Further assumption: Markovian (extend G^R integral to ∞) & ohmic spectral function $\sigma \sim \gamma\omega$

In FOURIER SPACE (neglecting non-local part of potential)

$$\left[-m\omega^2 + U'_{ext} - g^2 G^R(\omega)\right] q(\omega) = \xi(\omega) \qquad G^R(\omega) \overset{\omega \text{ "small"}}{=} G^R(0) + i\overbrace{\gamma\omega}^{\sigma(\omega)}$$

$\uparrow$ potential renormalization

$$\Rightarrow \left[-m\omega^2 + \tilde{U}'_{ext} + g^2\gamma\omega\right] q(\omega) = \xi(\omega)$$

$$\overset{\text{IFT}}{\Rightarrow} \quad m\ddot{q} + \tilde{U}' + \boxed{\overbrace{g^2\gamma}^{\eta}\,\dot{q}(t)} = \xi(\omega)$$

check thermal eq:

$$\langle \xi^2 \rangle = \frac{g^2}{2}\Delta(\omega) \qquad \Delta = 2\coth\left(\frac{\omega}{2T}\right)\sigma(\omega) \approx \frac{2T}{\omega}\gamma\omega = 4\gamma T$$

"ω indep"

thermal noise: $\langle \xi(\omega)\xi(\omega')\rangle = 2\pi\,\delta(\omega+\omega')\,\frac{g^2}{2}\Delta$

$$\approx 2\pi\,\delta(\omega+\omega')\,2g^2\gamma T$$

$$\Rightarrow \langle \xi(t)\xi(t')\rangle \approx \boxed{2\underbrace{g^2\gamma}_{\eta} T\,\delta(t-t')}$$

EINSTEIN RELATIONS FLUCTUATION vs DISSIP.

BRIEF NOTE ON SK for FERMIONS:

charged $\hat{\rho}_{th} = \frac{1}{Z} e^{-\beta(\hat{H}-\mu\hat{Q})}$ with $\hat{Q} = \int d^3x\, \bar{\psi}\gamma_0\psi$. In equilibrium difference ψ antisymmetric in Euclidean time $G_F^>(\omega) = -e^{\beta(\omega-\mu)} G^<(\omega)$ $\quad \Delta_F = \tanh\left(\frac{\omega}{2T}\right)\sigma_F(\omega)$

Note: $\sigma_F = \langle \{\psi(t), \bar{\psi}(t')\}\rangle \quad \Delta_F = \langle [\psi(t), \bar{\psi}(t')]\rangle$

FEYNMAN VERNON INFLUENCE FUNCTIONAL

"Integrate out the Environment" made concrete!

$$S_{tot} = S_{sys}[q] + S_{env}[X] + S_{int}[q,X] \qquad \hat{\rho}_{tot}(0) = \hat{\rho}_{sys} \otimes \hat{\rho}_{env}$$

Perform the PARTIAL TRACE, i.e. leave the q_1, q_2 index intact

$$\langle q_1|\hat{\rho}^{(t)}_{sys}|q_2\rangle = \mathrm{Tr}_E\left[\langle q_1|U^+(t,t_0)\,\hat{\rho}_{tot}\,U(t,t_0)|q_2\rangle\right] = \int dX_f\, \langle q_1, X_f|U^+\hat{\rho}\,U|q_2, X_f\rangle$$

with $\langle q_1 X_1|U|q_1' X_1'\rangle = \int_{q_1}^{q_1'} Dq \int_{X_1}^{X_1'} DX\; e^{iS_{tot}[q,X]}$

$\langle q_1' X_1'|\hat{\rho}|q_2' X_2'\rangle$ at $t = t_i$
$= \rho_s(q_1', q_2')\,\rho_{env}(X_1', X_2')$

write as PROPAGATOR

$$\langle q_1|\hat{\rho}_s|q_2\rangle = \int dq_1' \int dq_2'\; J(q_1, q_2, t\,|\,q_1', q_2', t_0)\; \rho_s(q_1', q_2', t_0)$$

$$\mathcal{J} = \int \mathcal{D}q_1 \int \mathcal{D}q_2 \; e^{iS_{sys}[q_1] - iS_{sys}[q_2]} \; e^{iS_{FV}[q_1, q_2]}$$

INFLUENCE FUNCTIONAL

$$e^{iS_{FV}} = \int dX_1' \, dX_2' \; \rho_{env}(x_1', x_2') \int \mathcal{D}x_1 \int \mathcal{D}x_2 \; e^{i(S_{env} - S_{int}[q_1, x_1])} \, e^{i(S_{env} - S_{int}[q_2, x_2])}$$

$$= \left\langle \tilde{T} e^{+i\int ds \, \hat{H}_{int}} \; T e^{-i\int ds \, \hat{H}_{int}} \right\rangle_E$$

SIMPLEST SETTING: GAUSSIAN ENVIRONMENT (not necc. thermal)

ASSUME LINEAR COUPLING $\quad H_{int} = g \cdot \hat{q} \cdot \hat{X}$

$$e^{iS_{FV}} = \left\langle T_{\mathcal{C}} \exp\left[-ig \oint_{\mathcal{C}} ds \; \hat{q}(s) \hat{X}(s)\right] \right\rangle_E \qquad \oint_{\mathcal{C}} = \int_{t_0}^{t} (\;\;)_1 - \int_{t_0}^{t} (\;\;)_2$$

Now expand into CUMMULANTS : $\log \langle e^A \rangle = \langle A \rangle + \frac{1}{2}\left(\langle A^2 \rangle - \langle A \rangle^2\right) + \dots$ vanishes beyond 2nd order in Gaussian environment. Assume $\langle X \rangle_E = 0$

$$iS_{FV} = -\frac{g^2}{2} \oint ds \oint ds' \; q(s) \langle T_{\mathcal{C}} X(s) X(s') \rangle q(s') = -\frac{g^2}{2} \int ds \, ds' \; (q_1, q_2)(s) \begin{pmatrix} G_X^F & -G_X^< \\ -G_X^> & G_X^{\tilde{F}} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}(s')$$

Decompose using identities $\quad G_X^F = \Theta \, G_X^> + \tilde{\Theta} \, G_X^< \qquad G_X^< = G_X^>(-s)$

$$= -g^2 \iint_{s>s'} \left[q_1(s) G_X^> q_1(s') + q_2(s) G_X^< q_2(s') - q_2(s) G^> q_1(s') - q_1(s) G^< q_2(s') \right]$$

Split into REAL and IMAGINARY parts $\quad G_X^> = \nu + i\mu$

$$\nu = \tfrac{1}{2} \langle \{ \hat{X}(t), \hat{X}(0) \} \rangle \qquad \mu = \tfrac{i}{2} \langle [\hat{X}(t), \hat{X}(0)] \rangle$$

for ν : $\quad [q_1 q_1 + q_2 q_2 - q_2 q_1 - q_1 q_2] = (q_1 - q_2)(s) \, (q_1 - q_2)(s') = q_-(s) \, q_-(s')$

for μ : $\quad [q_1 q_1 - q_2 q_2 + q_1 q_2 - q_2 q_1] = (q_1 - q_2)(s) \, (q_1 + q_2)(s') = 2 \, q_-(s) \, q_+(s')$

$$\Rightarrow \; S_{IF} = g^2 \iint_{s>s'} \left[\nu(s-s') \, q_-(s) \, q_-(s') - 2i \, \mu(s-s') \, q_-(s) \, q_+(s') \right)$$

Introduce theta functions to express as unrestricted integrals

$$= g^2 \int ds \int ds' \left[q_-(s) \, G_X^R(s-s') \, q_+(s') + \tfrac{i}{4} \, q_-(s) \, \Delta_X(s-s') \, q_-(s') \right.$$

FRICTION/DISSIPATION $\qquad$ FLUCTUATIONS/DECOHERENCE

An Iconic Open-Quantum System: Caldeira-Leggett Model

Non-relativistic POINT PARTICLE in a BATH of QHOs. PRL 46 (4) 211 (1981)
ENVIRONMENT IS THERMAL, particle initial state arbitrary ($\rho(0) = \rho_S \otimes \rho$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) + \sum_\alpha \left(\frac{\hat{p}_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} \hat{x}_\alpha^2 \right) + \hat{q} \sum_\alpha c_\alpha \hat{x}_\alpha + \hat{q}^2 \sum_\alpha \frac{c_\alpha^2}{2 m_\alpha \omega_\alpha^2}$$

per mode COUPLING

MEDIUM induced RENORMALIZATION COUNTERTERM

Since environment is GAUSSIAN we can use our FV influence functional with BATH CORRELATORS

① Heisenberg: $\hat{x}_\alpha(t) = \hat{x}_\alpha \cos(\omega_\alpha t) + \frac{\hat{p}_\alpha}{m_\alpha \omega_\alpha} \sin(\omega_\alpha t)$

② Thermal: $\langle \hat{x}_\alpha^2 \rangle = \frac{1}{2 m_\alpha \omega_\alpha} \coth \frac{\omega_\alpha}{2T} \qquad \langle \{ \hat{x}_\alpha, \hat{p}_\alpha \} \rangle = 0$

$$\langle p_\alpha x_\alpha \rangle = \tfrac{1}{2} \langle \{ x_\alpha, p_\alpha \} \rangle - \tfrac{1}{2} \langle [\hat{x}_\alpha, \hat{p}_\alpha] \rangle = -i/2$$

(c.f. WIGNER-WEYL transform: collect products of operators in terms of $W(\hat{x}^n \hat{p}^m) = 1/2^n \sum_{k=0}^{n} \binom{n}{k} \hat{x}^{n-k} \hat{p}^m \hat{x}^k$ and commutators. Since Gaussian bath $\langle W(\hat{x}^n \hat{p}^m) \rangle = 0$ for either n or m ODD)

$$\langle \hat{x}_\alpha(t) x_\alpha(0) \rangle = \cos(\omega_\alpha t) \langle x_\alpha^2 \rangle + \frac{\sin(\omega_\alpha t)}{m_\alpha \omega_\alpha} \langle \hat{p}_\alpha \hat{x}_\alpha \rangle$$
$$= \frac{1}{2 m_\alpha \omega_\alpha} \left[\coth\left(\frac{\omega_\alpha}{2T}\right) \cos(\omega_\alpha t) - i \sin(\omega_\alpha t) \right]$$

③ Introduce SPECTRAL REPRESENTATION (real-time)

$$\nu(t) = \frac{1}{\pi} \int_0^\infty d\omega\, J(\omega) \coth\left(\frac{\omega}{2T}\right) \cos(\omega t)$$

$$\mu(t) = \frac{1}{\pi} \int_0^\infty d\omega\, J(\omega) \sin(\omega t)$$

collection of δ-peaks

with $J(\omega) = \pi \sum_\alpha \frac{c_\alpha^2}{2 m_\alpha \omega_\alpha} \delta(\omega - \omega_\alpha)$

④ THERMODYNAMIC LIMIT for DISSIPATION to emerge

Assumptions: Ⓐ Ohmic spectral function $J(\omega) = \eta \cdot \omega$ (up to cutoff Ω)
Ⓑ Scale separation $\Omega \gg T \gg \omega_{sys}$

remember, bosonic J is ANTISYMMETRIC so linear at small ω

- $\omega \ll T$ $\coth \to 2T/\omega$ $\qquad \delta_\Omega(t) = 1/\pi \int_0^\Omega d\omega \cos(\omega t)$

$$\nu(t) = \frac{2T\eta}{\pi} \delta_\Omega(t) \qquad \mu(t) = \eta/\pi \int_0^\Omega d\omega\, \omega \sin(\omega t) = -\eta\, \delta_\Omega'(t)$$

- insert into S_{IF} $\quad i \iint_{s>s'} \nu\, q_- q_- = i \eta T \int ds\, q_-^2(s)$

$$2\int_t^t ds \int_{t_0}^{s} ds'\, q_-(s)\,\mu(s-s')\,q_+(s') \underset{\text{IBP}}{=} 2\int_{t_0}^{t} ds\, q_-(s)\,\eta\left[\delta_\Omega(0)\,q_+(s) - \delta_\Omega(s-t_0)\,q_+(t_0) - \tfrac{1}{2}\dot q_+(s)\right]$$

pot. RENORMALIZATION — TRANSIENT $\tau \sim 1/\Omega$ — DISSIPATION

$$\left(\frac{\partial}{\partial(s-s')}\delta(s-s') = -\frac{\partial}{\partial s'}\delta(s-s')\right)$$

$$\Rightarrow\quad S_{IF}^{\text{Markov}} = -\eta\int dt\, q_-(t)\,\dot q_+(t) + i\eta T\int dt\, q_-^2(t)$$

⑤ Setup the MASTER EQUATION from time-local action

$$\rho(q_1,q_2,t+\delta t) = \int dq_1'\,dq_2'\; J(q_1,q_2,t+\delta t\,|\,q_1',q_2',t)\,\rho(q_1',q_2',t)$$

For a SINGLE step δt Path integral just a SINGLE integration

$$\begin{aligned} q_1(t+\delta t) &= x & q_1(t) &= x' & \xi &= x-x' & & \dot q_1 = \frac{\xi}{\delta t} & q_- &= x-y \\ q_2(t+\delta t) &= y & q_2(t) &= y' & \zeta &= y-y' & \Rightarrow\; & \dot q_2 = \frac{\zeta}{\delta t} & \dot q_+ &= \frac{\xi+\zeta}{\delta t} \end{aligned}$$

REWRITE: $S_{kin} = \frac{M}{2}\left(\frac{\xi^2}{\delta t^2} - \frac{\zeta^2}{\delta t^2}\right)\delta t \qquad -\eta\int ds\, q_-\dot q_+ = -\frac{\eta}{2}(x-y)\frac{(\xi+\zeta)}{\delta t}\delta t$

$$-\eta T\int ds\, q_-^2 = -\eta T\,(x-y)^2\,\delta t$$

SWITCH coordinates from $x',y' \mapsto \xi,\zeta$ $\quad |\det J| = 1$

$$\rho(x,y,t+\delta t) = N\int d\xi\,d\zeta\; e^{i\frac{M}{2\delta t}(\xi^2-\zeta^2)}\; e^{-i\delta t(U(x)-U(y))}\; e^{-i\frac{\eta}{2}(x-y)(\xi+\zeta)}$$

$$\hookrightarrow \times\; e^{-\eta T(x-y)^2\delta t}\;\rho(x-\xi,y-\zeta,t)$$

⑥ GRADIENT EXPANSION $\quad \frac{\partial}{\partial x}f(x-y) = -\frac{\partial}{\partial y}f(x-y)$

$$\rho(x-\xi,y-\zeta,t) = \rho - \xi\frac{\partial\rho}{\partial x} - \zeta\frac{\partial\rho}{\partial y} + \frac{1}{2}\xi^2\frac{\partial^2\rho}{\partial x^2} + \frac{1}{2}\zeta^2\frac{\partial^2\rho}{\partial y^2}$$

NO MIXING TERMS due to form of action

Need to evaluate FRESNEL type integrals $\int dz\, e^{iaz^2+ibz}\{1,z,z^2\}$

$$\langle\xi\rangle = \frac{\eta\,\delta t}{2M}(x-y) \quad \langle\zeta\rangle = -\frac{\eta\,\delta t}{2M}(x-y) \quad \langle\xi^2\rangle = \frac{i\,\delta t}{M} \quad \langle\zeta^2\rangle = \frac{-i\,\delta t}{M} \quad \langle\xi\rangle\langle\zeta\rangle \sim O\delta t$$

⑦ collect all terms:

$$\frac{\partial \rho}{\partial t} = \frac{i}{2m}\left(\partial_x^2 - \partial_y^2\right)\rho - i\left[V(x) - V(y)\right]\rho - \overbrace{\gamma(x-y)(\partial_x - \partial_y)\rho}^{\text{FRICTION}}$$
$$- \underbrace{2M\gamma T\,(x-y)^2\rho}_{\text{DECOHERENCE}}$$

⑧ convert back to OPERATOR expression (found in TEXTBOOKS)

$$\hat{x}\hat{\rho} = x\rho(x,y) \qquad \hat{\rho}\hat{x} = \rho(x,y)\,y$$
$$\hat{p}\hat{\rho} = -i\partial_x \rho(x,y) \qquad \hat{\rho}\hat{p} = +i\partial_y \rho(x,y)$$

$\mapsto$

$$[\hat{x},\hat{\rho}] \to (x-y)\rho$$
$$[\hat{p},\hat{\rho}] \to -i(\partial_x+\partial_y)\rho$$
$$\{\hat{x},\hat{\rho}\} \to (x+y)\rho$$
$$\{\hat{p},\hat{\rho}\} \to -i(\partial_x-\partial_y)\rho$$

$$[\hat{x},[\hat{x},\hat{\rho}]] = (x-y)^2\rho$$
$$[\hat{x},\{\hat{p},\hat{\rho}\}] = -i(x-y)(\partial_x-\partial_y)\rho$$
$$[\hat{p},[\hat{p},\hat{\rho}]] = -(\partial_x^2+\partial_y^2)\rho$$

$$\Rightarrow \quad \frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] - i\gamma[\hat{x},\{\hat{p},\hat{\rho}\}] - \frac{\kappa}{2}[\hat{x},[\hat{x},\hat{\rho}]] \qquad \kappa = 4M\gamma T$$

CAVEAT: This master equation is NOT in LINDBLAD form.
i.e. positivity NOT guaranteed.

How to DIAGNOSE whether your master equation is positive?
Here $\hat{x}$ and $\hat{p}$ generate fluctuation and dissipation. Try

NOT A NUMBER

$$\frac{d\rho}{dt} = -i[\hat{H},\hat{\rho}] + \sum_{j,k=1}^{2} C_{jk}\left(\hat{A}_j\hat{\rho}\hat{A}_k - \frac{1}{2}\{\hat{A}_j\hat{A}_k,\hat{\rho}\}\right) \qquad \hat{A}\in(\hat{x},\hat{p})$$

Lindblad form ONLY if C is POSITIVE $\mapsto$ diagonalization REVEALS Lindblad operators $\hat{L}_j$.

$$C_{jj}\left(\hat{A}_j\hat{\rho}\hat{A}_j - \frac{1}{2}\{\hat{A}_j^2,\hat{\rho}\}\right) = -\frac{C_{jj}}{2}[\hat{A}_j,[\hat{A}_j,\hat{\rho}]]$$

$$C_{12}(\ldots) = \underbrace{C_{12}[\hat{x},\{\hat{p},\hat{\rho}\}]}_{\text{FRICTION}} + \underbrace{\frac{C_{12}}{2}[\{\hat{x},\hat{p}\},\hat{\rho}]}_{\text{HAMILTONIAN SHIFT}}$$

comparing to CL master equation

$$C_{CL} = \begin{pmatrix} \kappa & -i\gamma \\ i\gamma & 0 \end{pmatrix} \qquad \det C_{CL} = -\gamma^2 < 0$$

↯

To restore positivity MINIMAL modification (to get $\det C = 0$)

$$C_{22} = \frac{|C_{12}|^2}{C_{11}} = \frac{\gamma^2}{\kappa} = \frac{\gamma}{4MT} \Rightarrow C^{NEW} = \sigma^\dagger \sigma \qquad \sigma = \begin{pmatrix} \sqrt{\kappa} \\ i\gamma/\sqrt{\kappa} \end{pmatrix}$$

DIAGONALIZE for LINDBLAD operators $\quad L = \sigma_1 \hat{x} + \sigma_2 \hat{p} = \sqrt{\kappa}\left(\hat{x} + \frac{i\hat{p}}{4MT}\right)$

including shift

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -i[\hat{H}', \hat{\rho}] - i\gamma[\hat{x}, \{\hat{p}, \hat{\rho}\}] - \frac{\kappa}{2}[\hat{x}, [\hat{x}, \hat{\rho}]] - \frac{\gamma}{8MT}[\hat{p}, [\hat{p}, \hat{\rho}]]$$

$$H' = H + \gamma/2 \{\hat{x}, \hat{p}\} \qquad \kappa = 4M\gamma T$$

Reverse engineering the added term gives $\frac{\gamma}{8MT}(\partial_x + \partial_y)^2 \rho$

HIGHER ORDER in gradient expansion.

EHRENFEST RELATIONS and APPROACH TO EQUILIBRIUM

$$\frac{d}{dt}\langle O \rangle = i\langle [\hat{H}', \hat{O}] \rangle - i\gamma \langle \{\hat{p}, [\hat{O}, \hat{x}]\} \rangle - \frac{\kappa}{2}\langle [[\hat{O}, \hat{x}], \hat{x}] \rangle - \frac{\gamma}{8MT}\langle [[\hat{O}, \hat{p}], \hat{p}] \rangle$$

Relevant OBSERVABLES (free BROWNIAN motion $V=0$)

$\frac{d}{dt}\langle x \rangle = \langle \hat{p} \rangle / M \qquad \frac{d}{dt}\langle p \rangle = -\langle V'(x) \rangle - 2\gamma \langle p \rangle \qquad$ CLASSICAL damping rate $\tau_{rel} = 1/2\gamma$

$\frac{d}{dt}\langle p^2 \rangle = -\langle \{\hat{p}, V'(\hat{x})\} \rangle - 4\gamma \langle p^2 \rangle + \kappa \qquad$ MOMENTUM equilibration $\langle p^2 \rangle_\infty = \kappa/4\gamma = MT$

$\frac{d}{dt}\langle \{x,p\} \rangle = \frac{2\langle p^2 \rangle}{M} - \langle \{x, V'\} \rangle - 2\gamma \langle \{x,p\} \rangle \qquad$ DIFFUSION $\langle \{x,p\} \rangle \neq 0 = T/\gamma$

$\frac{d}{dt}\langle x^2 \rangle = \frac{1}{M}\langle \{\hat{x}, \hat{p}\} \rangle + \frac{\gamma}{4MT}$ small correction in approx $\qquad \langle x^2 \rangle \sim \frac{T}{M\gamma} t = 2D_s t$

$$D_s = \frac{T}{2M\gamma} = \frac{2T^2}{\kappa}$$

DIFFUSION COEFFICIENT

For the HARMONIC POTENTIAL STEADY STATE

$\langle p^2 \rangle_\infty = MT + O(\gamma/\omega_0, \omega_0/T^2)$

$\langle x^2 \rangle_\infty = \frac{T}{M\omega_0^2} \approx \frac{1}{2M\omega_0}\coth\left(\frac{\omega}{2T}\right) \qquad$ CORRECT to $O((\omega/T)^2)$

$\langle \{x,p\} \rangle_\infty = -\gamma/4T \neq 0 \qquad$ deviation from zero caused by Lindblad completion: SMALL order approx.

QCD analogy: single quark free Brownian, quarkonium in potential.

STOCHASTIC UNRAVELLING: For realistic systems in 3d space $\rho(\vec r, \vec r', t)$ is a 6-d object, master equation costly to solve.

Possible to setup STOCHASTIC EVOLUTION of an ENSEMBLE of WAVEFUNCTIONS (3d NOT 6d) that recovers $\rho(t) = E[|\psi_\theta(t)\rangle\langle\psi_\theta(t)|]$ NOISE

① Hermitean L (pure decoherence, no dissipation)

Amounts to random modification of Hamiltonian (STOCHASTIC POTENTIAL) $\hat H_\theta(t) = \hat H + \theta(t) L$ $\quad L = L^+$ $\quad E[\theta^2] = \kappa\,\delta(t-t')$

Fully hermitean $\hat H_\theta \mapsto$ UNITARY time evolution

To derive master equation introduce WIENER INCREMENT

$dW = \int_t^{t+dt} \theta(s)\,ds/\sqrt{\kappa}$ $\quad E[dW]=0$ $\quad E[dW^2] = dt$ $\quad$ i.e $dW \sim dt$

$$|\psi(t+dt)\rangle = \exp\left(-iH\,dt - i\sqrt{\kappa}\,L\,dW\right)|\psi(t)\rangle$$

CAREFUL: when expanding need to go to full order dt!

$$\exp[\;] = 1 - iH\,dt - i\sqrt{\kappa}\,L\,dW + \tfrac{1}{2}(-i\sqrt{\kappa}\,L\,dW)^2 + O(dt^{3/2})$$

$$\Rightarrow \quad d|\psi\rangle = -iH|\psi\rangle\,dt - i\sqrt{\kappa}\,L|\psi\rangle\,dW - \frac{\kappa}{2}L^2|\psi\rangle(dW)^2$$ LINEAR STOCHASTIC SCHRÖDINGER

check Norm: terms of $O(dt)$

$$d(\langle\psi|\psi\rangle) = \langle d\psi|\psi\rangle + |\psi\rangle\langle d\psi| + \langle d\psi|d\psi\rangle = -\kappa\langle L^2\rangle dW^2 + \kappa\langle L^2\rangle dW^2 = 0\ !$$

for density matrix

$$d(|\psi\rangle\langle\psi|) = -i[H, |\psi\rangle\langle\psi|] - \frac{\kappa}{2}\{L^2, |\psi\rangle\langle\psi|\}dW^2 + \kappa L|\psi\rangle\langle\psi|L\,dW^2 + (\ldots)\,dW$$ VANISH when E[

$$\Rightarrow \quad \frac{d\hat\rho}{dt} = -i[\hat H, \hat\rho] + \kappa\left(\hat L\hat\rho\hat L - \tfrac{1}{2}\{\hat L\hat L, \hat\rho\}\right)$$

How is decoherence different from a loss CHANNEL?

$$i\frac{d}{dt}\overbrace{E[|\psi\rangle]}^{\bar\psi} = \left(H - i\tfrac{\kappa}{2}L^2\right)\bar\psi$$

$$\langle\bar\psi(t)\,\psi(0)\rangle \sim e^{-\frac{\kappa}{2}L^2 t}$$

DAMPING of AVERAGE WAVEFUNCTION, DECORRELATION with INITIAL STATE but no prob. loss microscopically.

② For general NON-HERMITEAN Lindblad

QUANTUM STATE DIFFUSION: NON-LINEAR STOCH. SCHRÖDINGER

$$d|\psi\rangle = -iH|\psi\rangle dt + \sum_i \left(\langle L_i^+\rangle_\psi L_i - \tfrac{1}{2} L_i^+ L_i - \tfrac{1}{2}\langle L_i^+\rangle_\psi \langle L_i\rangle_\psi \right) |\psi\rangle dt$$

$$+ \sum_i \left(L_i - \langle L_i\rangle_\psi \right) |\psi\rangle d\xi_i \leftarrow \text{complex noise}$$

Here complex noise required due to fluct. in x & p

$$E[d\xi_i] = 0 \quad E[d\xi_i d\xi_j] = 0 \quad E[d\xi_i d\xi_j^*] = \delta_{ij}\, dt$$

EXACT MATCH to LINDBLAD EQUATION.

CONNECTION TO QCD: HEAVY QUARKONIUM

At high temperature due to ASYMPTOTIC FREEDOM we attempt to approximate the ENVIRONMENT as GAUSSIAN.

Range of validity: $v \sim \sqrt{\frac{T}{M}} \ll 1$ (NON-REL quarks) $\quad g \ll 1$ (perturbative medium) $\quad \tau_E \ll \tau_S, \tau_R$ (Markovian coarse graining)

Ⓐ NRQCD, the v/c expansion of the QCD Lagrangian (Foldy-Tani-Wouthuysen)

$$\psi^+\left(i\partial_t - gA_0^a t^a + \frac{D^2}{2M}\right)\psi \quad \text{coupling to medium via: } g\,\rho^a(t) A_0^a(t) \quad \rho^a = \psi^+ t^a \psi$$

LINEAR

Integrating out the medium gives

$$S_{IF} = -\frac{g^2}{2}\int d^4x \int d^4y\, (\rho_1^a, \rho_2^a) \begin{pmatrix} G^F_{ab,00} & -G^<_{ab,00} \\ -G^>_{ab,00} & G^{\tilde F}_{ab,00} \end{pmatrix} \begin{pmatrix} \rho_1^b \\ \rho_2^b \end{pmatrix} + O(g^3) \leftarrow \text{APPROX. GAUSSIAN}$$

Need to approximate into TIME LOCAL form

$t - t_0 \leftarrow$ Extend to ∞ since G decays $\tau_E \sim 1/gT$

$$\int dx^0 \int dy^0\, \rho(x^0) G(x^0 - y^0)\rho(y^0) = \int_{t_0} dt \int_0^{t-t_0} ds \left[\rho(t) G(s) \rho(t-s) + \rho(t-s) G(-s) \rho(t) \right]$$

Again ONLY G^R_{00} and Δ_{00} relevant.

TRICK: $\rho(t-s) = e^{-s\partial_t}\rho(t)$ FOURIER TRANSFORM

$$\int_0^\infty ds\, G^R(s)\,\rho(t)\,\rho(t-s) = \rho(t)\,\tilde{G}^R(\omega - i\partial_t)\,\rho(t)$$

Taylor expand: $\approx \sum_{n=0}^{2} \frac{1}{n!}\rho(t)\,(i\partial_t)^n\rho(t)\;\partial_\omega^n\,\tilde{G}^R(\omega)\big|_{\omega=0}$

$n=0$ $\quad \tilde{G}^R(0) \rightarrow \rho(t)\rho(t)\,\tilde{G}^R(0,\vec{r})$ $\quad$ REAL since $\mathrm{Im}\,\tilde{G}^R(0) = \sigma(0) = 0$ gluon spectral function ANTISYMM.

Branch DIAGONAL: POTENTIAL

$$V(\vec{r}) = -g^2\,\mathrm{Re}\,\tilde{G}^R_{00}(0,\vec{r}) = -g^2\, P\!\int \frac{d\omega}{2\pi}\,\frac{\sigma_{00}(\omega)}{\omega}$$

In perturbation theory: T=0 Coulomb, HTL Yukawa (screened)

$n=0$ $\quad \tilde{\Delta}(\omega) = \coth\left(\frac{\omega}{2T}\right)\sigma(\omega)$ $\quad$ small ω $\quad \sigma(\omega) \sim \partial_\omega\sigma(\omega)\big|_{\omega=0}\,\omega$

$\approx 2T\,\partial_\omega\sigma(0)$

NOISE CORRELATOR $\quad D(\vec{r}) = \frac{g^2}{2}\,\tilde{\Delta}_{00}(0,\vec{r}) = -g^2 T\,\frac{\partial\sigma_{00}}{\partial\omega}(0,\vec{r})$

From HTL understood to be related to LANDAU DAMPING (spacelike gluons absorbed & emitted)

$n=1$ $\quad$ provides DRAG FORCE $\quad \rho\, i\partial_t\rho\;\partial_\omega\tilde{G}^R(0)$ $\quad$ Real part turns out to be higher order

via $\partial_\omega\,\mathrm{Im}\,\tilde{G}^R(0) = \frac{1}{2}\partial_\omega\sigma(0)$ $\quad$ SAME CORRELATOR!

EINSTEIN RELATION $\quad$ FLUCTUATION & DISSIPATION

$n=2$ $\quad$ A contribution still related to $D(\vec{r})$ via $A(\vec{r}) = \frac{D(\vec{r})}{6T^2}$

Takes the role of LINDBLAD COMPLETION

In HTL: $\quad V(\vec{r}) = \mathrm{Re}[V_{Q\bar{Q}}] \quad C_F\,(D(\vec{0}) - D(\vec{r})) = \mathrm{Im}[V_{Q\bar{Q}}]$

Master equation for a single quark (BROWNIAN FREE)

$$\partial_t \rho(\vec{x},\vec{y},t) = i\,\frac{\nabla_x^2 - \nabla_y^2}{2M}\,\rho(\vec{x},\vec{y},t) + F_1(\vec{x}-\vec{y})\,t^a\rho\, t^a - C_F F_1(0)\,\rho$$

$$+ \vec{F}_2(\vec{x}-\vec{y})\,(\vec{\nabla}_x - \vec{\nabla}_y)\,t^a\rho\, t^a$$

$$+ F_3^{ij}(\vec{x}-\vec{y})\,\nabla_x^i\nabla_y^j\,t^a\rho\, t^a + C_F F_3^{ii}(0)\,\frac{\nabla_x^2+\nabla_y^2}{6}\,\rho$$

$$F_1 = -D + \frac{\nabla^2 D}{4MT} + \frac{(\nabla^2)^2 A}{8M^2}$$

$$\vec{F}_2 = -\vec{\nabla}\left(\frac{D}{4MT} + \frac{\nabla^2 A}{4M^2}\right) \qquad F_3^{ij} = \nabla^i\nabla^j\,\frac{A}{2M^2}$$