What can we learn about QCD and related physics from astrophysical observations?

DAY 1

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Lecture Plan

[Part 1]

Basics of Compact Stars
Mass, Radius
Equation of State (EoS)
Lane-Emden eq. / Stability*

👉 **Blackboard**

[Part 2]

EoS Inferences, Trace Anomaly
Gravitational Wave Implications
Recent Interpretation — Quarkyonic*

White Dwarfs / Neutron Stars

Typical Scales:

Average density

$$\bar{\rho}_{\mathrm{WD}} = \frac{3M}{4\pi R^3} \simeq 4 \times 10^5\ \mathrm{g\,cm^{-3}} \left(\frac{M}{0.6M_\odot}\right)\left(\frac{9{,}000\ \mathrm{km}}{R}\right)^3$$

$$\bar{\rho}_{\mathrm{NS}} = \frac{3M}{4\pi R^3} \simeq 3.8 \times 10^{14}\ \mathrm{g\,cm^{-3}} \left(\frac{M}{1.4M_\odot}\right)\left(\frac{12\ \mathrm{km}}{R}\right)^3$$

Compactness

$$\mathcal{C}_{\mathrm{WD}} = \frac{GM}{Rc^2} \simeq 1.0 \times 10^{-4} \left(\frac{M}{0.6M_\odot}\right)\left(\frac{9{,}000\ \mathrm{km}}{R}\right)$$

$$\mathcal{C}_{\mathrm{NS}} = \frac{GM}{Rc^2} \simeq 0.17 \left(\frac{M}{1.4M_\odot}\right)\left(\frac{12\ \mathrm{km}}{R}\right)$$

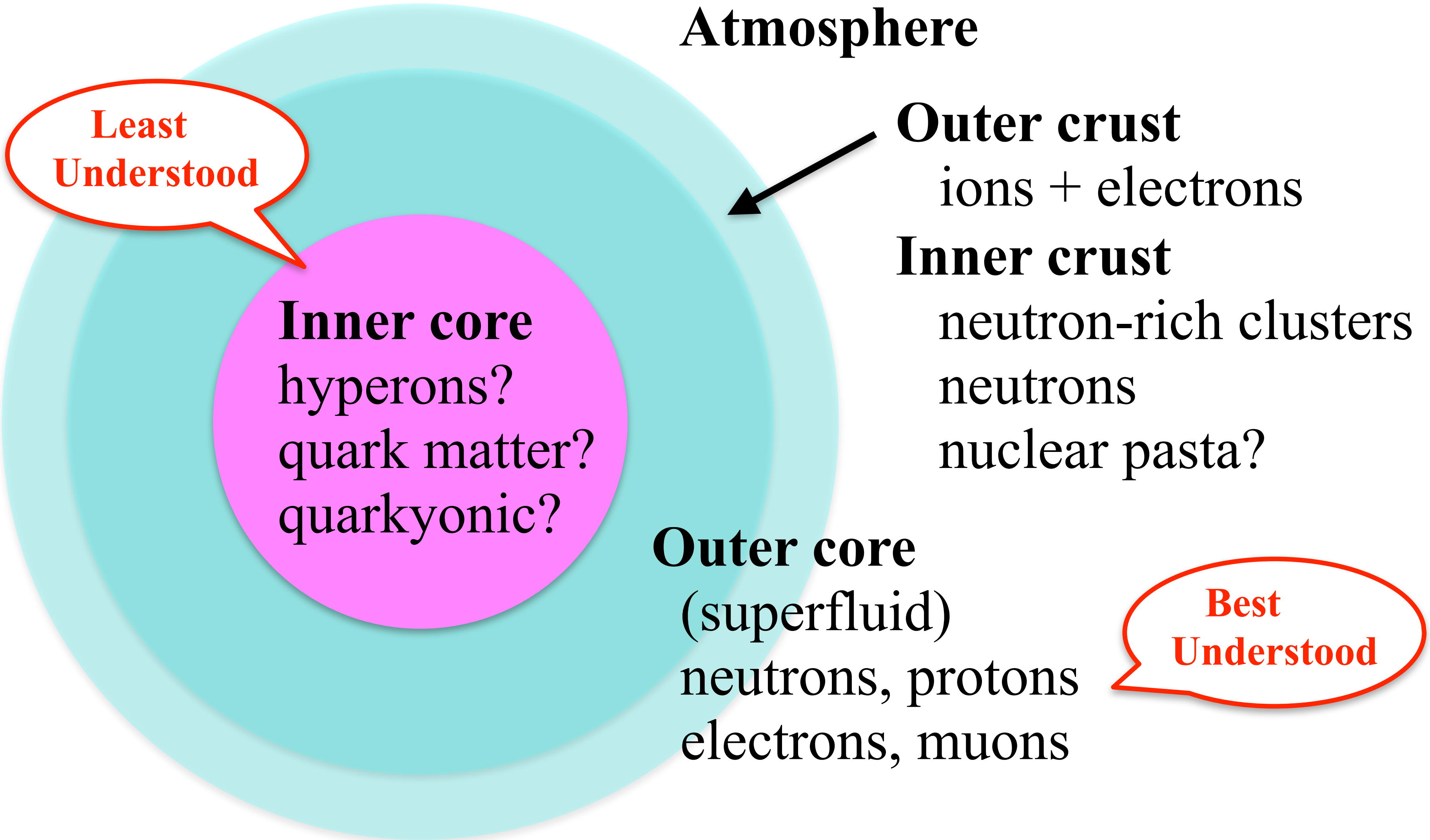
Often adopt the natural unit of $c = \hbar = 1$ and $G = 1$

White Dwarfs / Neutron Stars

	White Dwarfs	**Neutron Stars**
Mass	$0.5 \sim 1.3 M_\odot$	$1 \sim 2 M_\odot$
Radius	$5000 \sim 15000\,\mathrm{km}$	$10 \sim 14\,\mathrm{km}$
Composition	ionized carbon, oxygen electrons	neutron, proton electrons, muons
Magnetic Field	$10^3 \sim 10^9\,\mathrm{G}$	$10^{11} \sim 10^{13}\,\mathrm{G}$

Magnetars $10^{14} \sim 10^{15}\,\mathrm{G}$

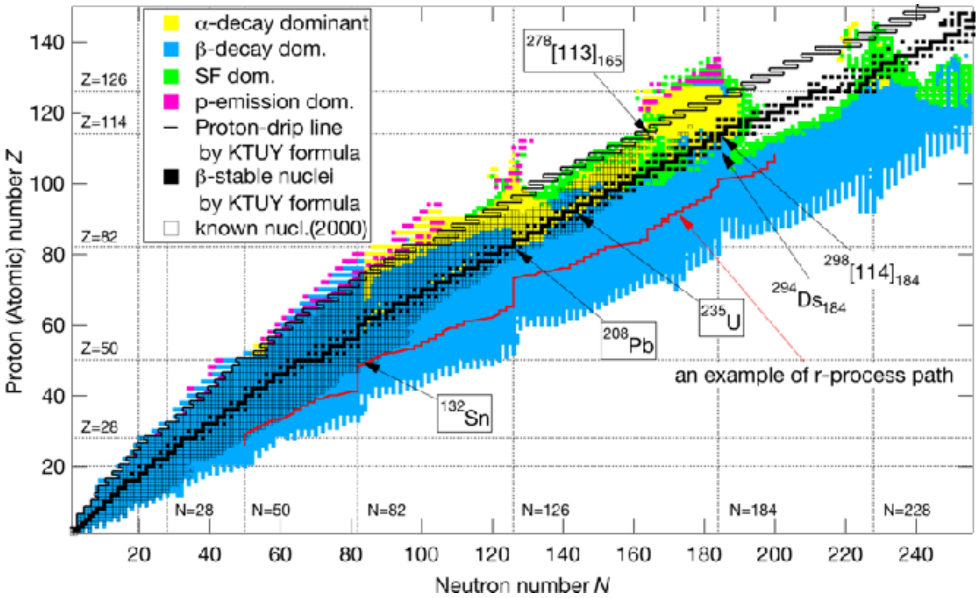
Neutron Stars



Neutron Stars as Gigantic Nuclei

Nuclear Chart

[From Wikipedia]



Along the stable(black) curve (Heisenberg's valley), the slope is getting smaller than 1.

For small A, the symmetric energy makes $N \sim Z$, while the Coulomb energy leads to $N \gg Z$ for large A.

If A is too large, heavy nuclei break up to smaller nuclei (nuclear fission).

Neutron Stars as Gigantic Nuclei

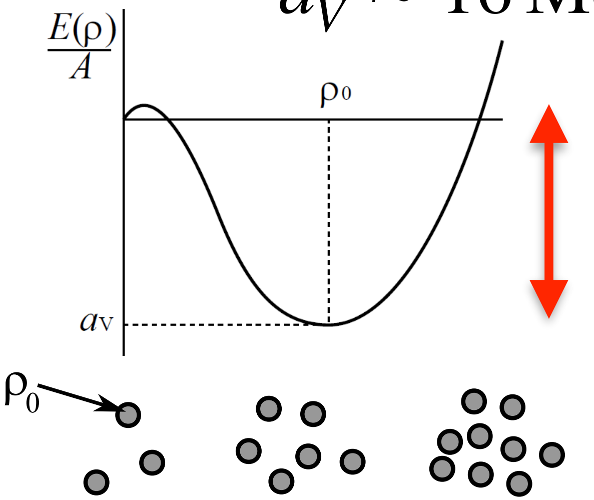
Mass Formula (binding energy)

$$m(Z,N)c^2 = Am_Nc^2 - B(Z,N)$$

$$B(Z,N) = \underbrace{a_V A}_{\text{Volume}} - \underbrace{a_S A^{2/3}}_{\text{Surface}} - \underbrace{a_C \frac{Z^2}{A^{1/3}}}_{\text{Coulomb}} - \underbrace{a_{\mathrm{sym}} \frac{(N-Z)^2}{A}}_{\text{Symmetry}}$$

$a_V \sim 16\,\mathrm{MeV}$

$a_{\mathrm{sym}} \sim 24\,\mathrm{MeV}$



For $N = Z$ and $A \to \infty$

$B/A \sim a_V \sim 16\,\mathrm{MeV}$

Neutron Stars as Gigantic Nuclei

Gigantic nuclei stabilized by Gravity

$$B_{\rm grav} = a_{\rm grav} A^{5/3} \qquad\qquad G = \hbar c / M_{\rm Planck}^2$$

$$a_{\rm grav} = \frac{3}{5}\frac{G(AM_n)^2}{1.2A^{1/3}} \cdot A^{-5/3} = 7 \times 10^{-37}\,\mathrm{MeV}$$

Stability Condition

$$(\mathrm{B.E}) = (a_V - a_{\rm sym})A + a_{\rm grav}A^{5/3} > 0$$

"Radius" of This Gigantic Nuclei

$$A_{\rm c} = \left(\frac{a_{\rm sym} - a_V}{a_{\rm grav}}\right)^{3/2} \simeq 3.8 \times 10^{55} \ \rightarrow\ R = 1.2A^{1/3}\,\mathrm{fm} \simeq \underline{4\,\mathrm{km}}$$

Maximum Mass

Simple argument for the NS maximum mass

$$p \sim \hbar n^{1/3} \;\to\; E_{\rm kin} \sim Apc \sim \frac{\hbar c A^{4/3}}{R}$$

$$E_{\rm total}(R) \sim \frac{\hbar c A^{4/3}}{R} - G\frac{A^2 M_n^2}{R}$$

If A is too large, the system collapses to $R \to 0$.

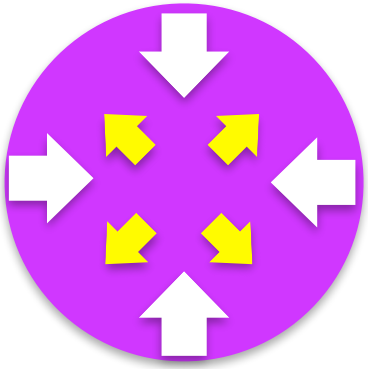
($R \to \infty$ is stabilized by $E_{\rm kin} \sim 1/R^2$ in the non-rela. limit.)

The order-estimated critical mass (precise calc. later):

$$M_c \sim \left(\frac{M_{\rm Planck}}{M_n}\right)^3 M_n \simeq \underline{2M_\odot}$$

Compact Star Basic Equations

Force Balance



Gravitational force is supported by the pressure from inside.

Hydrostatic condition for $r \sim r + dr$

$$\frac{dp(r)}{dr} = -G\frac{M(r)}{r^2}\varepsilon(r)$$

***M*(*r*) represents the integrated mass in *r*-sphere.**

$$\frac{dM(r)}{dr} = 4\pi r^2 \varepsilon(r)$$

(in Newtonian gravity)

Compact Star Basic Equations

$$\frac{dp(r)}{dr} = -G\frac{M(r)\,\varepsilon(r)}{r^2}$$

Tolman-Oppenheimer-Volkoff Eq.

General Relativistic extension

$$\frac{dp(r)}{dr} = -G\frac{M\varepsilon}{r^2}\left(1+\frac{p}{\varepsilon}\right)\left(1+\frac{4\pi r^3 p}{M}\right)\left(1-\frac{2GM}{r}\right)^{-1}$$

One condition still missing…

A relation between p and ε $\longrightarrow$ **Equation of State (EOS)**

$\boxed{\text{Initial Cond.}}$

$$r = 0$$

$$\varepsilon(r=0) = \varepsilon_c$$

free parameter

$$p(r=0) = p_c = p(\varepsilon_c)$$

$\boxed{\text{Final Cond.}}$

$$r = R$$

$$p(r=R) = 0$$

$$M = \int_0^R dr 4\pi r^2 \varepsilon(r)$$

Compact Star Basic Equations

Polytoropic EoS

$$p = K\rho^{1+1/n}$$

ρ: mass density

Hydrostatic Condition in Newtonian Gravity

$$\frac{dp(r)}{dr} = -G\frac{M(r)}{r^2}\rho(r) \qquad \frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

Lane-Emden Eq.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

$$\xi = r/r_0$$

$$\theta^n = \rho/\rho_c$$

$$\rho_c = \rho(r=0)$$

👉 **Blackboard**

Mass Determination

Useful information from binary systems with m_1, m_2 (thanks to the orbital motion)

Keplerian pulsar timing $\sim \dfrac{(m_2 \sin i)^3}{(m_1 + m_2)^2}$

Shapiro delay m_2 (precise)

Relativistic eccentric binary $m_1 + m_2$ + asymmetry

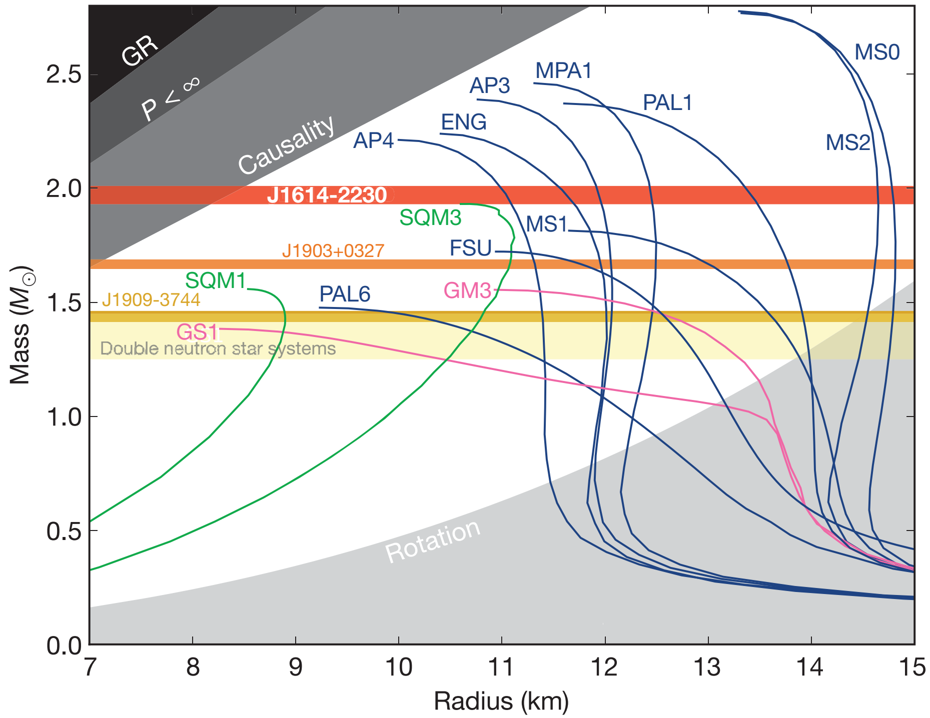
Two visible orbits m_1, m_2

Gravitational-wave inspiral $\sim \dfrac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$

Mass Determination

One Shapiro delay measurement changed the story…

[Very Famous Example]



Demorest et al. (2010-2016)

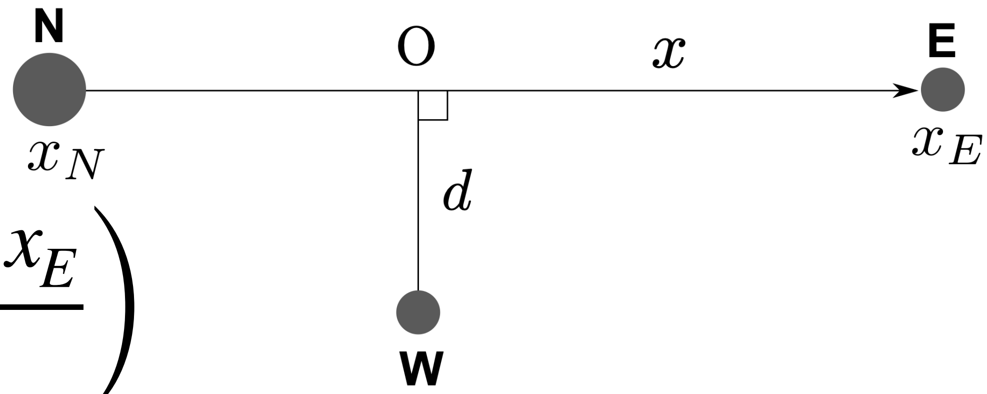
1.928(17) $M_{\rm sun}$ (J1614-2230)

Some models excluded from observations

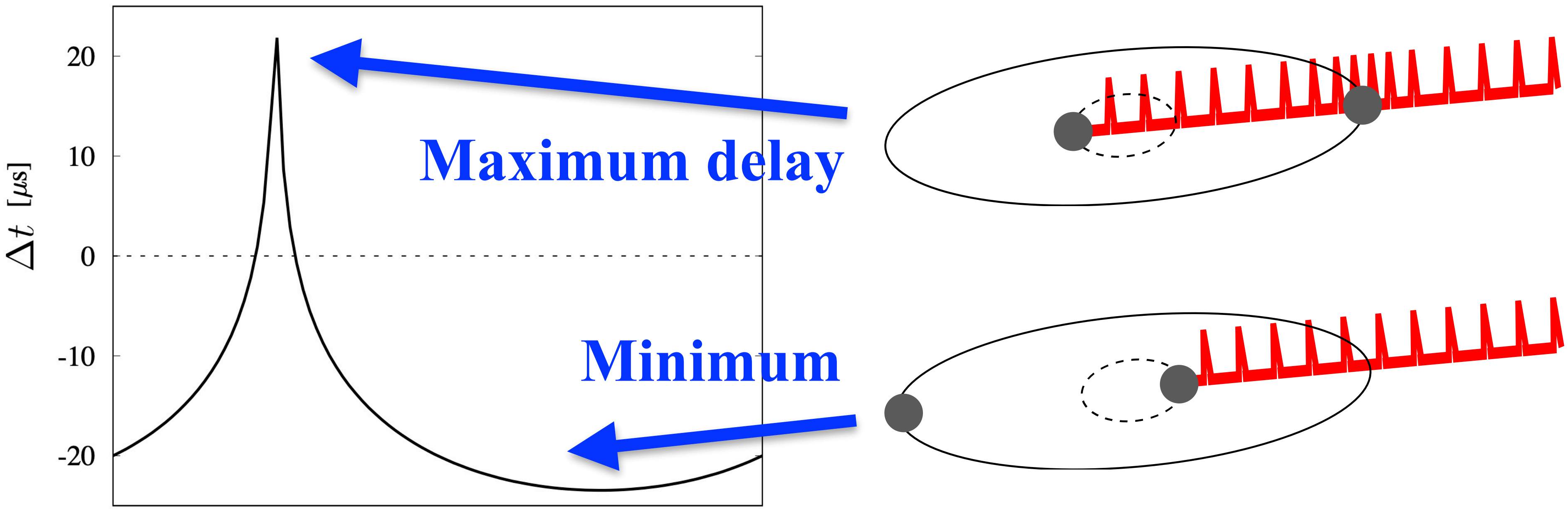
Even more massive NSs have been discovered later.

Mass Determination

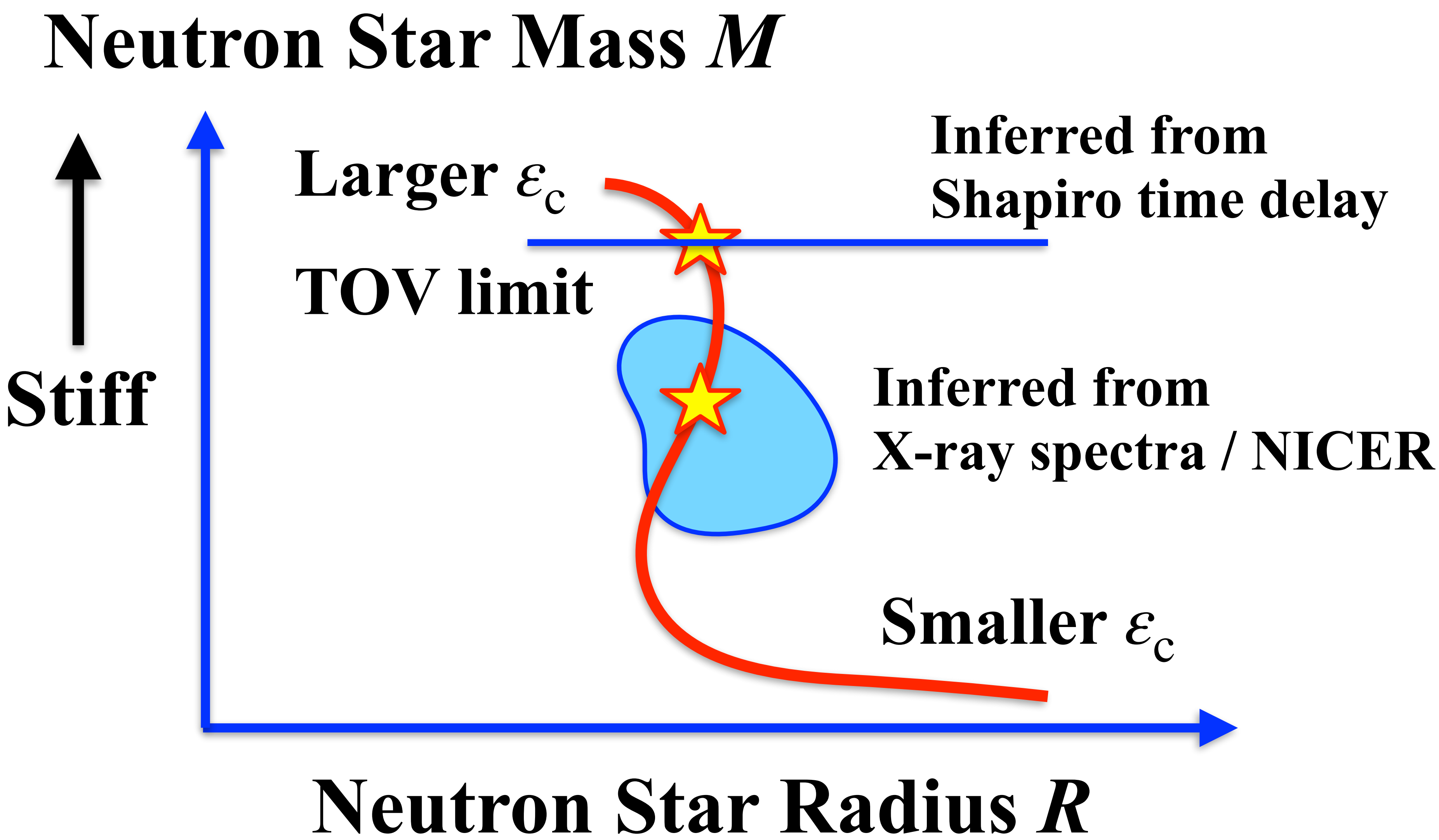
Shapiro time delay



$$\Delta t \approx 2 \times \frac{GM}{c^3} \ln\left(\frac{4|x_N|x_E}{d^2}\right)$$



Equation of State



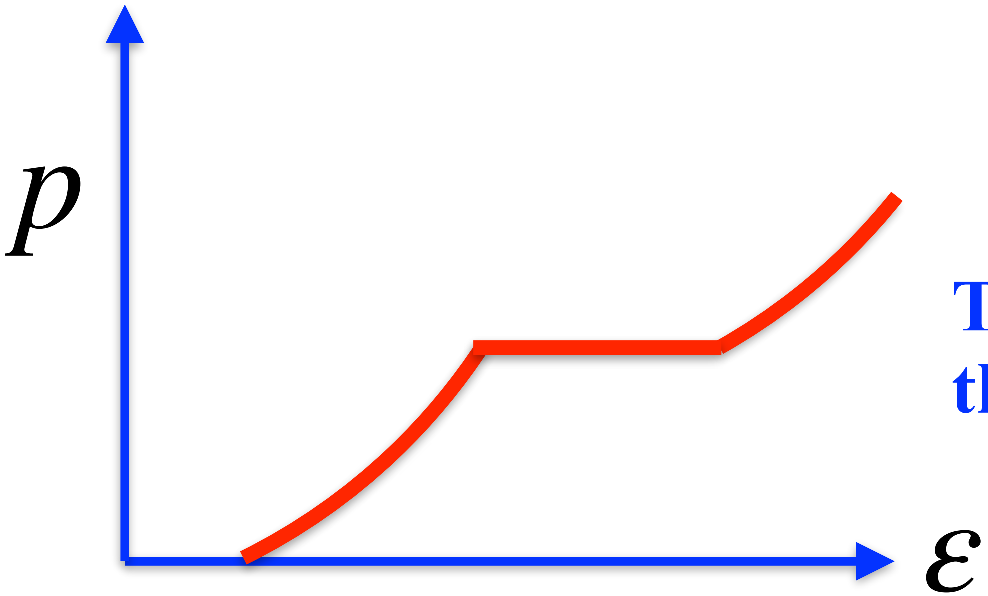
Equation of State

Mathematically proven:

$$p = p(\varepsilon) \longleftrightarrow M = M(R)$$

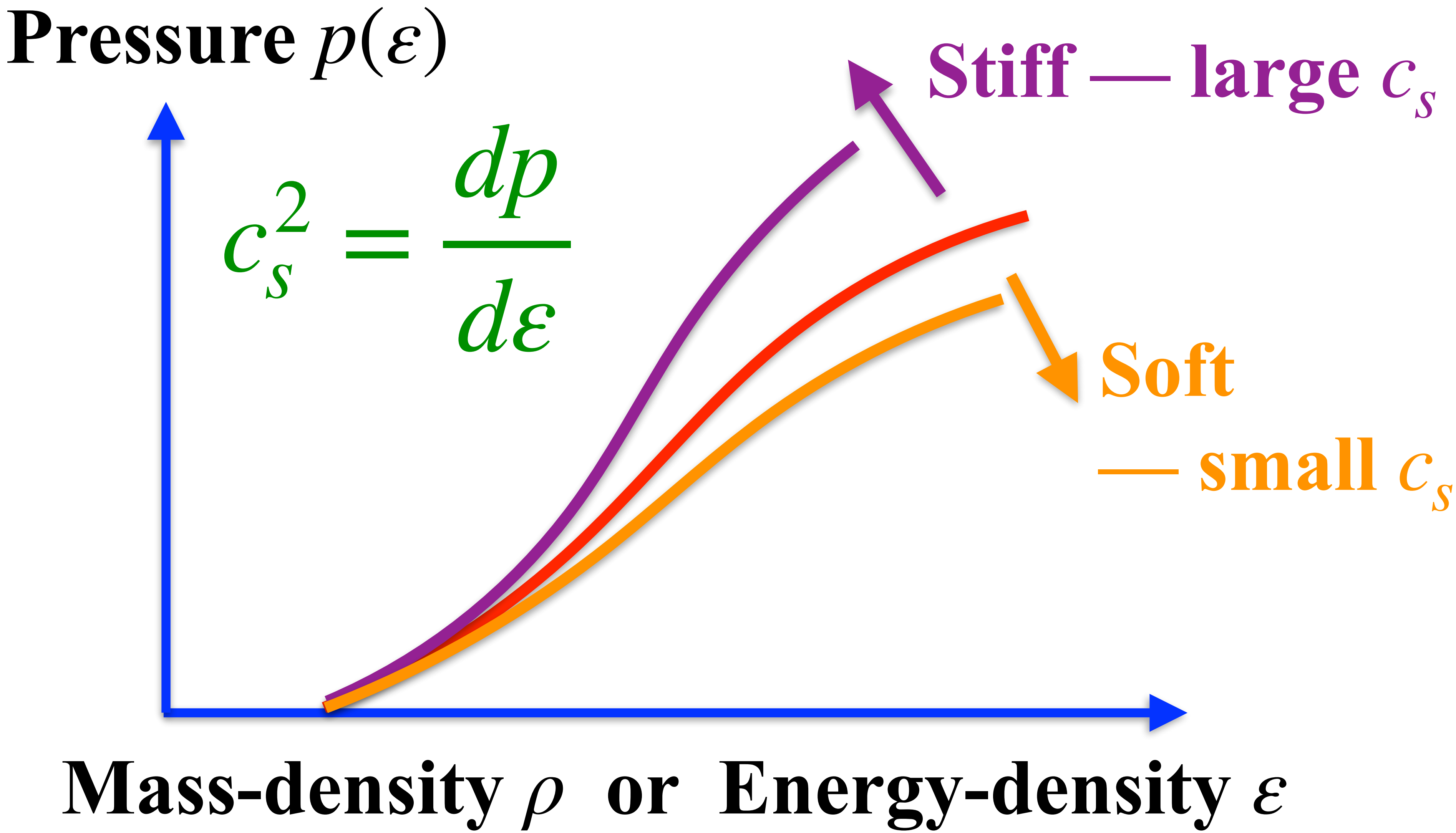
One-to-one Correspondence through TOV eq.

Lindblom (1992)

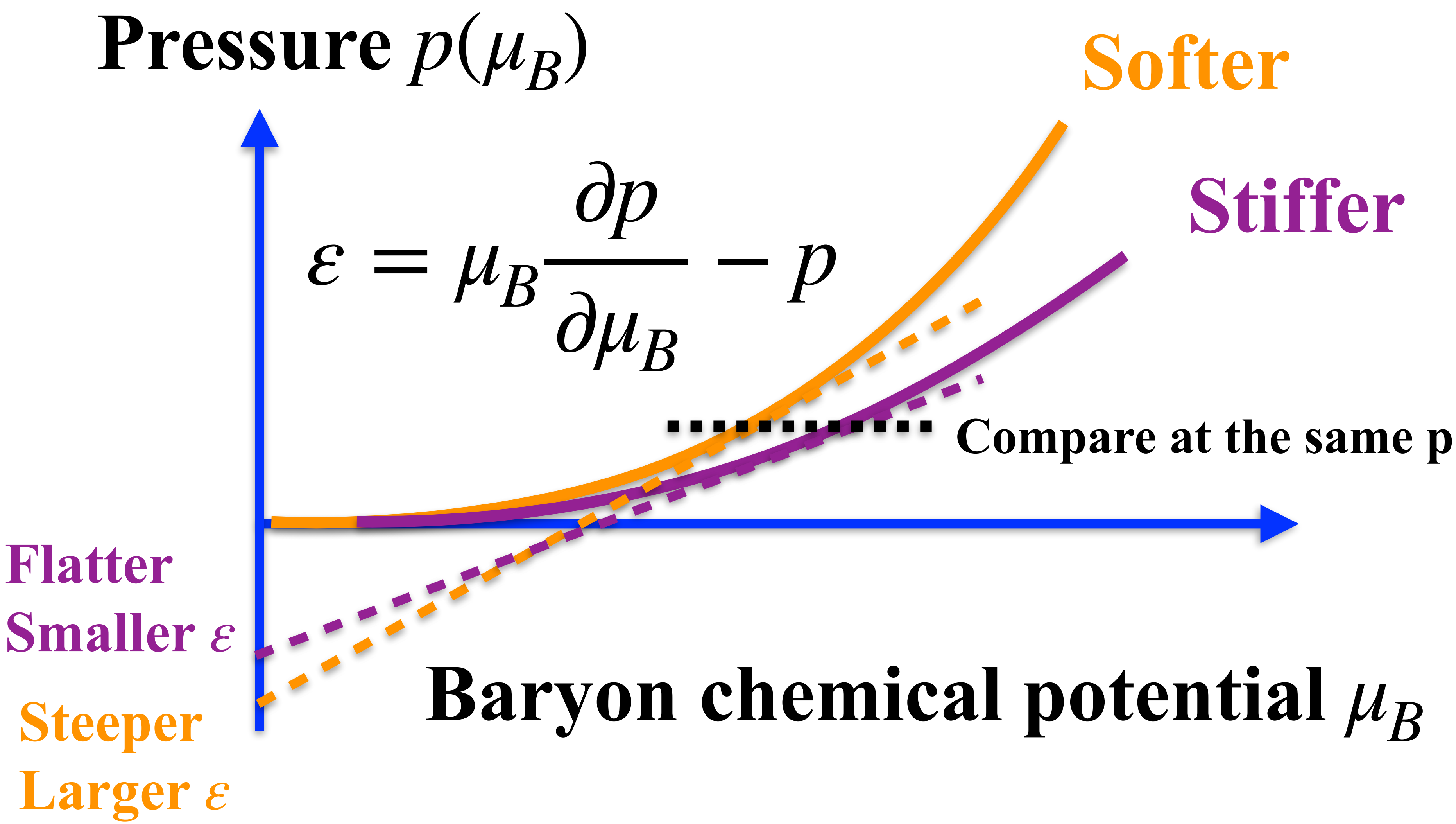


This is the case even with the 1st-order phase transition.

Equation of State



Equation of State



Blackboard Plan

Analysis of the Lane-Emden Equation

Derivation of the Chandrasekhar Mass

Stability Condition
(Sequence of One-parameter Solutions)

Third Family Scenario?