, hence

$$\Gamma^r{}_{\phi\phi} = \frac{1}{2} g^{rr} (-\partial_r g_{\phi\phi}) = -\frac{1}{2}(2r) = -r. \tag{35}$$

Step 3: Compute ${\Gamma^\phi}_{r\phi}$ and ${\Gamma^\phi}_{\phi r}$.
Taking $i = \phi$, $j = r$, $k = \phi$,

$$
{\Gamma^\phi}_{r\phi} = \frac{1}{2} g^{\phi\ell} \left(\partial_r g_{\phi\ell} + \partial_\phi g_{r\ell} - \partial_\ell g_{r\phi} \right). \tag{36}
$$

Only $\ell = \phi$ contributes, so

$$
{\Gamma^\phi}_{r\phi} = \frac{1}{2} g^{\phi\phi} \partial_r g_{\phi\phi} = \frac{1}{2} \left(\frac{1}{r^2} \right) (2r) = \frac{1}{r}. \tag{37}
$$

By symmetry of the Levi–Civita connection,

$$
{\Gamma^\phi}_{\phi r} = {\Gamma^\phi}_{r\phi} = \frac{1}{r}. \tag{38}
$$

Step 4: Result.
The non-vanishing Christoffel symbols are

$$
{\Gamma^r}_{\phi\phi} = -r, \qquad {\Gamma^\phi}_{r\phi} = {\Gamma^\phi}_{\phi r} = \frac{1}{r}. \tag{39}
$$

Step 5: Check that the Riemann tensor vanishes.

The Riemann curvature tensor is defined by

$$
{R^i}_{jkl} = \partial_k {\Gamma^i}_{jl} - \partial_l {\Gamma^i}_{jk} + {\Gamma^i}_{km} {\Gamma^m}_{jl} - {\Gamma^i}_{lm} {\Gamma^m}_{jk}. \tag{40}
$$

From the previous steps, the only non-vanishing Christoffel symbols are

$$
{\Gamma^r}_{\phi\phi} = -r, \qquad {\Gamma^\phi}_{r\phi} = {\Gamma^\phi}_{\phi r} = \frac{1}{r}. \tag{41}
$$

We now check explicitly that all components of ${R^i}_{jkl}$ vanish.
Step 5a: Component ${R^r}_{\phi r\phi}$.

This is the only potentially non-zero independent component. Using the definition,

$$
{R^r}_{\phi r\phi} = \partial_r {\Gamma^r}_{\phi\phi} - \partial_\phi {\Gamma^r}_{\phi r} + {\Gamma^r}_{rm} {\Gamma^m}_{\phi\phi} - {\Gamma^r}_{\phi m} {\Gamma^m}_{r\phi}. \tag{42}
$$

We evaluate each term:

$$
\partial_r {\Gamma^r}_{\phi\phi} = \partial_r(-r) = -1, \qquad \partial_\phi {\Gamma^r}_{\phi r} = 0. \tag{43}
$$

Next, since ${\Gamma^r}_{rm} = 0$ for all m, the third term vanishes. For the last term, the only non-zero contribution comes from $m = \phi$:

$$
{\Gamma^r}_{\phi\phi} {\Gamma^\phi}_{r\phi} = (-r) \left(\frac{1}{r} \right) = -1. \tag{44}
$$

Putting everything together,

$$R^r{}_{\phi r\phi} = (-1) - 0 + 0 - (-1) = 0. \tag{45}$$

Step 5b: Remaining components.

All other components of the Riemann tensor either vanish trivially or are related to $R^r{}_{\phi r\phi}$ by the symmetries of the Riemann tensor. Therefore,

$$R^i{}_{jkl} = 0 \quad \text{for all indices } i, j, k, l. \tag{46}$$

Although the Christoffel symbols are non-zero in polar coordinates, the Riemann tensor vanishes identically. This confirms that the metric

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{47}$$

describes flat space, and that the non-zero Christoffel symbols arise purely from the use of curvilinear coordinates.

(c)

Step 1: Components of the vector field.

$$V^r(r, \phi) = 1, \qquad V^\phi(r, \phi) = 0. \tag{48}$$

Step 2: Compute partial derivatives.
Since the components are constant,

$$\partial_r V^r = \partial_\phi V^r = \partial_r V^\phi = \partial_\phi V^\phi = 0, \tag{49}$$

or equivalently,

$$\partial_j V^i = 0 \quad \text{for all } i, j. \tag{50}$$

(d)
By definition,

$$\nabla_j V^i = \partial_j V^i + \Gamma^i{}_{jk} V^k. \tag{51}$$

Using part (c), $\partial_j V^i = 0$, so

$$\nabla_j V^i = \Gamma^i{}_{jk} V^k. \tag{52}$$

Since $V^k = (1, 0)$,

$$\nabla_j V^i = \Gamma^i{}_{jr}. \tag{53}$$

Step 1: Components with $i = r$.

$$\nabla_r V^r = \Gamma^r{}_{rr} = 0, \qquad \nabla_\phi V^r = \Gamma^r{}_{\phi r} = 0. \tag{54}$$

Step 2: Components with $i = \phi$.

$$\nabla_r V^\phi = \Gamma^\phi{}_{rr} = 0, \qquad \nabla_\phi V^\phi = \Gamma^\phi{}_{\phi r} = \frac{1}{r}. \tag{55}$$

Step 3: Matrix form.

$$(\nabla_j V^i) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix}. \tag{56}$$

3. **Solution to Exercise 3:**

a)
Step 1: Differentiate the equation with respect to $\bar{x}^i$.

By definition,

$$\bar{\partial}_i \bar{A}_{jk} \equiv \frac{\partial}{\partial \bar{x}^i} \bar{A}_{jk}. \tag{57}$$

Applying the product rule to the equation,

$$\bar{\partial}_i \bar{A}_{jk} = \bar{\partial}_i \left(\frac{\partial x^l}{\partial \bar{x}^j} \right) \frac{\partial x^m}{\partial \bar{x}^k} A_{lm} + \frac{\partial x^l}{\partial \bar{x}^j} \bar{\partial}_i \left(\frac{\partial x^m}{\partial \bar{x}^k} \right) A_{lm} + \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \bar{\partial}_i (A_{lm}). \tag{58}$$

Step 2: Simplify the first two terms (second derivatives of the coordinate map).

Since $x^l = x^l(\bar{x})$,

$$\bar{\partial}_i \left(\frac{\partial x^l}{\partial \bar{x}^j} \right) = \frac{\partial^2 x^l}{\partial \bar{x}^i \, \partial \bar{x}^j}, \qquad \bar{\partial}_i \left(\frac{\partial x^m}{\partial \bar{x}^k} \right) = \frac{\partial^2 x^m}{\partial \bar{x}^i \, \partial \bar{x}^k}. \tag{59}$$

Step 3: Apply the chain rule to the last term $\bar{\partial}_i (A_{lm})$.

The components A_{lm} are functions of the original coordinates x^n, and $x^n = x^n(\bar{x})$. Hence, by the chain rule,

$$\bar{\partial}_i (A_{lm}) = \frac{\partial x^n}{\partial \bar{x}^i} \partial_n A_{lm}. \tag{60}$$

Step 4: Substitute Eqs.

Plugging (59) and (60) into (58) gives

$$\bar{\partial}_i \bar{A}_{jk} = \frac{\partial^2 x^l}{\partial \bar{x}^i \, \partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} A_{lm} + \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial^2 x^m}{\partial \bar{x}^i \, \partial \bar{x}^k} A_{lm} + \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^i} \partial_n A_{lm}. \tag{61}$$

Reordering terms,

$$\boxed{\bar{\partial}_i \bar{A}_{jk} = \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^i} \partial_n A_{lm} + \left(\frac{\partial^2 x^l}{\partial \bar{x}^i \, \partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} + \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial^2 x^m}{\partial \bar{x}^i \, \partial \bar{x}^k} \right) A_{lm}.} \tag{62}$$

This is precisely the transformation law.

Step 5: Why $\partial_i A_{jk}$ is not a tensor.

If $\partial_i A_{jk}$ were a $(0,3)$ tensor, it would transform as

$$\bar{\partial}_i \bar{A}_{jk} \stackrel{?}{=} \frac{\partial x^n}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \partial_n A_{lm}. \tag{7}$$

But (6) contains an additional term involving *second derivatives* of the coordinate transformation, so (7) is false in general. Therefore $\partial_i A_{jk}$ does not transform tensorially.

b)

We want to prove that the covariant derivative of a $(0,2)$ tensor transforms tensorially:

$$\bar{\nabla}_i \bar{A}_{jk} = \frac{\partial x^n}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \nabla_n A_{lm}. \tag{63}$$

Step 1: Start from the definition of the covariant derivative.

For a $(0,2)$ tensor,

$$\nabla_n A_{lm} = \partial_n A_{lm} - \Gamma^p{}_{nl} A_{pm} - \Gamma^p{}_{nm} A_{lp}. \tag{64}$$

In barred coordinates,

$$\bar{\nabla}_i \bar{A}_{jk} = \bar{\partial}_i \bar{A}_{jk} - \bar{\Gamma}^p{}_{ij} \bar{A}_{pk} - \bar{\Gamma}^p{}_{ik} \bar{A}_{jp}. \tag{65}$$

Step 2: Use the tensor transformation law for A_{jk}.

We take as given

$$\bar{A}_{jk} = \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} A_{lm}. \tag{66}$$

Differentiating (66) with respect to $\bar{x}^i$ (product rule + chain rule) yields

$$\begin{aligned} \bar{\partial}_i \bar{A}_{jk} &= \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^i} \partial_n A_{lm} \\ &\quad + \left(\frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} + \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^k} \right) A_{lm}. \end{aligned} \tag{67}$$

Step 3: Use the (inhomogeneous) transformation law of the connection.

As seen in class, the Christoffel symbols transform as

$$\bar{\Gamma}^p{}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} \Gamma^q{}_{rs} + \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial^2 x^q}{\partial \bar{x}^i \partial \bar{x}^j}. \tag{68}$$

Step 4: Evaluate $\bar{\Gamma}^p{}_{ij} \bar{A}_{pk}$ and $\bar{\Gamma}^p{}_{ik} \bar{A}_{jp}$.

First, write

$$\bar{A}_{pk} = \frac{\partial x^a}{\partial \bar{x}^p} \frac{\partial x^b}{\partial \bar{x}^k} A_{ab}, \qquad \bar{A}_{jp} = \frac{\partial x^a}{\partial \bar{x}^j} \frac{\partial x^b}{\partial \bar{x}^p} A_{ab}. \tag{69}$$

Use the Jacobian identity

$$\frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^a}{\partial \bar{x}^p} = \delta^a{}_q. \tag{70}$$

Plugging (68) and (69) into $\bar{\Gamma}^p{}_{ij}\bar{A}_{pk}$ and simplifying with (70) gives

$$\bar{\Gamma}^p{}_{ij}\bar{A}_{pk} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^b}{\partial \bar{x}^k}\Gamma^a{}_{rs}\,A_{ab} + \frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^b}{\partial \bar{x}^k}A_{ab}. \tag{71}$$

Similarly,

$$\bar{\Gamma}^p{}_{ik}\bar{A}_{jp} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^k}\frac{\partial x^a}{\partial \bar{x}^j}\Gamma^b{}_{rs}\,A_{ab} + \frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial^2 x^b}{\partial \bar{x}^i \partial \bar{x}^k}A_{ab}. \tag{72}$$

Step 5: Substitute into the definition of $\bar{\nabla}_i\bar{A}_{jk}$ and show cancellation.

Insert (67), (71), and (72) into (65). The second-derivative terms in (67) cancel *exactly* against the second-derivative terms in (71) and (72) (after relabelling dummy indices).

What remains is

$$\bar{\nabla}_i\bar{A}_{jk} = \frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^m}{\partial \bar{x}^k}\frac{\partial x^n}{\partial \bar{x}^i}\Big(\partial_n A_{lm} - \Gamma^p{}_{nl}A_{pm} - \Gamma^p{}_{nm}A_{lp}\Big). \tag{73}$$

Step 6: Recognize $\nabla_n A_{lm}$.

Using the definition (64) inside (73), we obtain

$$\bar{\nabla}_i\bar{A}_{jk} = \frac{\partial x^n}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^m}{\partial \bar{x}^k}\,\nabla_n A_{lm}, \tag{74}$$

which is exactly the desired tensorial transformation law (63).

c)

The partial derivative $\partial_i A_{jk}$ describes how the *components* of the tensor A_{jk} vary with the coordinates. However, these components are defined with respect to a basis that itself depends on the coordinate system. In general coordinates, the basis vectors or one-forms change from point to point.

As a result, when taking a partial derivative, one is implicitly ignoring the variation of the basis. Under a general coordinate transformation, this leads to additional terms that depend on the chosen coordinates, and therefore the partial derivative of a tensor does not transform as a tensor itself.

The covariant derivative is introduced precisely to account for this effect. It modifies the partial derivative by incorporating information about how the basis changes from point to point. This ensures that the resulting object depends only on the geometric tensor field and not on the particular coordinate system used.

For this reason, the covariant derivative is essential in a generally covariant theory, where physical laws must have the same form in all coordinate systems.

4. **Solution Exercise 4:**

We start from the action with the square-root Lagrangian

$$S[x] = \int_{\lambda_1}^{\lambda_2} d\lambda\, L, \qquad L = \sqrt{g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu}, \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}. \tag{75}$$

Step 1: Vary the path.
We perform the variation

$$x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda), \qquad \delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0. \tag{76}$$

Then

$$\delta \dot{x}^\mu = \frac{d}{d\lambda}\big(\delta x^\mu\big). \tag{77}$$

Step 2: Compute δL.
Define

$$A \equiv g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu, \qquad \text{so that} \qquad L = \sqrt{A}. \tag{78}$$

Then

$$\delta L = \frac{1}{2\sqrt{A}}\,\delta A. \tag{79}$$

Since $g_{\mu\nu}$ depends on x, its variation is

$$\delta g_{\mu\nu}(x) = \partial_\rho g_{\mu\nu}(x)\,\delta x^\rho. \tag{80}$$

Therefore

$$\delta A = \delta g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu + g_{\mu\nu}\,\delta \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu}\,\dot{x}^\mu \delta \dot{x}^\nu. \tag{81}$$

Using the symmetry $g_{\mu\nu} = g_{\nu\mu}$, the last two terms combine:

$$\delta A = \partial_\rho g_{\mu\nu}\,\delta x^\rho\,\dot{x}^\mu \dot{x}^\nu + 2\,g_{\mu\nu}\,\dot{x}^\mu\,\delta \dot{x}^\nu. \tag{82}$$

Hence

$$\delta L = \frac{1}{2L}\,\partial_\rho g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu\,\delta x^\rho + \frac{1}{L}\,g_{\mu\nu}\,\dot{x}^\mu\,\delta \dot{x}^\nu. \tag{83}$$

Step 3: Compute δS and integrate by parts.
We have

$$\delta S = \int_{\lambda_1}^{\lambda_2} d\lambda\,\delta L = \int_{\lambda_1}^{\lambda_2} d\lambda\,\left[\frac{1}{2L}\,\partial_\rho g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu\,\delta x^\rho + \frac{1}{L}\,g_{\mu\nu}\,\dot{x}^\mu\,\delta \dot{x}^\nu\right]. \tag{84}$$

Integrate the second term by parts, using $\delta \dot{x}^\nu = \frac{d}{d\lambda}(\delta x^\nu)$:

$$\int_{\lambda_1}^{\lambda_2} d\lambda\,\frac{1}{L}g_{\mu\nu}\dot{x}^\mu\,\delta \dot{x}^\nu = \left[\frac{1}{L}g_{\mu\nu}\dot{x}^\mu\,\delta x^\nu\right]_{\lambda_1}^{\lambda_2} - \int_{\lambda_1}^{\lambda_2} d\lambda\,\frac{d}{d\lambda}\left(\frac{1}{L}g_{\mu\nu}\dot{x}^\mu\right)\delta x^\nu. \tag{85}$$

The boundary term vanishes because $\delta x^\nu(\lambda_1) = \delta x^\nu(\lambda_2) = 0$. Thus

$$\delta S = \int_{\lambda_1}^{\lambda_2} d\lambda\,\left[\frac{1}{2L}\,\partial_\nu g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta - \frac{d}{d\lambda}\left(\frac{1}{L}g_{\mu\nu}\dot{x}^\mu\right)\right]\delta x^\nu. \tag{86}$$

Step 4: Euler–Lagrange equations.
Since δx^ν is arbitrary (with fixed endpoints), $\delta S = 0$ implies

$$\frac{d}{d\lambda}\left(\frac{1}{L} g_{\mu\nu}\dot{x}^\mu\right) - \frac{1}{2L}\,\partial_\nu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = 0. \tag{87}$$

Multiplying by L gives

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\dot{x}^\mu\right) - \frac{\dot{L}}{L}\,g_{\mu\nu}\dot{x}^\mu - \frac{1}{2}\,\partial_\nu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = 0. \tag{88}$$

Expanding the total derivative,

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\dot{x}^\mu\right) = \partial_\rho g_{\mu\nu}\,\dot{x}^\rho\dot{x}^\mu + g_{\mu\nu}\ddot{x}^\mu, \tag{89}$$

so the equation becomes

$$g_{\mu\nu}\ddot{x}^\mu + \partial_\rho g_{\mu\nu}\,\dot{x}^\rho\dot{x}^\mu - \frac{1}{2}\,\partial_\nu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = \frac{\dot{L}}{L}\,g_{\mu\nu}\dot{x}^\mu. \tag{90}$$

Raising the index with $g^{\nu\sigma}$,

$$\ddot{x}^\sigma + \Gamma^\sigma{}_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = \frac{\dot{L}}{L}\,\dot{x}^\sigma, \tag{91}$$

where

$$\Gamma^\sigma{}_{\alpha\beta} = \frac{1}{2}\,g^{\sigma\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right). \tag{92}$$

Step 5: Affine parameter and the standard geodesic equation.
If λ is chosen to be an affine parameter (e.g. proportional to proper length/time), then L is constant along the curve, so $\dot{L} = 0$. In that case the equation reduces to

$$\ddot{x}^\sigma + \Gamma^\sigma{}_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = 0, \tag{93}$$

which is the standard geodesic equation.

5. **Solution Exercise 5:**

Step 1: Write the definition of the covariant derivative of a covariant rank-2 tensor.
For any $T_{\mu\nu}$,

$$\nabla_\alpha T_{\mu\nu} = \partial_\alpha T_{\mu\nu} - \Gamma^\rho{}_{\alpha\mu} T_{\rho\nu} - \Gamma^\rho{}_{\alpha\nu} T_{\mu\rho}. \tag{94}$$

Setting $T_{\mu\nu} = g_{\mu\nu}$, we have

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\rho{}_{\alpha\mu} g_{\rho\nu} - \Gamma^\rho{}_{\alpha\nu} g_{\mu\rho}. \tag{95}$$

Step 2: Lower the upper index of Γ using the metric.
Define

$$\Gamma_{\lambda\mu\nu} \equiv g_{\lambda\rho} \Gamma^\rho{}_{\mu\nu}. \tag{96}$$

Then the last two terms become

$$\Gamma^\rho{}_{\alpha\mu} g_{\rho\nu} = \Gamma_{\nu\alpha\mu}, \qquad \Gamma^\rho{}_{\alpha\nu} g_{\mu\rho} = \Gamma_{\mu\alpha\nu}. \tag{97}$$

So

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_{\nu\alpha\mu} - \Gamma_{\mu\alpha\nu}. \tag{98}$$

Step 3: Compute $\Gamma_{\lambda\mu\nu}$ from the Levi-Civita expression.
Start from

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{99}$$

Multiply both sides by $g_{\lambda\rho}$:

$$\Gamma_{\lambda\mu\nu} = g_{\lambda\rho} \Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\, g_{\lambda\rho} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{100}$$

Using $g_{\lambda\rho} g^{\rho\sigma} = \delta^\sigma{}_\lambda$, we get

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right). \tag{101}$$

Step 4: Substitute $\Gamma_{\nu\alpha\mu}$ and $\Gamma_{\mu\alpha\nu}$.
From the formula above,

$$\Gamma_{\nu\alpha\mu} = \frac{1}{2} \left(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\alpha\mu} \right), \tag{102}$$

$$\Gamma_{\mu\alpha\nu} = \frac{1}{2} \left(\partial_\alpha g_{\nu\mu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu} \right). \tag{103}$$

Step 5: Plug into $\nabla_\alpha g_{\mu\nu}$ and simplify.
Recall

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma_{\nu\alpha\mu} - \Gamma_{\mu\alpha\nu}. \tag{104}$$

Substitute:

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \frac{1}{2} \left(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\alpha\mu} \right) - \frac{1}{2} \left(\partial_\alpha g_{\nu\mu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu} \right).$$

Use the symmetry $g_{\mu\nu} = g_{\nu\mu}$, hence $\partial_\alpha g_{\nu\mu} = \partial_\alpha g_{\mu\nu}$:

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \frac{1}{2}\left(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\alpha\mu}\right) - \frac{1}{2}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu}\right).$$

Now expand and collect terms:

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \frac{1}{2}\partial_\alpha g_{\mu\nu} - \frac{1}{2}\partial_\mu g_{\alpha\nu} + \frac{1}{2}\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\alpha g_{\mu\nu} - \frac{1}{2}\partial_\nu g_{\alpha\mu} + \frac{1}{2}\partial_\mu g_{\alpha\nu}.$$

Everything cancels pairwise:

$$\partial_\alpha g_{\mu\nu} - \frac{1}{2}\partial_\alpha g_{\mu\nu} - \frac{1}{2}\partial_\alpha g_{\mu\nu} = 0, \tag{105}$$

$$-\frac{1}{2}\partial_\mu g_{\alpha\nu} + \frac{1}{2}\partial_\mu g_{\alpha\nu} = 0, \qquad \frac{1}{2}\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\nu g_{\alpha\mu} = 0. \tag{106}$$

Therefore,

$$\boxed{\nabla_\alpha g_{\mu\nu} = 0.} \tag{107}$$

This property is called **metric compatibility** and is one of the defining features of the Levi-Civita connection. This also means that nonmetricity is zero (angles and norms are preserved under infinitesimal parallel transportation)

6. **Solution Exercise 6:**

Spherical symmetry means invariance under spatial rotations, i.e. the spacetime admits three Killing vectors generating SO(3), and the metric is invariant under their flows. In coordinates (t, r, θ, ϕ), the rotational Killing vectors act only on (θ, ϕ).

Step 1: Write the rotational Killing vectors on the 2-sphere
A convenient basis of Killing vectors generating rotations is

$$\xi_{(1)} = \partial_\phi,$$
$$\xi_{(2)} = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi,$$
$$\xi_{(3)} = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi.$$

They satisfy the $\mathfrak{so}(3)$ commutation relations and have no t or r components.

Step 2: Use the Killing vector implies $\mathcal{L}_\xi g_{\mu\nu} = 0$.
The Killing equation $\nabla_{(\mu}\xi_{\nu)} = 0$ is equivalent to

$$(\mathcal{L}_\xi g)_{\mu\nu} = 0, \tag{108}$$

where the Lie derivative is

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu\xi^\rho + g_{\mu\rho}\partial_\nu\xi^\rho. \tag{109}$$

We will impose this for the three rotational $\xi_{(i)}$.

Step 3: First constrain the θ and ϕ dependence using $\xi_{(1)} = \partial_\phi$.
For $\xi = \partial_\phi$ we have $\xi^\rho\partial_\rho = \partial_\phi$ and $\partial_\mu\xi^\rho = 0$. Thus

$$(\mathcal{L}_{\partial_\phi} g)_{\mu\nu} = \partial_\phi g_{\mu\nu} = 0. \tag{110}$$

Hence all metric components are independent of ϕ:

$$\partial_\phi g_{\mu\nu} = 0. \tag{111}$$

Step 4: Use the remaining rotations to eliminate mixed angular terms.
Now impose $\mathcal{L}_{\xi_{(2)}} g = 0$ and $\mathcal{L}_{\xi_{(3)}} g = 0$. These vectors mix θ and ϕ and generate all rotations on the sphere.

A key consequence is:

There is no non-zero rotationally invariant 1-form on the 2-sphere.

The objects $g_{t\theta}\,d\theta + g_{t\phi}\,d\phi$ and $g_{r\theta}\,d\theta + g_{r\phi}\,d\phi$ transform as 1-forms on the sphere. If the metric is invariant under all rotations, these 1-forms must be invariant under SO(3). Therefore they must vanish:

$$g_{t\theta} = g_{t\phi} = g_{r\theta} = g_{r\phi} = 0. \tag{112}$$

Similarly, the mixed angular piece $g_{\theta\phi}\,d\theta\,d\phi$ is not invariant under all rotations (it would pick out preferred directions on the sphere), so spherical symmetry forces

$$g_{\theta\phi} = 0. \tag{113}$$

Step 5: Determine the form of the angular 2-metric.
After Step 4, the metric splits into a (t, r) block and an angular block:

$$ds^2 = g_{ab}(t, r, \theta)\, dx^a dx^b + g_{AB}(t, r, \theta)\, dx^A dx^B, \tag{114}$$

with $a, b \in \{t, r\}$ and $A, B \in \{\theta, \phi\}$, and no cross terms.

Spherical symmetry means that the angular part g_{AB} must be invariant under all rotations on the sphere. But the only SO(3)-invariant rank-2 symmetric tensor on S^2 is proportional to the unit-sphere metric γ_{AB}:

$$\gamma_{AB}\, dx^A dx^B = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{115}$$

Therefore,

$$g_{AB}(t, r, \theta, \phi) = D(t, r)\, \gamma_{AB}(\theta, \phi), \tag{116}$$

i.e.

$$g_{\theta\theta} = D(t, r), \qquad g_{\phi\phi} = D(t, r) \sin^2\theta. \tag{117}$$

In particular, D cannot depend on (θ, ϕ), otherwise the metric would not be invariant under rotations.

Step 6: Determine the remaining components.
The remaining nonzero components live in the (t, r) block:

$$g_{tt}(t, r), \quad g_{tr}(t, r), \quad g_{rr}(t, r). \tag{118}$$

They cannot depend on θ or ϕ because any angular dependence would break rotational invariance. Thus we rename

$$g_{tt} = -A(t, r), \qquad g_{tr} = B(t, r), \qquad g_{rr} = C(t, r), \tag{119}$$

where the minus sign in g_{tt} is conventional for Lorentzian signature.

Putting everything together, the most general spherically symmetric metric is

$$\boxed{ds^2 = -A(t, r)\, dt^2 + 2B(t, r)\, dt\, dr + C(t, r)\, dr^2 + D(t, r)\, (d\theta^2 + \sin^2\theta\, d\phi^2).} \tag{120}$$

Optional Step 7: Show that $B(t, r)$ can be removed locally.
Consider a coordinate redefinition

$$t' = t'(t, r), \qquad r' = r. \tag{121}$$

One can choose $t'(t, r)$ so that the $dt'\, dr$ cross term vanishes (this is a standard diagonalization of the 2D metric in the (t, r) subspace). Hence locally one may set $B = 0$, giving

$$ds^2 = -A(t, r)\, dt^2 + C(t, r)\, dr^2 + D(t, r)\, (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{122}$$

Optional Step 8: Show that in the static case one can choose $D(r) = r^2$.

After Optional Step 7, the metric can be written (locally) as

$$ds^2 = -A(r)\, dt^2 + C(r)\, dr^2 + D(r)\, (d\theta^2 + \sin^2\theta\, d\phi^2), \tag{123}$$

where we have used that the spacetime is static, so all metric functions depend only on r.

Consider now a redefinition of the radial coordinate

$$\tilde{r} \equiv \sqrt{D(r)}. \tag{124}$$

Since $D(r) > 0$ and depends only on r, this transformation is purely radial and invertible (at least locally).

In terms of the new coordinate $\tilde{r}$, the angular sector becomes

$$D(r)\,(d\theta^2 + \sin^2\theta\, d\phi^2) = \tilde{r}^{\,2}(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{125}$$

The radial part of the metric transforms as

$$C(r)\, dr^2 = \tilde{C}(\tilde{r})\, d\tilde{r}^{\,2}, \tag{126}$$

where the new function $\tilde{C}(\tilde{r})$ absorbs the Jacobian factor

$$\tilde{C}(\tilde{r}) = C(r)\left(\frac{dr}{d\tilde{r}}\right)^2. \tag{127}$$

Therefore, without loss of generality, one can always choose coordinates such that

$$\boxed{D(r) = r^2.} \tag{128}$$

This choice is known as the *areal radius*, since the area of the 2-spheres of constant r is $A = 4\pi r^2$.

7. **Solution Exercise 7:**

Step 0: Metric components and inverse metric.
In coordinates (t, r, θ, ϕ),

$$g_{\mu\nu} = \text{diag}\left(-e^{\nu},\, e^{a},\, r^2,\, r^2\sin^2\theta\right), \tag{129}$$

$$g^{\mu\nu} = \text{diag}\left(-e^{-\nu},\, e^{-a},\, \frac{1}{r^2},\, \frac{1}{r^2\sin^2\theta}\right). \tag{130}$$

Step 1: Non-vanishing derivatives of the metric.
Since $\nu = \nu(r)$ and $a = a(r)$,

$$\partial_r g_{tt} = -e^{\nu}\nu', \qquad \partial_r g_{rr} = e^{a}a', \tag{131}$$

$$\partial_r g_{\theta\theta} = 2r, \qquad \partial_r g_{\phi\phi} = 2r\sin^2\theta, \qquad \partial_\theta g_{\phi\phi} = 2r^2\sin\theta\cos\theta. \tag{132}$$

—

a)

Step 2: Compute the independent Christoffel symbols.

We use

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right), \qquad \Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu}. \tag{133}$$

$\Gamma^{t}{}_{tr}$:

$$\Gamma^{t}{}_{tr} = \frac{1}{2}g^{tt}\partial_r g_{tt} = \frac{1}{2}(-e^{-\nu})(-e^{\nu}\nu') = \frac{1}{2}\nu'. \tag{134}$$

$\Gamma^{r}{}_{tt}$:

$$\Gamma^{r}{}_{tt} = \frac{1}{2}g^{rr}(-\partial_r g_{tt}) = \frac{1}{2}e^{-a}e^{\nu}\nu' = \frac{1}{2}\nu' e^{\nu-a}. \tag{135}$$

$\Gamma^{r}{}_{rr}$:

$$\Gamma^{r}{}_{rr} = \frac{1}{2}g^{rr}\partial_r g_{rr} = \frac{1}{2}a'. \tag{136}$$

$\Gamma^{r}{}_{\theta\theta}$:

$$\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}g^{rr}(-\partial_r g_{\theta\theta}) = -re^{-a}. \tag{137}$$

$\Gamma^{r}{}_{\phi\phi}$:

$$\Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta\, e^{-a}. \tag{138}$$

$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\phi}{}_{r\phi}$:

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\phi}{}_{r\phi} = \frac{1}{r}. \tag{139}$$

$\Gamma^{\phi}{}_{\theta\phi}$:

$$\Gamma^{\phi}{}_{\theta\phi} = \cot\theta. \tag{140}$$

$\Gamma^{\theta}{}_{\phi\phi}$:

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta. \tag{141}$$

The independent non-vanishing Christoffel symbols are

$$\boxed{\Gamma^t{}_{tr} = \frac{1}{2}\nu', \quad \Gamma^r{}_{tt} = \frac{1}{2}\nu' e^{\nu - a}, \quad \Gamma^r{}_{rr} = \frac{1}{2}a', \quad \Gamma^r{}_{\theta\theta} = -re^{-a}, \quad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta\, e^{-a},} \tag{142}$$

$$\boxed{\Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi{}_{\theta\phi} = \cot\theta, \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta,} \tag{143}$$

plus symmetry in the lower indices.

—

b)

Step 1: Non-vanishing components of $R_{\mu\nu}$.

Using the definition

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\mu\nu}\Gamma^\sigma{}_{\lambda\sigma} - \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma}, \tag{144}$$

and substituting the Christoffel symbols obtained above, one finds after direct calculation:

$$\boxed{R_{tt} = \frac{1}{2}e^{\nu-a}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}a'\nu' + \frac{2}{r}\nu'\right),} \tag{145}$$

$$\boxed{R_{rr} = -\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}a'\nu' + \frac{1}{r}a',} \tag{146}$$

$$\boxed{R_{\theta\theta} = 1 - e^{-a}\left(1 - \frac{r}{2}a' + \frac{r}{2}\nu'\right), \qquad R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}.} \tag{147}$$

All remaining components vanish by symmetry.

—

c)

Step 1: Substitute the Schwarzschild functions.
Let

$$e^\nu = f(r), \qquad e^a = \frac{1}{f(r)}, \qquad f(r) = 1 - \frac{2M}{r}. \tag{148}$$

Then

$$\nu' = \frac{f'}{f}, \qquad a' = -\frac{f'}{f}, \qquad \nu'' = \frac{f''}{f} - \frac{(f')^2}{f^2}. \tag{149}$$

with

$$f' = \frac{2M}{r^2}, \qquad f'' = -\frac{4M}{r^3}. \tag{150}$$

Step 2: Vanishing of R_{tt} and R_{rr}.
Both components contain the combination

$$f'' + \frac{2}{r}f' = 0, \tag{151}$$

hence

$$R_{tt} = 0, \qquad R_{rr} = 0. \tag{152}$$

Step 3: Vanishing of $R_{\theta\theta}$ and $R_{\phi\phi}$.

$$R_{\theta\theta} = 1 - f - rf' = 1 - \left(1 - \frac{2M}{r}\right) - \frac{2M}{r} = 0, \tag{153}$$

and therefore

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta} = 0. \tag{154}$$

Therefore, for the Schwarzschild solution:

$$\boxed{R_{\mu\nu} = 0.} \tag{155}$$

This means that it is a solution of Einstein's field equations in vacuum.

8. **Solution Exercise 8:**

(a)

Step 1: Start from the given invariant.
We are told that for Schwarzschild

$$K \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}. \tag{156}$$

Since K is a scalar, its value does not depend on the coordinate system.

Step 2: Evaluate K at the horizon $r = 2M$.
Substitute $r = 2M$:

$$K(2M) = \frac{48M^2}{(2M)^6}. \tag{157}$$

Compute the power:

$$(2M)^6 = 2^6 M^6 = 64M^6. \tag{158}$$

Therefore,

$$K(2M) = \frac{48M^2}{64M^6} = \frac{48}{64}\frac{1}{M^4} = \frac{3}{4M^4}, \tag{159}$$

which is finite.

Step 3: Check the behavior as $r \to 0$.
As $r \to 0$,

$$K = \frac{48M^2}{r^6} \longrightarrow \infty, \tag{160}$$

so curvature diverges at the origin.

Step 4: Interpret the result.
Because K is finite at $r = 2M$, the hypersurface $r = 2M$ is *not* a curvature singularity. Because K diverges at $r = 0$, the point $r = 0$ is a *true* (physical) curvature singularity.

(b)

Step 1: Rewrite the metric using $f(r)$.
Define

$$f(r) \equiv 1 - \frac{2M}{r}. \tag{161}$$

Then

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \equiv -f\,dt^2 + f^{-1}\,dr^2 + r^2 d\Omega^2. \tag{162}$$

Step 2: Define the tortoise coordinate r_*.
We define

$$r_* \equiv r + 2M \ln\left|\frac{r}{2M} - 1\right|. \tag{163}$$

Step 3: Differentiate r_* to obtain dr_*.
Differentiate with respect to r:

$$\frac{dr_*}{dr} = 1 + 2M\frac{d}{dr}\left[\ln\left|\frac{r}{2M} - 1\right|\right]. \tag{164}$$

Using $\frac{d}{dr}\ln|u| = \frac{u'}{u}$ with $u = \frac{r}{2M} - 1$, we get

$$u' = \frac{1}{2M}, \qquad \frac{d}{dr}\ln|u| = \frac{1/(2M)}{\frac{r}{2M} - 1} = \frac{1}{r - 2M}. \tag{165}$$

Hence

$$\frac{dr_*}{dr} = 1 + 2M\frac{1}{r - 2M} = \frac{r - 2M + 2M}{r - 2M} = \frac{r}{r - 2M} = \frac{1}{1 - \frac{2M}{r}} = \frac{1}{f(r)}. \tag{166}$$

Therefore,

$$\boxed{dr_* = \frac{dr}{f}.} \tag{167}$$

Step 4: Define advanced time v and compute dt.
Define

$$v \equiv t + r_*. \tag{168}$$

Differentiate:

$$dv = dt + dr_*. \tag{169}$$

Using $dr_* = \frac{dr}{f}$,

$$dt = dv - dr_* = dv - \frac{dr}{f}. \tag{170}$$

Step 5: Substitute dt into the metric and expand.
Start with the t-part:

$$-f\,dt^2 = -f\left(dv - \frac{dr}{f}\right)^2. \tag{171}$$

Expand the square:

$$\left(dv - \frac{dr}{f}\right)^2 = dv^2 - \frac{2}{f}dv\,dr + \frac{1}{f^2}dr^2. \tag{172}$$

Multiply by $-f$:

$$-f\,dt^2 = -f\,dv^2 + 2\,dv\,dr - \frac{1}{f}dr^2. \tag{173}$$

Step 6: Add the radial term $f^{-1}dr^2$.
Now include $+f^{-1}dr^2$:

$$-f\,dt^2 + f^{-1}dr^2 = \left(-f\,dv^2 + 2\,dv\,dr - \frac{1}{f}dr^2\right) + \frac{1}{f}dr^2 = -f\,dv^2 + 2\,dv\,dr. \tag{174}$$

Step 7: Write the final Eddington–Finkelstein metric.
Thus, in coordinates (v, r, θ, ϕ),

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\,d\phi^2).} \tag{175}$$

Step 8: Check regularity at $r = 2M$.
At $r = 2M$,

$$g_{vv} = -\left(1 - \frac{2M}{r}\right) \to 0, \qquad g_{vr} = 1, \qquad g_{rr} = 0, \tag{176}$$

and all angular terms are finite. There is no diverging factor like f^{-1}. Hence the metric is regular at $r = 2M$ in Eddington–Finkelstein coordinates.

(c)

Combine the two results.
From part (a), K is finite at $r = 2M$, so there is no curvature singularity there.
From part (b), we found coordinates where the metric is manifestly regular at $r = 2M$.

$$\boxed{\begin{array}{l} r = 2M \text{ is a coordinate singularity (the event horizon)}, \\ r = 0 \text{ is a true curvature singularity.} \end{array}}$$

9. **Solution Exercise 9:**

(a)

Using

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \tag{177}$$

one finds the standard identity

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2. \tag{178}$$

Therefore the metric becomes

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right), \tag{179}$$

which is manifestly homogeneous and isotropic.

(b)

Step 1: Metric and inverse metric.

$$g_{00} = -1, \qquad g_{ij} = a(t)^2\delta_{ij}, \tag{180}$$

$$g^{00} = -1, \qquad g^{ij} = a(t)^{-2}\delta^{ij}. \tag{181}$$

Step 2: Non-vanishing derivatives.

$$\partial_0 g_{ij} = 2a\dot{a}\,\delta_{ij}, \qquad \partial_k g_{\mu\nu} = 0. \tag{182}$$

Step 3: Compute the nonzero Christoffels.

$$\Gamma^0{}_{ij} = \frac{1}{2}g^{00}\left(-\partial_0 g_{ij}\right) = a\dot{a}\,\delta_{ij} = \frac{\dot{a}}{a}g_{ij}, \tag{183}$$

$$\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \frac{1}{2}g^{ik}\partial_0 g_{jk} = \frac{\dot{a}}{a}\delta^i{}_j. \tag{184}$$

All other Christoffel symbols vanish.

(c)

Define the Hubble parameter

$$H \equiv \frac{\dot{a}}{a}. \tag{185}$$

The Ricci tensor is

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda{}_{\mu\nu} - \partial_\nu\Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\mu\nu}\Gamma^\sigma{}_{\lambda\sigma} - \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma}. \tag{186}$$

Step 1: R_{00}

$$\Gamma^\lambda{}_{00} = 0, \qquad \Gamma^\lambda{}_{0\lambda} = 3H. \tag{187}$$

Thus

$$R_{00} = -\partial_0(3H) - 3H^2 = -3(\dot{H} + H^2) = -3\frac{\ddot{a}}{a}. \tag{188}$$

Step 2: R_{ij}.
Using ${\Gamma^0}_{ij} = a\dot{a}\,\delta_{ij}$ and ${\Gamma^i}_{0j} = H{\delta^i}_j$, one finds

$$R_{ij} = (a\ddot{a} + 3\dot{a}^2)\delta_{ij} = \left(\frac{\ddot{a}}{a} + 2H^2\right) g_{ij}. \tag{189}$$

Step 3: Ricci scalar.

$$R = g^{\mu\nu} R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + H^2\right). \tag{190}$$

(d)

We use

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{191}$$

Step 1: Stress-energy tensor components.
With $u^\mu = (1, 0, 0, 0)$ and $g_{00} = -1$,

$$T_{00} = \rho, \qquad T_{0i} = 0, \qquad T_{ij} = p\, g_{ij}. \tag{192}$$

Step 2: Einstein tensor components.
From part (c),

$$G_{00} = 3H^2, \qquad G_{ij} = -\left(2\frac{\ddot{a}}{a} + H^2\right) g_{ij}. \tag{193}$$

Step 3: 00-component (first Friedmann equation).
For $\mu\nu = 00$,

$$G_{00} + \Lambda g_{00} = 8\pi G\, T_{00}. \tag{194}$$

Substituting $G_{00} = 3H^2$, $g_{00} = -1$, $T_{00} = \rho$,

$$3H^2 - \Lambda = 8\pi G\rho, \tag{195}$$

hence

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}.} \tag{196}$$

Step 4: Spatial ij-component (acceleration equation).
For $\mu\nu = ij$,

$$G_{ij} + \Lambda g_{ij} = 8\pi G\, T_{ij} = 8\pi G\, p\, g_{ij}. \tag{197}$$

Substitute $G_{ij} = -(2\ddot{a}/a + H^2) g_{ij}$ and cancel $g_{ij} \neq 0$:

$$-2\frac{\ddot{a}}{a} - H^2 + \Lambda = 8\pi G\, p. \tag{198}$$

Using $H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{199}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}.} \tag{200}$$

(e)

We work with the flat FRW metric in Cartesian coordinates

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \qquad H \equiv \frac{\dot{a}}{a}. \tag{201}$$

For a perfect fluid at rest in comoving coordinates,

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\, g^{\mu\nu}, \qquad u^\mu = (1, 0, 0, 0). \tag{202}$$

Step 1: Components of $T^{\mu\nu}$.
Since $g^{00} = -1$ and $g^{ij} = a^{-2}\delta^{ij}$,

$$T^{00} = \rho, \qquad T^{0i} = 0, \qquad T^{ij} = p\, a^{-2} \delta^{ij}. \tag{203}$$

Step 2: Write $\nabla_\mu T^{\mu 0} = 0$.
By definition,

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda} T^{\mu\lambda}. \tag{204}$$

Setting $\nu = 0$,

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu{}_{\mu\lambda} T^{\lambda 0} + \Gamma^0{}_{\mu\lambda} T^{\mu\lambda}. \tag{205}$$

Step 3: Evaluate each term.

<u>(i) Derivative term:</u>

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} = \dot{\rho}. \tag{206}$$

<u>(ii) Trace-connection term:</u>

$$\Gamma^\mu{}_{\mu\lambda} T^{\lambda 0} = \Gamma^\mu{}_{\mu 0}\, T^{00}. \tag{207}$$

Since

$$\Gamma^\mu{}_{\mu 0} = \Gamma^0{}_{00} + \Gamma^i{}_{i0} = 3H, \tag{208}$$

we get

$$\Gamma^\mu{}_{\mu\lambda} T^{\lambda 0} = 3H\rho. \tag{209}$$

<u>(iii) Last term:</u>

$$\Gamma^0{}_{\mu\lambda} T^{\mu\lambda} = \Gamma^0{}_{ij} T^{ij}. \tag{210}$$

Using $\Gamma^0{}_{ij} = a\dot{a}\, \delta_{ij}$ and $T^{ij} = p\, a^{-2} \delta^{ij}$,

$$\Gamma^0{}_{ij} T^{ij} = 3Hp. \tag{211}$$

Step 4: Conservation equation.

$$\nabla_\mu T^{\mu 0} = \dot{\rho} + 3H(\rho + p) = 0, \tag{212}$$

therefore

$$\boxed{\dot{\rho} + 3H(\rho + p) = 0.} \tag{213}$$

Even in the presence of a cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{214}$$

the matter conservation law $\nabla_\mu T^{\mu\nu} = 0$ holds because $\nabla_\mu G^{\mu\nu} = 0$ and $\nabla_\mu g^{\mu\nu} = 0$.