Pole structure of $T_{cs0}(2870)$ in a coupled channel quark model with three-body forces

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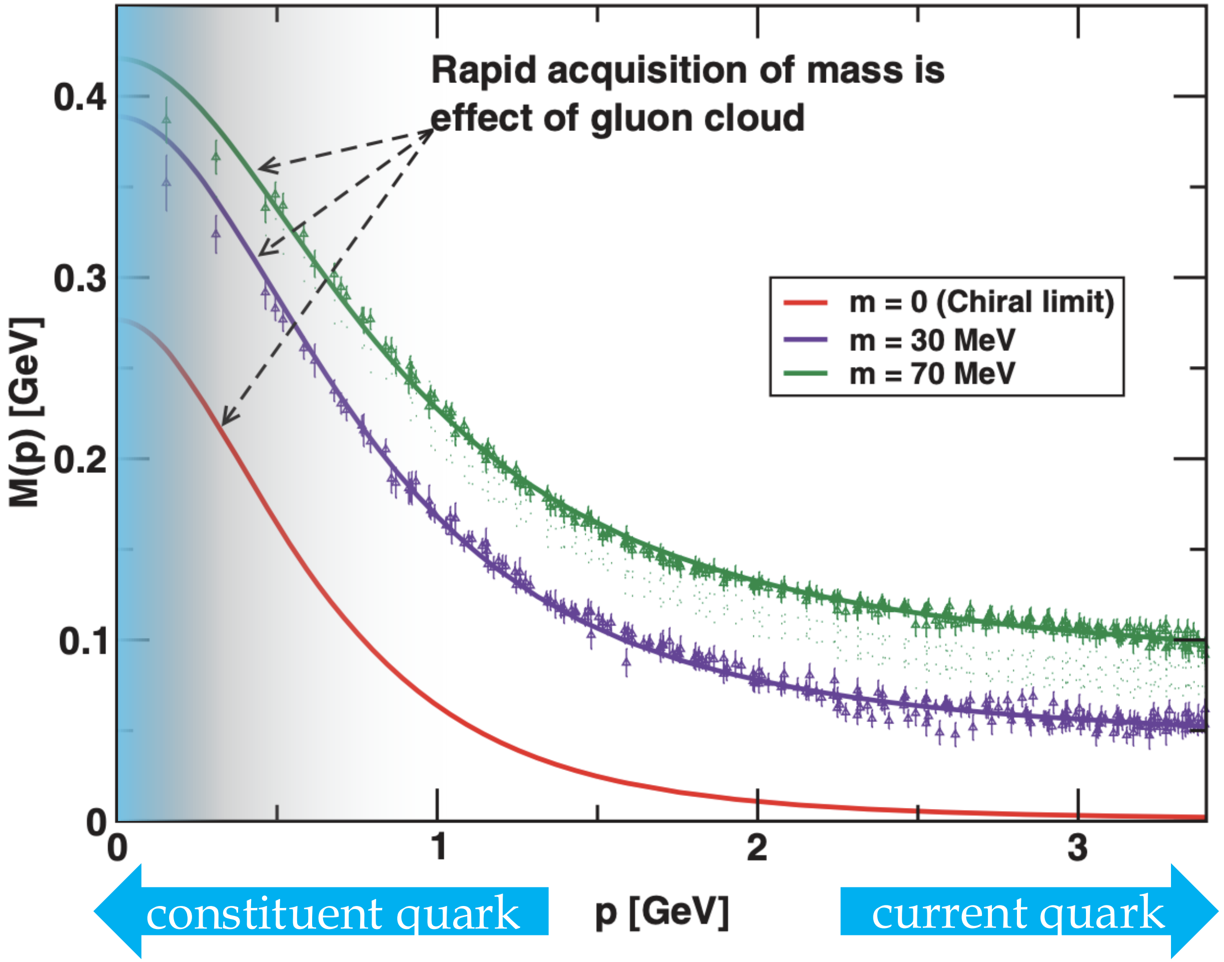
Quark model

Constituent (dressed) quark



S.-J. Brodsky et al, Phys. Rev. C **85**, 065202 (2012)

Large *p*

pQCD works ($\alpha_s < 1$)

small gluon dressing

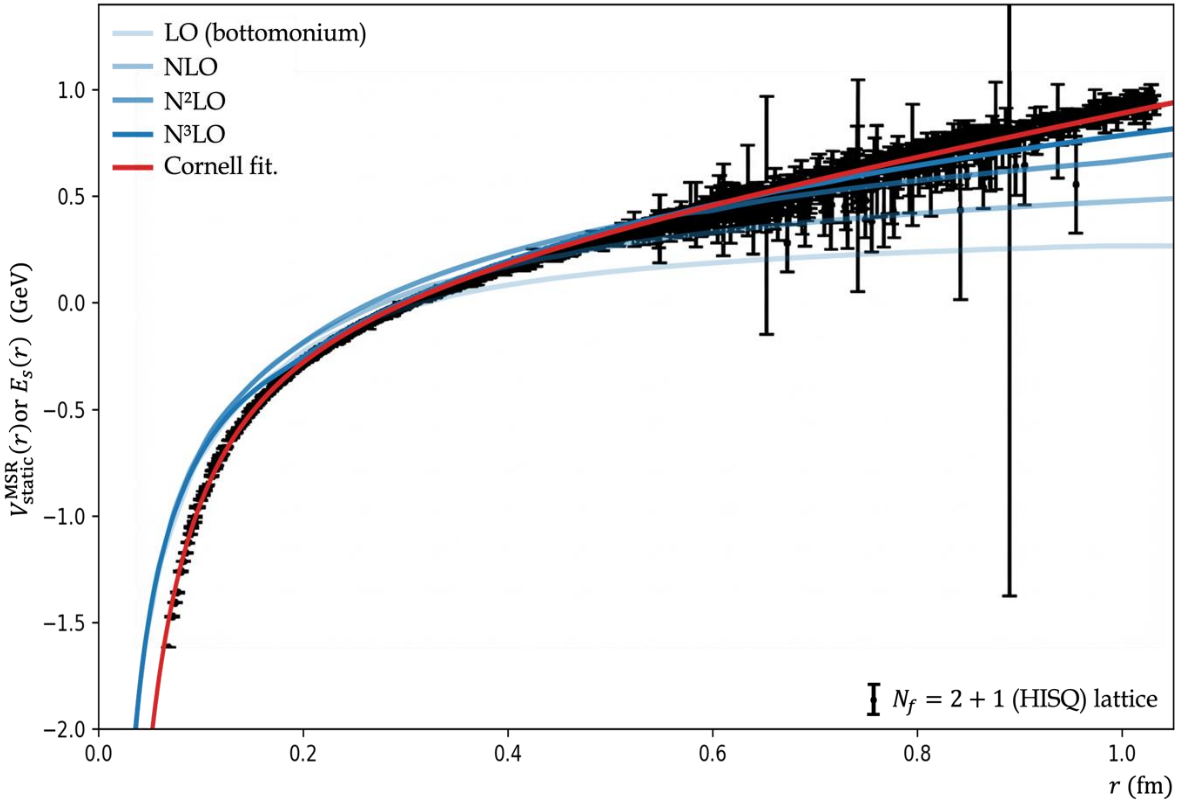
Small *p*

non-perturbative ($\alpha_s \gg 1$) → solve on lattice?

large gluon dressing → constituent quark!

Quark model

Quark-quark interaction (central piece; color type)



$$V^C(r) = -\frac{3}{4}F_i^c F_j^c\left(-\frac{\kappa}{r} + \sigma r - D\right), \qquad (\kappa,\ \sigma,\ D > 0)$$

Coulomb: one-gluon exchange (OGE) of pQCD (QED-like)

Confinement: long distance flux-tube from non-perturbative $SU(3)_C$ gauge dynamics

Constant: vacuum energy / self-energy / non-perturbative background effects

Quark model

Quark-quark interaction (hyperfine; color-spin type)

$$V^C(r) = -\frac{3}{4}F_i^c F_j^c\left(-\frac{\kappa}{r} + \sigma r - D\right), \qquad (\kappa,\ \sigma,\ D > 0)$$

$$\Big\downarrow \text{ Breit Fermi reduction}$$

$$V^{CS}_{\text{cont}}(\mathbf{r}_{ij}) = -\frac{8\pi\,\alpha_s}{3\,m_i m_j}\,F_i^c F_j^c\,\mathbf{S}_i\cdot\mathbf{S}_j\,\delta^{(3)}(\mathbf{r}_{ij}),$$

Color-magnetic piece of pQCD
($\leftrightarrow \boldsymbol{\mu}_1 \cdot \mathbf{B}_2$ in QED)

$$V^T(\mathbf{r}_{ij}) = -\frac{\alpha_s}{m_i m_j}\,F_i^c F_j^c\,\frac{1}{r_{ij}^3}\,[3(\mathbf{S}_i\cdot\hat{\mathbf{r}}_{ij})(\mathbf{S}_j\cdot\hat{\mathbf{r}}_{ij}) - \mathbf{S}_i\cdot\mathbf{S}_j]$$

Tensor force
(nearly negligible in ground state mesons/baryons)

$$\Big\downarrow \text{ unresolved short-distance QCD dressing effects}$$

$$V^{CS}_{ij} = -\frac{3}{4}F_i^c F_j^c\,\frac{\mu_{ij}^2}{m_i m_j c^4}\,\frac{\hbar c}{r_{ij}}\,e^{-(\mu_{ij} r_{ij}/\hbar c)^2}\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j \qquad \mu_{ij} = \left(\alpha + \frac{\beta\, m_i\, m_j}{m_i + m_j}\right)$$

(fitted on lattice) Y. Koma, M. Koma, Nucl. Phys. B **769**, 79 (2007)

Quark model Hamiltonian under comparison

$$H_X = \sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} - T_{\mathrm{CM}} + \sum_{i<j} V_{ij}^C + V_X^{\mathrm{SD}}$$

$X \in \{2\mathrm{BD}, \mathrm{III}, \mathrm{TGE}\}$

$$V_{ij}^C = -\frac{3}{4} F_i^c F_j^c \left(-\frac{\kappa}{r_{ij}} + \sigma r_{ij} - D \right) \qquad V_{ij}^{CS} = -\frac{3}{4} F_i^c F_j^c \frac{\mu_{ij}^2}{m_i m_j} \frac{\hbar c}{r_{ij}} e^{-(\mu_{ij} r_{ij}/\hbar c)^2} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$

Two-body model (2BD)

$$V_{\mathrm{2BD}}^{\mathrm{SD}} = \sum_{i<j} V_{ij}^{CS}$$

Instanton model (III)

$$V_{\mathrm{III}}^{\mathrm{SD}} = \sum_{i<j} \left[(1 - p_{ij}) V_{ij}^{CS} + p_{ij} V_{\mathrm{III},ij}^{(2)} \right] + \sum_{i<j<k} p_{ijk} V_{\mathrm{III},ijk}^{(3)}$$

Two-gluon exchange model (TGE)

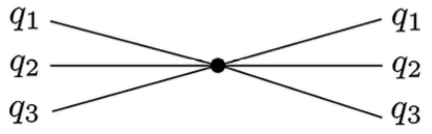
$$V_{\mathrm{TGE}}^{\mathrm{SD}} = \sum_{i<j} V_{ij}^{CS} + V_{\mathrm{TGE}}^{(3)}$$

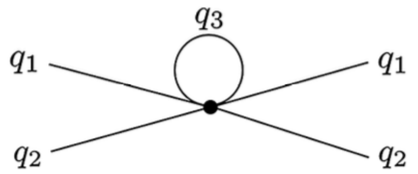
Three-body forces in the quark model

Three-body spin-dependent forces: $\mathrm{U}(1)_A$ anomaly; Instanton-induced interaction (III)

$$\mathcal{L}_{\mathrm{KMT}} = G_D\Big[\det_{f,f'=u,d,s}\big(\bar{q}_f(1+\gamma_5)q_{f'}\big) + \det_{f,f'=u,d,s}\big(\bar{q}_f(1-\gamma_5)q_{f'}\big)\Big]$$

G. 't Hooft, Phys. Rev. D **14**, 3432 (1976), Phys. Rev. D **18**, 2199 (1978)
D. Diakonov, Prog. Part. Nucl. Phys. **51**, 173 (2003)





Non-relativistic reduction
S. Takeuchi, M. Oka, Phys. Rev. Lett. **66**, 1271 (1991)

Instanton two-body force:

$$V_{\mathrm{III}}^{ij} = \frac{U_0^{(2)}}{m_i m_j}\Gamma_{ij}\,\delta^{(3)}(\mathbf{r}), \qquad \Gamma_{ij} \equiv 1 + \frac{3}{32}s(i)s(j)\lambda_i\cdot\lambda_j + \frac{9}{32}\lambda_i\cdot\lambda_j\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j \qquad (i\neq j)$$

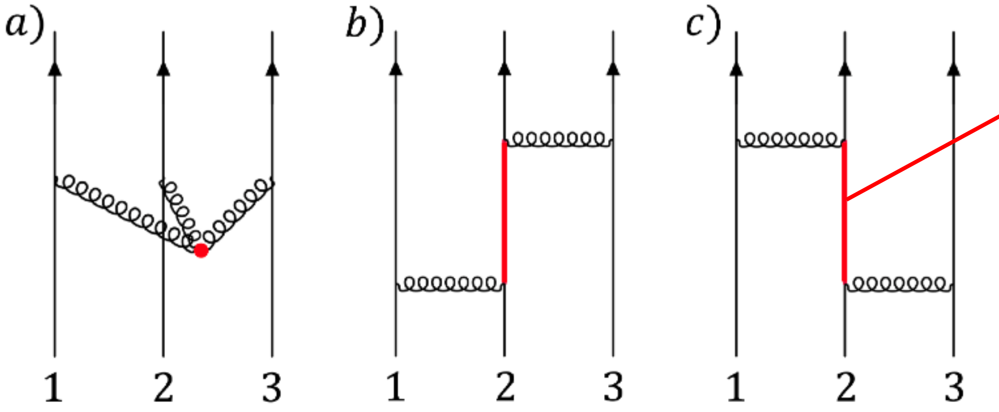
Instanton three-body force:

$$V_{\mathrm{III},ijk}^{(3)} = V_0\,\mathcal{O}_{ijk}^{(3)}\,\frac{1}{3}\left(\delta^{(3)}(\mathbf{r}_{ij})\,\delta^{(3)}(\mathbf{r}_{jk}) + \delta^{(3)}(\mathbf{r}_{jk})\,\delta^{(3)}(\mathbf{r}_{kl}) + \delta^{(3)}(\mathbf{r}_{kl})\,\delta^{(3)}(\mathbf{r}_{ij})\right), \qquad (i,j,k)\sim(u,d,s)$$

$$\begin{aligned}\mathcal{O}_{ijk}^{(3)} =\; & 1 + \frac{3}{32}\sum_{\mathrm{cyc}(i,j,k)}\lambda_i^c\lambda_j^c - \frac{9}{320}s(i)s(j)s(k)\,d_{abc}\,\lambda_i^a\lambda_j^b\lambda_k^c + \frac{9}{32}\sum_{\mathrm{cyc}(i,j,k)}\lambda_i^c\lambda_j^c\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j \\ & + \frac{27}{320}\,d_{abc}\,\lambda_i^a\lambda_j^b\lambda_k^c\Big(s(k)\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + s(i)\,\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k + s(j)\,\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_i\Big) - \frac{9}{64}\,f_{abc}\,\lambda_i^a\lambda_j^b\lambda_k^c\,(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j)\cdot\boldsymbol{\sigma}_k\end{aligned}$$

Three-body forces in the quark model

Three-body spin-dependent forces: Two-gluon exchange (TGE)



For example, a possible configuration is $j_{qg}^{P} = 1/2^{+}, 3/2^{+}$

$$|[(q_i q_j)_{\bar{\mathbf{3}}_C}(q_k g_{\mathbf{8}_C})_{\mathbf{3}_C}]_{\mathbf{1}_C}\rangle \longleftrightarrow |[(q_i q_j q_k)_{\mathbf{8}_C} g_{\mathbf{8}_C}]_{\mathbf{1}_C}\rangle$$

Same analogy as Fujita-Miyazawa mechanism (3N force) in nuclear physics

$$V_{3N}^{2\pi} \sim \sum_{\text{cyc}} \frac{(\boldsymbol{\sigma}_i \cdot \mathbf{q}_i)(\boldsymbol{\sigma}_k \cdot \mathbf{q}_k)}{(q_i^2 + m_\pi^2)(q_k^2 + m_\pi^2)} (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k) \frac{f_{\pi N\Delta}^2 f_{\pi NN}^2}{E_\Delta - E_N}$$

$$L_{123}^{C-C} = \frac{4}{3} \sum_{\text{cyc}(l,i,j)} \frac{\lambda_i^c \lambda_j^c}{M_l} + 2\, d^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c \sum_{l=1}^{3} \frac{\text{sgn}(q_l)}{M_l}, \tag{4.6}$$

$$L_{123}^{S-S} = \sum_{\text{cyc}(l,i,j)} \frac{4\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\, \lambda_i^c \lambda_j^c}{3\, m_l M_l (m_1 m_2 m_3)} + \frac{2\, d^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c}{m_1 m_2 m_3} \sum_{\text{cyc}(l,i,j)} \frac{\text{sgn}(q_l)}{m_l M_l} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - \frac{2\, f^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c\, \boldsymbol{\sigma}_1 \cdot (\boldsymbol{\sigma}_2 \times \boldsymbol{\sigma}_3)}{m_1 m_2 m_3} \sum_{l=1}^{3} \frac{1}{m_l M_l}$$

$$L_{123}^{C-S} = \frac{4}{3} \sum_{\text{cyc}(l,i,j)} \frac{\lambda_i^c \lambda_j^c}{M_l} \left(\frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_i}{m_l m_i} + \frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_j}{m_l m_j} \right) + 2\, d^{abc}\, \lambda_1^a \lambda_2^b \lambda_3^c \sum_{\text{cyc}(l,i,j)} \frac{\text{sgn}(q_l)}{m_l M_l} \left(\frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_i}{m_i} + \frac{\boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_j}{m_j} \right).$$

$$\boxed{L_{\text{Y}}^{3Q} = \sum_{i<j<k} \left(A L_{ijk}^{C-C} + B L_{ijk}^{S-S} + C L_{ijk}^{C-S} \right) \exp\left[-\frac{c}{\hbar} (m_{ij} r_{ij} + m_{jk} r_{jk} + m_{ki} r_{ki}) \right]}$$

J. Baek et. al., *“Three-body forces in the quark model”*, Prog. Part. Nucl. Phys., **under revision,** hep-ph 2608.18949

Quark model Hamiltonian under comparison

meson+baryon fit (u, d, s, c sectors)

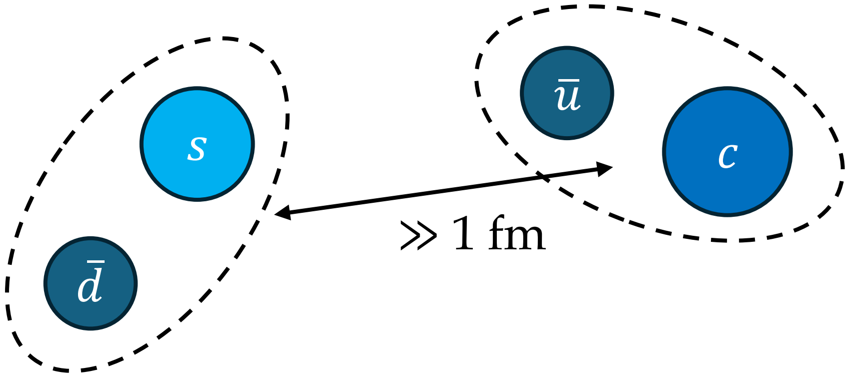
State	M_{exp}	2BD (RMSE = 13.33 MeV)			III (RMSE = 11.97 MeV)			TGE (RMSE = 2.54 MeV)		
		M_{fit}	ΔM	δM [%]	M_{fit}	ΔM	δM [%]	M_{fit}	ΔM	δM [%]
$\eta_c\,(0^{-+})$	2983.600	2982.400	-1.200	-0.040	2985.453	$+1.853$	$+0.062$	2983.800	$+0.200$	$+0.007$
$J/\psi\,(1^{--})$	3096.900	3097.594	$+0.694$	$+0.022$	3100.939	$+4.039$	$+0.130$	3096.923	$+0.023$	$+0.001$
$D\,(0^-)$	1864.800	1866.695	$+1.895$	$+0.102$	1862.838	-1.962	-0.105	1865.572	$+0.772$	$+0.041$
$D^*\,(1^-)$	2010.300	2008.277	-2.023	-0.101	2021.199	$+10.899$	$+0.542$	2008.584	-1.716	-0.085
$\pi\,(0^{-+})$	139.570	127.706	-11.864	-8.500	137.169	-2.401	-1.720	138.067	-1.503	-1.077
$\rho\,(1^{--})$	775.110	736.238	-38.872	-5.015	737.885	-37.225	-4.803	772.203	-2.907	-0.375
$K\,(0^-)$	493.680	495.649	$+1.969$	$+0.399$	499.406	$+5.726$	$+1.160$	496.975	$+3.295$	$+0.667$
$K^*\,(1^-)$	891.660	879.773	-11.887	-1.333	879.585	-12.075	-1.354	895.045	$+3.385$	$+0.380$
$p\,(1/2^+)$	938.270	960.693	$+22.423$	$+2.390$	950.705	$+12.435$	$+1.325$	940.736	$+2.466$	$+0.263$
$\Delta\,(3/2^+)$	1232.000	1250.249	$+18.249$	$+1.481$	1248.941	$+16.941$	$+1.375$	1232.207	$+0.207$	$+0.017$
$\Sigma\,(1/2^+)$	1197.450	1198.417	$+0.967$	$+0.081$	1194.789	-2.661	-0.222	1192.263	-5.187	-0.433
$\Lambda\,(1/2^+)$	1115.680	1123.369	$+7.689$	$+0.689$	1114.070	-1.610	-0.144	1115.374	-0.306	-0.027
$\Sigma_c\,(1/2^+)$	2453.970	2441.428	-12.542	-0.511	2446.484	-7.486	-0.305	2450.682	-3.288	-0.134
$\Lambda_c\,(1/2^+)$	2286.460	2290.620	$+4.160$	$+0.182$	2279.999	-6.461	-0.283	2288.864	$+2.404$	$+0.105$
$\Xi_{cc}\,(1/2^+)$	3621.460	3630.469	$+9.009$	$+0.249$	3615.784	-5.676	-0.157	3619.489	-1.971	-0.054
$\Sigma^*\,(3/2^+)$	1383.700	1382.418	-1.282	-0.093	1390.861	$+7.161$	$+0.518$	1385.074	$+1.374$	$+0.099$
$\Sigma_c^*\,(3/2^+)$	2518.480	2510.667	-7.813	-0.310	2527.959	$+9.479$	$+0.376$	2522.749	$+4.269$	$+0.170$

$T_{cs0}(2870)$ resonance in the coupled channel quark model

Quark models well describe <u>compact</u> (~1 fm) **mesons and baryons.**

Then what if some exotic state is <u>not compact</u>?

e.g., $\bar{u}\bar{d}cs$ tetraquark states (T_{cs}): a resonance pole!



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$$V_{ij}^{CS} = -\frac{3}{4} F_i^c F_j^c \frac{\mu_{ij}^2}{m_i m_j} \frac{\hbar c}{r_{ij}} e^{-(\mu_{ij} r \ldots}$$

Two-body model (2BD) $V \ldots$

Instanton model (III) $V_{\mathrm{III}}^{\mathrm{SD}} = \sum_{i<j} \left[(1 - p_{ij}) V_{ij} \ldots \right] + \sum_{\ldots i<k} p_{ijk} V_{\mathrm{III},ijk}^{(3)}$

Two-gluon exchange model (TGE) $V_{\mathrm{TGE}}^{\mathrm{SD}} = \sum_{i<j} V_{ij}^{CS} + V_{\mathrm{TGE}}^{(3)}$

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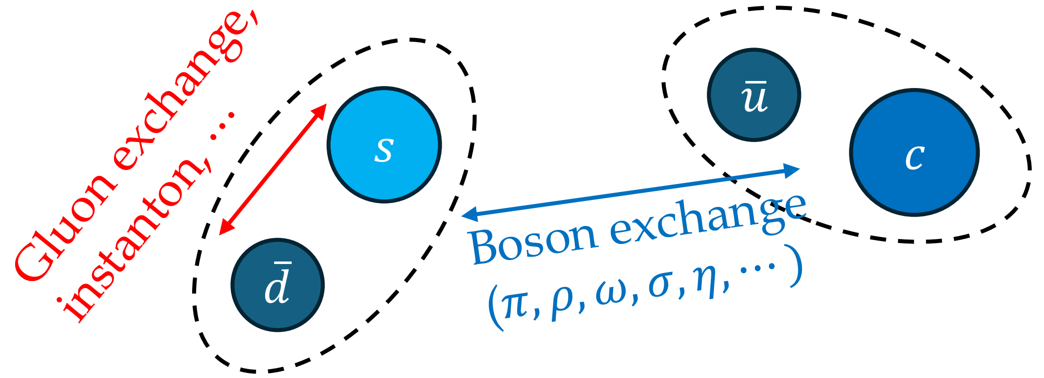
Compact region (short-range): quark-level interaction ⟶ "gauge dynamics (quark model)"
Continuum region (long-range): hadron-level interaction ⟶ "chiral dynamics"

e.g., $\bar{u}\bar{d}cs$ tetraquark states (T_{cs}): a resonance pole!



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We should consider both compact and continuum space.

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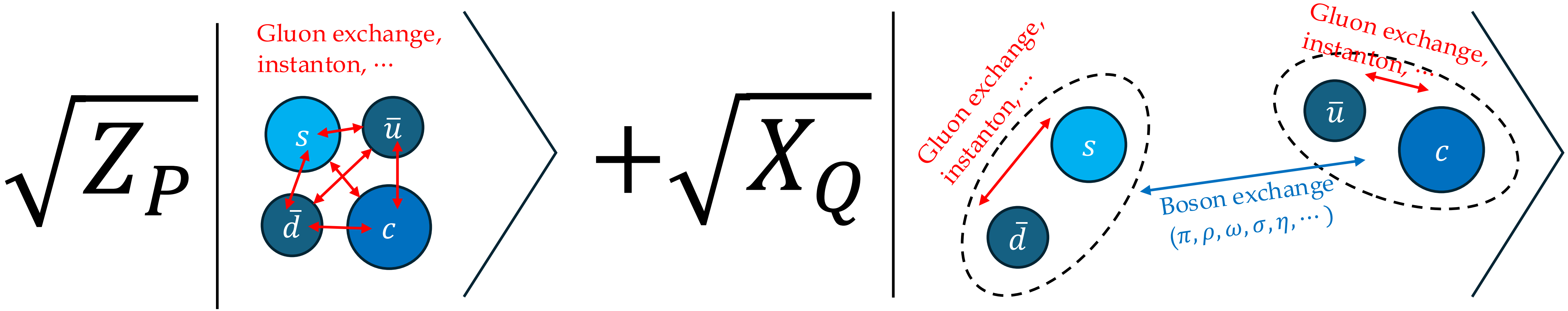


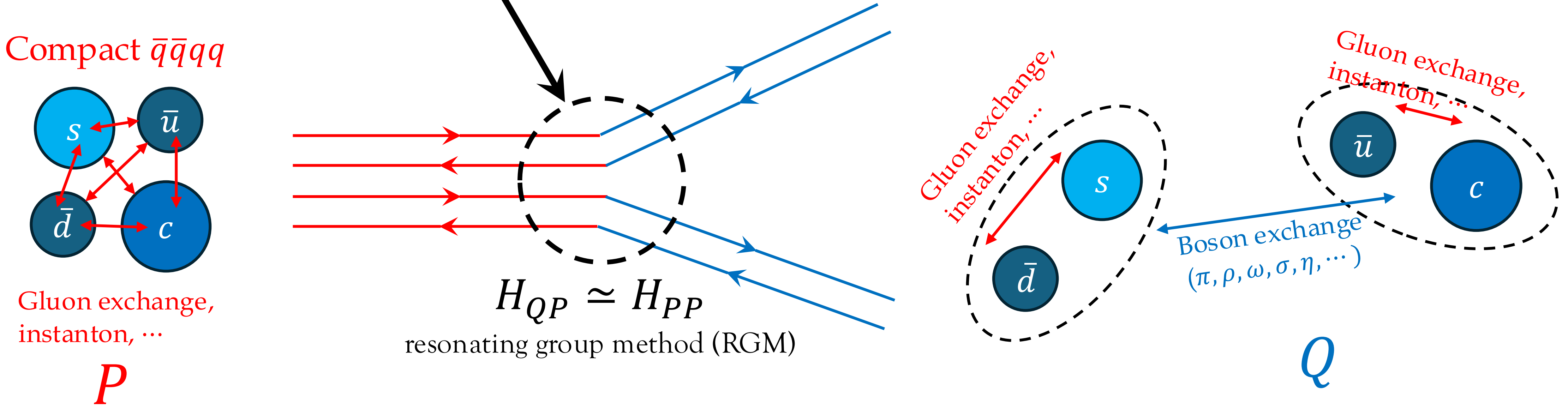
$\in$ compact space (P) $\in$ continuum space (Q)

$T_{cs0}(2870)$ resonance in the coupled channel quark model

We should consider both compact and continuum space.

$$\begin{array}{cc} P & Q \end{array}$$

$$H = \begin{pmatrix} H_{PP}^{\text{bare}} & H_{PQ} \\ H_{QP} & H_{QQ}^{(0)} + V_{QQ}^{\text{long-range}} \end{pmatrix}$$



$T_{cs0}(2870)$ resonance in the coupled channel quark model

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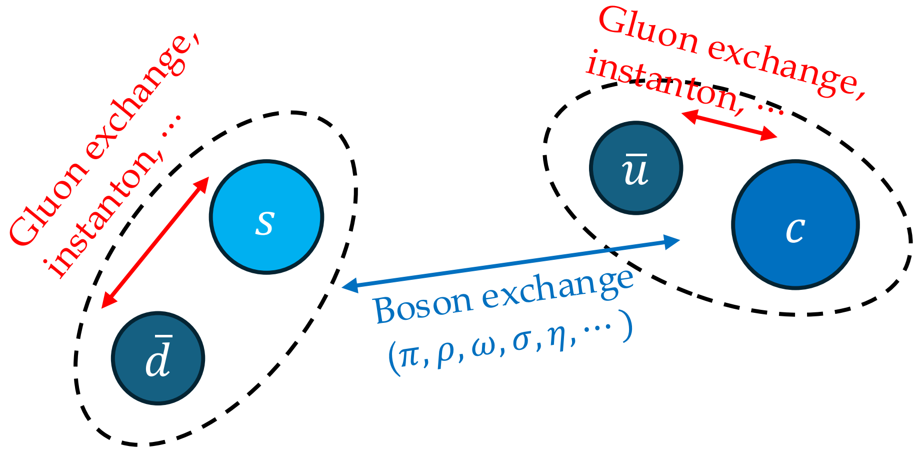
$$\begin{array}{ccc} & P & Q \\ H = & \left(\begin{array}{c} H_{PP}^{\text{bare}} \\ H_{QP} \end{array} \right. & \left. \begin{array}{c} H_{PQ} \\ H_{QQ}^{(0)} + V_{QQ}^{\text{long-range}} \end{array} \right) \end{array}$$



Continuum one-boson $(\pi, \rho, \omega, \sigma, \eta)$ **exchange potential**

1. $\rho, \omega, \sigma, \eta$ boson-exchanges are all intermediate or short-ranged.
2. These potentials require and depend largely on short-range cutoff. Also these potentials require unknown parameters.

So for the continuum space, we use pion-exchange potential only.

$T_{cs0}(2870)$ resonance in the coupled channel quark model

***Q*-space Hamiltonian** with $Q = \{DK, D^*K^*\}$

$$H_{QQ} = \begin{pmatrix} H^{(0)}_{DK} & 0 \\ 0 & H^{(0)}_{D^*K^*} \end{pmatrix} + \begin{pmatrix} 0 & V^{\pi}_{DK\to D^*K^*} \\ V^{\pi}_{D^*K^*\to DK} & V^{\pi}_{D^*K^*\to D^*K^*} \end{pmatrix}$$

Relevant effective Lagrangian from heavy quark sym, chiral sym, SU(2) flavor symmetry:

$$\mathcal{L}_{\tilde{D}_{(*)}\tilde{D}^{(*)}P} = +\frac{2g}{f_\pi}\left(\tilde{D}_b\tilde{D}^{*\dagger}_{a\lambda} + \tilde{D}^{*}_{b\lambda}\tilde{D}^{\dagger}_{a}\right)\partial^\lambda P_{ba} + i\frac{2g}{f_\pi}v^\alpha\epsilon_{\alpha\mu\nu\lambda}\tilde{D}^{*\mu}_b\tilde{D}^{*\dagger\lambda}_a\partial^\nu P_{ba}.$$

M. B. Wise, Phys. Rev. D **45**, R2188 (1992)

Effective potential for $DK \to D^*K^*$ is

$$V^{\pi}_{DK\to D^*K^*}(\mathbf{q}) = \frac{g_{DD^*\pi}g_{KK^*\pi}}{\mathbf{q}^2+m_\pi^2}(\boldsymbol{\tau}_D\cdot\boldsymbol{\tau}_K)(\boldsymbol{\epsilon}^*_{D^*}\cdot\mathbf{q})(\boldsymbol{\epsilon}^*_{K^*}\cdot\mathbf{q})F^2(q) \qquad V^{\pi}_{D^*K^*\to D^*K^*}(\mathbf{q}) = \frac{g_{D^*D^*\pi}g_{K^*K^*\pi}}{\mathbf{q}^2+m_\pi^2}(\boldsymbol{\tau}_D\cdot\boldsymbol{\tau}_K)(\mathbf{S}_{D^*}\cdot\mathbf{q})(\mathbf{S}_{K^*}\cdot\mathbf{q})F^2(q)$$

form factor

Couplings

$$g_{DD^*\pi} = 3.084\times 10^{-3}\ \mathrm{MeV}^{-1}, \qquad g_{KK^*\pi} = 6.061\times 10^{-3}\ \mathrm{MeV}^{-1}.$$

$g_{DD^*\pi} \simeq g_{D^*D^*\pi}$ and $g_{KK^*\pi} \simeq g_{K^*K^*\pi}$.

B Wang and S. L. Zhu , Eur. Phys. J. C **82**, 419 (2022)

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$$F(q) = \frac{\Lambda^2 - m_\pi^2}{\Lambda^2+q^2}$$ monopole form factor with $\Lambda = 1$ GeV

***S*-wave, spin singlet projection**

$$q_iq_j = \underbrace{\frac{\mathbf{q}^2}{3}\delta_{ij}}_{\text{scalar part }(L=0)} + \underbrace{\left(q_iq_j - \frac{\mathbf{q}^2}{3}\delta_{ij}\right)}_{\text{traceless symmetric }(L=2)},$$

$$\epsilon^*_{D^*,i}\epsilon^*_{K^*,j} = \underbrace{\frac{1}{3}\delta_{ij}\left(\boldsymbol{\epsilon}^*_{D^*}\cdot\boldsymbol{\epsilon}^*_{K^*}\right)}_{S=0} + \underbrace{\frac{1}{2}\left(\epsilon^*_{D^*,i}\epsilon^*_{K^*,j} - \epsilon^*_{D^*,j}\epsilon^*_{K^*,i}\right)}_{S=1} + \underbrace{\left[\frac{1}{2}\left(\epsilon^*_{D^*,i}\epsilon^*_{K^*,j} + \epsilon^*_{D^*,j}\epsilon^*_{K^*,i}\right) - \frac{1}{3}\delta_{ij}\left(\boldsymbol{\epsilon}^*_{D^*}\cdot\boldsymbol{\epsilon}^*_{K^*}\right)\right]}_{S=2},$$

Long-range contraction

1. Short-range components only shifts phase of the scattering wavefunction
2. Short-range physics are governed by gluon-exchange or instanton-induced interactions

$$\frac{\mathbf{q}^2}{\mathbf{q}^2+m_\pi^2} = 1 - \frac{m_\pi^2}{\mathbf{q}^2+m_\pi^2}$$

long-range force

contact interaction $\delta(\mathbf{R}_{DK})$: change the phase

$$\longrightarrow \int \frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2+m_\pi^2} = \frac{e^{-m_\pi r}}{4\pi r}$$ Yukawa-tail

$T_{cs0}(2870)$ resonance in the coupled channel quark model

The continuum(Q)-dressed effective Hamiltonian in the compact(P) space is

$$H_{\text{eff}}(E) = H_{PP}^{\text{bare}} + H_{PQ}\frac{1}{E - H_{QQ} + i0}H_{QP}$$

If we solve $H\Psi_{\mathcal{J}^P M} = E\mathcal{O}\Psi_{\mathcal{J}^P M}$ (diagonalize H) in $\Psi_{\mathcal{J}^P M} \in P_{\text{eff}}$ basis, we can get eigenenergies.

Pole equation

$$\det\left[E\mathbf{1} - M^{(0)} - \Sigma(E)\right] = 0$$

Compact bare seed eigenmass

$= \text{diag}\left(M_0^{(0)}, M_1^{(0)}, M_2^{(0)}, \cdots\right)$

Self-energy correction

$$\Sigma_{mn}^{(\eta)}(E) = \sum_{\alpha,\beta\in Q}\int_0^\infty \frac{p^2dp}{2\pi^2}\int_0^\infty \frac{p'^2dp'}{2\pi^2}\, g_{m\alpha}(p)\, G_{Q,\alpha\beta}^{(\eta)}(E;p,p')\, g_{n\beta}(p').$$

$T_{cs0}(2870)$ resonance in the coupled channel quark model

Pole equation

$$\det\left[E\mathbf{1} - M^{(0)} - \Sigma(E)\right] = 0$$

Compact bare seed eigenmass

$= \mathrm{diag}\left(M_0^{(0)}, M_1^{(0)}, M_2^{(0)}, \cdots\right)$

Self-energy correction

$$\Sigma_{mn}^{(\boldsymbol{\eta})}(E) = \sum_{\alpha,\beta\in Q} \int_0^\infty \frac{p^2 dp}{2\pi^2} \int_0^\infty \frac{p'^2 dp'}{2\pi^2}\, g_{m\alpha}(p)\, G_{Q,\alpha\beta}^{(\boldsymbol{\eta})}(E;p,p')\, g_{n\beta}(p').$$

Possible continuum bases (similar for D^*K^*)

$$|\Phi_{DK,p}^{(13)(24)}\rangle = |D(13)K(24);p\rangle = \phi_D^{(13)}(\mathbf{r}_{13})\,\phi_K^{(24)}(\mathbf{r}_{24})\,j_0(pR_{(13)(24)}) \otimes |\mathbf{1}_{13}\mathbf{1}_{24}\rangle_C \otimes |((13)_0(24)_0)_0\rangle_S \otimes |D(13)K(24)\rangle_F$$

$$|\Phi_{DK,p}^{(23)(14)}\rangle = |D(23)K(14);p\rangle = \phi_D^{(23)}(\mathbf{r}_{23})\,\phi_K^{(14)}(\mathbf{r}_{14})\,j_0(pR_{(23)(14)}) \otimes |\mathbf{1}_{23}\mathbf{1}_{14}\rangle_C \otimes |((23)_0(14)_0)_0\rangle_S \otimes |D(23)K(14)\rangle_F$$

Isospin projected bases $|\Phi_{\alpha,p}^{Q,I=0}\rangle = \frac{1}{\sqrt{2}}\left[|\Phi_{\alpha,p}^{(13)(24)}\rangle - |\Phi_{\alpha,p}^{(23)(14)}\rangle\right],\quad |\Phi_{\alpha,p}^{Q,I=1}\rangle = \frac{1}{\sqrt{2}}\left[|\Phi_{\alpha,p}^{(13)(24)}\rangle + |\Phi_{\alpha,p}^{(23)(14)}\rangle\right]$

Compact bases

$$|\Psi_P\rangle = \sum_{\{\nu_i,\mathcal{C}\}} \phi_{\nu_1}(\mathbf{x}_\mathcal{C})\,\phi_{\nu_2}(\mathbf{y}_\mathcal{C})\phi_{\nu_3}(\mathbf{z}_\mathcal{C}) \otimes \boxed{|I\rangle \otimes |\mathbf{3}_{12}\bar{\mathbf{3}}_{34}\rangle_C \otimes |((12)_0(34)_0)_0\rangle_S}$$

$$+ \sum_{\{\nu_i',\mathcal{C}\}} \phi_{\nu_1'}(\mathbf{x}_\mathcal{C})\,\phi_{\nu_2'}(\mathbf{y}_\mathcal{C})\phi_{\nu_3'}(\mathbf{z}_\mathcal{C}) \otimes \boxed{|I\rangle \otimes |\bar{\mathbf{6}}_{12}\mathbf{6}_{34}\rangle_C \otimes |((12)_1(34)_1)_0\rangle_S .}$$

Two limited bases cannot reproduce DK pseudostates.
It make eigenstates compact

$T_{cs}(\bar{u}\bar{d}cs; I = 0, J^P = 0^+)$ tetraquark resonance

Lowest 3 bare seed states in the 2BD model

seed	$M_n^{(0)}$	T	V^C	V^{CS}	w_{CSI1}	w_{CSI4}	r_{13-24}	$r_{\max}$
0	2586.25	1047.39	-1015.85	-502.37	0.363	0.637	0.546	0.883
1	2927.97	791.76	-797.24	-123.63	0.611	0.389	0.623	0.979
2	3087.92	918.15	-579.29	-308.02	0.300	0.700	0.921	1.200

➡ $M_{DK} < M_0^{(0)} < M_{D^*K^*}$: $(-,+)$
➡ $M_{D^*K^*} < M_1^{(0)}$: $(-,-)$
➡ $M_{D^*K^*} < M_2^{(0)}$: $(-,-)$

Riemann sheet jump term (for seed 1)

$$\Sigma_{mn}^{(--)}(E) = \Sigma_{mn}^{\rm grid}(E) + J_{mn}^{DK}(E) + J_{mn}^{D^*K^*}(E)$$

$$J_{mn}^{\alpha}(E) = -\,i\,\frac{\mu_\alpha k_\alpha(E)}{\pi(\hbar c)^3}\Gamma_{m\alpha}(k_\alpha(E))\,\Gamma_{n\alpha}(k_\alpha(E))\,, \quad k_\alpha(E) = \sqrt{2\mu_\alpha\,(E - M_\alpha)}.$$

$$\Gamma_{n\alpha}(E;k) = h_{n\alpha}(k) + \sum_\beta \int_0^\infty \frac{p^2\,dp}{2\pi^2(\hbar c)^3} V_{\alpha\beta}^{\pi,\rm long}(k,p)\frac{1}{E - E_\beta^{(0)}(p)}\Gamma_{n\beta}(E;p),$$

Integral equation: recursive!

$T_{cs}(\bar{u}\bar{d}cs; I = 0, J^P = 0^+)$ tetraquark resonance

$$\Gamma_{n\alpha}(E;k) = h_{n\alpha}(k) + \sum_{\beta} \int_0^{\infty} \frac{p^2\,dp}{2\pi^2(\hbar c)^3} V_{\alpha\beta}^{\pi,\text{long}}(k,p) \frac{1}{E - E_{\beta}^{(0)}(p)} \Gamma_{n\beta}(E;p),$$

Integral equation: recursive!

This expression is recursive. But on momentum grids $p \to p_i$ and $k \to k_i$ with weights $w_i = \Delta p_i\, p_i^2/(2\pi^2(\hbar c)^3)$, the integral equation becomes a finite-dimensional linear equation,

$$\Gamma_{n,\alpha i}(E) = h_{n,\alpha i} + \sum_{\beta,j} V_{\alpha i,\beta j}^{\pi,\text{long}} \frac{w_j}{E - E_{\beta j}^{(0)}} \Gamma_{n,\beta j}(E), \tag{70}$$

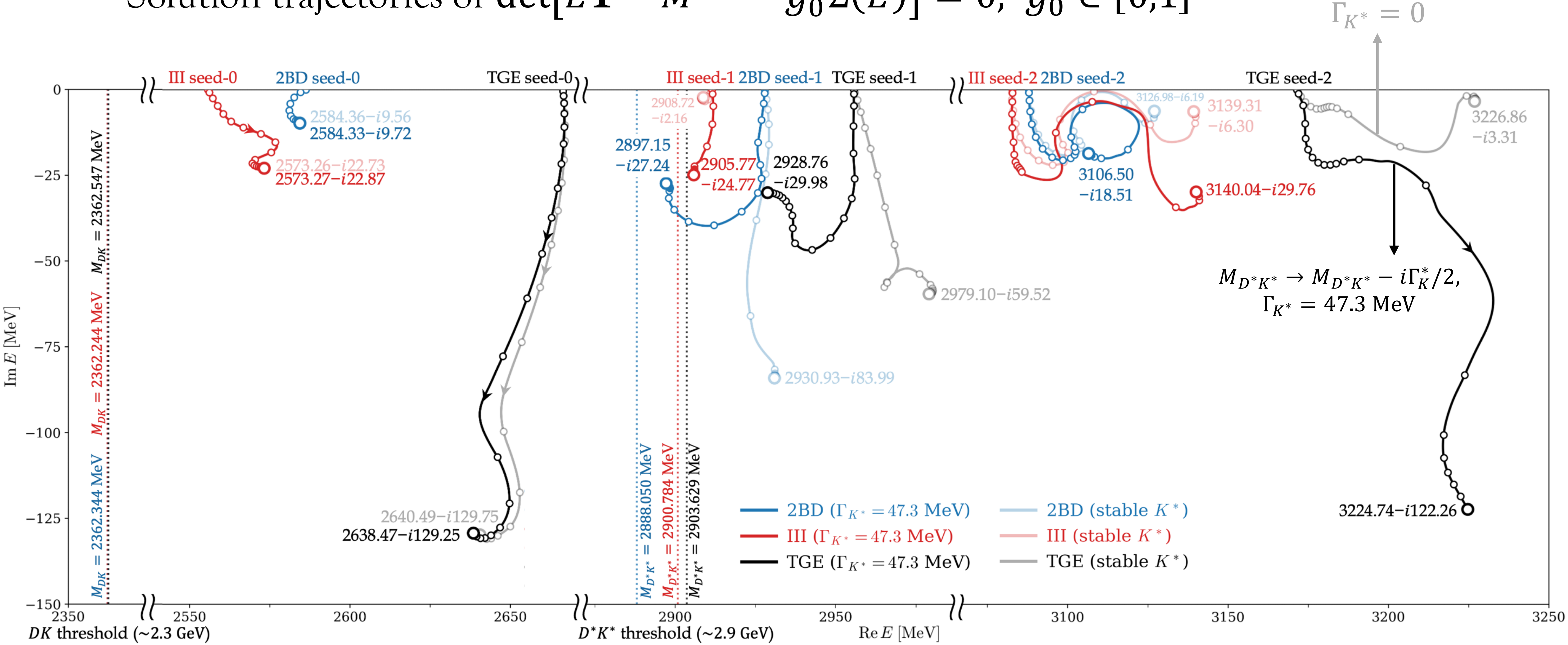
where $h_{n,\alpha i} \equiv h_{n\alpha}(p_i)$, $\Gamma_{n,\alpha i}(E) \equiv \Gamma_{n\alpha}(E;p_i)$, $E_{\beta j}^{(0)} = M_\beta + p_j^2/2\mu_\beta$, and $V_{\alpha i,\beta j}^{\pi,\text{long}} \equiv V_{\alpha\beta}^{\pi,\text{long}}(p_i,p_j)$. Equivalently,

$$\sum_{\beta,j} \left[\delta_{\alpha\beta}\delta_{ij} - V_{\alpha i,\beta j}^{\pi,\text{long}} \frac{w_j}{E - E_{\beta j}^{(0)}}\right] \Gamma_{n,\beta j}(E) = h_{n,\alpha i}. \quad \textbf{Linear equation now.} \tag{71}$$

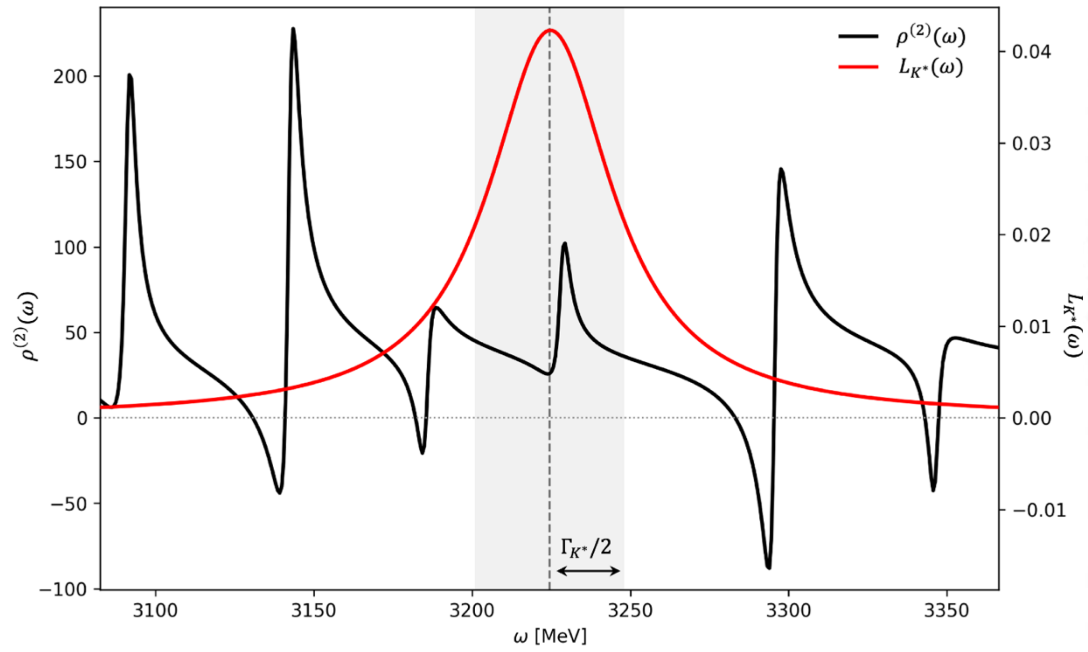
Thus the recursive expression is solved by a matrix inversion on the finite momentum grid.

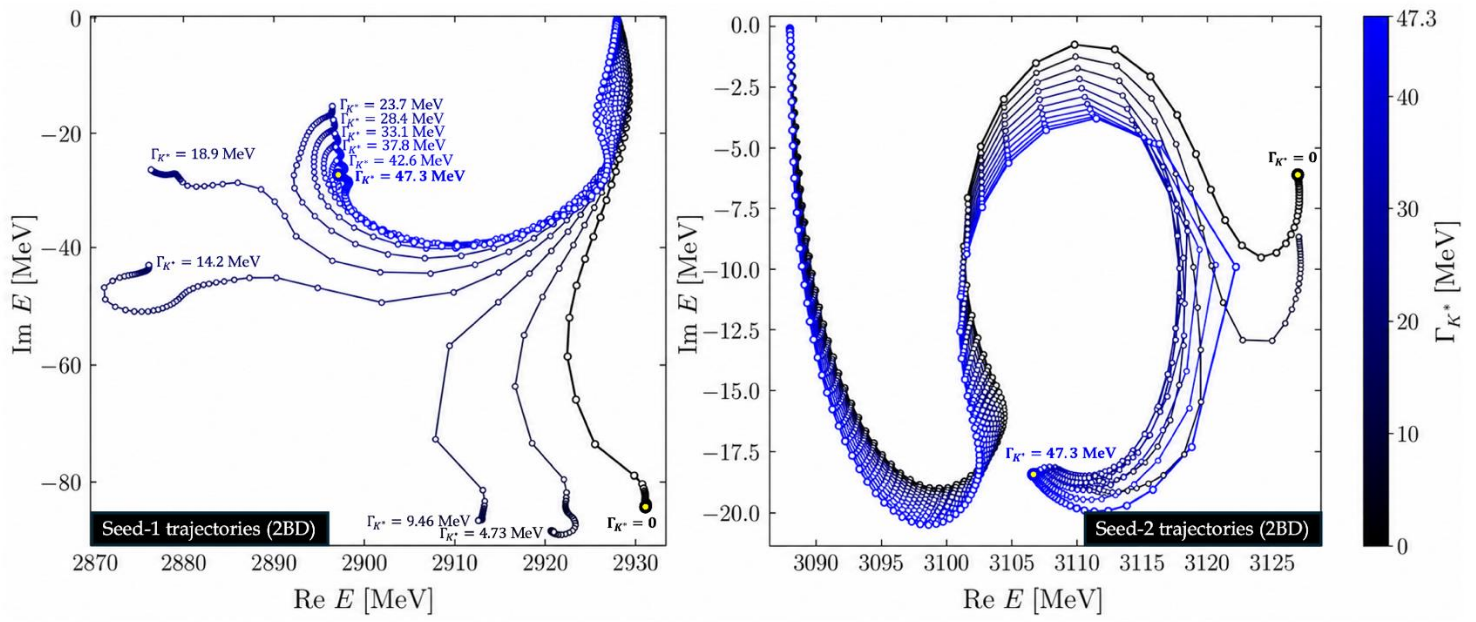
$T_{cs}(\bar{u}\bar{d}cs; I = 0, J^P = 0^+)$ tetraquark resonance

Solution trajectories of $\det\left[E\mathbf{1} - M^{(0)} - g_0^2\Sigma(E)\right] = 0,\ g_0 \in [0,1]$



$T_{cs}(\bar{u}\bar{d}cs; I = 0, J^P = 0^+)$ tetraquark resonance





On the real axis, the corresponding spectral strength is

$$\rho^{(2)}(\omega) = -\,\mathrm{Re}\left(\frac{1}{2\pi i}\left[\Sigma^{(2)}(\omega + i0) - \Sigma^{(2)}(\omega - i0)\right]\right) = \sum_{\pm} \frac{(-1)^{\mp}}{2\pi}\,\mathrm{Im}\left[g_0^2 \Psi_L^{(2)T} \mathcal{W}^{PQ} G_Q \mathcal{W}^{QP} \Psi_R^{(2)}\right]\Big|_{E=\omega\pm i0}.$$

$$-\,\mathrm{Im}\,\Sigma^{(2)}_{\mathrm{fixed}}(M_R) \supset \int d\omega\, \rho^{(2)}(\omega) L_{K^*}(\omega; M_R), \quad L_{K^*}(\omega; M_R) = \frac{\Gamma_{K^*}/2}{(M_R - \omega)^2 + (\Gamma_{K^*}/2)^2}$$

$T_{cs}(\bar{u}\bar{d}cs; I = 0, J^P = 0^+)$ tetraquark resonance

Residue decomposition

$$D^{(\eta)}(E_{\text{pole}})\Psi_R = 0, \qquad \Psi_L^T D^{(\eta)}(E_{\text{pole}}) = 0. \qquad D^{(\eta)}(E) \equiv E\mathbf{1} - M^{(0)} - g_0^2\Sigma(E)$$

$$Z_{P,i} = \mathcal{N}^{-1}\Psi_L^T P_i \Psi_R,$$ "elementariness"

$$X_\alpha = \mathcal{N}^{-1} g_0^2 \Psi_L^T \mathcal{W}^{PQ} G_Q P_\alpha G_Q \mathcal{W}^{QP}\Psi_R,$$ "compositeness" (propagating piece)

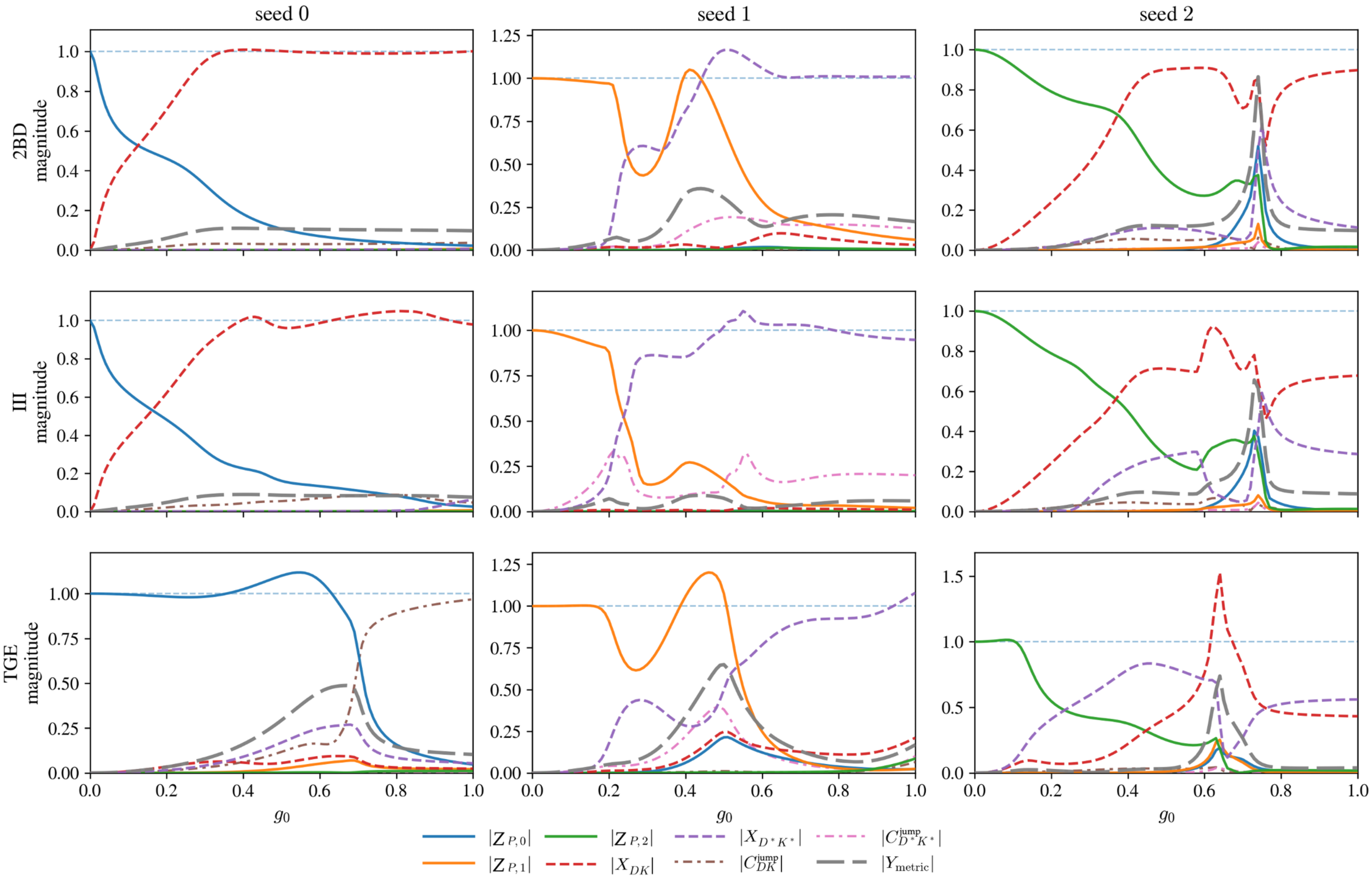
$$Y_{\text{metric}} = \mathcal{N}^{-1} g_0^2 \Psi_L^T \left[\mathcal{O}^{PQ} G_Q \mathcal{W}^{QP} + \mathcal{W}^{PQ} G_Q \mathcal{O}^{QP}\right]\Psi_R,$$ ~ come from non-orthogonality

$$C_\alpha^{\text{jump}} = -\mathcal{N}^{-1} g_0^2 \Psi_L^T J^{\alpha\prime}(E_{\text{pole}})\Psi_R.$$ "compositeness" (Sheet jump piece)

$$\Rightarrow \quad 1 = \sum_i Z_{P,i} + \sum_{\alpha\in Q}\left(X_\alpha + C_\alpha^{\text{jump}}\right) + Y_{\text{metric}}.$$

$T_{cs}(\bar{u}\bar{d}cs; I = 0, J^P = 0^+)$ tetraquark resonance

Residue decomposition



Summary

Seed	Model	$M_n^{(0)}$	$E_{\rm pole}$	Z_P	X_{DK}	$X_{D^*K^*}$	$C_{DK}^{\rm jump}$	$C_{D^*K^*}^{\rm jump}$	$Y_{\rm metric}$
0	2BD	2586.25	$2584.33 - i\,9.72$	$-0.012 + i\,0.020$	$0.997 - i\,0.084$	$0.007 + i\,0.000$	$-0.025 - i\,0.027$	–	$0.033 + i\,0.092$
1	2BD	2927.97	$2897.15 - i\,27.24$	$0.008 - i\,0.060$	$0.027 + i\,0.014$	$1.009 + i\,0.015$	$-0.003 + i\,0.003$	$0.125 + i\,0.022$	$-0.166 + i\,0.006$
2	2BD	3087.92	$3106.50 - i\,18.51$	$-0.011 + i\,0.020$	$0.897 + i\,0.027$	$0.049 - i\,0.102$	$-0.012 - i\,0.005$	$-0.001 + i\,0.002$	$0.079 + i\,0.058$
0	III	2555.73	$2573.27 - i\,22.87$	$-0.013 + i\,0.024$	$0.975 - i\,0.093$	$0.060 + i\,0.022$	$-0.041 - i\,0.025$	–	$0.019 + i\,0.073$
1	III	2911.51	$2905.77 - i\,24.77$	$0.009 - i\,0.018$	$0.006 + i\,0.008$	$0.937 - i\,0.140$	$-0.001 + i\,0.001$	$0.103 + i\,0.172$	$-0.054 - i\,0.023$
2	III	3082.80	$3140.04 - i\,29.76$	$-0.012 + i\,0.017$	$0.678 + i\,0.012$	$0.274 - i\,0.084$	$-0.008 - i\,0.004$	$0.000 + i\,0.005$	$0.068 + i\,0.056$
0	TGE	2666.40	$2638.47 - i\,129.25$	$0.066 - i\,0.012$	$0.010 - i\,0.022$	$0.038 - i\,0.039$	$0.969 + i\,0.012$	–	$-0.083 + i\,0.061$
1	TGE	2955.67	$2928.76 - i\,29.98$	$-0.096 + i\,0.022$	$-0.104 + i\,0.033$	$1.074 - i\,0.011$	$-0.023 - i\,0.060$	$0.029 - i\,0.091$	$0.120 + i\,0.106$
2	TGE	3171.45	$3224.74 - i\,122.26$	$-0.001 + i\,0.013$	$0.420 + i\,0.103$	$0.549 - i\,0.111$	$-0.007 - i\,0.001$	$0.000 + i\,0.003$	$0.039 - i\,0.006$

EXP masses

$$E_{\rm BW}^{2020} = (2866 \pm 7 \pm 2) - i(28.5 \pm 6 \pm 2)\ \text{MeV},$$
$$E_{\rm BW}^{2024} = (2914 \pm 11 \pm 15) - i(64 \pm 11 \pm 11.5)\ \text{MeV},$$
$$E_{\rm BW}^{2025} = (2883 \pm 11 \pm 8) - i(43.5^{+11}_{-23.5} \pm 8.5)\ \text{MeV}.$$

Mass estimate: $M_{\rm EXP} \simeq M_{\rm 2BD} < M_{\rm III} < M_{\rm TGE}$

TGE can explain why seed-0 and -2 do not appear in experiments: too broad (noise-like)