Nuclear liquid-gas transition in strong coupling QCD

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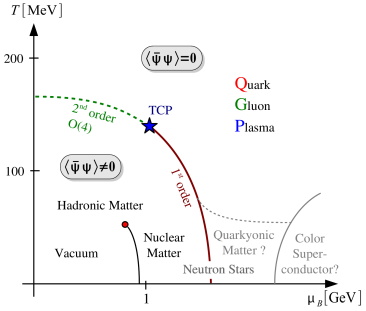
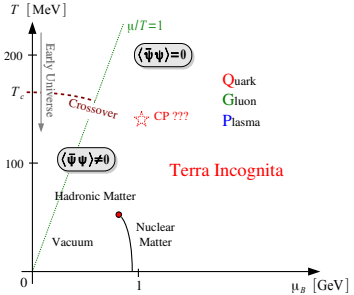
QCD phase diagram





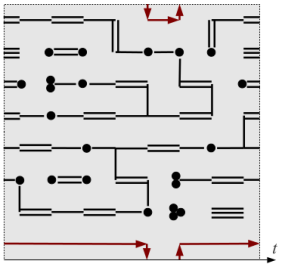
- Finite baryon density is hard for standard lattice QCD because the fermion determinant becomes complex.
- Strong-coupling lattice QCD gives a discrete dual representation that can be sampled directly.
- The low-temperature region is the key target for separating nuclear and chiral transitions.

QCD dual representation in the strong coupling limit

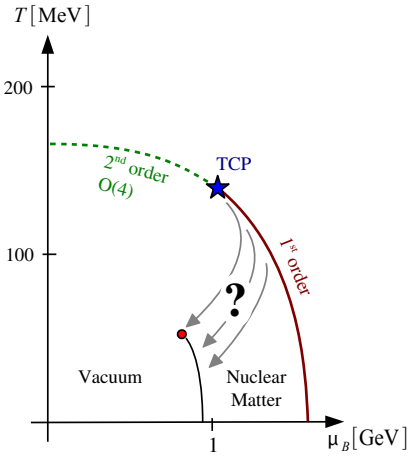
$$\beta = \frac{2N_c}{g^2} \to 0, \qquad S_g + S_f \to S_f. \tag{1}$$

- Gauge fields can be integrated out analytically.
- The remaining degrees of freedom are integer monomers, dimers, and baryon loops.
- Every site must satisfy the local Grassmann constraint,

$$n_x + \sum_{\pm\hat{\mu}} \left(k_{x,\hat{\mu}} + \frac{N_c}{2} |\ell_{x,\hat{\mu}}| \right) = N_c. \tag{2}$$



Low temperature region and gauge corrections



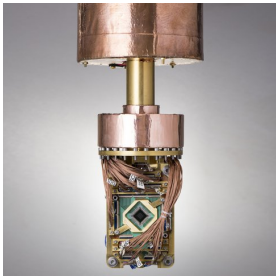
- Worm sampling suffers from critical slowing down at low temperatures.
- Gauge corrections can split the nuclear and chiral transitions.

D-Wave annealer as a proposal sampler

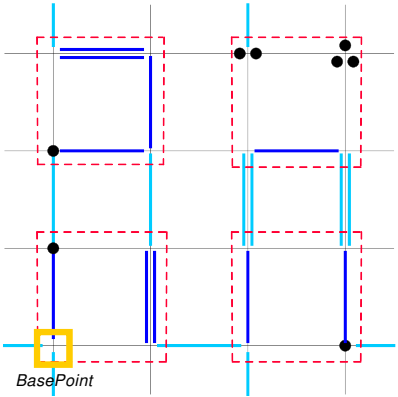
$$E_{\mathrm{QUBO}}(x) = \sum_{i<j} Q_{ij} x_i x_j + \sum_i Q_{ii} x_i, \qquad x_i \in \{0,1\}. \tag{3}$$

- D-Wave returns binary vectors from a Boltzmann-like distribution.
- We do not rely on the annealer as an exact thermal sampler.
- Instead, QPU outputs are used as proposal samples in a corrected Markov chain.

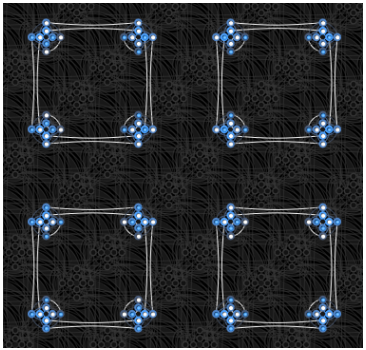
$$P_{\mathrm{acc}} = \min\left[1, \frac{\pi(C')q(C|C')}{\pi(C)q(C'|C)}\right]. \tag{4}$$



Hybrid subvolume update



Classical code selects a subvolume and handles the Markov-chain bookkeeping.



The QUBO block is embedded into the QPU connectivity.

- Boundary conditions are fixed by the surrounding lattice.
- The annealer proposes updates inside the chosen block.
- Block-diagonal QUBOs allow many local proposals to be generated in one QPU call.

QUBO construction I: monomer-dimer $U(3)$

Partition function with explicit monomers and dimers

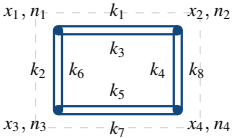
$$Z = \sum_{\{k,n\}}^{GC} \prod_{b=(x,\hat{\mu})} \frac{(3-k_b)!}{3!k_b!} \gamma^{2k_b\delta_{0,\hat{\mu}}} \prod_x \frac{3!}{n_x!} (2am_q)^{n_x}.$$

$$S = -\sum_b \log W_b(k_b) - \sum_x \log W_x(n_x), \qquad W_b(k) = \frac{(3-k)!}{3!k!}\gamma^{2k\delta_{0,\hat{\mu}}}, \quad W_x(n) = \frac{3!}{n!}(2am_q)^n.$$

- Occupations are $k_b, n_x \in \{0,1,2,3\}$.
- Two binary variables can encode each integer:

$$0 \mapsto (0,0), \qquad 1 \mapsto (0,1),$$
$$2 \mapsto (1,0), \qquad 3 \mapsto (1,1).$$

Direct encoding on a 2×2 block uses 16 qubits for monomers and dimers.



x_i, n_i: sites; $k_1, \ldots, k_8$: bond variables.

- The local constraint is imposed by a quadratic penalty:

$$\sum_{\mu=\pm 0,\cdots,\pm d} k_\mu(x) + n_x = 3.$$

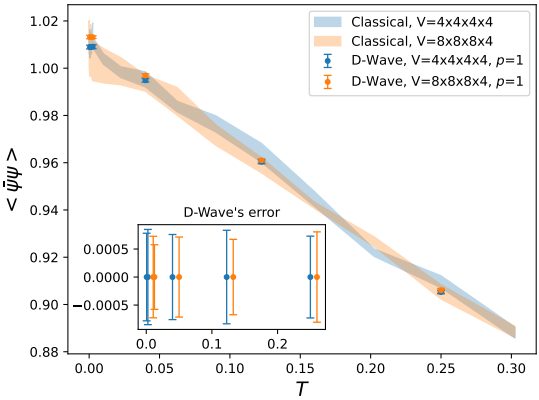
$$S = x^T W x + \text{const.},$$

$$D_{\hat{\mu}} = \begin{pmatrix} \log 12 - 4\delta_{0,\hat{\mu}} \log\gamma & 0 \\ 0 & \log 3 - 2\delta_{0,\hat{\mu}} \log\gamma \end{pmatrix},$$

$$W = \begin{pmatrix} \oplus_b D_{\hat{\mu}(b)} & 0 \\ 0 & \oplus_x M \end{pmatrix},$$

$$M = \begin{pmatrix} \log 2 - 2\log(2am_q) & \log 3 \\ 0 & -\log(2am_q) \end{pmatrix}.$$

Results of $U(3)$ gauge group



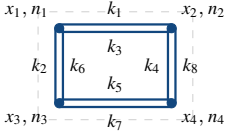
PRD, (2023), (2025), JK, T. Luu, W. Unger

- Classical simulation (worm algorithm) is inefficient: the statistical uncertainty grows as the temperature decreases.
- D-Wave simulation scales independently of the physical parameter (temperature): statistical errors remain nearly constant.

QUBO construction II: eliminate monomers

Partition function after eliminating monomers

$$Z = \sum_{\{k\}}^{GC} \prod_{b=(x,\hat{\mu})} \frac{(3-k_b)!}{3!k_b!} \gamma^{2k_b\delta_{0,\hat{\mu}}} \prod_x \frac{3!}{\left(3-\sum_\mu \tilde{k}_{x,\mu}\right)!} (2am_q)^{3-\sum_\mu \tilde{k}_{x,\mu}}. \tag{5}$$



Incident k's determine the reconstructed n_x.

- Reconstruct monomers from dimers:

$$n_x(\vec{k}) = 3 - \sum_\mu \tilde{k}_{x,\mu}, \tag{6}$$

$$\tilde{k}_{x,\mu} = k_b^{(1)} + 2k_b^{(2)} + 3k_b^{(3)}. \tag{7}$$

- Each dimer occupation is one-state encoded:

$$k_b = (k_b^{(1)}, k_b^{(2)}, k_b^{(3)}) \tag{8}$$

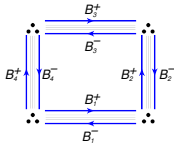
$$0 \mapsto k_b = (0,0,0), \qquad 1 \mapsto k_b = (1,0,0), \tag{9}$$

$$2 \mapsto k_b = (0,1,0), \qquad 3 \mapsto k_b = (0,0,1). \tag{10}$$

- P_{occ}: forbids multiple occupations on one bond.
- P_{neighbor}: penalizes $n_x(\vec{k}) < 0$, i.e. $\sum_\mu \tilde{k}_{x,\mu} > 3$.

$$S_{\text{QUBO}} = S_{\text{dim}} + S_{\text{mon}} + P_{\text{occ}} + P_{\text{neighbor}}. \tag{11}$$

Adding $SU(3)$ baryon loops



- A baryon loop is an oriented closed loop.
- Two binary variables per bond encode the positive and negative directions:

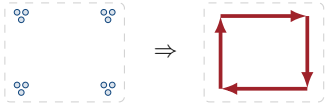
$$B_b = (B_b^+, B_b^-), \qquad B_b^+ B_b^- = 0.$$

- A subvolume proposal can create a closed baryon loop or reroute an existing baryon line.
- A diagonal corner reroute can flip the baryon-loop sign:

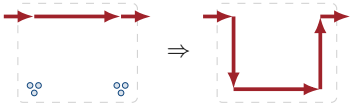
$$\sigma_{\text{new}} / \sigma_{\text{old}} = -1.$$

Static baryons + sublattice updates are currently stable.

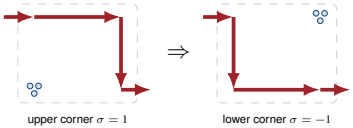
Create a local baryon loop



Reroute an existing baryon line



Corner reroute changes sign



Baryon sampling

Why local updates are not enough

- 2×2 sublattice sampling can locally create or reroute baryon segments.
- It can change local baryon signs, but it cannot change the winding number W.
- Therefore baryon sampling also needs a global proposal that moves between winding sectors.

Local QPU proposals are useful inside a fixed baryon sector; winding-sector sampling is the remaining global problem.

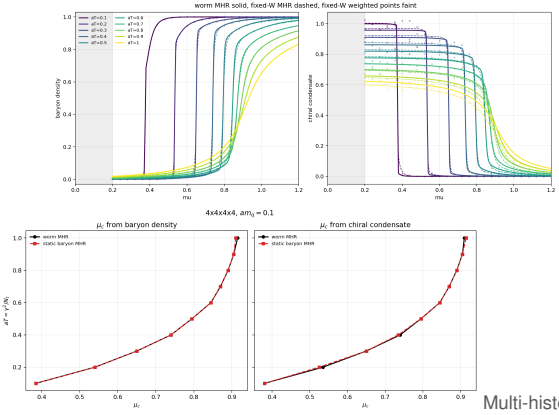
Update strategies tested

- ✗ Temporal baryon-line insertion + sublattice updates
- ✗ Baryon worm + sublattice updates
- ✗ Tube update: insert one temporal baryon loop, fill the surrounding 2×2 block with a QPU suggestion, and accept/reject the full proposal
- ✓ Static baryons + 2×2 sublattice sampling

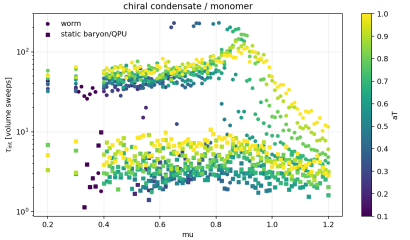
Current working strategy

- Sample 2×2 sublattice with QUBO proposals on several fixed static baryons.

Static-baryon results using baryon worm distributions



Multi-histogram reweighting comparison between worm and static-baryon results.



Autocorrelation-time comparison for the tested update strategies.

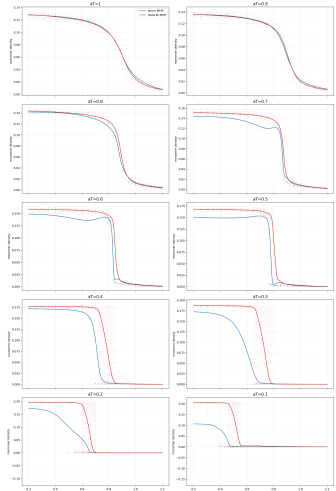
- Static-baryon sectors are obtained from worm-sampled baryon distributions.
- Given these fixed-baryon backgrounds, monomer-dimer(and baryons) are sampled with 2×2 QUBO proposals.
- The two methods agree well at high temperatures, while at low temperature the worm algorithm already suffers from severe sector-changing problems.

Extending to larger volumes without worm input

Recursive static-baryon strategy

- The worm simulation on $6^3 \times 4$ is used only as a benchmark.
- Static-baryon sampling starts from a 4^4 baryon distribution used as a prior.
- For several values of μ, multi-histogram reweighting reconstructs an updated baryon distribution.
- Static baryons are then added according to the updated distribution, and MHR is repeated.
- The results agree well at high temperatures, but at low temperature the worm algorithm already suffers from severe sector-changing problems.

The recursive MHR procedure starts from 4^4 information and iteratively moves toward the $6^3 \times 4$ target distribution.



Progress and perspectives with gauge corrections

Gauge group	Number of Vertices	Classical Simulations (3+1 dim.)	Number of qubit for 2×2 sublattice	Quantum Simulations (3+1 dim.)
U(3)	165	complete	12	complete
SU(3)	221	complete	20	in progress
$SU(3) + O(\beta)$	3,485	complete	22	
$SU(3) + O(\beta^2)$	51,125	impractical		
$SU(3) + O(\beta^3)$	681,013			
$SU(3) + O(\beta^4)$	8,620,741			

Summary and outlook

- The Grassmann constraint can be used constructively to reduce the QUBO size.
- QPU samples enter as Metropolis-Hastings proposals, so exact accept/reject correction is retained.
- For $SU(3)$, global baryon-sector moves are required beyond local subvolume updates.
- For large volumes, we are studying diffusion models and related generative approaches to improve baryon sampling.
- Next steps: With gauge corrections.