Stringy Constraints on Modular Flavor Models

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FLASY 2026 in Gyeongju, 2026/08/18

In collaboration with

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Based on arXiv:2508.12392 [hep-ph] (accepted by PTEP)

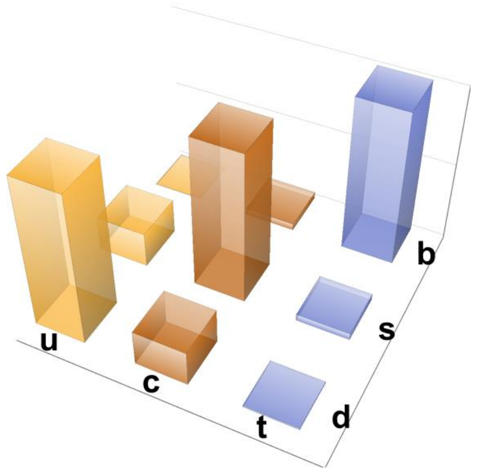
Flavor structure and modular invariance

Feruglio, 1706.08749

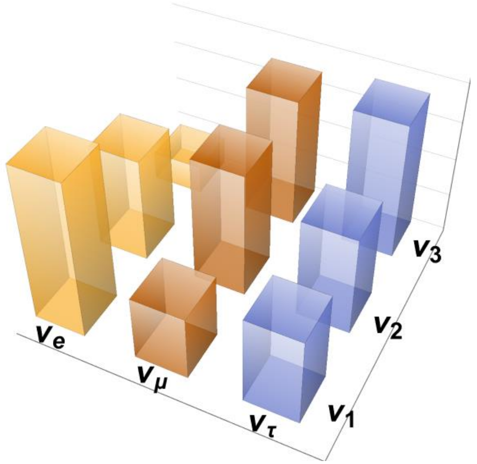
Flavor puzzle

What is the origin of the parameters in the flavor sector?

Quark



Lepton



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One possibility : **Modular symmetry**

Modular invariance can provide **rich structures** for the flavor sector

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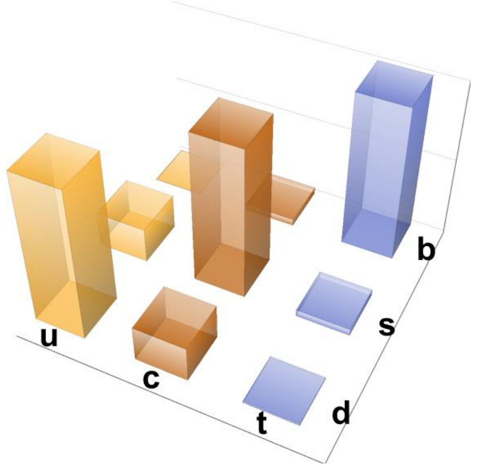

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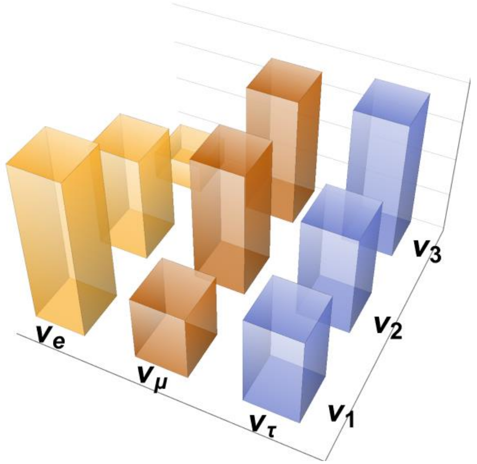

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Representations of flavor group → **Modular group**

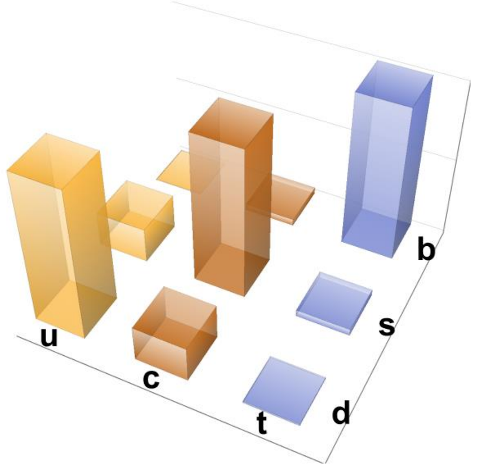
$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

What determines the flavor parameters → **Modular forms**

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} f(\tau), \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}(N)\right)$$

In this way, all the flavor parameters become functions of a complex parameter

Quark



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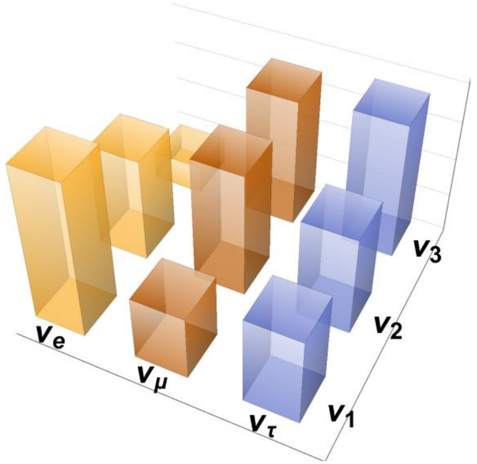

Flavor structure and modular invariance

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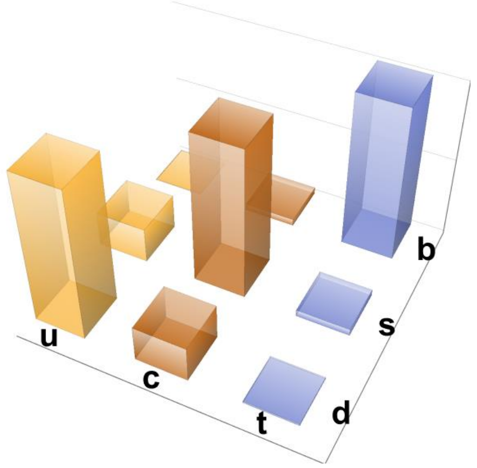
<u>Top-down approach</u> : string theory and extra-dimensional theories

<u>Quark</u>



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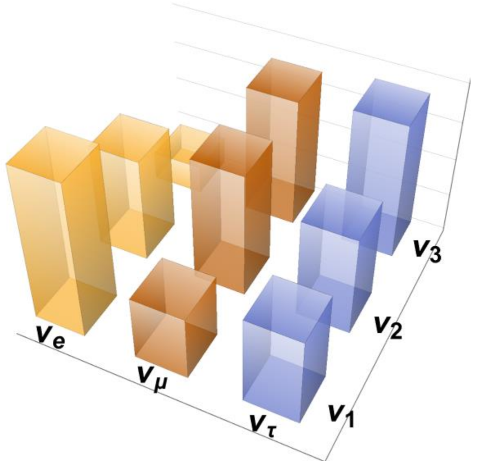


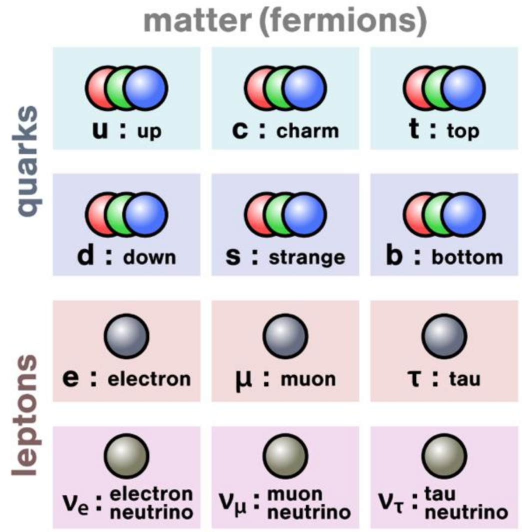
Extra-dimensional space

Strominger, Witten, *Commun. Math. Phys.* **101** (1985) 341

Candelas, *Nuclear Physics B* **298** (1988) 458

Ishiguro, Kobayashi, Nishimura, Otsuka, *JHEP* **08** (2024) 088

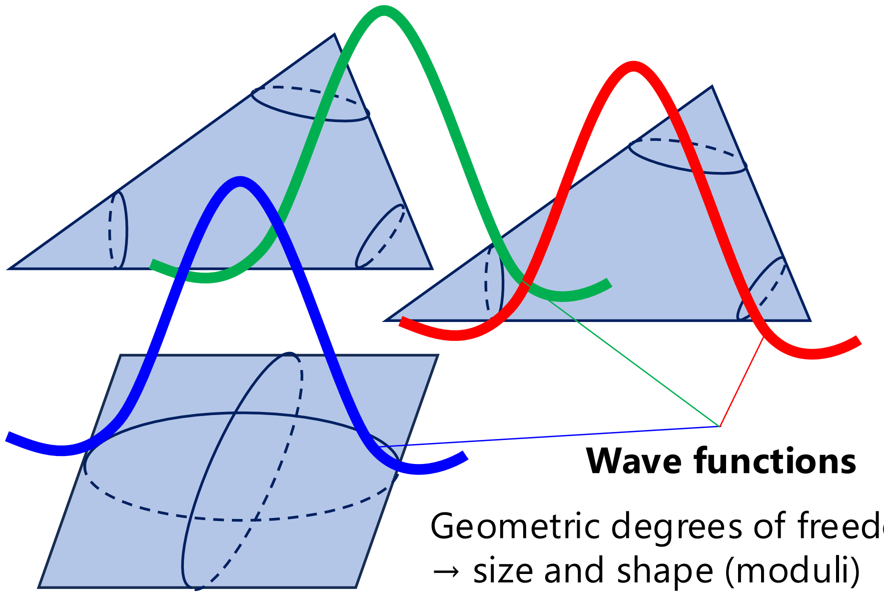
The flavor structure in SM



higgstan.com

<u>**Yukawa coupling**</u>

$Y_{ijk}(\tau)$



Geometric degrees of freedom
→ size and shape (moduli)

→ **The input parameters** can be determined through the Yukawa couplings described by <u>geometric degrees of freedom and wave functions</u>

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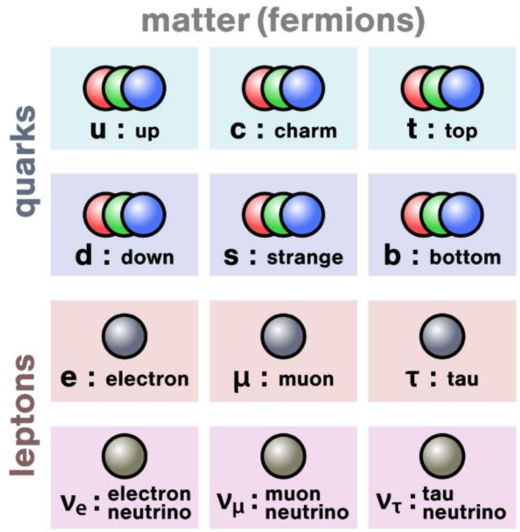

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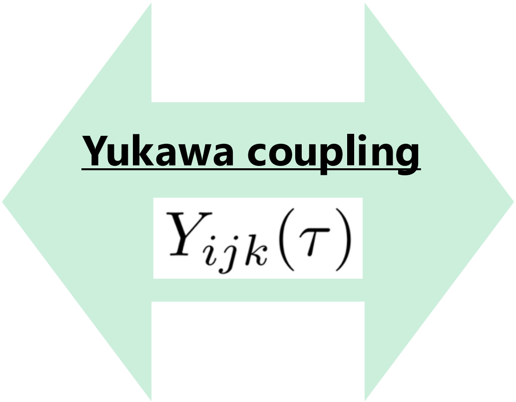
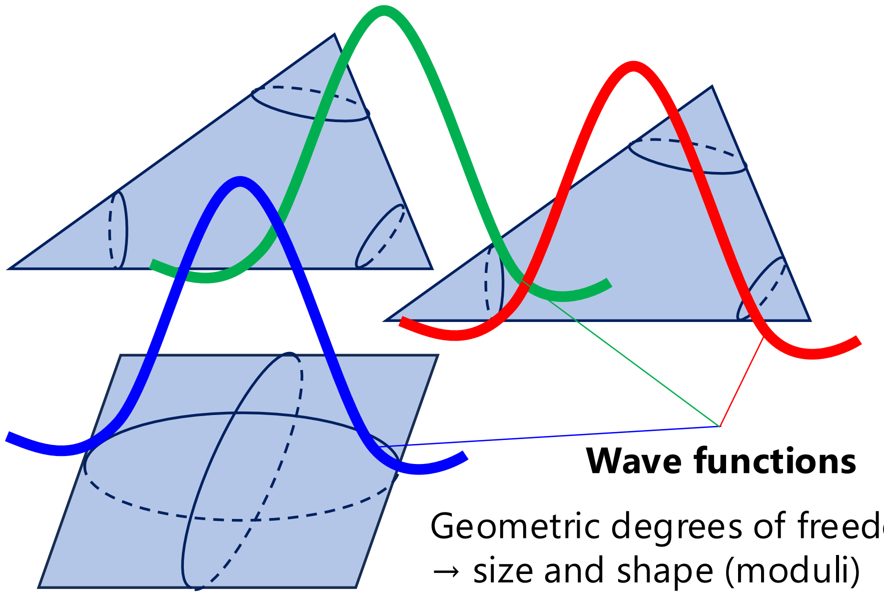




Geometric degrees of freedom
→ size and shape (moduli)

→ **The input parameters** can be determined through the Yukawa couplings described by <u>geometric degrees of freedom and wave functions</u>

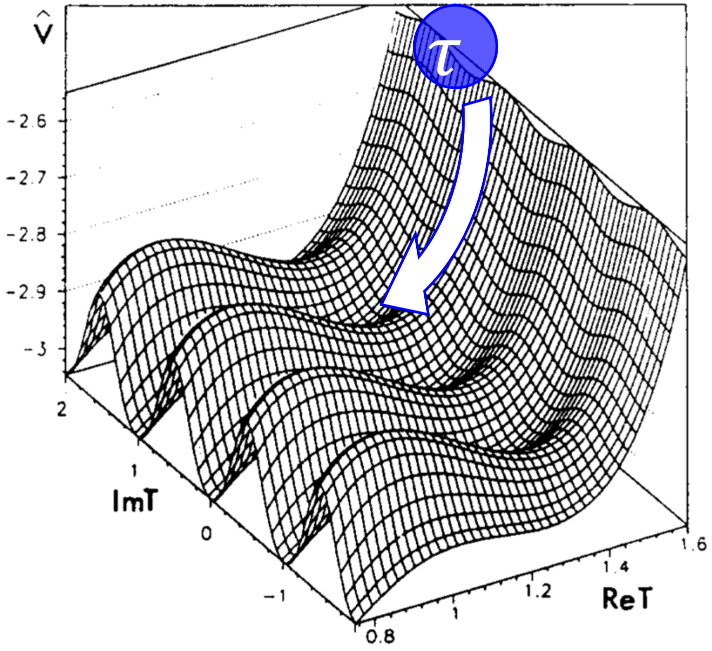
<u>In top-down model building based on heterotic string theory</u>, the Yukawa coupling can be described as **modular forms**.

Symmetry breaking

Font, Ibáñez, Lüst, Quevedo , *Phys.Lett.B* **245** (1990) 401-408

Constructing a potential for the moduli in the effective theory of heterotic string vacua

→ It is possible to **dynamically determine** the VEVs of the modulus



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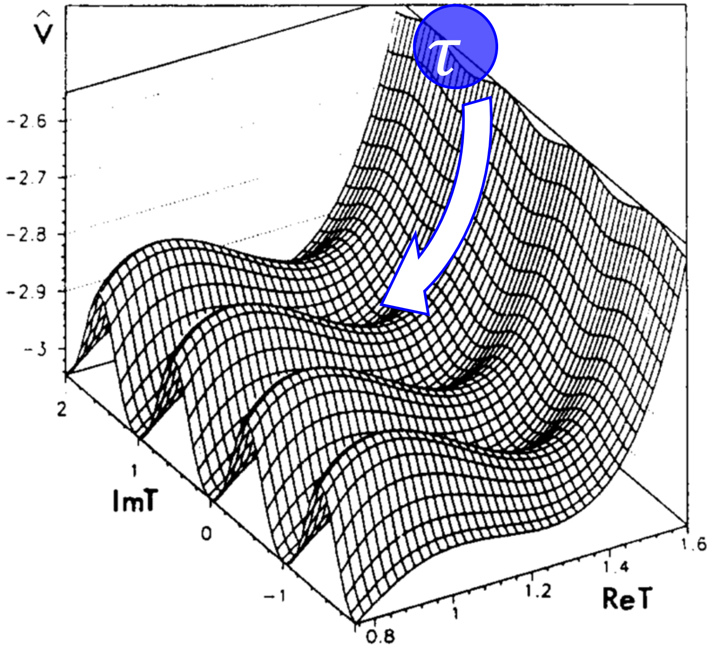


When the moduli acquire VEVs, **the modular symmetry is spontaneously broken**

The VEVs of the modulus directly characterize the values of masses and flavor mixings

Flavor structure of the lepton

Problems

Penedo, Petcov, *JHEP* **10** (2024) 172

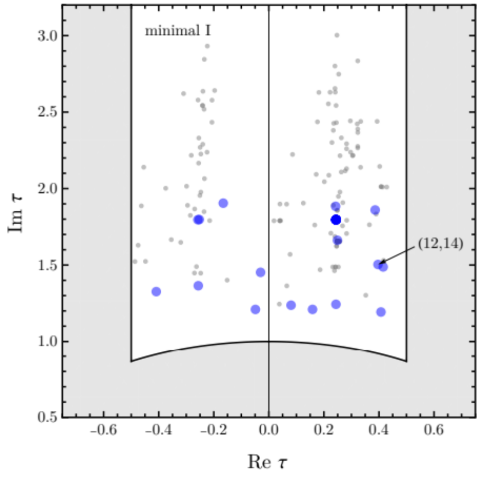
Petcov, Tanimoto, *Eur.Phys.J.C* **84** (2024) 9, 914

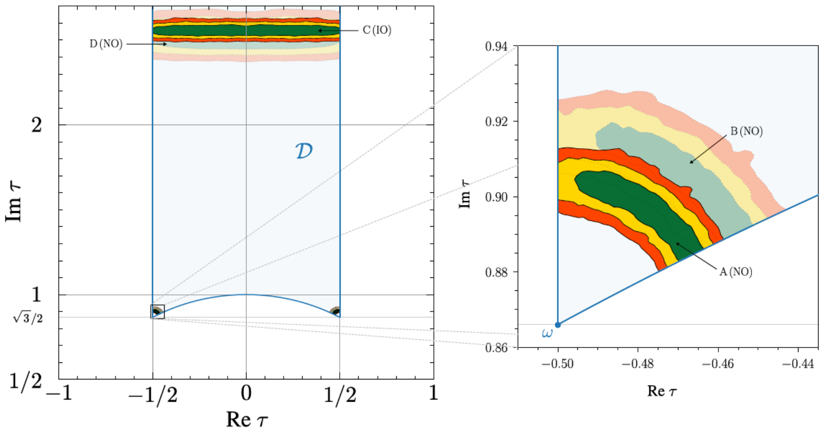
Granelli, Meloni, Parriciatu, Penedo, Petcov, 2505.21405

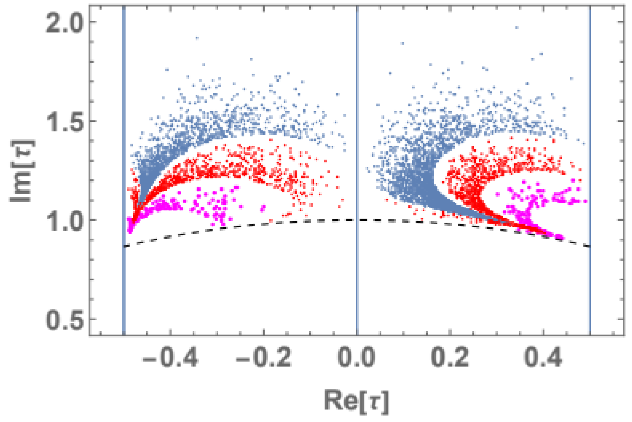
Various modular flavor models have been proposed

In these analyses, the VEV of the modulus is essentially a free parameter, which is fixed by fitting the data

There is **no criterion** coming from string theory itself, which is independent of both the flavor models and the stabilization mechanism







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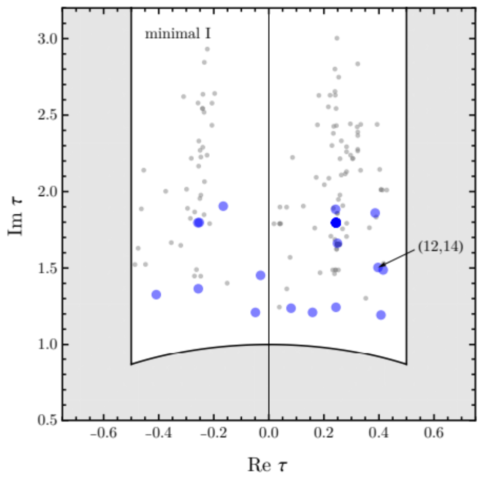
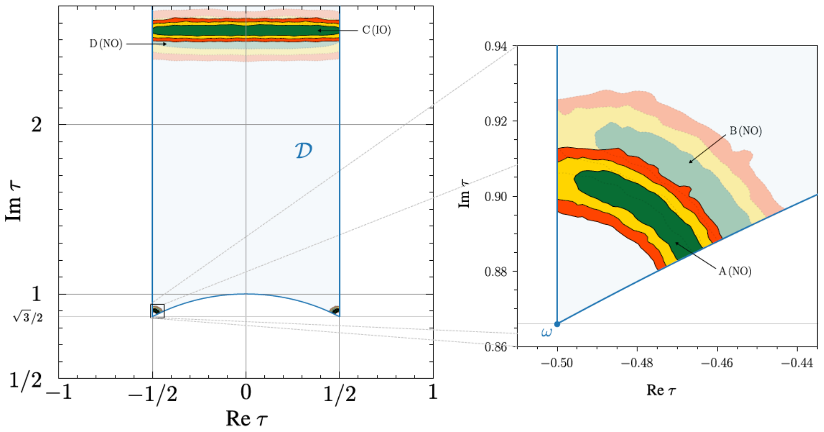
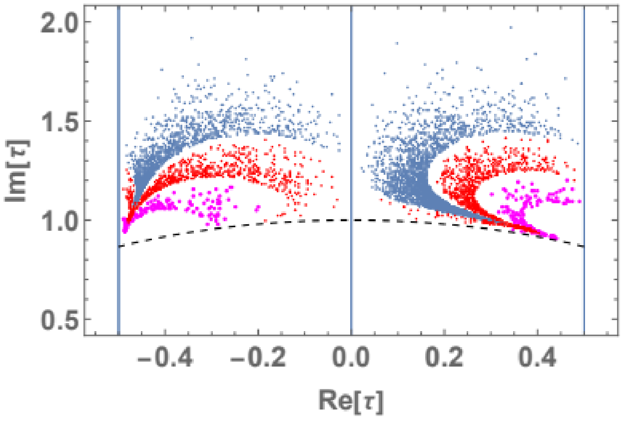
Motivation

Using the implications derived from heterotic string theory, we would like to constrain allowed ranges within **the fundamental domain**.







Stringy constraints

Dixon, Kaplunovsky, Louis, *Nucl. Phys. B* **355** (1991) 649

At the one-loop level, **the renormalized gauge couplings** of the effective theory regarding heterotic string vacua

$$\frac{16\pi^2}{g_a^2(\mu)} = k_a \; \frac{16\pi^2}{g_{\rm string}^2} + b_a \; \log\frac{M_{\rm string}^2}{\mu^2} + \Delta_a$$

μ : renormalization scale
b_a : beta-function coefficient
k_a : level of Kač-Moody algebra
$g_{\rm string}$: string coupling
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For conventional modular flavor models, **the perturbation theory** in the string vacua is necessary

Tree level > One-loop level

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Research purpose

Considering this inequality and moduli-dependent threshold correction, we propose **new constraints** on **the moduli space**, which guarantee the perturbation theory of heterotic string vacua

Outline

1. **Introduction**
2. **Moduli-dependent threshold correction**
    - Explicit threshold correction
3. **Stringy constraints for moduli**
    - Analysis of stringy constraints
    - Various analyses
    - Numerical analysis
4. **Summary**

Explicit threshold correction

$(1)\ PSL_T(2,\mathbb{Z}) \times PSL_U(2,\mathbb{Z})$

$$\Delta_a = -b_a^{\mathcal{N}=2} \ln\left[\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}}\ T_2|\eta(T)|^4\ U_2|\eta(U)|^4\right]$$

$b_a^{\mathcal{N}=2}$: beta-function coefficient
γ_{E} : Euler-Mascheroni constant

Kähler modulus : $T = T_1 + i\,T_2$, **Complex-structure modulus** : $U = U_1 + i\,U_2$

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Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(6n-1)^2}{24}} = q^{\frac{1}{24}}[1-q+\ldots], \qquad (q \equiv e^{2\pi i\tau})$$

Modular transformation of the eta function

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau), \quad \eta(\tau+1) = e^{\pi i/12}\eta(\tau)$$

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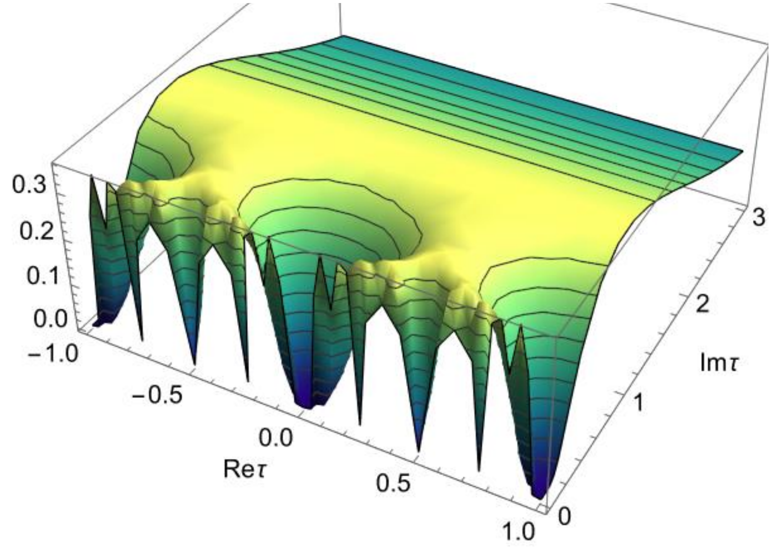
Tree level = Threshold correction → **Contour line**

Shape of the function $\mathrm{Im}\tau\,|\eta(\tau)|^4$



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Analysis of stringy constraints

Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

The inequality to guarantee the perturbation theory

$$16\pi^2 k_a \mathrm{Im} S > |\Delta_a|$$

Axio-dilaton : $S = C_0 + ie^{-\phi}$
$g_{\mathrm{string}}^{-2} \sim \mathrm{Im}S + \mathcal{O}(1)$

Left hand side: The tree-level contribution, which is proportional to g_{string}^{-2}

Right hand side: the one-loop contribution, including the threshold correction

Requiring that the tree-level contribution is larger than the absolute value of the threshold correction, we obtain **this simple inequality**

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Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

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The case of large imaginary part: $\mathrm{Im}U \gg 1$ ($T = e^{2\pi i/3}$)

Dedekind eta function : $\eta(U) \sim e^{\frac{2\pi i U}{24}}$

We rewrite the inequality **approximately**

$$\frac{16\pi^2 \mathrm{Im}S}{b_a^{\mathcal{N}=2}} > \frac{\pi}{3}U_2 - \ln U_2 - \ln\left[\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_{\mathrm{E}}}\,\frac{\sqrt{3}}{2}|\eta(\omega)|^4\right]$$

Solving this inequality for the imaginary part of U, we obtain an analytic expression written in terms of **the Lambert W function**

Analysis of stringy constraints

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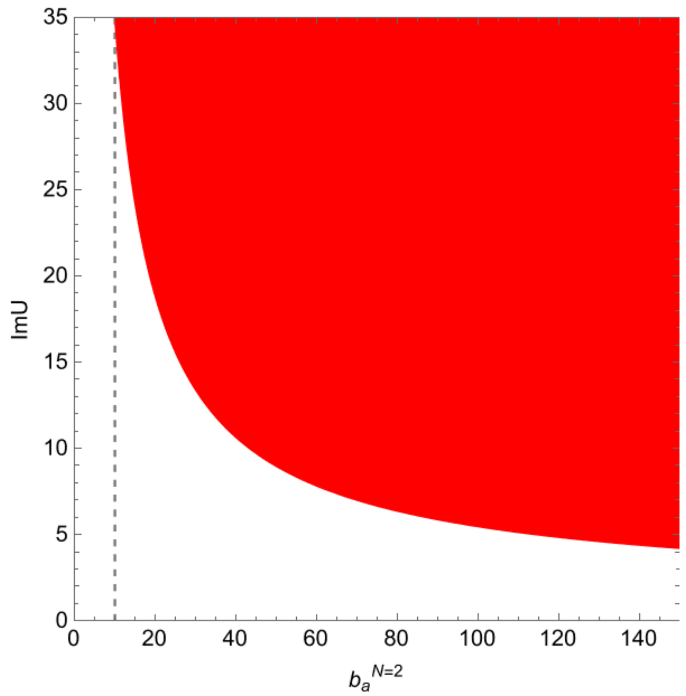


Stringy constraints on the complex-structure modulus

Dilaton : $\mathrm{Im}S = 2$

The constraints for the imaginary part of the CS modulus

$$U_2 < -\frac{3}{\pi} W_{-1}\left(-\frac{e^{-1+\gamma_E}}{4|\eta(\omega)|^4} e^{-\frac{16\pi^2 \mathrm{Im}S}{b_a^{\mathcal{N}=2}}}\right)$$

$W_{-1}(z)$: **Lambert W function**

(-1 branch)

In the red region, the threshold correction is not only comparable to the tree-level contribution, but also larger than it

In other words, **the one-loop term dominates** over the tree-level term, so the perturbative expansion is no longer **under control** and the effective theory loses its predictive power

Various analyses ($T = U$)

Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

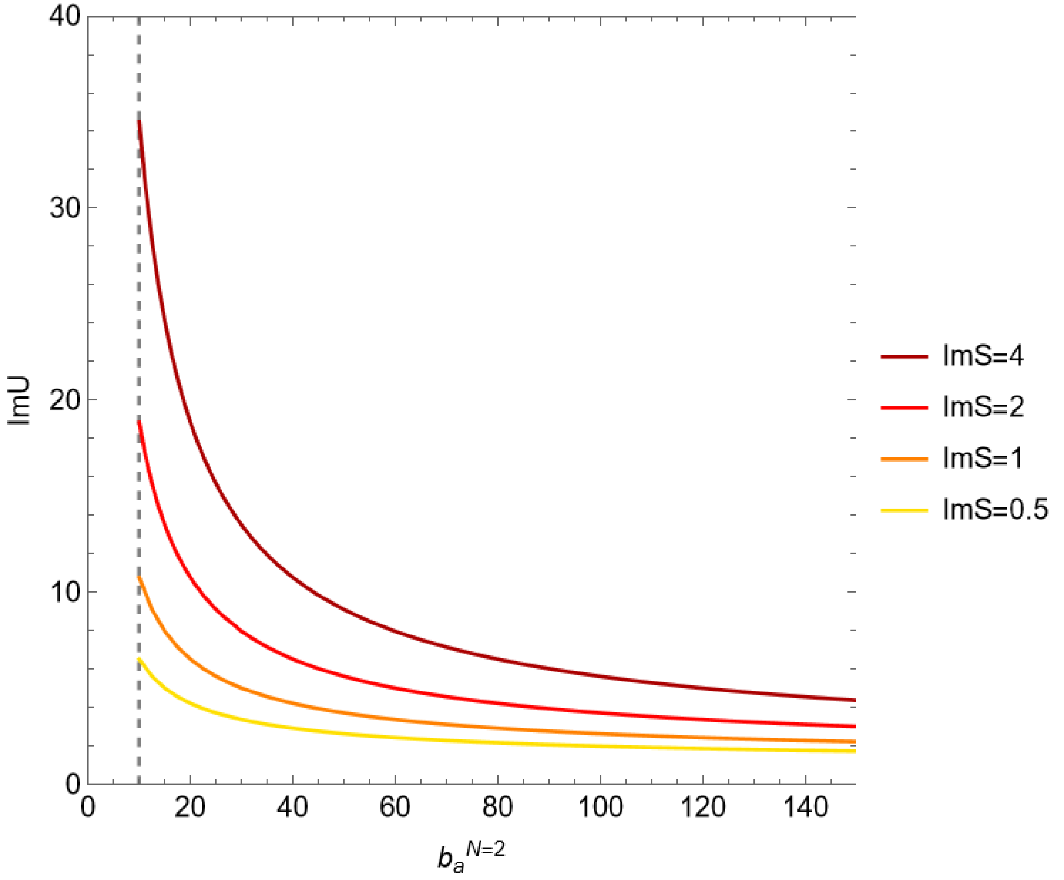
<u>Stringy constraints</u>

$$U_2 < -\frac{3}{\pi} W_{-1}\left(-\frac{\pi}{3}\left(\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}}\right)^{-\frac{1}{2}} e^{-\frac{8\pi^2 \mathrm{Im} S}{b_a^{\mathcal{N}=2}}}\right)$$

the boundaries concerning $T = U$ when the dilaton is varied over the range of typical values $\mathcal{O}(0.1) - \mathcal{O}(1)$

The larger $b_a^{\mathcal{N}=2}$, the stronger the stringy constraints

The smaller ImS, <u>the stronger the stringy constraints</u>



Various analyses ($T = aU$)

Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

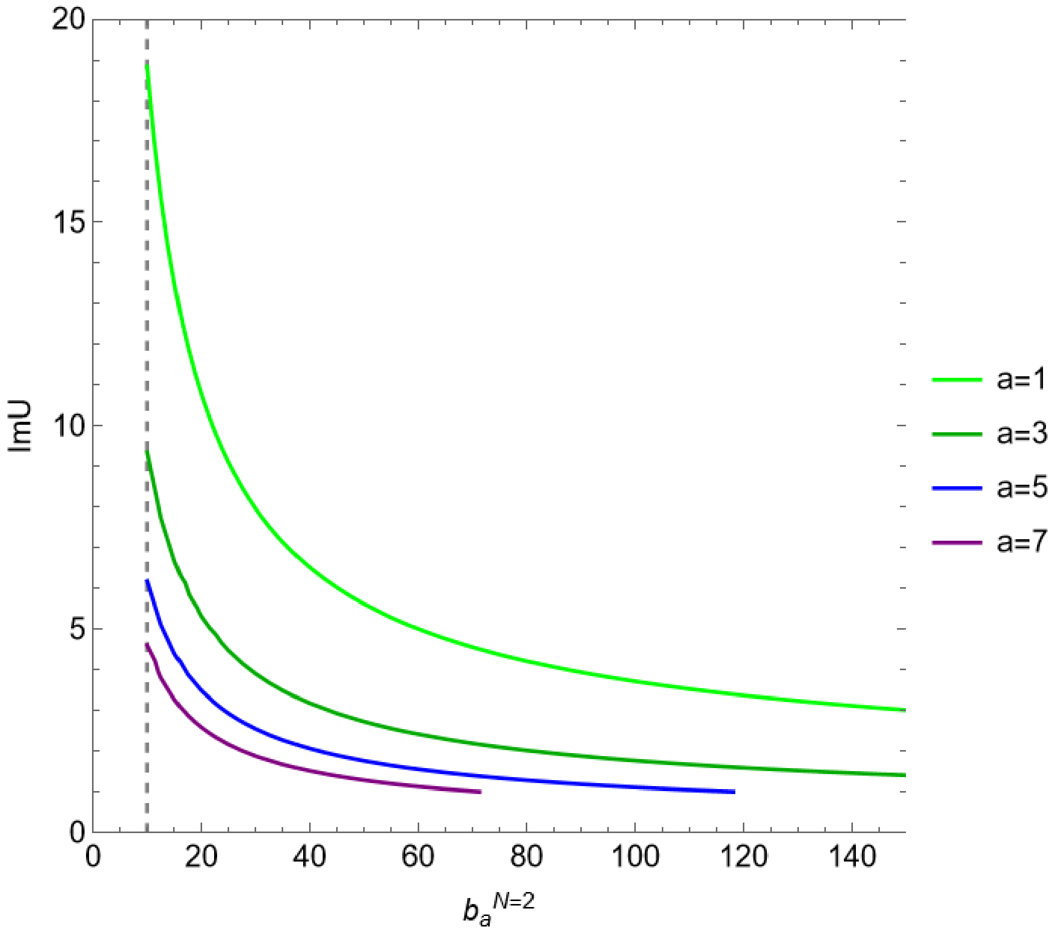
<u>Stringy constraints</u>

$$U_2 < -\frac{6}{(a+1)\pi} W_{-1}\left(-\frac{(a+1)\pi}{6\sqrt{a}}\left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_{\mathrm{E}}}\right)^{-\frac{1}{2}} e^{-\frac{8\pi^2 \mathrm{Im}S}{b_a^{\mathcal{N}=2}}}\right)$$

Boundaries concerning $T = aU$ and $\mathrm{Im}S = 2$ when the parameter is varied over the range from $a = 1$ to $a = 7$

The larger $b_a^{\mathcal{N}=2}$, the stronger the stringy constraints

The larger the ratio between T and U, <u>the stronger the stringy constraints</u>



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Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

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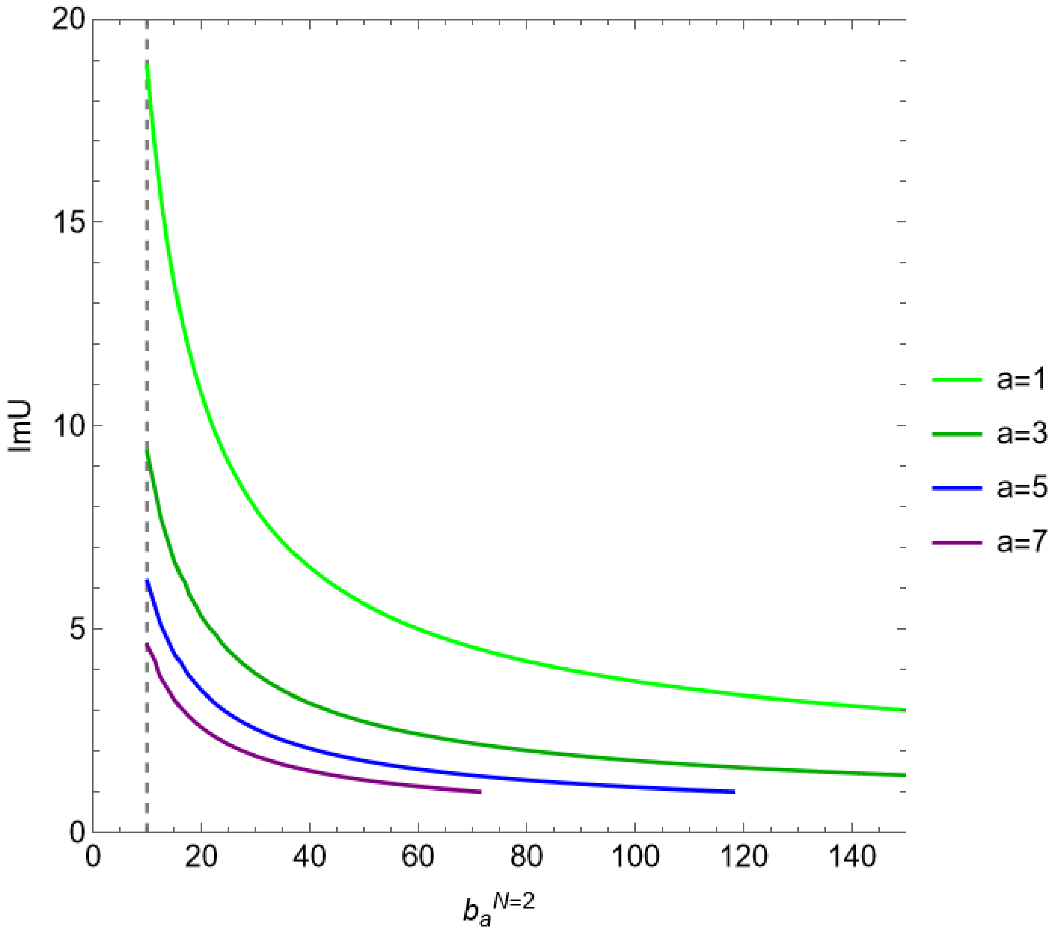
Boundaries concerning $T = aU$ and $\mathrm{Im}S = 2$ when the parameter is varied over the range from $a = 1$ to $a = 7$

The larger $b_a^{\mathcal{N}=2}$, the stronger the stringy constraints

The larger the ratio between T and U, the stronger the stringy constraints

For the region $\mathrm{Im}U < 1$, **the numerical analysis** is necessary to investigate the stringy constraints



Numerical analyses

Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

Stringy constraints

$$\frac{3\sqrt{3}}{8\pi}e^{-1+\gamma_E}\exp\left[-\frac{16\pi^2 k_a \mathrm{Im}S}{b_a^{\mathcal{N}=2}}\right] < 7U_2|\eta(7U)|^4\ U_2|\eta(U)|^4$$

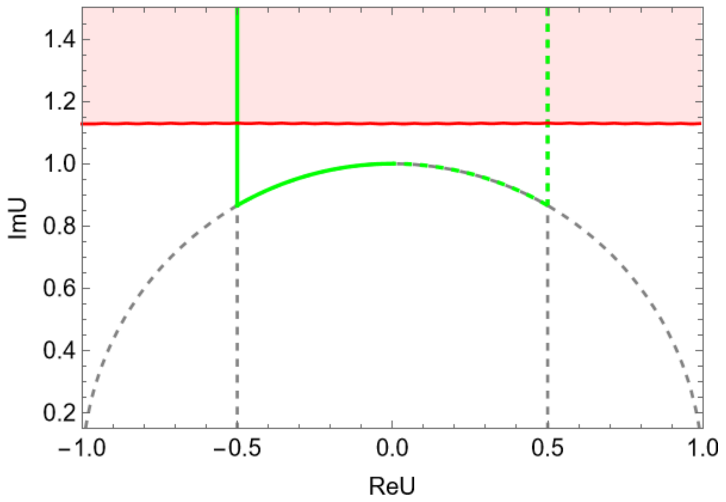
$T = 7U$

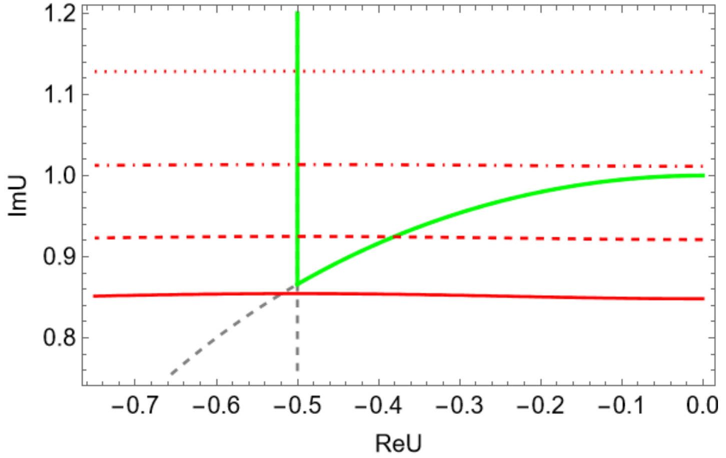
Dilaton : $\mathrm{Im}S = 2$

Dotted line : $b_a^{\mathcal{N}=2} = 60$
Dash-dotted line : $b_a^{\mathcal{N}=2} = 70$
Dashed line : $b_a^{\mathcal{N}=2} = 80$
Solid line : $b_a^{\mathcal{N}=2} = 90$

The red region is the parameter space where the perturbative theory cannot be guaranteed.

The region framed by the green curve is the fundamental region of $PSL(2,\mathbb{Z})$.





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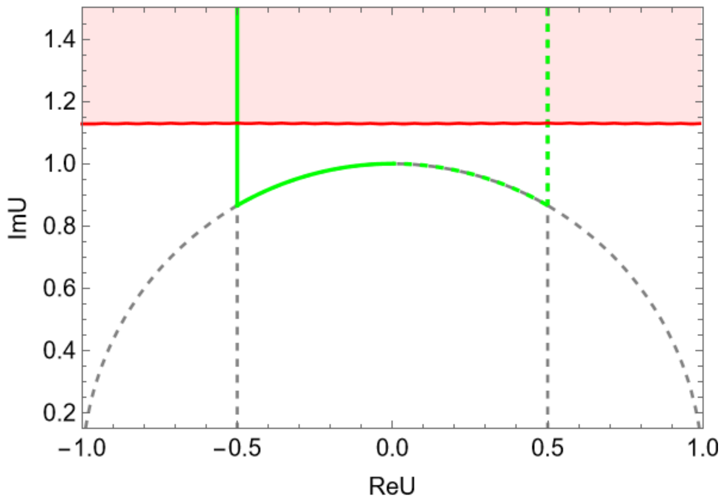
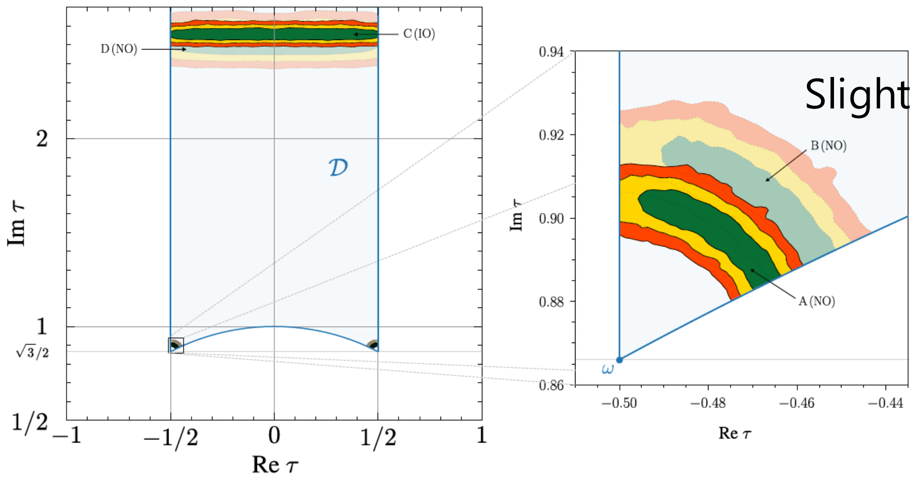


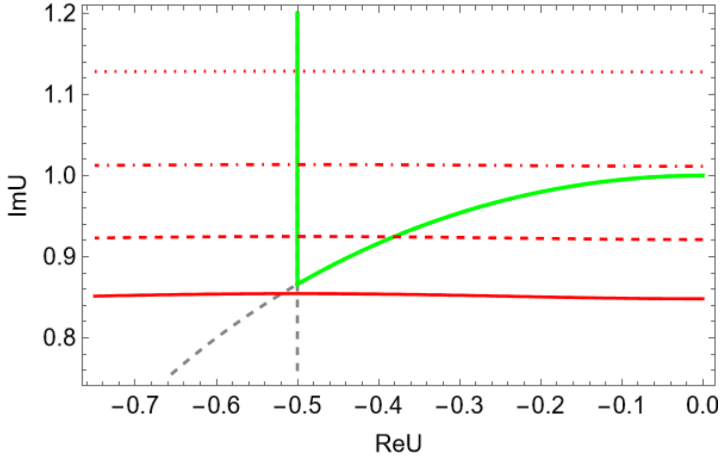
<u>Comparing our result with modular flavor models</u>

They can produce the lepton data equally well, on the other hand, the bottom-up approach gives **no principle** to tell <u>which fixed point should be chosen</u>

<u>Examples</u>



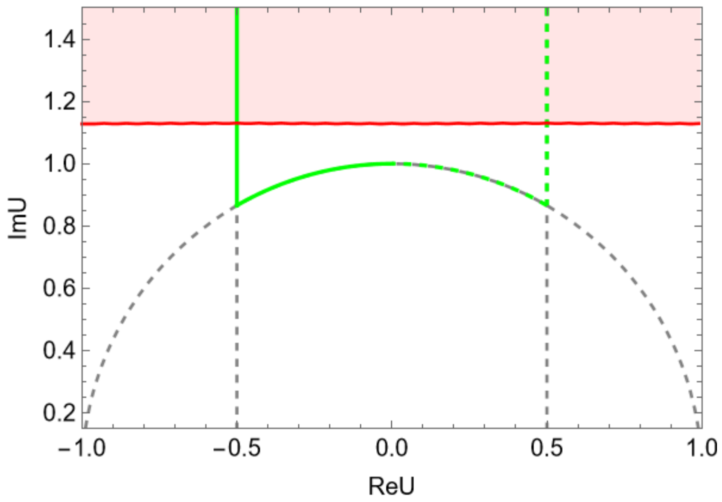
Slight deviation



Numerical analyses

Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

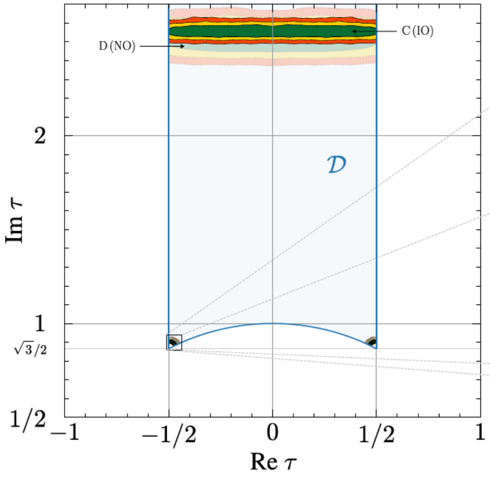
Granelli, Meloni, Parriciatu, Penedo, Petcov, 2505.21405

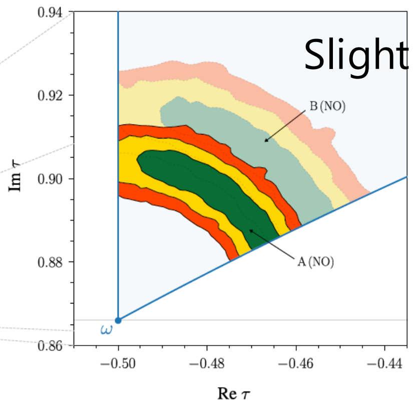


Comparing our result with modular flavor models

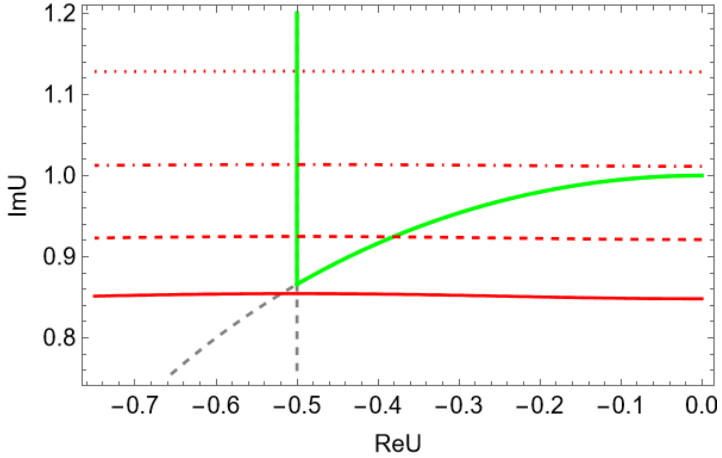
If they restrict $\mathrm{Im}U$ from above, the region with a **large imaginary part** is **the first one to be affected**, while the region close to $\boldsymbol{\tau \sim \omega}$ survives up to a larger value of the beta-function coefficient

Examples





Slight deviation



Outline

Summary

Main idea

For modular flavor models, **the perturbation theory** in the string vacua is necessary

Considering the renormalized gauge couplings of the effective theory regarding **heterotic string,**

Tree level > One-loop level

Conclusion

1. The larger $b_a^{\mathcal{N}=2}$, the stronger the constraints
2. The smaller $\mathrm{Im}S$, the stronger the constraints
3. The larger the ratio between T and U, the stronger the constraints

The stringy constraints can provide interesting insights when we construct modular flavor models

Future work

Construction of the modular flavor models and the modular inflation models,
which are consistent with **these stringy constraints**

Back up

Target space duality symmetries

Here, we introduce **the target space duality symmetry** (or modular symmetry) to classify the threshold correction systematically

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$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ S^4 = 1, (ST)^3 = 1$$

$\boldsymbol{PSL(2,\mathbb{Z})}$ **group** : $SL(2,\mathbb{Z})/\{\mathbb{I}, -\mathbb{I}\}$

Linear fractional transformation : $\tau \to \dfrac{a\tau+b}{c\tau+d}$, $\quad S : \tau \to -\dfrac{1}{\tau},\ T : \tau \to \tau+1, \quad S^2 = 1,\ (ST)^3 = 1$

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Congruence subgroup

$$\bar{\Gamma}_0(n) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \middle| c \equiv 0 \pmod n \right\}$$

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e.g. $\bar{\Gamma}_T^0(2) \times (\bar{\Gamma}_U)_0(2)$

$$T \to T+2, \quad T \to \frac{T}{T+1},$$

$$U \to U+1, \quad U \to -\frac{U}{2U-1},$$

We can classify the threshold corrections by <u>invariance</u> under **each modular transformation**

General structure

Ishiguro, TK, Kobayashi, Otsuka, 2508.12392

The threshold corrections based on other modular symmetries

(2) $\Gamma^0_T(2) \times (\Gamma_U)_0(2)$

$$\Delta_a = -b_a^{\mathcal{N}=2} \ln \left[\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}} \; T_2 |\eta(T)|^2 \left| \eta\left(\frac{T}{2}\right) \right|^2 \; U_2 |\eta(U)|^2 |\eta(2U)|^2 \right]$$

(3) $\Gamma^0_T(2) \times PSL(2,\mathbb{Z})_U$

$$\Delta_a = -b_a^{\mathcal{N}=2} \ln \left[\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}} \; \frac{1}{4^{2/3}} \; T_2 \left| \eta\left(\frac{T}{2}\right) \right|^4 \; U_2 |\eta(U)|^4 \right]$$

(4) $\Gamma^0_T(3) \times (\Gamma_U)_0(3)$ and $PSL_{\hat{T}}(2,\mathbb{Z}) \times PSL_{\hat{U}}(2,\mathbb{Z})$

$$\Delta_a = -b_a^{\mathcal{N}=2} \ln \left[\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}} \; T_2 |\eta(T)|^2 \left| \eta\left(\frac{T}{3}\right) \right|^2 \; U_2 |\eta(U)|^2 |\eta(3U)|^2 \right]$$
$$-b_a^{\mathcal{N}=2} \ln \left[\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}} \; \hat{T}_2 |\eta(\hat{T})|^4 \hat{U}_2 |\eta(\hat{U})|^4 \right]$$

(5) $\Gamma^0_T(3) \times (\Gamma_U)_0(3)$ and $\Gamma^0_{\hat{T}}(2)$

$$\Delta_a = -b_a^{\mathcal{N}=2} \ln \left[\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}} \; T_2 |\eta(T)|^2 \left| \eta\left(\frac{T}{3}\right) \right|^2 \; U_2 |\eta(U)|^2 |\eta(3U)|^2 \right]$$
$$-b_a^{\mathcal{N}=2} \ln \left[\frac{8\pi}{3\sqrt{3}} e^{1-\gamma_{\mathrm{E}}} \; \frac{1}{4^{2/3}} \; \hat{T}_2 \left| \eta\left(\frac{\hat{T}}{2}\right) \right|^4 \frac{\sqrt{3}}{2} |\eta(\omega)|^4 \right]$$

Target space duality symmetry (or modular symmetry) can classify the threshold corrections systematically

Large volume limit

Large volume regime $T \gg 1$ is very interesting in the context of string compactification and modular flavor model

Abe, Kobayashi, Otsuka, *JHEP* **04** (2015) 160

Ding, Jiang, Zhao, *JCAP* **10** (2024) 016

King, Wang, *JCAP* **07** (2024) 073

Stringy constraints

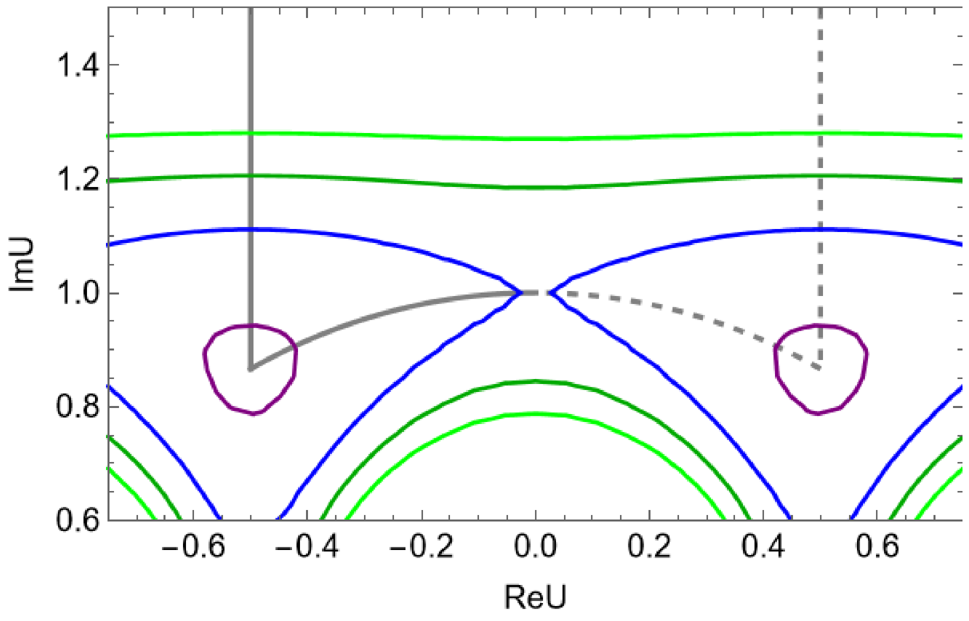
$$\frac{3\sqrt{3}}{8\pi}e^{-1+\gamma_{\mathrm{E}}}\exp\left[-\frac{16\pi^2 k_a \mathrm{Im}S}{b_a^{\mathcal{N}=2}}\right] < T_2|\eta(T)|^4\ U_2|\eta(U)|^4$$

Dilaton : $\mathrm{Im}S = 2$

Beta-function coefficient : $b_a^{\mathcal{N}=2} = 60$

The allowed region of the complex-structure modulus is **severely restricted**

Ishiguro, TK, Kobayashi, Otsuka, 2508.12392



Green, dark green, blue and purple curves are respectively the boundaries with $T = 10i$, $T = 10.02i$, $T = 10.04i$ and $T = 10.06i$. The perturbative description is valid inside of these curves.

Constraints based on other modular symmetries

- $T = aU$

$$U_2 < -\frac{8}{(a+2)\pi} W_{-1}\left(-\frac{(a+2)\pi}{8\sqrt{a}}\left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_{\mathrm{E}}}\right)^{-\frac{1}{2}} \quad e^{-\frac{8\pi^2 \mathrm{Im}S}{b_a^{\mathcal{N}=2}}}\right),$$

- $aT = U$

$$T_2 < -\frac{8}{(1+2a)\pi} W_{-1}\left(-\frac{(1+2a)\pi}{8\sqrt{a}}\left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_{\mathrm{E}}}\right)^{-\frac{1}{2}} \quad e^{-\frac{8\pi^2 \mathrm{Im}S}{b_a^{\mathcal{N}=2}}}\right),$$

Constraints based on other modular symmetries

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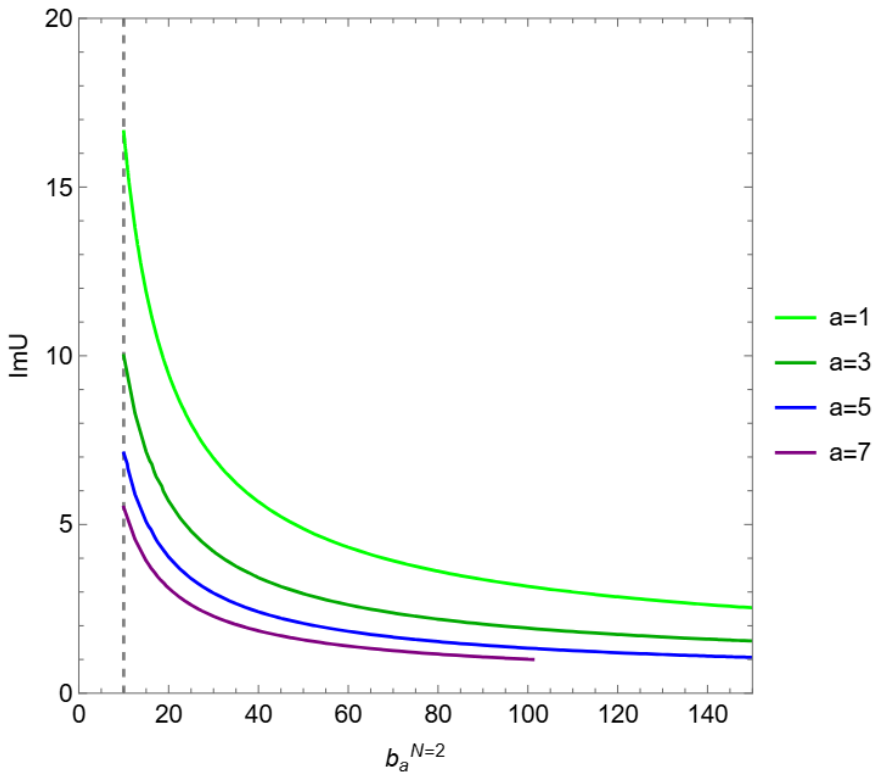
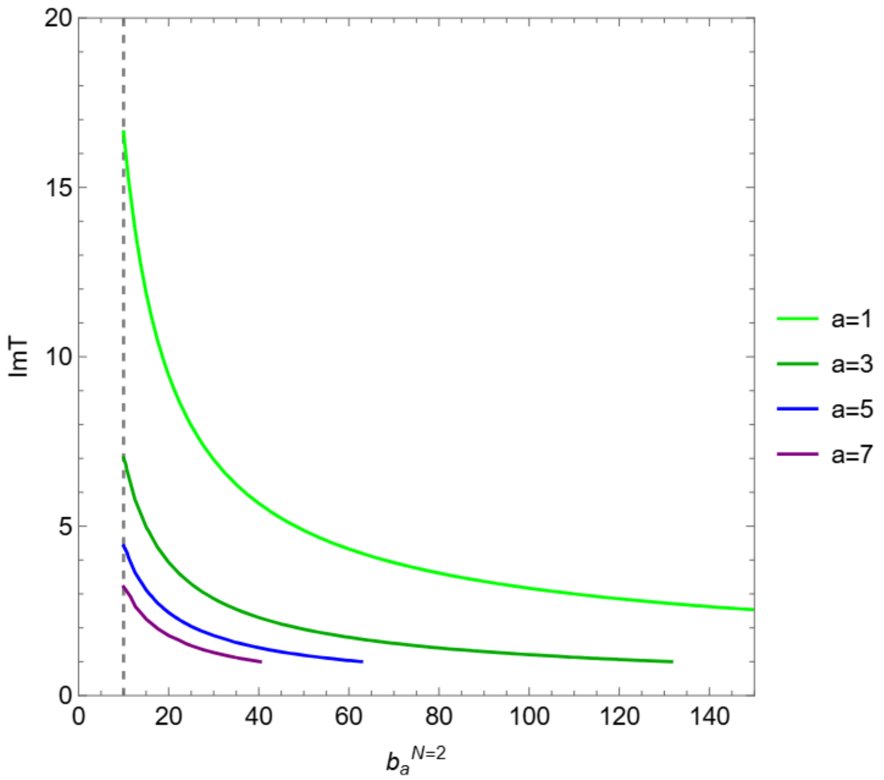


$aT = U$



Constraints based on other modular symmetries

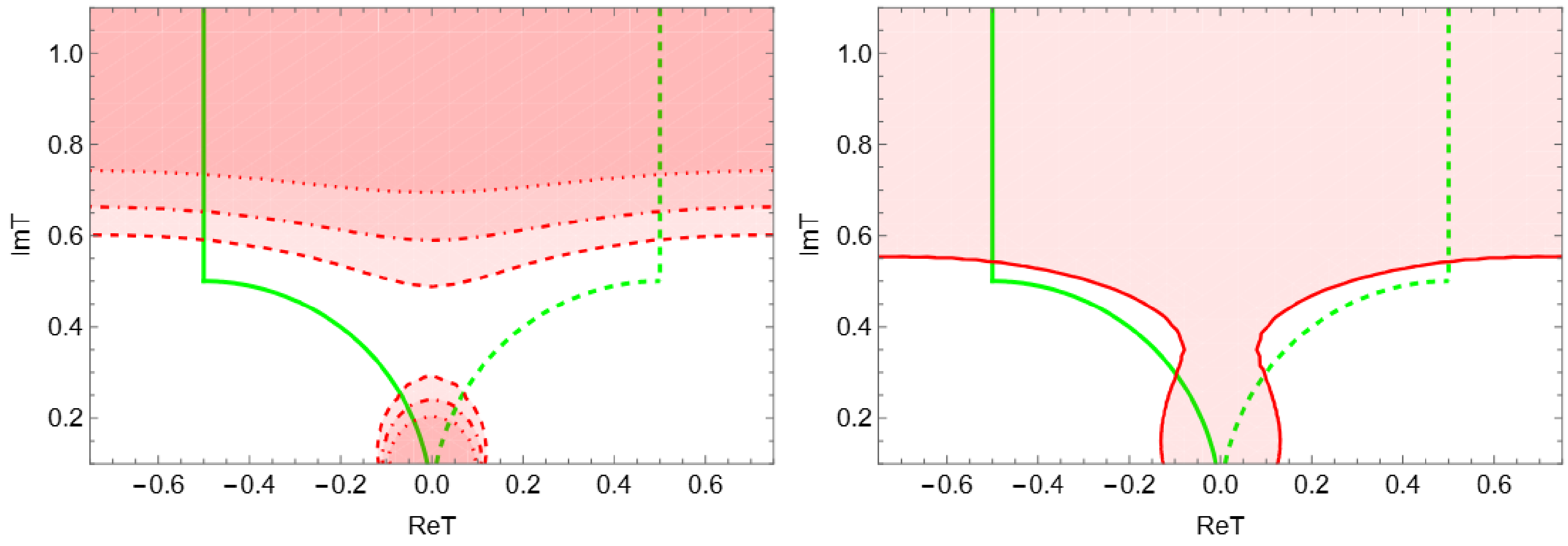


Figure 10: These are the boundaries concerning $7T = U$ and $\mathrm{Im}S = 2$. Dotted red, dash-dotted red, dashed red curves are respectively the boundaries with $b_a^{\mathcal{N}=2} = 60$, $b_a^{\mathcal{N}=2} = 70$, $b_a^{\mathcal{N}=2} = 80$ in the left panel, and solid red curve is the boundary with $b_a^{\mathcal{N}=2} = 90$ in the right panel. In both panels, the shaded regions are outside the perturbative theory, and in the left panel, when the $b_a^{\mathcal{N}}$ increases, the lighter the color is. The region framed by the green curve is the fundamental region of $\Gamma_0(2)$.

Constraints based on other modular symmetries

- $T = aU,\ \hat{T} = bU,\ \hat{U} = bcU$

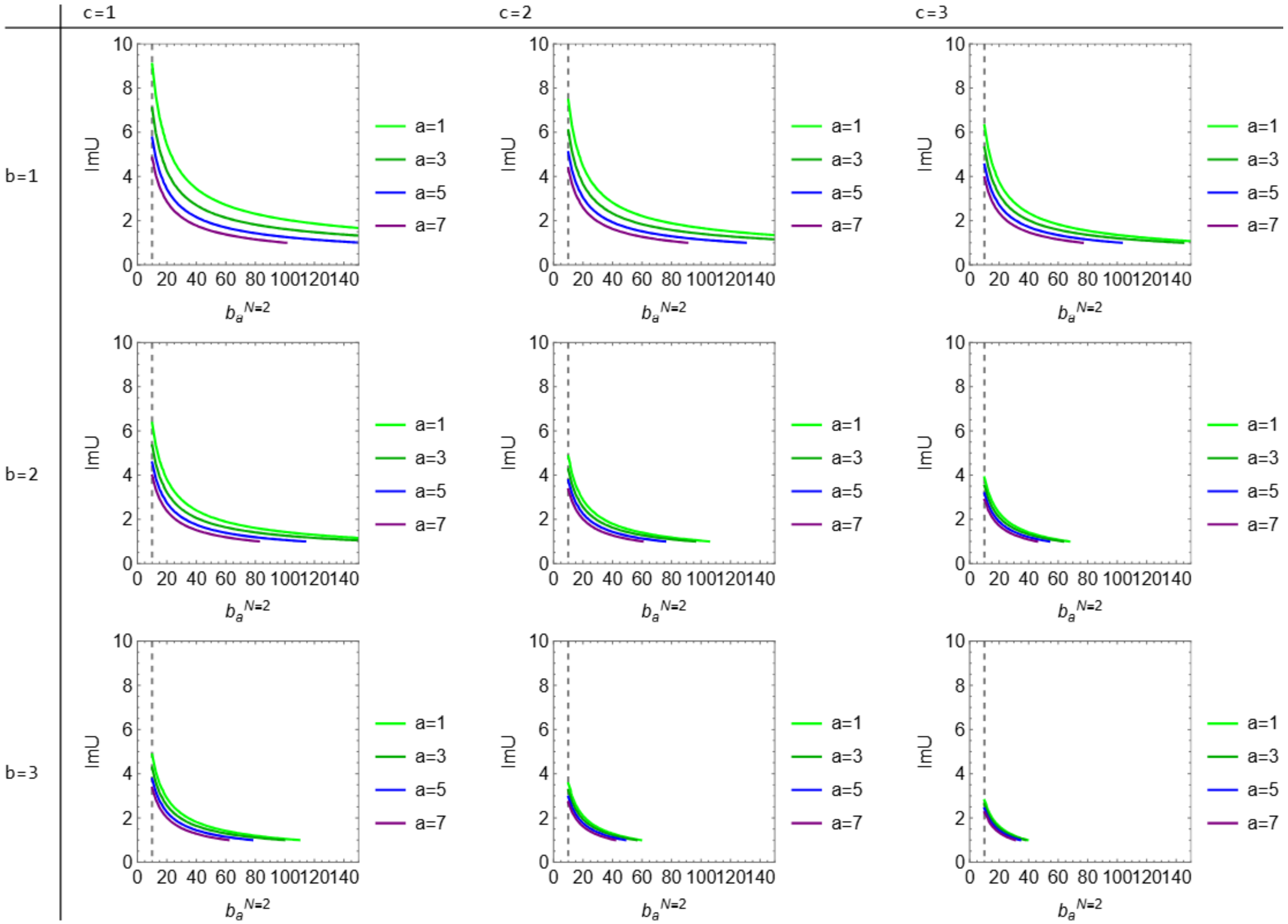
$$U_2 < -\frac{36}{(2a+6+3b(c+1))\pi} \times W_{-1}\left(-\frac{(2a+6+3b(c+1))\pi}{36a^{\frac{1}{4}}b^{\frac{1}{2}}c^{\frac{1}{4}}}\left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_{\mathrm{E}}}\right)^{-\frac{1}{2}} e^{-\frac{4\pi^2 \mathrm{Im}S}{b_a^{\mathcal{N}=2}}}\right), \tag{5.17}$$

- $aT = U,\ \hat{T} = bT,\ \hat{U} = bcT$

$$T_2 < -\frac{36}{(2+6a+3b(c+1))\pi} \times W_{-1}\left(-\frac{(2+6a+3b(c+1))\pi}{36a^{\frac{1}{4}}b^{\frac{1}{2}}c^{\frac{1}{4}}}\left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_{\mathrm{E}}}\right)^{-\frac{1}{2}} e^{-\frac{4\pi^2 \mathrm{Im}S}{b_a^{\mathcal{N}=2}}}\right), \tag{5.18}$$

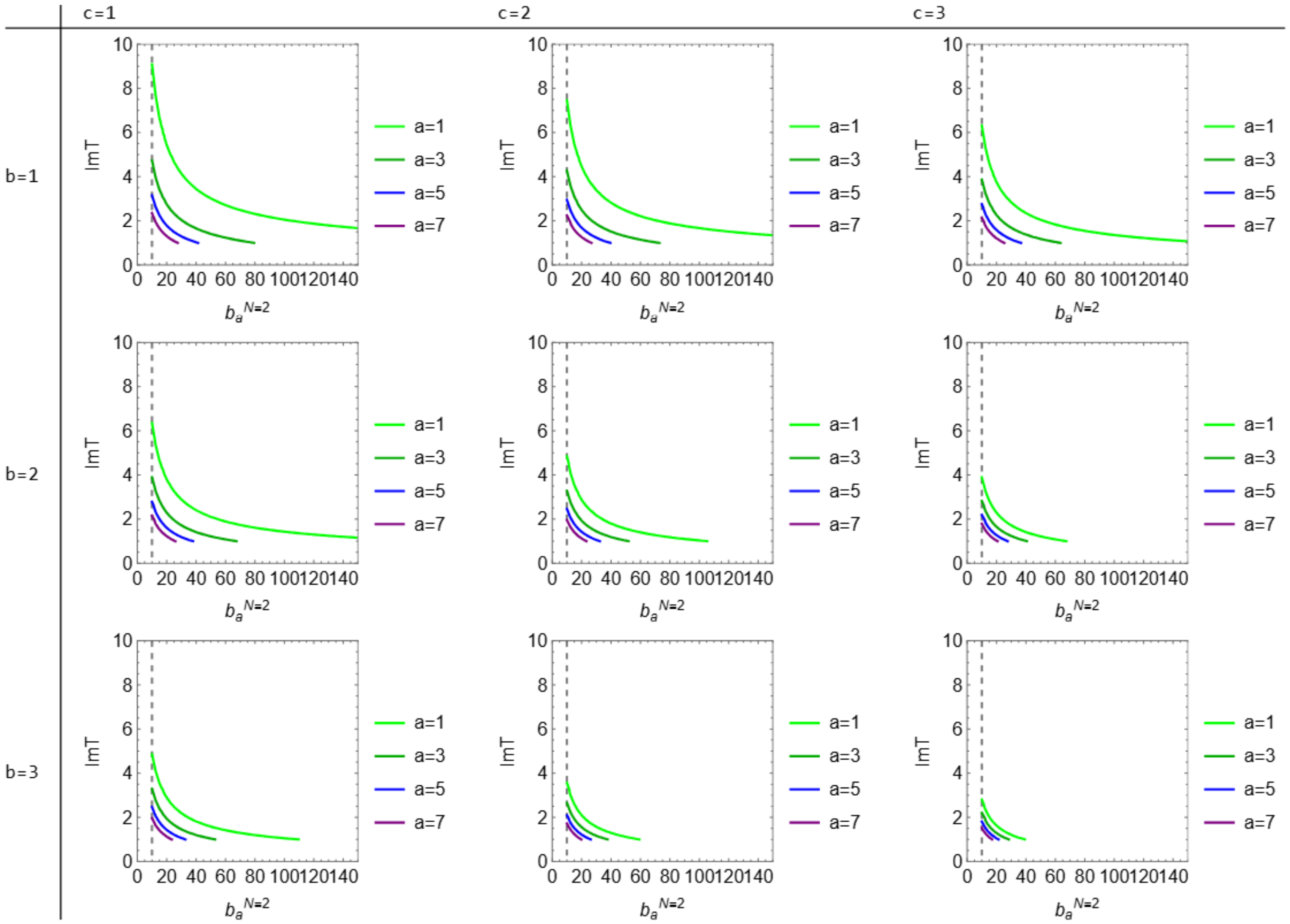
Constraints based on other modular symmetries

$T = aU,$
$\hat{T} = bU,$
$\hat{U} = bcU$



Constraints based on other modular symmetries

$aT = U,$
$\hat{T} = bU,$
$\hat{U} = bcU$



Constraints based on other modular symmetries

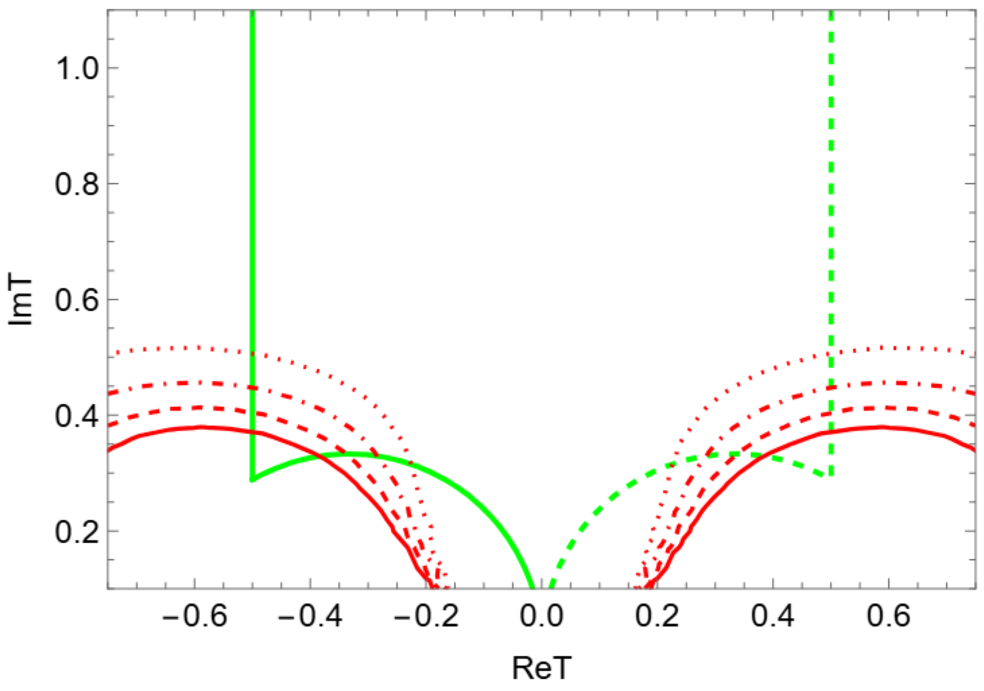


Figure 13: These are the boundaries concerning $7T = U$, $\hat{T} = T$, $\hat{U} = T$ and $\mathrm{Im}S = 2$. Dotted red, dash-dotted red, dashed red and solid red curves are respectively the boundaries with $b_a^{\mathcal{N}=2} = 60$, $b_a^{\mathcal{N}=2} = 70$, $b_a^{\mathcal{N}=2} = 80$ and $b_a^{\mathcal{N}=2} = 90$. The region framed by the green curve is the fundamental region of $\Gamma_0(3)$. The perturbative description is valid below these curves.