Effective-model analysis of topological susceptibility and meson susceptibilities in hot and dense matter

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Topological susceptibility

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Topological charge density

$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- U(1) axial anomaly
- Hadron mass
  (e.g., Witten-Veneziano relation: $m_{\eta'} \leftrightarrow \chi_{\rm top}^{\rm (YM)}$)
- Chiral symmetry breaking and vacuum structure
  → QCD phase structure at finite T and μ_B

Actively studied using lattice QCD simulations at T

- **Kanamori et al. / JLQCD (2026)**
  Physical-point $N_f = 2 + 1$domain-wall study
  Discretization and fermion-action effects in $\chi_{\rm top}(T)$
- **Kotov, Lombardo, Trunin (2025)**
  Physical $N_f = 2 + 1 + 1$ Wilson twisted-mass fermions
  θ-dependence, and DIGA behavior in high- T
- **Brandt et al. (2024)**
  Physical $N_f = 2 + 1$stout-improved staggered quarks
  Strong magnetic field effects on $\chi_{\rm top}(T)$
- **Aoki et al. / JLQCD (2024)**
  $N_f = 2 + 1$Möbius domain-wall fermions
  Dirac spectrum, $U(1)_A$ susceptibility, and $\chi_{\rm top}$
- **Athenodorou et al. (2022)**
  High-temperature $\chi_{\rm top}$using staggered spectral projectors.

⋮ (Many other studies in lattice QCD and other approaches)

Topological susceptibility

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Topological charge density

$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- ✓ $\chi_{\rm top} = 0$ in chiral limit
- ✓ Closely related to SU(2) chiral symmetry

???

Ward–Takahashi identities (WTIs)

Analyses based on WTIs have not been conducted in sufficient detail....

Using effective model approaches, we investigate $\chi_{\rm top}$ through WTIs.

Ward–Takahashi identities

$$\chi_{\mathrm{top}} = -i\int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Using the following relation

$\langle \bar{q}q\rangle = -im_l\chi_\pi$

$$\chi_{\mathrm{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

Meson susceptibilities

$$\left(\begin{array}{l} \chi_\eta = \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\psi)(x)(i\bar{\psi}\gamma_5\psi)(0)|0\rangle \\ \chi_\pi\delta^{ab} = \int d^4x \langle 0|T(i\bar{\psi}\gamma_5\tau_f^a\psi)(x)(i\bar{\psi}\gamma_5\tau_f^b\psi)(0)|0\rangle \end{array}\right)$$

Related to U(1) axial anomaly

$$\partial_\mu j_A^\mu = 2m_l\bar{\psi}i\gamma_5\psi + 2N_fQ$$

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}(i\gamma^\mu D_\mu - m_l)\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + \boxed{\theta Q}$$

$\mathrm{U(1)_A}$ transformation

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}i\gamma^\mu D_\mu\psi - \boxed{m_l\bar{\psi}\exp\left(i\theta/2\,\gamma_5\right)\psi} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}$$

$$\chi_{\mathrm{top}} = -\int d^4x \left.\frac{\delta^2\Gamma_{\mathrm{QCD}}}{\delta\theta(x)\delta\theta(0)}\right|_{\theta=0}$$

Ward–Takahashi identities and chiral limit

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

$$\chi_{\rm top} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

$\chi_{\rm top}$ is proportional to m_l^2.

→ In the chiral limit, $\chi_{\rm top} = 0$.

(θ-parameter becomes redundant.)

Related to U(1) axial anomaly

$$\partial_\mu j_A^\mu = 2m_l\bar{\psi}i\gamma_5\psi + 2N_f Q$$

$$\mathcal{L}_{\rm QCD} = \bar{\psi}(i\gamma^\mu D_\mu - m_l)\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + \boxed{\theta Q}$$

U(1)$_A$ transformation

$$\mathcal{L}_{\rm QCD} = \bar{\psi}i\gamma^\mu D_\mu\psi - \cancel{m_l\bar{\psi}\exp\left(i\theta/2\,\gamma_5\right)\psi} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}$$

$$\chi_{\rm top} = -\int d^4x \left.\frac{\delta^2\Gamma_{\rm QCD}}{\delta\theta(x)\delta\theta(0)}\right|_{\theta=0}$$

χ_{top} in 2 flavor Linear Sigma Model (LSM)

Let's look at typical results from LSM analyses.

LSM is constructed by
- Chiral field: $\Phi = (S^a + iP^a)T^a \quad (a = 0, 1, 2, 3)$
- VEV of Φ : $\bar{\Phi} = \frac{1}{2}\sigma_0^{\text{vev}}\, 1_{2\times 2}$

U(1)$_A$ anomaly is described by det Φ term:

$$\mathcal{L}_{\text{int}}^{(\text{anom})} = B(\det\Phi + \det\Phi^\dagger)$$

U(1) axial anomaly:

$$\partial_\mu j_A^\mu = (m_l \text{ part}) - 2iN_f B\big(\det\Phi - \det\Phi^\dagger\big)$$

Introduce θ-parameter

$$\mathcal{L}_{\text{int}}^{(\text{anom})} = B(e^{i\theta}\det\Phi + e^{-i\theta}\det\Phi^\dagger)$$

$$\chi_{\text{top}} = \lim_{\theta\to 0}\frac{d^2 V_{\text{eff}}}{d\theta^2} = \frac{B}{2}(\sigma_0^{\text{vev}})^2$$

In the chiral limit, $\chi_{\text{top}} \neq 0$…

$\neq$

In the chiral limit, $\chi_{\text{top}} = 0.$

???

χ_{top} in 2 flavor Linear Sigma Model (LSM)

Let's look at typical results from LSM analyses.

LSM is constructed by
- Chiral field: $\Phi = (S^a + iP^a)T^a \quad (a = 0, 1, 2, 3)$
- VEV of Φ : $\overline{\Phi} = \frac{1}{2}\sigma_0^{\text{vev}}\, 1_{2\times 2}$

U(1)$_\text{A}$ anomaly is described by det Φ term:

$$\mathcal{L}_{\text{int}}^{(\text{anom})} = B(\det\Phi + \det\Phi^\dagger)$$

U(1) axial anomaly:

$$\partial_\mu j_A^\mu = (m_l \text{ part}) - 2iN_f B\big(\det\Phi - \det\Phi^\dagger\big)$$

Introduce θ-parameter and perform U(1)$_\text{A}$ transformation ($\Phi \to e^{i\alpha}\,\Phi$)

$$\mathcal{L}_{\text{int}}^{(\text{anom})} + \mathcal{L}_{\text{int}}^{(\text{SB})} = B\big(\det\Phi + \det\Phi^\dagger\big) + cm_l \text{tr}\big[e^{-i\theta/2}\Phi + e^{i\theta/2}\Phi^\dagger\big]$$

$$\chi_{\text{top}} = \frac{f_\pi^2 m_\pi^4}{2}\left(\frac{1}{m_\pi^2} - \frac{1}{m_\eta^2}\right) = \chi_{\text{top}} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

Does $\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$ really work?

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

This holds for any N_f.

- 3 flavor NJL: $\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta) = 0.025/\text{fm}^4$

  Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki, M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031

- 3 flavor LSM: $\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta) = 0.0263/\text{fm}^4$

  M. K., S. Matsuzaki and A. Tomiya, PRD 103 (2021) 054034.

χ_{top} is fixed by meson properties, so similar results are expected in other models.

$$\chi_{\text{top}} = -i\int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

2+1+1 flavor lattice QCD: $\chi_{\text{top}} = 0.0245/\text{fm}^4$

S. Borsanyi et al., Nature 539, no. 7627, 69 (2016).

χ_{top} and symmetry breaking/restoration

$$\chi_{\text{top}} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle$$

Topological charge density

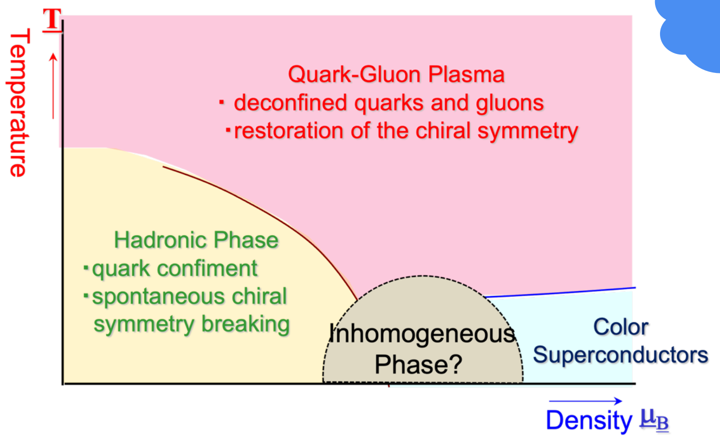
$$Q = (g^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- ✓ $\chi_{\text{top}} = 0$ in chiral limit
- ✓ Closely related to SU(2) chiral symmetry

WTIs

$$\chi_{\text{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

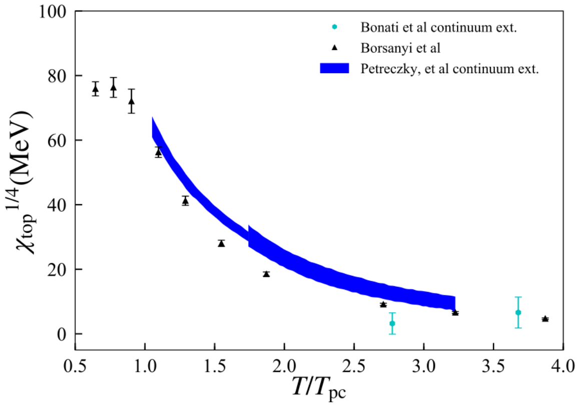
???



- How does χ_{top} behave at finite T and μ_B?
- Is χ_{top} related to SU(2) chiral restoration?

Lattice observations (T and μ_q dep.)

In hot QCD matter

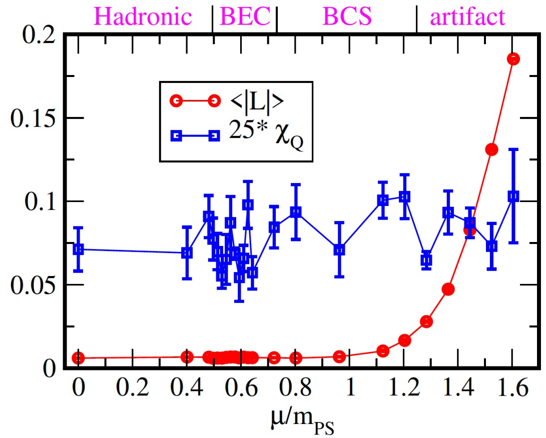


- C. Bonati et al, JHEP 11, 170 (2018), 1807.07954.
- *S. Borsanyi et al., Nature 539, no. 7627, 69 (2016).*
- P. Petreczky et al, Phys. Lett. B 762, 498-505 (2016)

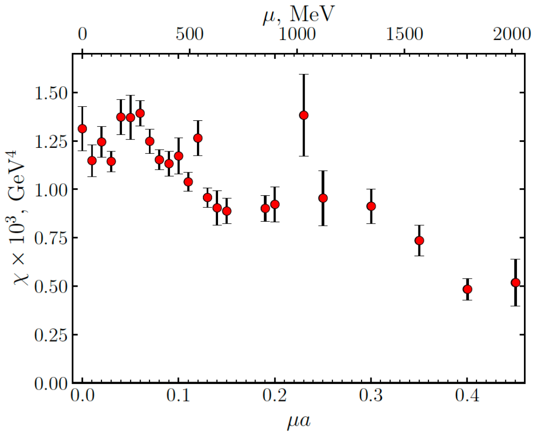
χ_{top} becomes smaller at high T.

Suppression of $U(1)_A$ anomaly

In dense QC_2D matter



Itou et al. JHEP01(2020)181



N. Astrakhantsev et al.
PRD 102 (2020) 7, 074507

μ_B does not affect χ_{top}.

χ_{top} is suppressed.

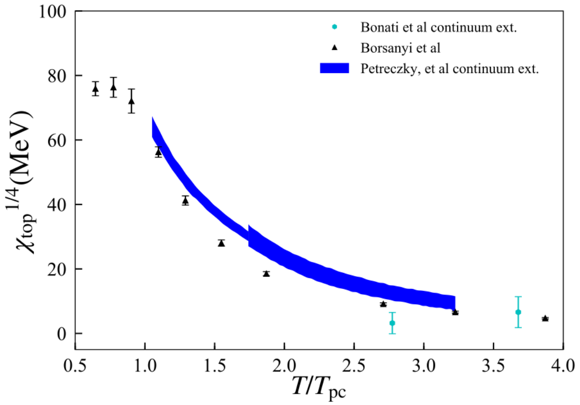
?

?

Fate of χ_{top} at high density is controversial …

Lattice observations (T) and effective model

In hot QCD matter



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$\chi_{\rm top}$ becomes smaller at high T.

Suppression of $U(1)_A$ anomaly

NJL model (linear sigma model)

$$\mathcal{L} = \bar{q}(i\gamma_\mu \partial^\mu - \boxed{\mathbf{m}})q + \mathcal{L}_{4f} + \boxed{\mathcal{L}_{\rm KMT}}$$

$$\mathcal{L}_{4f} = \frac{g_s}{2}\sum_{a=0}^{8}[(\bar{q}\lambda^a q)^2 + (\bar{q}i\gamma_5\lambda^a q)^2]$$

$$\mathcal{L}_{\rm KMT} = g_D[\det_{i,j}\bar{q}_i(1+\gamma_5)q_j + \text{h.c.}]$$

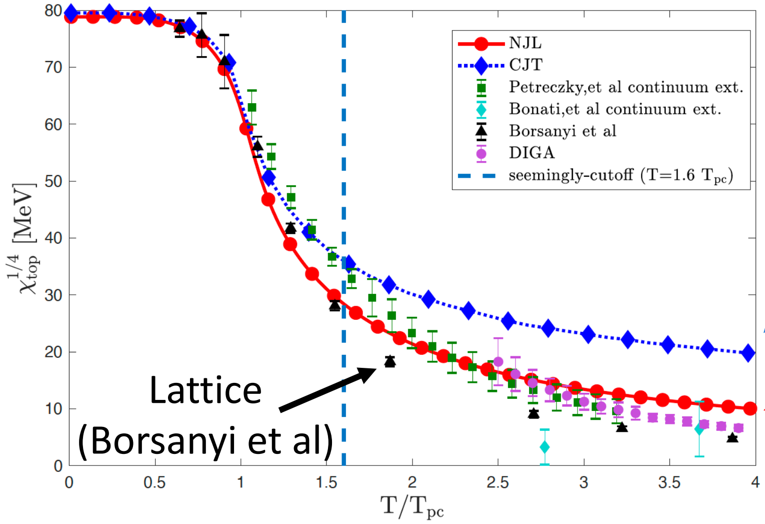
*Perform analysis within mean-field approximation.

*Model parameters are fixed to provide physical meson masses.

- Reproduce the WT identity of underlying QCD.
- Evaluate meson susceptibilities.

$$\chi_{\rm top} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

Model results and lattice observations



LSM (CJT approach) w/ $\chi_{\rm top} = \frac{i m_l^2}{4}(\boxed{\chi_\pi} - \boxed{\cancel{\chi_\eta}})$

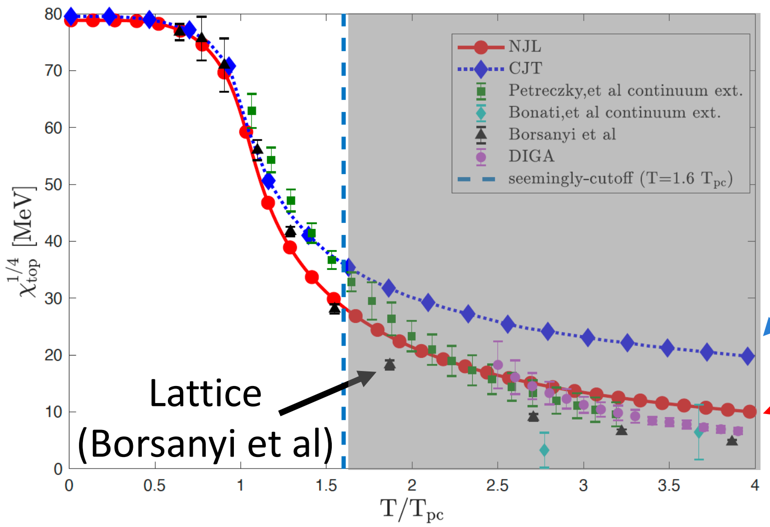
M.K., S. Matsuzaki, A. Tomiya,
PRD 103 (2021) 5, 054034L

NJL mean-field approach w/ $\chi_{\rm top} = \frac{i m_l^2}{4}(\boxed{\chi_\pi - \chi_\eta})$

Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki,
M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031

Model results and lattice observations

Model results are in good agreement with lattice observations.



LSM (CJT approach) w/ $\chi_{\rm top} = \frac{im_l^2}{4}(\boxed{\chi_\pi} - \boxed{\cancel{\chi_\eta}})$

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NJL mean-field approach w/ $\chi_{\rm top} = \frac{im_l^2}{4}(\boxed{\chi_\pi - \chi_\eta})$

Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki,
M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031

χ_π contribution dominates T dependence: $\chi_\pi \gg \chi_\eta$.

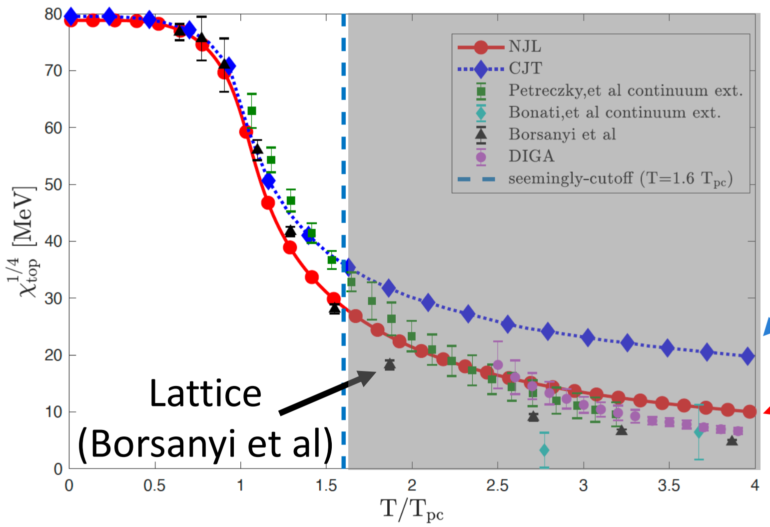
Using the WT identity
$\langle \bar{q}q \rangle = -im_l\chi_\pi$

$$\chi_{\rm top} \cong \frac{im_l^2}{4}\chi_\pi = -\frac{m_l}{4}\langle \bar{q}q \rangle$$

Suppression of $\chi_{\rm top}$ is linked with chiral restoration.

χ_{top} and chiral restoration at T

Model results are in good agreement with lattice observations.



LSM (CJT approach) w/ $\chi_{\text{top}} = \frac{im_l^2}{4}(\boxed{\chi_\pi} - \boxed{\cancel{\chi_\eta}})$

M.K., S. Matsuzaki, A. Tomiya,
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NJL mean-field approach w/ $\chi_{\text{top}} = \frac{im_l^2}{4}(\boxed{\chi_\pi - \chi_\eta})$

Chuan-Xin Cui, Jin-Yang Li, Shinya Matsuzaki,
M.K., Akio Tomiya, *PRD* 105 (2022) 11, 114031

- ✓ U(1)_A is effectively restored.
- ✓ Related to chiral restoration.

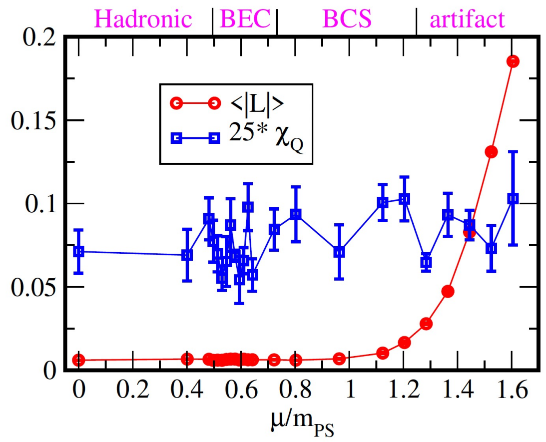
χ_π contribution dominates T dependence: $\chi_\pi \gg \chi_\eta$.

$$\chi_{\text{top}} \cong \frac{im_l^2}{4}\chi_\pi = -\frac{m_l}{4}\langle\bar{q}q\rangle$$

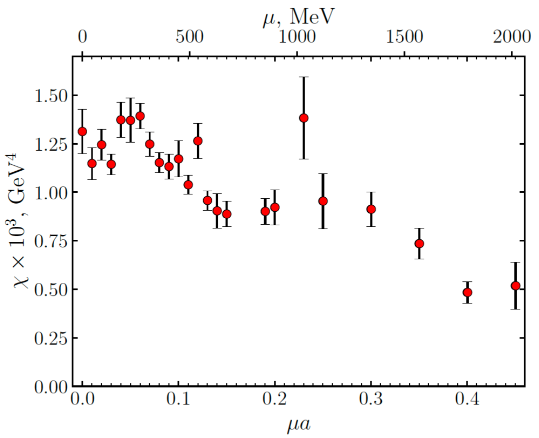
Suppression of χ_{top} is linked with chiral restoration.

Lattice observations (μ_q)

χ_{top} in dense QC_2D lattice simulations



Itou et al. JHEP01(2020)181



N. Astrakhantsev et al.
PRD 102 (2020) 7, 074507

μ_B does not affect χ_{top}.

χ_{top} is suppressed.

Fate of χ_{top} at high density is controversial …

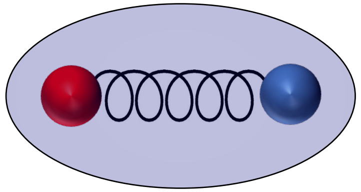
Effective model analysis provides useful benchmark.

Unique feature of dense QC_2D

Before analyzing $\chi_{\rm top}$…

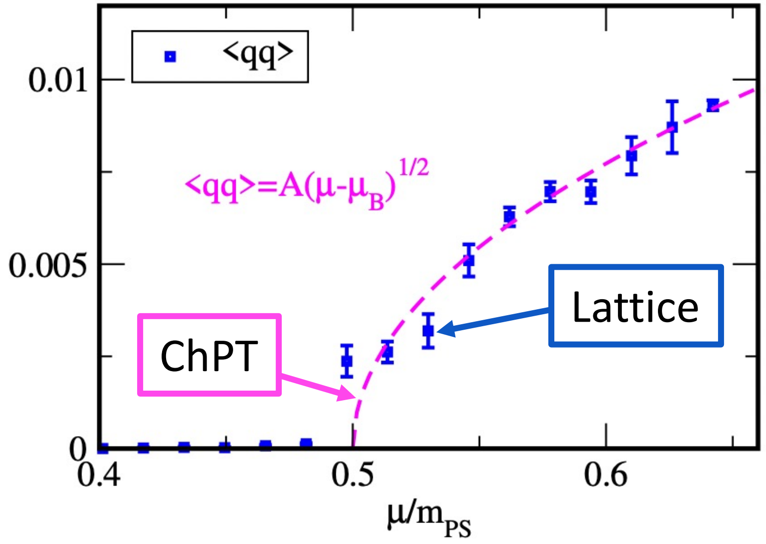
2-color QCD

Baryon is formed from two quarks.



"Bosonic baryon"

Baryon superfluid phase transition



K. Iida, E. Itou and T. G. Lee, JHEP 01, 181 (2020)

This feature can be described by …

$$\Sigma = \frac{1}{2}\begin{pmatrix} 0 & -B' + iB & \frac{\sigma - i\eta + a^0 - i\pi^0}{\sqrt{2}} & a^+ - i\pi^+ \\ B' - iB & 0 & a^- - i\pi^- & \frac{\sigma - i\eta - a^0 + i\pi^0}{\sqrt{2}} \\ -\frac{\sigma - i\eta + a^0 - i\pi^0}{\sqrt{2}} & -a^- + i\pi^- & 0 & -\bar{B}' + i\bar{B} \\ -a^+ + i\pi^+ & -\frac{\sigma - i\eta - a^0 + i\pi^0}{\sqrt{2}} & \bar{B}' - i\bar{B} & 0 \end{pmatrix}$$

Linear sigma model:

$$\mathcal{L}_{\rm LSM} = {\rm tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - V_0 - V_{\rm sp} - V_{\rm anom}$$

Transition to baryon superfluid phase

Linear sigma model:

$$\mathcal{L}_{\rm LSM} = \mathrm{tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - V_0 - V_{\rm sp} - V_{\rm anom}$$

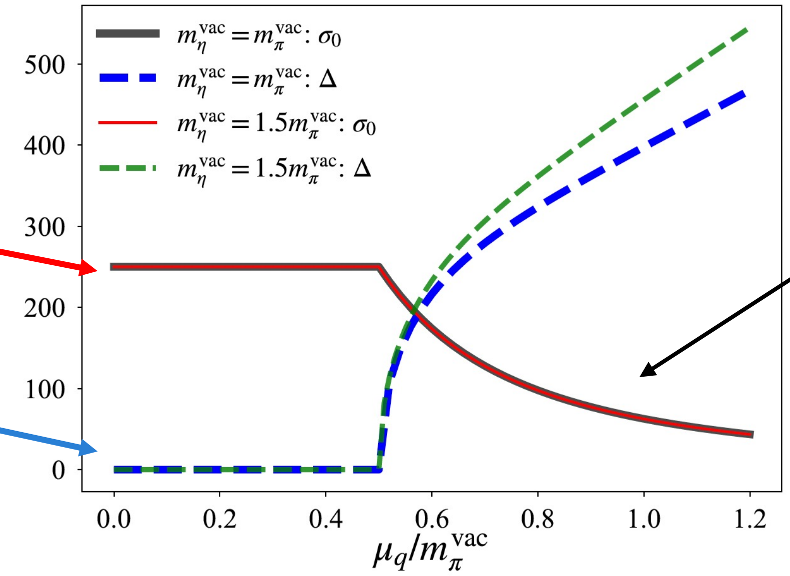
Can describe phase transition from hadron phase to baryon superfluid phase.

Sigma meson:

$\sigma_0 = \langle \sigma \rangle \sim \langle \bar{q} q \rangle$

Positive parity baryon:

$\Delta = \langle B \rangle \sim \langle qq \rangle$



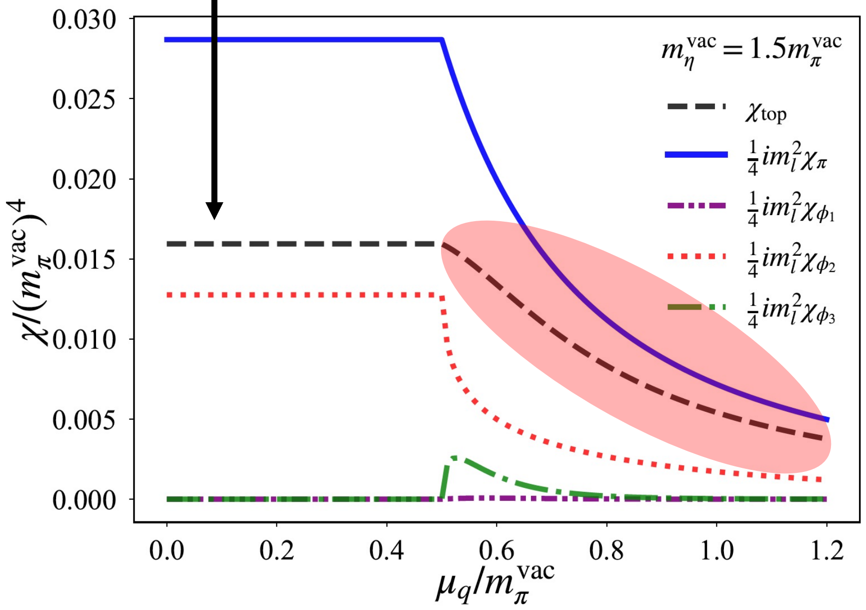
Chiral condensate scales with

$$\sigma_0 \sim \mu_q^{-2}$$

χ_{top} in superfluid phase

$$\chi_{\mathrm{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

used in LSM analysis



M. K., D. Suenaga *JHEP 08 (2023) 189*

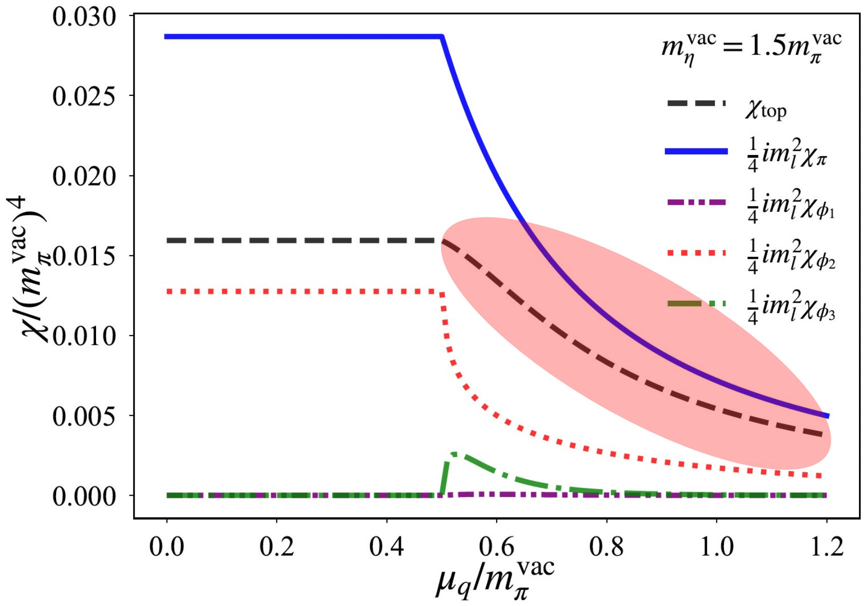
LSM result:

χ_{top} is suppressed in superfluid phase.

χ_{top} in superfluid phase

$$\chi_{\mathrm{top}} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)$$

used in LSM analysis



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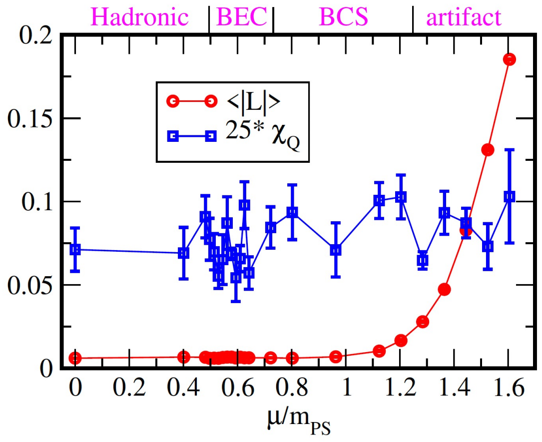
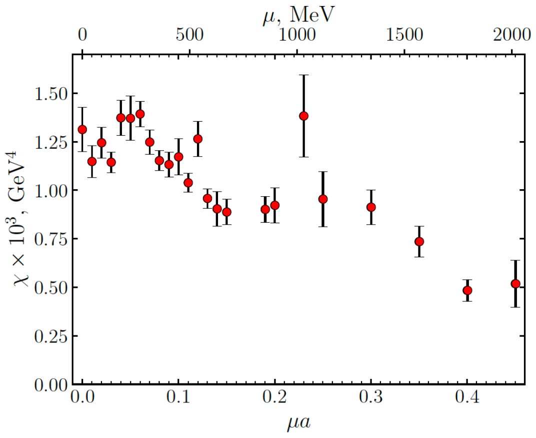
Lattice observations



Itou et al. JHEP01(2020)181



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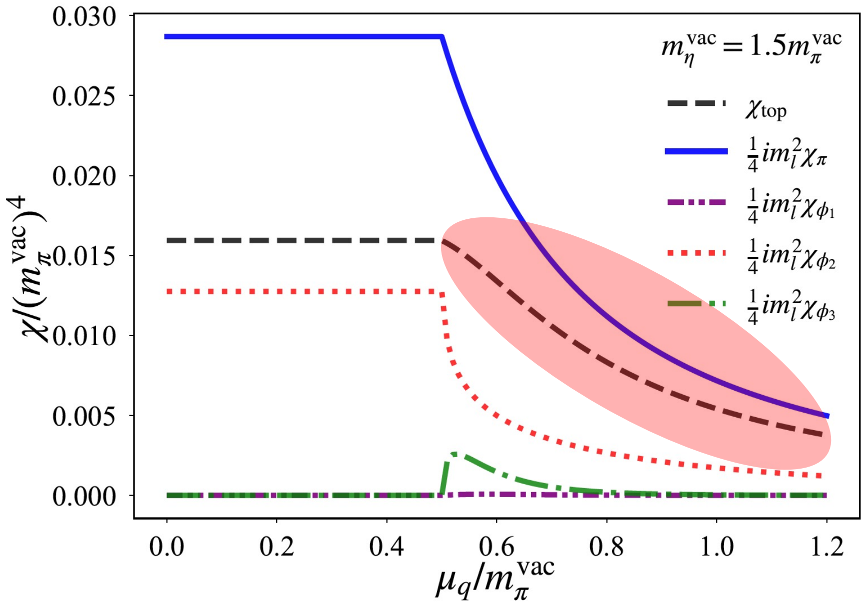
μ_B does not affect χ_{top}.

χ_{top} is suppressed.

χ_{top} and chiral restoration at finite μ_B

$$\chi_{\mathrm{top}} = \frac{i m_l^2}{4}(\chi_\pi - \chi_\eta)$$

used in LSM analysis



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μ scaling coincides with chiral condensate.

$$\frac{\chi_{\mathrm{top}}}{(m_\pi^{\mathrm{vac}})^4} \sim \frac{(f_\pi^{\mathrm{vac}})^2}{12}\boxed{\mu_q^{-2}} \quad \Longleftrightarrow \quad \sigma_0 \sim \boxed{\mu_q^{-2}}$$

At finite μ_B..

$$\chi_{\mathrm{top}} \cong -\frac{m_l}{4}\langle \bar{q} q \rangle$$

✓ $\mathrm{U(1)_A}$ is effectively restored.
✓ Related to chiral restoration.

Summary

We have investigated $\chi_{\rm top}$ using effective models based on Ward–Takahashi identities.

$$\chi_{\rm top} = -i \int d^4x \langle 0|TQ(x)Q(0)|0\rangle \quad \xleftrightarrow{\text{WTIs}} \quad \boxed{\chi_{\rm top} = \frac{im_l^2}{4}(\chi_\pi - \chi_\eta)}$$

At low- energy $\chi_{\rm top} = \frac{f_\pi^2 m_\pi^4}{2}\left(\frac{1}{m_\pi^2} - \frac{1}{m_\eta^2}\right)$ for 2 flavor

At finite T/μ_B $\chi_{\rm top} \cong -\frac{m_l}{4}\langle \bar{q}q \rangle$

✓ $\mathrm{U(1)_A}$ is effectively restored.
✓ Related to chiral restoration.

For quantitative studies and higher-T/μ regions, other approaches are needed.
(instanton picture may become more relevant.)

Schwinger–Dyson and FRG analyses will be important.