IBS Summer School Preparation + Follow-up Packet

Prerequisites For The IBS Summer School on QC for QFT

Lie algebras, Schwinger bosons, angular-momentum addition

Prepared as pre-requisites, mid-school and post-school material for Indrakshi Raychowdhury's four-lecture module on

Hamiltonian formulation of lattice gauge theories reformulated in terms of prepotentials and Loop–String–Hadron variables

Contains notes and pre-school assignments

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Contents

Chapter 1

Orientation:

Sections 2, 3 and 4 are prerequisites for the school. Sections 5 and 6 are left as optional post-school readings to bridge the gap between the school's learning outcomes and the attempt to simulate QCD on quantum computers.

Learning goals.

- Revise the representation-theory language that will be used without hesitation in the school.
- Learn the Schwinger-boson construction of SU(2) angular momentum from first principles.
- Practice adding two, three, and four angular momenta using invariant oscillator contractions.
- Understand why the same idea is easy for SU(2) but nontrivial for SU(3).
- Enter the lectures ready for the prepotential and loop–string–hadron constructions.

The school lectures will introduce the prepotential formulation and the loop–string–hadron (LSH) formulation of Hamiltonian lattice gauge theory. The technical obstacle for students is usually not the Hamiltonian itself; it is the local representation theory. A non-Abelian Gauss law asks us to add several color angular momenta at every site and retain only singlets or controlled external charges. The Schwinger-boson construction makes this addition algebraic.

This packet is therefore deliberately focused. It does not introduce the full Kogut–Susskind Hamiltonian, the full prepotential construction, or the IBM implementation. Those are lecture topics. The purpose here is to make sure that the following statements are already familiar before the school begins:

- SU(2) irreducible representations are labeled by one number j.
- A spin-j representation can be realized by two bosonic oscillators with fixed total occupation $N = 2j$.
- Coupling angular momenta can be done by creating invariant singlet pairs with $\epsilon_{\alpha\beta}$.
- Multi-leg SU(2) intertwiners are oscillator states built from invariant contractions, subject to simple Mandelstam/Schouten identities.
- SU(3) irreducible representations are labeled by two Dynkin labels (p, q), and tensor products carry Littlewood–Richardson multiplicities.

- Naive SU(3) Schwinger bosons contain a trace/multiplicity problem. The Mukunda–Chaturvedi analysis and the irreducible Schwinger-boson construction of Anishetty–Mathur–Raychowdhury are the correct entry points.

Expected preparation time

A well-prepared student should spend about one day on the notes and one day on the assignment. The assignment is written to force the Schwinger-boson manipulation, because the same algebra reappears in the LSH construction.

Notation used throughout

For SU(2), spinor indices are $\alpha, \beta = 1, 2$. We write

$$\epsilon_{12} = 1, \qquad \epsilon_{21} = -1, \tag{1.1}$$

and raise/lower indices only when needed. For SU(3), fundamental indices are $\alpha, \beta = 1, 2, 3$. The fundamental representation is denoted $\mathbf{3}$ and the antifundamental by $\overline{\mathbf{3}}$. Dynkin labels are written as (p, q), with

$$(1, 0) = \mathbf{3}, \qquad (0, 1) = \overline{\mathbf{3}}, \qquad (1, 1) = \mathbf{8}. \tag{1.2}$$

Minimal completion target

Before Lecture 1, students should be able to do the following without consulting a table:

1. derive $[J^a, J^b] = \mathrm{i}\epsilon^{abc}J^c$ from $J^a = \frac{1}{2}a^\dagger\sigma^a a$;
2. identify the spin-j subspace as the fixed-number subspace $N = 2j$;
3. construct the singlet of two spin-$\frac{1}{2}$ particles as $\epsilon_{\alpha\beta}a^{\dagger\alpha}[1]a^{\dagger\beta}[2]\,|0\rangle$;
4. construct highest-weight states in $j_1 \otimes j_2$ using powers of $A^\dagger_{12}$;
5. count the number of four-valent SU(2) singlet intertwiners in simple examples;
6. explain why $\mathbf{8} \otimes \mathbf{8}$ in SU(3) contains two octets.

Chapter 2

Lie-algebra revision: SU(2) and SU(3)

2.1 Compact Lie algebras and normalization

A compact gauge group G has Hermitian generators T^a in a chosen representation. In physics convention,

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c, \tag{2.1}$$

where f^{abc} are real antisymmetric structure constants. In the fundamental representation of $\mathrm{SU}(N)$ we use

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}. \tag{2.2}$$

The quadratic Casimir in an irreducible representation R is defined by

$$\sum_a T^a_R T^a_R = C_2(R)\mathbb{1}_R. \tag{2.3}$$

The number $C_2(R)$ is the non-Abelian analogue of $j(j+1)$ for angular momentum.

2.2 SU(2): generators, ladder operators, and irreducible representations

The algebra $\mathfrak{su}(2)$ is generated by J_1, J_2, J_3 satisfying

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k. \tag{2.4}$$

It is convenient to define

$$J_\pm = J_1 \pm \mathrm{i} J_2. \tag{2.5}$$

Then

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3. \tag{2.6}$$

The Casimir operator is

$$J^2 = J_1^2 + J_2^2 + J_3^2 = J_- J_+ + J_3(J_3 + 1) = J_+ J_- + J_3(J_3 - 1). \tag{2.7}$$

An irreducible representation is labeled by $j = 0, \frac{1}{2}, 1, \ldots$. The standard basis is $|j, m\rangle$, with

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle, \tag{2.8}$$

$$J_3 |j, m\rangle = m |j, m\rangle, \qquad m = -j, -j+1, \ldots, j, \tag{2.9}$$

$$J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle. \tag{2.10}$$

The dimension is $2j + 1$.

2.3 The SU(2) fundamental representation

In the fundamental representation,

$$T^a = \frac{\sigma^a}{2}, \tag{2.11}$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.12}$$

They obey

$$\sigma^a \sigma^b = \delta^{ab}\mathbb{1} + \mathrm{i}\epsilon^{abc}\sigma^c, \qquad \mathrm{tr}(\sigma^a \sigma^b) = 2\delta^{ab}. \tag{2.13}$$

The invariant tensor in the product of two doublets is $\epsilon_{\alpha\beta}$. If u^α and v^β transform as doublets, then

$$\epsilon_{\alpha\beta} u^\alpha v^\beta \tag{2.14}$$

is invariant under SU(2). This one-line fact is the algebraic origin of the Schwinger-boson singlet creation operators used later.

2.4 SU(2) tensor products

The tensor product of two SU(2) irreducible representations decomposes as

$$j_1 \otimes j_2 = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} J, \tag{2.15}$$

where J increases in integer steps. There is no outer multiplicity: each allowed J appears once. The simplest examples are

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0, \tag{2.16}$$

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}, \tag{2.17}$$

$$1 \otimes 1 = 2 \oplus 1 \oplus 0. \tag{2.18}$$

The absence of outer multiplicity is one reason why SU(2) LSH variables are considerably simpler than their SU(3) analogues.

2.5 SU(3): generators and Cartan subalgebra

The algebra $\mathfrak{su}(3)$ has eight generators. In the fundamental representation,

$$T^a = \frac{\lambda^a}{2}, \qquad a = 1, \ldots, 8, \tag{2.19}$$

where the Gell-Mann matrices satisfy

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c, \qquad \{T^a, T^b\} = \frac{1}{3}\delta^{ab}\mathbb{1} + d^{abc}T^c. \tag{2.20}$$

A standard Cartan subalgebra is generated by T^3 and T^8:

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.21}$$

States in an SU(3) representation may be labeled by their two weights (m_3, m_8), but these weights alone do not always distinguish states inside tensor products. This is one source of the additional bookkeeping absent in SU(2).

2.6 SU(3) irreducible representations

An SU(3) irrep is labeled by two Dynkin labels (p, q), equivalently by a Young diagram with row lengths

$$r_1 = p + q, \qquad r_2 = q. \tag{2.22}$$

The dimension and quadratic Casimir are

$$\dim(p, q) = \frac{1}{2}(p + 1)(q + 1)(p + q + 2), \tag{2.23}$$

$$C_2(p, q) = \frac{1}{3}\left(p^2 + q^2 + pq + 3p + 3q\right). \tag{2.24}$$

Important examples are

Dynkin label	common name	dimension	Young shape
$(1, 0)$	$\mathbf{3}$	3	one box
$(0, 1)$	$\overline{\mathbf{3}}$	3	one column of two boxes
$(1, 1)$	$\mathbf{8}$	8	two boxes in first row, one in second row
$(2, 0)$	$\mathbf{6}$	6	two symmetric fundamental boxes
$(3, 0)$	$\mathbf{10}$	10	three symmetric fundamental boxes

2.7 Basic SU(3) tensor products

The first products are

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}, \tag{2.25}$$

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}, \tag{2.26}$$

$$\overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = \overline{\mathbf{6}} \oplus \mathbf{3}, \tag{2.27}$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{2.28}$$

The last line is already a warning: the octet occurs twice. Unlike SU(2), an SU(3) irrep may occur with multiplicity greater than one in a tensor product. The decomposition must therefore include an additional outer-multiplicity label.

2.8 What students should remember

For the school, the following distinction is essential:

- SU(2) tensor products are controlled by triangle inequalities and contain no outer multiplicity.
- SU(3) tensor products are controlled by Young diagrams and Littlewood–Richardson coefficients; outer multiplicities occur and must be tracked.
- The prepotential and LSH constructions solve local Gauss laws. Solving a Gauss law is a problem in representation theory at every lattice site.

Chapter 3

SU(2) Schwinger bosons

Learning goals.

- Realize every SU(2) spin-j irrep as a fixed-number sector of two harmonic oscillators.
- Derive the $\mathfrak{su}(2)$ commutation relations directly from oscillator algebra.
- Derive $J^2 = \frac{N}{2}\left(\frac{N}{2}+1\right)$ on the fixed-number sector.
- Identify the invariant contraction that creates singlet pairs.

3.1 Two oscillator modes and the Schwinger map

Introduce two bosonic oscillators

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0, \qquad \alpha, \beta = 1, 2. \tag{3.1}$$

Define the number operator

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 \tag{3.2}$$

and define

$$J^a = \frac{1}{2} a_\alpha^\dagger (\sigma^a)_{\alpha\beta} a_\beta. \tag{3.3}$$

In matrix notation, if $a = (a_1, a_2)^T$, this is simply

$$J^a = \frac{1}{2} a^\dagger \sigma^a a. \tag{3.4}$$

The operators J^a are bilinears, so they conserve N.

3.2 The general bilinear identity

Let

$$X_M = a_\alpha^\dagger M_{\alpha\beta} a_\beta, \qquad X_N = a_\alpha^\dagger N_{\alpha\beta} a_\beta, \tag{3.5}$$

where M, N are ordinary matrices. Then

$$[X_M, X_N] = a^\dagger [M, N] a. \tag{3.6}$$

Derivation.

Step 1. Use
$$[a_\alpha^\dagger a_\beta, a_\gamma^\dagger a_\delta] = a_\alpha^\dagger [a_\beta, a_\gamma^\dagger] a_\delta - a_\gamma^\dagger [a_\delta, a_\alpha^\dagger] a_\beta. \tag{3.7}$$

Step 2. Therefore
$$[a_\alpha^\dagger a_\beta, a_\gamma^\dagger a_\delta] = \delta_{\beta\gamma} a_\alpha^\dagger a_\delta - \delta_{\delta\alpha} a_\gamma^\dagger a_\beta. \tag{3.8}$$

Step 3. Multiply by $M_{\alpha\beta} N_{\gamma\delta}$ and sum repeated indices:
$$\begin{aligned}[X_M, X_N] &= M_{\alpha\beta} N_{\gamma\delta} \left(\delta_{\beta\gamma} a_\alpha^\dagger a_\delta - \delta_{\delta\alpha} a_\gamma^\dagger a_\beta \right) \\ &= a_\alpha^\dagger (MN)_{\alpha\delta} a_\delta - a_\gamma^\dagger (NM)_{\gamma\beta} a_\beta = a^\dagger [M, N] a. \end{aligned} \tag{3.9}$$

Putting $M = \sigma^a/2$ and $N = \sigma^b/2$ gives
$$[J^a, J^b] = \frac{1}{4} a^\dagger [\sigma^a, \sigma^b] a = \frac{1}{4} (2\mathrm{i}\epsilon^{abc}) a^\dagger \sigma^c a = \mathrm{i}\epsilon^{abc} J^c. \tag{3.10}$$

Thus the Schwinger map realizes $\mathfrak{su}(2)$ exactly.

3.3 Fixed occupation number gives fixed spin

The normalized oscillator basis is
$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |0\rangle. \tag{3.11}$$

The number operator has eigenvalue
$$N |n_1, n_2\rangle = (n_1 + n_2) |n_1, n_2\rangle. \tag{3.12}$$

The J_3 operator is
$$J_3 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \tag{3.13}$$
so
$$J_3 |n_1, n_2\rangle = \frac{n_1 - n_2}{2} |n_1, n_2\rangle. \tag{3.14}$$

Thus the identification is
$$j = \frac{N}{2} = \frac{n_1 + n_2}{2}, \qquad m = \frac{n_1 - n_2}{2}. \tag{3.15}$$

Equivalently,
$$|j, m\rangle = \frac{(a_1^\dagger)^{j+m} (a_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}} |0\rangle. \tag{3.16}$$

The fixed-N subspace has dimension $N + 1 = 2j + 1$, exactly the dimension of spin j.

3.4 Derivation of the Casimir

We now derive
$$J^2 = \frac{N}{2} \left(\frac{N}{2} + 1 \right) \tag{3.17}$$
on a fixed-number sector.

Derivation.

Step 1. Write

$$J^2 = \frac{1}{4} a^\dagger_\alpha (\sigma^a)_{\alpha\beta} a_\beta a^\dagger_\gamma (\sigma^a)_{\gamma\delta} a_\delta. \tag{3.18}$$

Step 2. Normal-order the middle oscillators:

$$a_\beta a^\dagger_\gamma = \delta_{\beta\gamma} + a^\dagger_\gamma a_\beta. \tag{3.19}$$

Hence

$$\begin{aligned} J^2 &= \frac{1}{4} a^\dagger_\alpha (\sigma^a)_{\alpha\beta} (\delta_{\beta\gamma} + a^\dagger_\gamma a_\beta)(\sigma^a)_{\gamma\delta} a_\delta \\ &= \frac{1}{4} a^\dagger_\alpha (\sigma^a \sigma^a)_{\alpha\delta} a_\delta + \frac{1}{4} a^\dagger_\alpha a^\dagger_\gamma (\sigma^a)_{\alpha\beta} (\sigma^a)_{\gamma\delta} a_\beta a_\delta. \end{aligned} \tag{3.20}$$

Step 3. Use the Pauli completeness relation

$$(\sigma^a)_{\alpha\beta} (\sigma^a)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{3.21}$$

Also $\sum_a \sigma^a \sigma^a = 3\mathbb{1}$.

Step 4. The first term is $\frac{3}{4}N$. The second term is

$$\frac{1}{4}\left(2a^\dagger_\alpha a^\dagger_\gamma \delta_{\alpha\delta}\delta_{\beta\gamma} a_\beta a_\delta - a^\dagger_\alpha a^\dagger_\gamma \delta_{\alpha\beta}\delta_{\gamma\delta} a_\beta a_\delta\right) = \frac{1}{4}\left(2a^\dagger_\alpha a^\dagger_\beta a_\beta a_\alpha - a^\dagger_\alpha a^\dagger_\gamma a_\alpha a_\gamma\right). \tag{3.22}$$

Because bosonic creation and annihilation operators commute among themselves, the two quartic terms are the same type, leaving

$$\frac{1}{4} a^\dagger_\alpha a^\dagger_\beta a_\alpha a_\beta. \tag{3.23}$$

Step 5. Finally,

$$a^\dagger_\alpha a^\dagger_\beta a_\alpha a_\beta = N(N-1). \tag{3.24}$$

Therefore

$$J^2 = \frac{3}{4}N + \frac{1}{4}N(N-1) = \frac{1}{4}N(N+2) = \frac{N}{2}\left(\frac{N}{2}+1\right). \tag{3.25}$$

3.5 Ladder operators in oscillator language

The ladder operators are

$$J_+ = a^\dagger_1 a_2, \qquad J_- = a^\dagger_2 a_1. \tag{3.26}$$

Their action follows directly:

$$J_+ \left|n_1, n_2\right\rangle = \sqrt{(n_1+1)n_2}\,\left|n_1+1, n_2-1\right\rangle, \tag{3.27}$$

$$J_- \left|n_1, n_2\right\rangle = \sqrt{(n_2+1)n_1}\,\left|n_1-1, n_2+1\right\rangle. \tag{3.28}$$

In terms of j, m, this reproduces the usual angular-momentum formulas.

3.6 Invariant singlet contraction

For two independent Schwinger doublets $a^\dagger_\alpha[1]$ and $a^\dagger_\alpha[2]$, define

$$A^\dagger_{12} = \epsilon_{\alpha\beta} a^{\dagger\alpha}[1] a^{\dagger\beta}[2] = a^\dagger_1[1] a^\dagger_2[2] - a^\dagger_2[1] a^\dagger_1[2]. \tag{3.29}$$

Let $J^a_{\text{tot}} = J^a[1] + J^a[2]$. Then

$$[J^a_{\text{tot}}, A^\dagger_{12}] = 0. \tag{3.30}$$

Thus $A^\dagger_{12}$ creates an SU(2) singlet pair. This object is the basic building block for Schwinger-boson angular momentum addition.

Remark 3.1. *In the prepotential formulation of* SU(2) *lattice gauge theory, one places such oscillator doublets at the ends of links. The non-Abelian part of the Gauss law is solved by forming local invariant contractions, and the remaining Abelian occupation matching equates the flux entering and leaving a link. The school will introduce this in detail. This packet only prepares the oscillator algebra.*

Chapter 4

Angular-momentum addition with Schwinger bosons

Learning goals.

- Convert $j_1 \otimes j_2$ addition into an oscillator construction.
- Build highest-weight states for two, three, and four angular momenta.
- See explicitly how multi-leg singlet intertwiners arise from invariant contractions.
- Recognize the Schouten/Mandelstam identity that removes redundant SU(2) loop states.

4.1 Multiple copies of Schwinger bosons

For each angular momentum j_i, introduce an independent pair of oscillators

$$a_\alpha[i], \qquad a_\alpha^\dagger[i], \qquad \alpha = 1, 2, \tag{4.1}$$

with

$$[a_\alpha[i], a_\beta^\dagger[j]] = \delta_{ij}\delta_{\alpha\beta}. \tag{4.2}$$

The spin at leg i is fixed by

$$N_i = a_\alpha^\dagger[i] a_\alpha[i] = 2j_i. \tag{4.3}$$

For any pair of legs define the invariant singlet creator

$$A_{ij}^\dagger = \epsilon_{\alpha\beta} a^{\dagger\alpha}[i] a^{\dagger\beta}[j], \qquad A_{ji}^\dagger = -A_{ij}^\dagger. \tag{4.4}$$

It commutes with the total SU(2) generator of all legs:

$$\left[\sum_i J^a[i], A_{jk}^\dagger\right] = 0. \tag{4.5}$$

Thus $A_{ij}^\dagger$ removes one unit of oscillator occupation from each of the two legs and packages it into a singlet.

4.2 Addition of two angular momenta

Take two legs with spins j_1 and j_2. A highest-weight state in the total spin-J sector can be written as

$$|J, J; j_1, j_2\rangle \propto \left(A_{12}^\dagger\right)^k \left(a_1^\dagger[1]\right)^{2j_1-k} \left(a_1^\dagger[2]\right)^{2j_2-k} |0\rangle\,, \tag{4.6}$$

where

$$k = j_1 + j_2 - J. \tag{4.7}$$

The allowed values are

$$k = 0, 1, 2, \ldots, 2\min(j_1, j_2). \tag{4.8}$$

Therefore

$$J = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|. \tag{4.9}$$

This reproduces the usual SU(2) tensor product.

Derivation of (4.6).

Step 1. The operator $A^\dagger_{12}$ is an SU(2) singlet, so it does not change the total spin carried by the unpaired oscillators.

Step 2. Each factor $A^\dagger_{12}$ uses one oscillator from leg 1 and one oscillator from leg 2. Therefore the unpaired occupations are $2j_1 - k$ and $2j_2 - k$.

Step 3. The unpaired oscillators are all chosen to be $a^\dagger_1$, so the state is a highest-weight state: J^{tot}_+ annihilates it.

Step 4. The total number of unpaired spin-up oscillators is

$$(2j_1 - k) + (2j_2 - k) = 2(j_1 + j_2 - k) = 2J. \tag{4.10}$$

Hence $M = J$.

Step 5. Nonnegative occupations require $k \leq 2j_1$ and $k \leq 2j_2$. Thus $k \leq 2\min(j_1, j_2)$.

Lower-weight states are obtained by applying

$$J^{\text{tot}}_- = J_-[1] + J_-[2]. \tag{4.11}$$

Thus the entire multiplet follows from one oscillator expression.

4.3 Examples: two angular momenta

4.3.1 Two spin-$\frac{1}{2}$ particles

Here $2j_1 = 2j_2 = 1$. For $k = 0$, $J = 1$ and the highest-weight state is

$$a^\dagger_1[1]a^\dagger_1[2]\,|0\rangle = |{\uparrow\uparrow}\rangle\,. \tag{4.12}$$

For $k = 1$, $J = 0$ and the singlet is

$$A^\dagger_{12}\,|0\rangle = \Big(a^\dagger_1[1]a^\dagger_2[2] - a^\dagger_2[1]a^\dagger_1[2]\Big)\,|0\rangle\,. \tag{4.13}$$

After normalization,

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle\right). \tag{4.14}$$

4.3.2 Spin 1 plus spin $\frac{1}{2}$

Take $j_1 = 1$ and $j_2 = \frac{1}{2}$. For $J = \frac{3}{2}$ one has $k = 0$:

$$\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle \propto (a_1^\dagger[1])^2 a_1^\dagger[2] \left|0\right\rangle . \tag{4.15}$$

For $J = \frac{1}{2}$ one has $k = 1$:

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \propto A_{12}^\dagger a_1^\dagger[1] \left|0\right\rangle . \tag{4.16}$$

Expanding gives

$$A_{12}^\dagger a_1^\dagger[1] \left|0\right\rangle = \sqrt{2} \left|1,1\right\rangle \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \left|1,0\right\rangle \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle . \tag{4.17}$$

Normalizing,

$$\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left|1,1\right\rangle \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \sqrt{\frac{1}{3}} \left|1,0\right\rangle \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle . \tag{4.18}$$

The relative sign is a convention fixed by the definition of ϵ_{12}.

4.4 Addition of three angular momenta

For three legs, define the three invariant contractions

$$A_{12}^\dagger, \qquad A_{23}^\dagger, \qquad A_{31}^\dagger. \tag{4.19}$$

A general highest-weight oscillator state can be written as

$$\left|\{l_{ij}\}; J, J\right\rangle \propto \left(A_{12}^\dagger\right)^{l_{12}} \left(A_{23}^\dagger\right)^{l_{23}} \left(A_{31}^\dagger\right)^{l_{31}} \prod_{i=1}^{3} \left(a_1^\dagger[i]\right)^{n_i} \left|0\right\rangle , \tag{4.20}$$

where

$$n_1 = 2j_1 - l_{12} - l_{31}, \tag{4.21}$$
$$n_2 = 2j_2 - l_{12} - l_{23}, \tag{4.22}$$
$$n_3 = 2j_3 - l_{23} - l_{31}. \tag{4.23}$$

The integers $l_{ij} \geq 0$ must be chosen so that $n_i \geq 0$. The total spin is

$$J = \frac{1}{2}(n_1 + n_2 + n_3) = j_1 + j_2 + j_3 - (l_{12} + l_{23} + l_{31}). \tag{4.24}$$

Remark 4.1. *For three angular momenta, the same total J may occur more than once. The extra label is usually chosen as an intermediate spin, for example j_{12} in the coupling scheme $(j_1 \otimes j_2) \otimes j_3$. In the oscillator language, different choices of the integers l_{ij} encode different ways of placing singlet contractions among the legs.*

4.4.1 Three spin-$\frac{1}{2}$ particles

The decomposition is

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}. \tag{4.25}$$

The quartet highest-weight state is

$$a_1^\dagger[1] a_1^\dagger[2] a_1^\dagger[3] \left|0\right\rangle . \tag{4.26}$$

Two doublet highest-weight states may be represented by

$$D_{12}^\dagger = A_{12}^\dagger a_1^\dagger[3], \tag{4.27}$$
$$D_{23}^\dagger = A_{23}^\dagger a_1^\dagger[1]. \tag{4.28}$$

They span the two-dimensional space of doublet copies. A different coupling scheme gives a different basis of the same two-dimensional multiplicity space.

4.5 Addition of four angular momenta

For four legs one may write oscillator highest-weight states using six pairwise invariants,

$$A^\dagger_{ij}, \qquad 1 \le i < j \le 4. \tag{4.29}$$

A general state has the form

$$|\{l_{ij}\}; J, J\rangle \propto \prod_{1\le i<j\le 4} (A^\dagger_{ij})^{l_{ij}} \prod_{i=1}^{4} (a^\dagger_1[i])^{n_i} |0\rangle, \tag{4.30}$$

with

$$n_i = 2j_i - \sum_{j\neq i} l_{ij}, \qquad J = \sum_{i=1}^{4} j_i - \sum_{i<j} l_{ij}. \tag{4.31}$$

Again all l_{ij} and n_i must be nonnegative integers.

4.6 The Schouten identity and Mandelstam redundancy

For four spinor doublets, the invariant contractions are not all independent. They satisfy the Schouten identity

$$A^\dagger_{12}A^\dagger_{34} - A^\dagger_{13}A^\dagger_{24} + A^\dagger_{14}A^\dagger_{23} = 0. \tag{4.32}$$

This is the oscillator version of the elementary spinor identity

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} = 0. \tag{4.33}$$

Remark 4.2. *In loop formulations of lattice gauge theory, the same kind of identity appears as a Mandelstam constraint among loop states. The prepotential and LSH constructions are powerful because they give local gauge-invariant variables and a controlled way of handling these redundancies.*

4.7 Four-valent singlet counting

A practical way to count four-valent SU(2) singlets is to choose a coupling tree. Couple legs 1 and 2 to an intermediate j_{12}, and legs 3 and 4 to an intermediate j_{34}. A total singlet requires

$$j_{12} = j_{34}. \tag{4.34}$$

Thus the singlet dimension is the number of common values in the two sets

$$j_{12} \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2\}, \tag{4.35}$$

$$j_{34} \in \{|j_3 - j_4|, |j_3 - j_4| + 1, \ldots, j_3 + j_4\}. \tag{4.36}$$

For four spin-$\frac{1}{2}$ legs, both pairs can couple to 0 or 1, so there are two singlets. For four spin-1 legs, both pairs can couple to $0, 1, 2$, so there are three singlets.

Chapter 5

Mid school reading: First look at $\mathrm{SU}(3)$ Schwinger bosons and the multiplicity problem

Learning goals.

- Understand the naive oscillator construction of SU(3) generators.
- See why two species of oscillators are needed for generic (p, q) representations.
- Identify the $\mathrm{Sp}(2, \mathbb{R})$ invariant algebra that causes and organizes the multiplicity problem.
- Understand the role of the Mukunda–Chaturvedi paper and the irreducible Schwinger-boson papers of Anishetty–Mathur–Raychowdhury.

5.1 One triplet of bosons: symmetric representations only

Start with three bosonic oscillators

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3. \tag{5.1}$$

Define

$$Q^a = a_\alpha^\dagger (T^a)_{\alpha\beta} a_\beta, \qquad T^a = \frac{\lambda^a}{2}. \tag{5.2}$$

Using the same bilinear identity as in the SU(2) case,

$$[Q^a, Q^b] = \mathrm{i} f^{abc} Q^c. \tag{5.3}$$

The fixed-number sector with $N_a = p$ creates the completely symmetric representation $(p, 0)$:

$$(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} (a_3^\dagger)^{n_3} |0\rangle, \qquad n_1 + n_2 + n_3 = p. \tag{5.4}$$

This realizes the sequence

$$(0,0), \quad (1,0), \quad (2,0), \quad (3,0), \ldots \tag{5.5}$$

but it does not realize generic (p, q) representations. For generic SU(3) irreps one needs both fundamental and antifundamental oscillator species.

5.2 Two oscillator species for generic (p, q)

Introduce a triplet oscillator $a^\dagger_\alpha$ and an anti-triplet oscillator $b^{\dagger\alpha}$, with

$$[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}, \qquad [b^\alpha, b^\dagger_\beta] = \delta^\alpha{}_\beta. \tag{5.6}$$

A convenient convention for the SU(3) generators is

$$Q^a = a^\dagger_\alpha (T^a)_{\alpha\beta} a_\beta - b^\dagger_\alpha (T^a)^*_{\alpha\beta} b_\beta. \tag{5.7}$$

The minus sign implements the conjugate action on the $b^\dagger$ oscillators. Again

$$[Q^a, Q^b] = \mathrm{i} f^{abc} Q^c. \tag{5.8}$$

Naively, one might expect the fixed-number sector

$$N_a = p, \qquad N_b = q \tag{5.9}$$

to give the irrep (p, q). This is not quite true. It contains trace pieces. The reason is that a fundamental and an antifundamental index can be contracted with the invariant tensor $\delta^\alpha{}_\beta$.

5.3 The invariant trace algebra

Define the SU(3) invariant operators

$$k_+ = a^\dagger \cdot b^\dagger = a^\dagger_\alpha b^{\dagger\alpha}, \qquad k_- = a \cdot b = a_\alpha b^\alpha, \qquad k_0 = \frac{1}{2}(N_a + N_b + 3). \tag{5.10}$$

They commute with all Q^a:

$$[Q^a, k_\pm] = [Q^a, k_0] = 0. \tag{5.11}$$

They satisfy

$$[k_-, k_+] = 2k_0, \qquad [k_0, k_\pm] = \pm k_\pm, \tag{5.12}$$

which is the $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{su}(1, 1)$ algebra.

Derivation of the multiplicity mechanism.

Step 1. A state with p a-bosons and q b-bosons has p fundamental indices and q antifundamental indices.

Step 2. Contracting one fundamental with one antifundamental creates a trace. The trace creation operator is k_+.

Step 3. Since k_+ commutes with all SU(3) generators, applying powers of k_+ to a state does not change the SU(3) irrep label.

Step 4. Therefore the naive Schwinger-boson Fock space contains repeated copies of the same SU(3) irrep:

$$|\psi\rangle, \quad k_+ |\psi\rangle, \quad k_+^2 |\psi\rangle, \ldots \tag{5.13}$$

all transform in the same SU(3) representation, but live in different oscillator-number sectors.

Step 5. The traceless or lowest-weight condition

$$k_- |\psi\rangle = 0 \tag{5.14}$$

selects the irreducible representative without trace contamination.

5.4 Mukunda–Chaturvedi reading point

The paper by Chaturvedi and Mukunda analyzes precisely this SU(3) Schwinger construction. Its central lesson for this school is not the full induced-representation machinery. The required lesson is this:

> The SU(3) Schwinger-boson construction naturally carries an $\mathrm{Sp}(2,\mathbb{R})$ algebra commuting with SU(3); this algebra organizes the infinite repeated occurrences of each SU(3) irrep. The physically useful generating representation is obtained by selecting one traceless representative of every irrep.

This is the representation-theory problem that irreducible Schwinger bosons solve in a computationally useful way.

5.5 Irreducible SU(3) Schwinger bosons: idea and formula

Anishetty, Mathur, and Raychowdhury introduced irreducible SU(3) Schwinger bosons that build states satisfying the tracelessness condition directly. A commonly used ordered form is

$$A^\dagger_\alpha = a^\dagger_\alpha - \frac{1}{N_a + N_b + 1} k_+ b_\alpha, \tag{5.15}$$

$$B^{\dagger\alpha} = b^{\dagger\alpha} - \frac{1}{N_a + N_b + 1} k_+ a^\alpha. \tag{5.16}$$

The point of this definition is not cosmetic. If $|\psi\rangle$ satisfies

$$k_- |\psi\rangle = 0, \tag{5.17}$$

then the states created by $A^\dagger$ and $B^\dagger$ also remain in the irreducible/traceless sector. The monomial

$$A^\dagger_{\alpha_1} \cdots A^\dagger_{\alpha_p} B^{\dagger\beta_1} \cdots B^{\dagger\beta_q} |0\rangle \tag{5.18}$$

then carries the correct Young-tableau symmetries of the SU(3) irrep (p, q), without adding spurious trace copies.

Remark 5.1. *Different papers use slightly different index placements and normal-ordering conventions for the irreducible bosons. The invariant content is the same: the irreducible bosons are corrected oscillator operators designed to preserve the $k_- |\psi\rangle = 0$ subspace.*

5.6 Connection to prepotentials

In the prepotential formulation of lattice gauge theory, harmonic oscillators are attached locally to link ends. For SU(2), the Schwinger bosons are already irreducible; fixed occupation gives a unique spin. For SU(3), the naive oscillator construction contains the extra trace/multiplicity problem. Hence the SU(3) prepotential formulation uses the irreducible Schwinger-boson technology to construct local color-flux Hilbert spaces and local gauge-invariant vertex operators.

The school will introduce prepotentials as dynamical lattice variables. The prerequisite point is simpler:

$$\text{prepotentials} = \text{Schwinger-boson variables adapted to lattice gauge links.} \tag{5.19}$$

The SU(2) case prepares the algebra. The SU(3) case shows why irreducibility and multiplicity control are unavoidable.

Chapter 6

Post school reading: Preliminary note on $\mathrm{SU}(3)$ tensor products, Littlewood–Richardson coefficients, and outer multiplicity

Learning goals.

- Translate SU(3) Dynkin labels into Young diagrams.
- Understand tensor-product decomposition as a Littlewood–Richardson problem.
- Recognize outer multiplicity as a genuine label in SU(3) Clebsch–Gordan theory.
- See why SU(3) LSH is structurally harder than SU(2) LSH.

6.1 From Dynkin labels to Young diagrams

For SU(3) an irrep (p, q) corresponds to a Young diagram with row lengths

$$r_1 = p + q, \qquad r_2 = q. \tag{6.1}$$

Examples:

$$\begin{aligned}
&(1,0) = \mathbf{3}, && \text{one fundamental box,} && (6.2)\\
&(0,1) = \overline{\mathbf{3}}, && \text{one column of two boxes,} && (6.3)\\
&(1,1) = \mathbf{8}, && \text{row lengths } (2,1), && (6.4)\\
&(2,0) = \mathbf{6}, && \text{two boxes symmetrized,} && (6.5)\\
&(0,2) = \overline{\mathbf{6}}, && \text{two anti-fundamentals symmetrized.} && (6.6)
\end{aligned}$$

The conjugate of (p, q) is (q, p).

6.2 Littlewood–Richardson decomposition

The tensor product of two SU(3) irreps is written as

$$(p_1, q_1) \otimes (p_2, q_2) = \bigoplus_{(p,q)} c^{(p,q)}_{(p_1,q_1),(p_2,q_2)} (p, q), \tag{6.7}$$

where

$$c^{(p,q)}_{(p_1,q_1),(p_2,q_2)} \in \{0, 1, 2, \ldots\} \tag{6.8}$$

is the Littlewood–Richardson coefficient. It counts how many independent copies of (p, q) occur in the product.

For SU(2) the analogous coefficient is always zero or one. For SU(3) it can exceed one. This is called outer multiplicity.

6.3 Elementary decompositions

The following decompositions should be memorized:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}, \tag{6.9}$$

$$(1, 0) \otimes (1, 0) = (2, 0) \oplus (0, 1), \tag{6.10}$$

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}, \tag{6.11}$$

$$(1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0), \tag{6.12}$$

$$\mathbf{3} \otimes \mathbf{6} = \mathbf{10} \oplus \mathbf{8}, \tag{6.13}$$

$$(1, 0) \otimes (2, 0) = (3, 0) \oplus (1, 1), \tag{6.14}$$

$$\mathbf{3} \otimes \mathbf{8} = \mathbf{15} \oplus \overline{\mathbf{6}} \oplus \mathbf{3}, \tag{6.15}$$

$$(1, 0) \otimes (1, 1) = (2, 1) \oplus (0, 2) \oplus (1, 0). \tag{6.16}$$

The dimensions check:

$$3 \times 8 = 24 = 15 + 6 + 3. \tag{6.17}$$

6.4 Three fundamentals and the first multiplicity

The product of three fundamentals decomposes as

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} &= (\mathbf{6} \oplus \overline{\mathbf{3}}) \otimes \mathbf{3} \\ &= (\mathbf{10} \oplus \mathbf{8}) \oplus (\mathbf{8} \oplus \mathbf{1}) \\ &= \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \end{aligned} \tag{6.18}$$

There are two independent octets. Physically, these two octets correspond to two inequivalent internal coupling channels. Algebraically, the same final irrep has Littlewood–Richardson coefficient two:

$$c^{(1,1)}_{(1,0),(1,0),(1,0)} = 2. \tag{6.19}$$

This is analogous in spirit to having two spin-$\frac{1}{2}$ copies in

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}, \tag{6.20}$$

but in SU(3) such multiplicities appear more pervasively and interact with trace constraints.

6.5 The adjoint product and explicit outer multiplicity

A canonical example is

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8}_S \oplus \mathbf{8}_A \oplus \mathbf{1}. \tag{6.21}$$

Dimension check:

$$64 = 27 + 10 + 10 + 8 + 8 + 1. \tag{6.22}$$

The two octets are not the same vector space copy. They are distinguished by an outer multiplicity label, commonly indicated by S and A in this product. In general SU(3) Clebsch–Gordan coefficients must carry labels beyond weights and irrep labels.

6.6 Invariant tensors in SU(3)

The fundamental invariant tensors are

$$\delta^{\alpha}{}_{\beta}, \qquad \epsilon_{\alpha\beta\gamma}, \qquad \epsilon^{\alpha\beta\gamma}. \tag{6.23}$$

They allow three kinds of singlet contractions:

$$\mathbf{3} \otimes \overline{\mathbf{3}} \to \mathbf{1} \qquad \text{through } \delta^{\alpha}{}_{\beta}, \tag{6.24}$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \to \mathbf{1} \qquad \text{through } \epsilon_{\alpha\beta\gamma}, \tag{6.25}$$

$$\overline{\mathbf{3}} \otimes \overline{\mathbf{3}} \otimes \overline{\mathbf{3}} \to \mathbf{1} \qquad \text{through } \epsilon^{\alpha\beta\gamma}. \tag{6.26}$$

The δ contraction is the trace operation responsible for the k_+ and k_- operators in the SU(3) Schwinger-boson construction. The ϵ contractions create baryon-like singlets. Both become important in SU(3) LSH.

6.7 Why this matters for LSH

At a lattice site, Gauss law requires the total color charge to combine to a singlet. In SU(2), one can organize the problem using pairwise spin addition and invariant ϵ contractions. In SU(3), one must handle:

- two Dynkin labels (p, q) instead of one spin j;
- trace constraints from $\mathbf{3} \otimes \overline{\mathbf{3}}$ contractions;
- baryonic ϵ contractions;
- outer multiplicity in tensor products;
- local Mandelstam-type relations among gauge-invariant contractions.

This is why the irreducible Schwinger-boson papers are not optional technical background. They supply the algebra needed to keep the SU(3) local Hilbert space nonredundant.

Chapter 7

Pre-school assignment: Schwinger-boson angular-momentum addition

Instructions

This assignment is designed as a bridge into the prepotential and LSH lectures. Work through it using oscillator algebra, not only by quoting standard Clebsch–Gordan tables. The main skill is to translate angular-momentum addition into invariant Schwinger-boson contractions.

Use the convention

$$A^\dagger_{ij} = \epsilon_{\alpha\beta} a^{\dagger\alpha}[i] a^{\dagger\beta}[j] = a^\dagger_1[i]a^\dagger_2[j] - a^\dagger_2[i]a^\dagger_1[j]. \tag{7.1}$$

Exercise 7.1 (Pauli identity and oscillator bilinears)**.** *Prove the bilinear identity*

$$[a^\dagger M a, a^\dagger N a] = a^\dagger [M, N] a. \tag{7.2}$$

Then use it to prove

$$[J^a, J^b] = \mathrm{i}\epsilon^{abc} J^c, \qquad J^a = \frac{1}{2} a^\dagger \sigma^a a. \tag{7.3}$$

Exercise 7.2 (Spin from occupation number)**.** *Using two Schwinger bosons, show that the fixed-number sector $N = n_1 + n_2 = 2j$ is the spin-j irrep. Derive*

$$J^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right), \qquad m = \frac{n_1 - n_2}{2}. \tag{7.4}$$

Exercise 7.3 (Two spin-$\frac{1}{2}$ particles)**.** *Using Schwinger bosons, construct the triplet and singlet states in*

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0. \tag{7.5}$$

Normalize all four states.

Exercise 7.4 (Spin 1 plus spin $\frac{1}{2}$)**.** *Use*

$$|J, J\rangle \propto (A^\dagger_{12})^k (a^\dagger_1[1])^{2j_1 - k} (a^\dagger_1[2])^{2j_2 - k} |0\rangle \tag{7.6}$$

to construct the highest-weight states in

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}. \tag{7.7}$$

Expand the $J = \frac{1}{2}$ highest-weight state in the product basis and normalize it.

Exercise 7.5 (General two-spin addition). *For arbitrary j_1, j_2, prove that the Schwinger-boson expression*

$$(A_{12}^\dagger)^k (a_1^\dagger[1])^{2j_1-k} (a_1^\dagger[2])^{2j_2-k} |0\rangle \tag{7.8}$$

has total spin

$$J = j_1 + j_2 - k. \tag{7.9}$$

Derive the allowed range of k and recover the usual triangle rule.

Exercise 7.6 (Three spin-$\frac{1}{2}$ particles). *Decompose*

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}. \tag{7.10}$$

Construct the $J = \frac{3}{2}$ highest-weight state and two independent $J = \frac{1}{2}$ highest-weight states using the contractions $A_{ij}^\dagger$. Explain why the two $J = \frac{1}{2}$ states represent a multiplicity space.

Exercise 7.7 (Three arbitrary angular momenta). *For three angular momenta j_1, j_2, j_3, consider the state*

$$(A_{12}^\dagger)^{l_{12}} (A_{23}^\dagger)^{l_{23}} (A_{31}^\dagger)^{l_{31}} \prod_{i=1}^{3} (a_1^\dagger[i])^{n_i} |0\rangle . \tag{7.11}$$

Find n_i in terms of j_i and l_{ij}. Derive the total spin J. State the nonnegativity constraints.

Exercise 7.8 (Four spin-$\frac{1}{2}$ particles and the Schouten identity). *Construct the three pairwise singlet states*

$$S_{12|34} = A_{12}^\dagger A_{34}^\dagger |0\rangle , \tag{7.12}$$

$$S_{13|24} = A_{13}^\dagger A_{24}^\dagger |0\rangle , \tag{7.13}$$

$$S_{14|23} = A_{14}^\dagger A_{23}^\dagger |0\rangle . \tag{7.14}$$

Show that they satisfy one linear relation. How many independent singlets are there in $(\frac{1}{2})^{\otimes 4}$?

Exercise 7.9 (Four-valent SU(2) singlet counting). *Show that the number of singlets in $j_1 \otimes j_2 \otimes j_3 \otimes j_4$ is the number of common values in the two intermediate-spin sets*

$$\mathcal{I}_{12} = \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2\}, \tag{7.15}$$

$$\mathcal{I}_{34} = \{|j_3 - j_4|, |j_3 - j_4| + 1, \ldots, j_3 + j_4\}. \tag{7.16}$$

Evaluate the answer for four spin-$\frac{1}{2}$ legs and for four spin-1 legs.

Exercise 7.10 (SU(3) tensor-product warm-up). *Using Young diagrams or standard* SU(3) *tensor-product rules, derive*

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}, \tag{7.17}$$

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}, \tag{7.18}$$

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{7.19}$$

Identify the first occurrence of outer multiplicity.

Exercise 7.11 (SU(3) Schwinger trace problem). *For the* SU(3) *Schwinger construction with triplet $a^\dagger$ and anti-triplet $b^\dagger$ oscillators, define*

$$k_+ = a^\dagger \cdot b^\dagger, \qquad k_- = a \cdot b. \tag{7.20}$$

Show that k_+ and k_- commute with all SU(3) *generators. Explain why powers of k_+ create repeated copies of the same* SU(3) *irrep and why the condition $k_- |\psi\rangle = 0$ is a tracelessness condition.*

Appendix A

Reading material and pre-school study plan

A.1 Required reading before the school

The following reading is targeted. Students should not try to read every paper in full before arriving. The goal is to understand the representation-theory bottleneck that the lectures will solve.

SU(2) **Schwinger bosons and prepotentials**

Reading task.

- Review the Schwinger-boson realization of SU(2) angular momentum from these notes.
- Read Mathur's harmonic-oscillator prepotential paper for the idea that SU(2) link electric fields can be represented with local oscillator doublets at link ends [3].
- Focus on why the prepotential variables transform locally as SU(2) doublets and why an additional Abelian link-number matching appears.

A.2 Post-school reading to appreciate the bottlenecks towards quantum simulating QCD

SU(3) **Schwinger construction and multiplicity**

Reading task.

- Read Chaturvedi and Mukunda, "The Schwinger SU(3) construction I: multiplicity problem and relation to induced representations" [5].
- Focus on the $\mathrm{Sp}(2,\mathbb{R})$ algebra commuting with SU(3), the repeated occurrence of irreps, and the generating representation.
- Do not spend much time on induced representation details unless already comfortable with them.

Irreducible Schwinger bosons

Reading task.

- Read Anishetty–Mathur–Raychowdhury, "Irreducible SU(3) Schwinger bosons" [6].
- Then read Mathur–Raychowdhury–Anishetty, "SU(N) irreducible Schwinger bosons" [8].
- Optional but useful: Mathur–Raychowdhury–Sreeraj on invariants, projection operators, and SU(N) $\times$ SU(N) irreducible Schwinger bosons [9].

SU(3) **prepotentials**

Reading task.

- Read Anishetty–Mathur–Raychowdhury, "Prepotential formulation of SU(3) lattice gauge theory" [7].
- Focus on three items: irreducible prepotential operators, the local form of Mandelstam constraints, and reconstruction of link operators.

A.3 What to bring to the school

Students should bring:

- the completed assignment;
- the formula sheet from this packet;
- a working familiarity with oscillator commutators and ϵ-tensor contractions.

Appendix B

Formula sheet

B.1 SU(2)

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \tag{B.1}$$
$$J_\pm = J_1 \pm \mathrm{i}J_2, \tag{B.2}$$
$$[J_3, J_\pm] = \pm J_\pm, \tag{B.3}$$
$$[J_+, J_-] = 2J_3, \tag{B.4}$$
$$J^2 \left|j, m\right\rangle = j(j+1) \left|j, m\right\rangle, \tag{B.5}$$
$$J_3 \left|j, m\right\rangle = m \left|j, m\right\rangle, \tag{B.6}$$
$$J_\pm \left|j, m\right\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \left|j, m \pm 1\right\rangle. \tag{B.7}$$

B.2 SU(2) Schwinger bosons

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \tag{B.8}$$
$$J^a = \frac{1}{2} a_\alpha^\dagger (\sigma^a)_{\alpha\beta} a_\beta, \tag{B.9}$$
$$N = a_1^\dagger a_1 + a_2^\dagger a_2, \tag{B.10}$$
$$j = \frac{N}{2}, \qquad m = \frac{n_1 - n_2}{2}, \tag{B.11}$$
$$J^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right), \tag{B.12}$$
$$\left|j, m\right\rangle = \frac{(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}} \left|0\right\rangle, \tag{B.13}$$
$$A_{ij}^\dagger = \epsilon_{\alpha\beta} a^{\dagger\alpha}[i] a^{\dagger\beta}[j]. \tag{B.14}$$

B.3 Schwinger-boson addition

For two angular momenta,

$$\left|J, J; j_1, j_2\right\rangle \propto (A_{12}^\dagger)^k (a_1^\dagger[1])^{2j_1 - k} (a_1^\dagger[2])^{2j_2 - k} \left|0\right\rangle, \tag{B.15}$$
$$J = j_1 + j_2 - k, \tag{B.16}$$
$$k = 0, 1, \ldots, 2\min(j_1, j_2). \tag{B.17}$$

For three angular momenta,

$$|\{l_{ij}\}; J, J\rangle \propto (A_{12}^\dagger)^{l_{12}}(A_{23}^\dagger)^{l_{23}}(A_{31}^\dagger)^{l_{31}} \prod_{i=1}^{3}(a_1^\dagger[i])^{n_i} |0\rangle\,, \tag{B.18}$$

$$n_1 = 2j_1 - l_{12} - l_{31}, \tag{B.19}$$

$$n_2 = 2j_2 - l_{12} - l_{23}, \tag{B.20}$$

$$n_3 = 2j_3 - l_{23} - l_{31}, \tag{B.21}$$

$$J = j_1 + j_2 + j_3 - l_{12} - l_{23} - l_{31}. \tag{B.22}$$

Schouten identity:

$$A_{12}^\dagger A_{34}^\dagger - A_{13}^\dagger A_{24}^\dagger + A_{14}^\dagger A_{23}^\dagger = 0. \tag{B.23}$$

B.4 SU(3)

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c, \tag{B.24}$$

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \tag{B.25}$$

$$\dim(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2), \tag{B.26}$$

$$C_2(p,q) = \frac{1}{3}(p^2+q^2+pq+3p+3q), \tag{B.27}$$

$$\mathbf{3}\otimes\mathbf{3} = \mathbf{6}\oplus\overline{\mathbf{3}}, \tag{B.28}$$

$$\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{8}\oplus\mathbf{1}, \tag{B.29}$$

$$\mathbf{3}^{\otimes 3} = \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}, \tag{B.30}$$

$$\mathbf{8}\otimes\mathbf{8} = \mathbf{27}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{8}_S\oplus\mathbf{8}_A\oplus\mathbf{1}. \tag{B.31}$$

B.5 SU(3) Schwinger bosons

$$Q^a = a_\alpha^\dagger (T^a)_{\alpha\beta} a_\beta - b_\alpha^\dagger (T^a)^*_{\alpha\beta} b_\beta, \tag{B.32}$$

$$k_+ = a^\dagger\cdot b^\dagger, \qquad k_- = a\cdot b, \qquad k_0 = \frac{1}{2}(N_a + N_b + 3), \tag{B.33}$$

$$[k_-, k_+] = 2k_0, \qquad [k_0, k_\pm] = \pm k_\pm, \tag{B.34}$$

$$[Q^a, k_\pm] = 0. \tag{B.35}$$

Irreducibility/tracelessness condition:

$$k_- |\psi\rangle = 0. \tag{B.36}$$

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