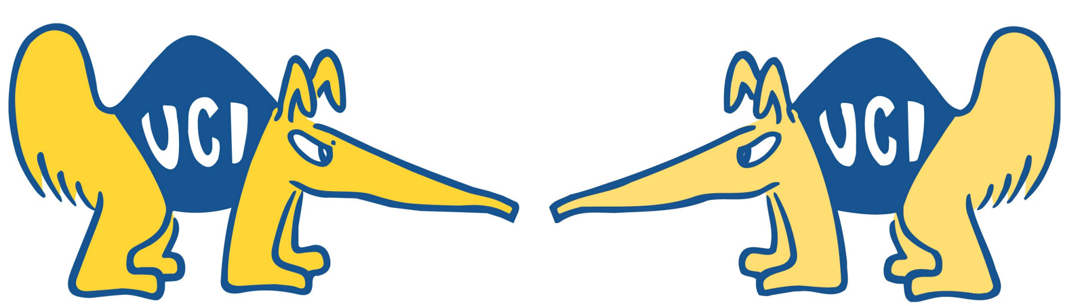
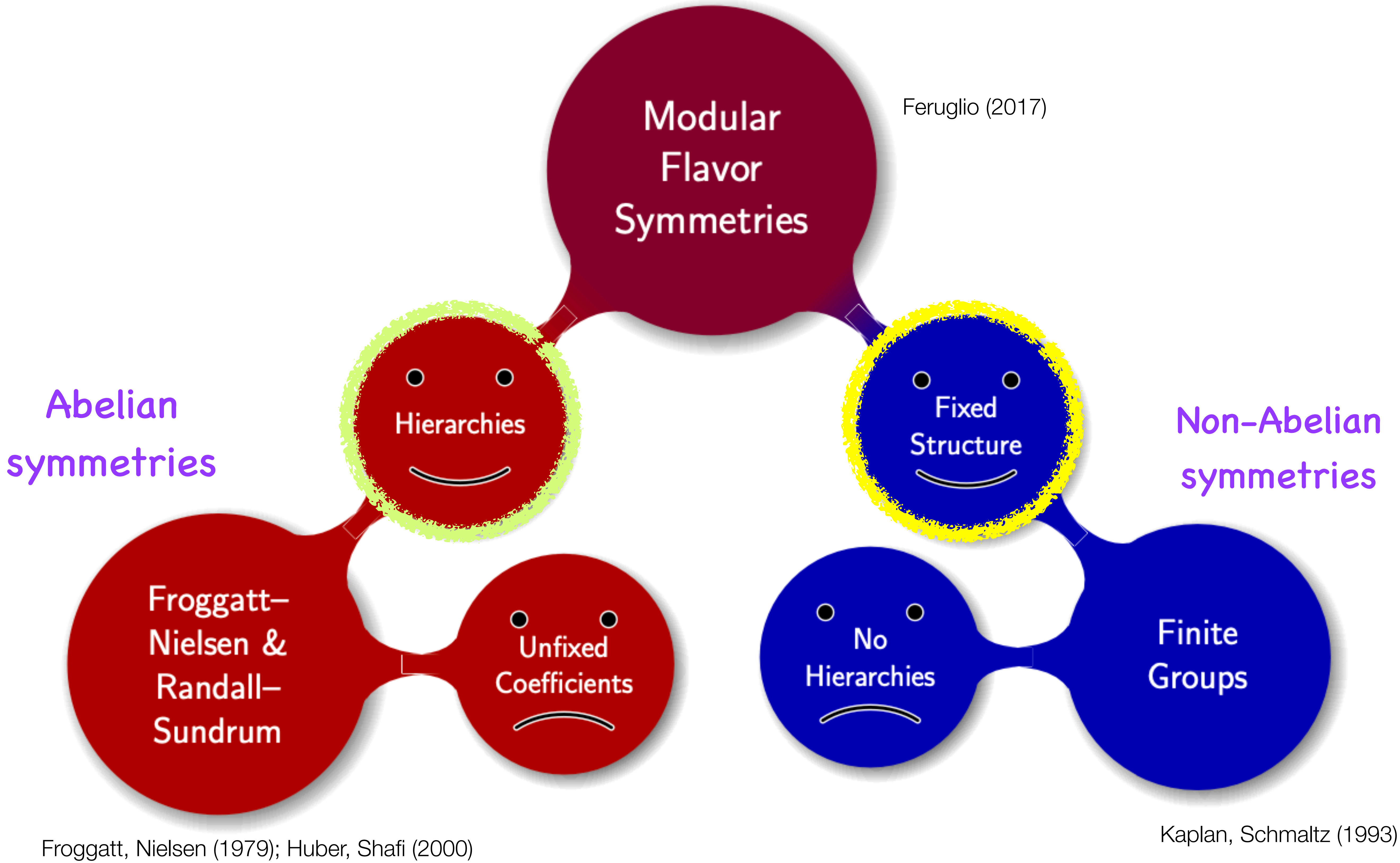
Modular Flavor Symmetries and Fermion Mass Hierarchy

Mu-Chun Chen, University of California at Irvine



FLASY 2026, GyeongJu, South Korea, August 19, 2026

Theories of Flavor



Modular Flavor Symmetries

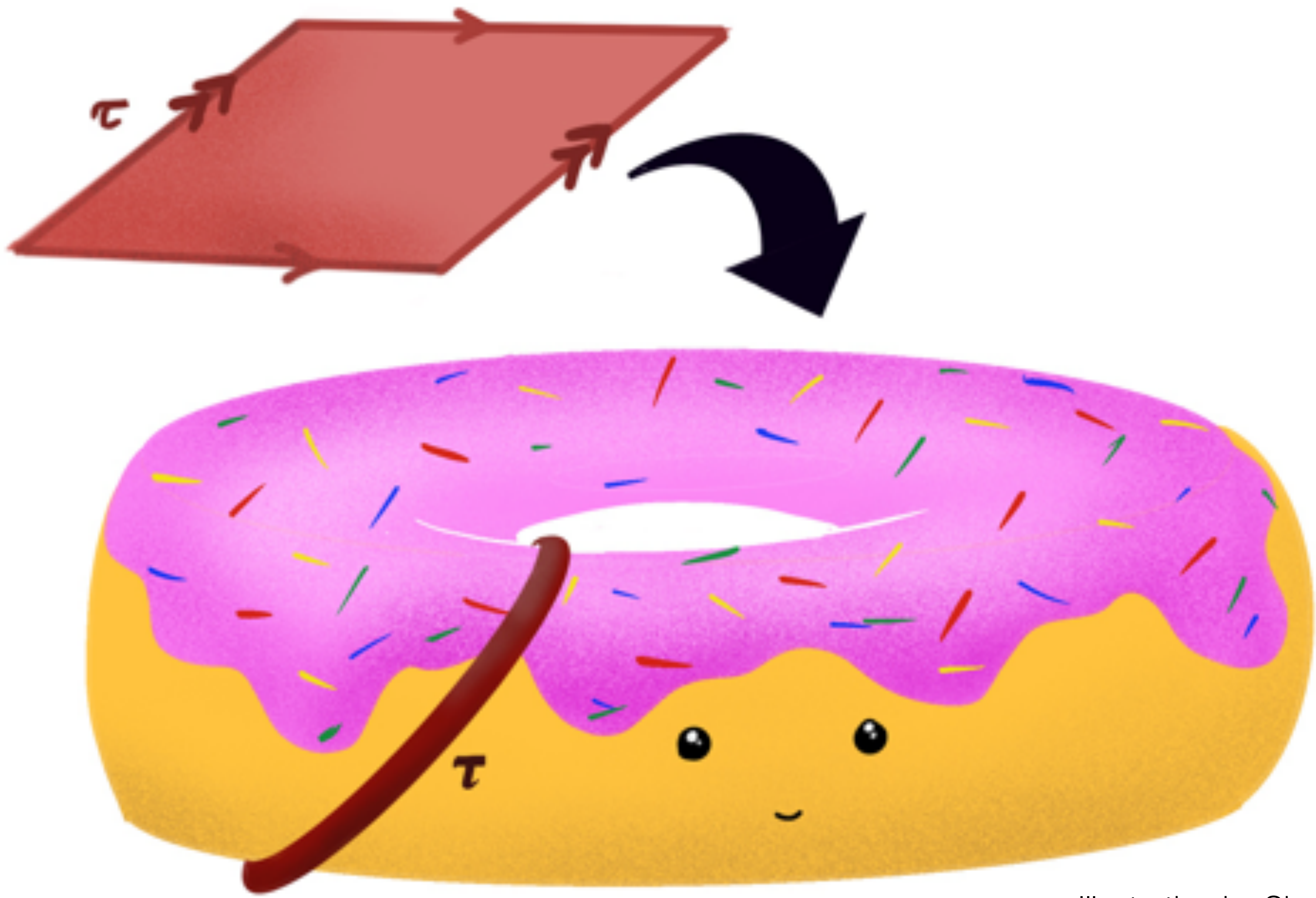
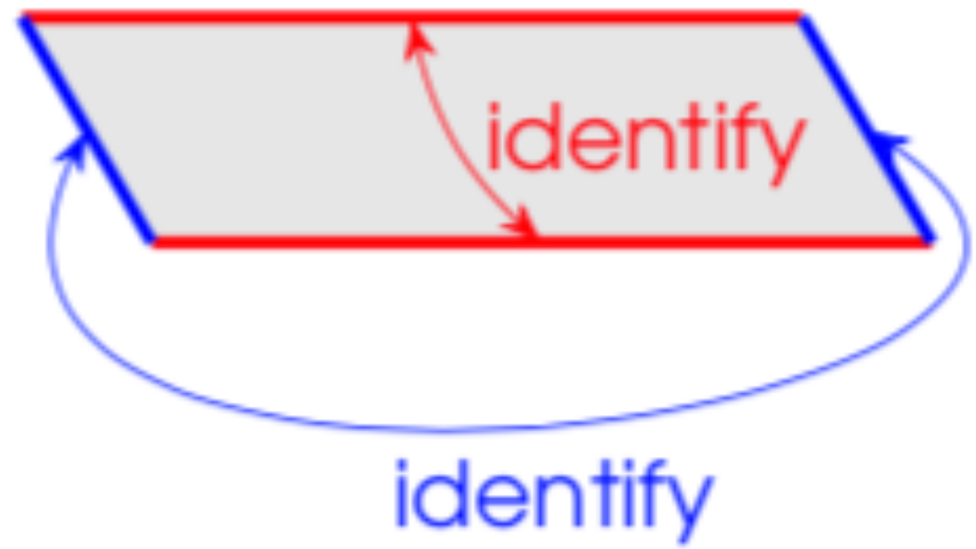
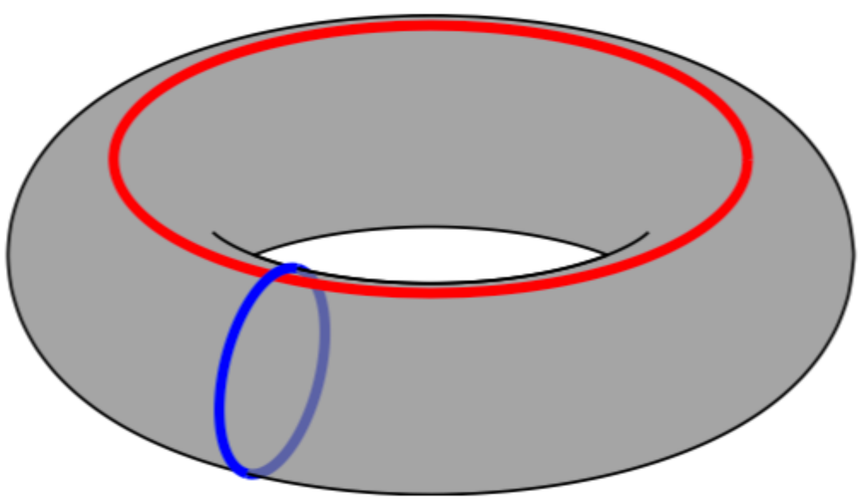


Illustration by Shreya Shukla

Donuts = TORI

constructed
from
parallelogram

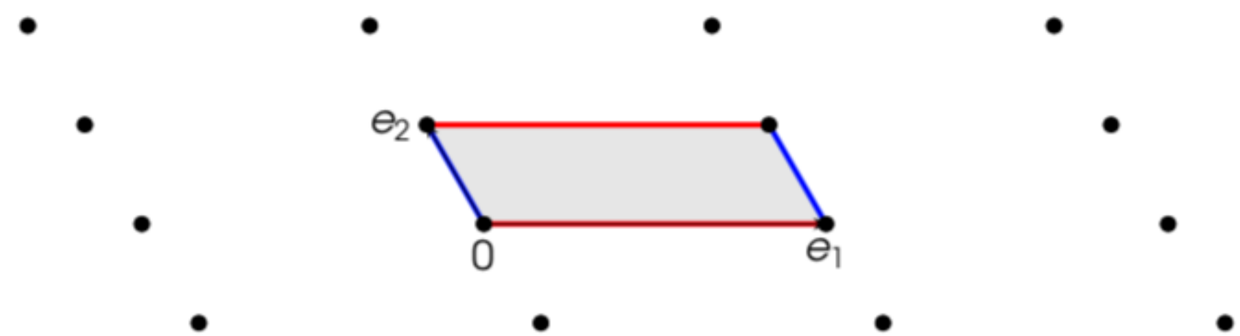




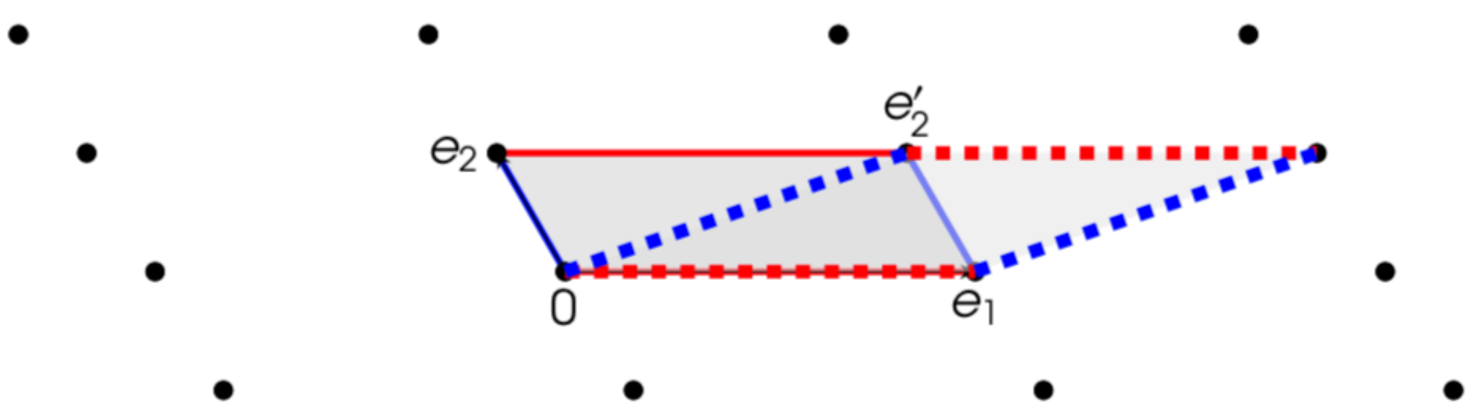
two cycles

Modular Symmetries

edges ⇒ lattice basis vectors



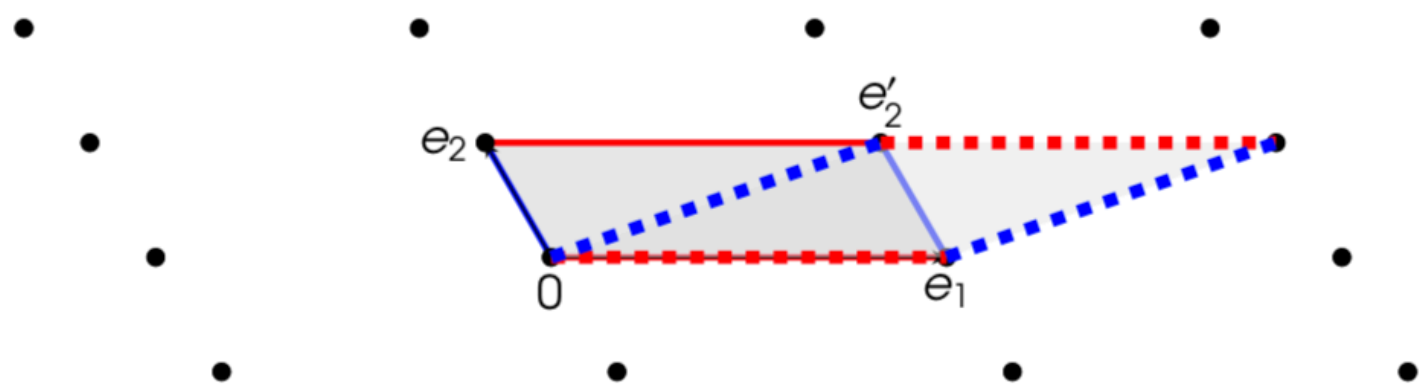
points in plane identified if differ by a lattice translation



Equivalent TORI related by Modular Symmetries

Modular Symmetries

- TORI: fundamental domain not unique



- Basis Vectors are related: $\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \overset{\gamma}{\mapsto} \begin{pmatrix} e_2' \\ e_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} =: \gamma \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}$

$a, b, c, d \in \mathbb{Z}$

- Volume of fundamental domain the same $\Rightarrow$ $\det \gamma = 1$

Modular Symmetries

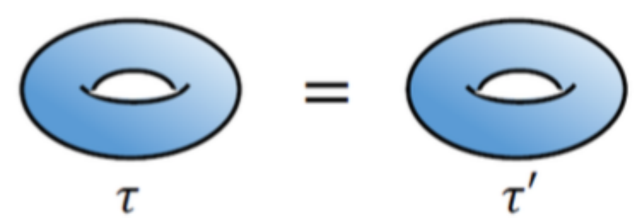
- Two basic transformations:

$$T \; : \; e_2 \mapsto e_2' = e_2 + e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T$$

$$S \; : \; e_1 \mapsto e_1' = e_2 \quad \text{and} \quad e_2 \mapsto e_2' = -e_1 \qquad \curvearrowright \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: S$$

- In complex coordinates: modulus $\tau = e_2/e_1$

$$\tau \xmapsto{S} \frac{-1}{\tau} \quad \text{and} \quad \tau \xmapsto{T} \tau + 1$$



- S and T generate $\mathrm{SL}(2, \mathbb{Z})$ and satisfy

$$\boxed{S^2 = (S\,T)^3 = \mathbb{1}}$$

Modular Symmetries

- **Finite Modular Group** (quotient group): $\Gamma_N := \Gamma/\Gamma(N)$ where principal congruence group Γ(N) is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \underbrace{SL(2,\mathbb{Z})/\mathbb{Z}_2}_{\Gamma} ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

- Generators of the quotient group Γ_N satisfy

$$\boxed{S^2 = 1, \quad (ST)^3 = 1, \quad T^N = 1}$$

- Some examples

$$\Gamma_2 \simeq S_3, \quad \Gamma_3 \simeq A_4, \quad \Gamma_4 \simeq S_4, \quad \Gamma_5 \simeq A_5$$

Modular Flavor Symmetries

Feruglio (2017)

- Imposing modular symmetry Γ on Lagrangian:

$$\mathscr{L} \supset \sum Y_{i_1,\, i_2,\, \ldots\ldots\, i_n} \Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_n}$$

For non-SUSY case, see talk by Takaaki Nomura

$$\tau \xrightarrow{\gamma} \gamma\tau := \frac{a\tau + b}{c\tau + d},$$

$$\Phi_j \xrightarrow{\gamma} (c\tau + d)^{k_j} \boxed{\rho_{r_j}(\gamma)} \Phi_j\,, \quad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

k_i : integers

representation matrix of Γ_N

- Yukawa Couplings = Modular Forms at level "N" w/ weight "k"

$$f_i(\gamma\tau) = (c\tau + d)^{-k} \boxed{[\rho_N(\gamma)]_{ij}} f_j(\tau)$$

$k = k_{i_1} + k_{i_2} + \ldots + k_{i_n}$

representation matrix of Γ_N

A Toy Modular A_4 Model

Feruglio (2017)

- Weinberg Operator $$\mathscr{W}_\nu = \frac{1}{\Lambda}[(H_u \cdot L)\ Y\ (H_u \cdot L)]_{\mathbf{1}}$$

L: 3 generations of lepton doubles

- Traditional A4 Flavor Symmetry
  - Yukawa Coupling Y $\rightarrow$ Flavon VEVs (A_4 triplet, 6 real parameters)

$$Y \rightarrow \langle \phi \rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Longrightarrow \quad m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2a & -c & -b \\ -c & 2b & -a \\ -b & -a & 2c \end{pmatrix}$$

- Modular A4 Flavor Symmetry
  - Yukawa Coupling Y $\rightarrow$ Modular Forms (A4 triplet, 2 real parameters)

$$Y \rightarrow \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} \quad \Longrightarrow \quad m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

Modular Forms

Feruglio (2017)

- Known mathematical functions:
  - Level (N) = 3, Weight (k) = 2

$$\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} X_2^2(\tau) \\ \sqrt{2}X_1(\tau)X_2(\tau) \\ -X_1^2(\tau) \end{pmatrix}$$

$$X_1(\tau) = 3\sqrt{2}\frac{\eta^3(3\tau)}{\eta(\tau)}$$

$$X_2(\tau) = -3\frac{\eta^3(3\tau)}{\eta(\tau)} - \frac{\eta^3(\tau/3)}{\eta(\tau)}$$

Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q \equiv e^{i2\pi\tau}$$

A Toy Modular A_4 Model

Feruglio (2017)

- Input Parameters:

$$\boxed{\tau = 0.0111 + 0.9946i}$$

$$v_u^2/\Lambda$$

3 free parameters ⇒

3 mass,
3 angles,
3 CP phases

- Predictions: inverted ordering

$$\frac{\Delta m^2_{sol}}{|\Delta m^2_{atm}|} = 0.0292$$

$$\sin^2\theta_{12} = 0.295 \qquad \boxed{\sin^2\theta_{13} = 0.0447} \qquad \sin^2\theta_{23} = 0.651$$

$$\frac{\delta_{CP}}{\pi} = 1.55 \qquad \frac{\alpha_{21}}{\pi} = 0.22 \qquad \frac{\alpha_{31}}{\pi} = 1.80 \quad .$$

$$m_1 = 4.998 \times 10^{-2}\ eV \qquad m_2 = 5.071 \times 10^{-2}\ eV \qquad m_3 = 7.338 \times 10^{-4}\ eV$$

Predictive Power of Modular Symmetries

- Modular Invariance:

  modular weights + representation assignments ⇒ extremely constrained framework

  - Much fewer input parameters ⇒ predictions consistent with data

- Constraints from UV theories

Stringy Constraints: see talk by Takafumi Kai

Predictive Power of Modular Symmetries

- This Talk: Two generic predictions from properties of modular forms
  - Invariants ⇒ robust sum rules among mixing parameters: modular invariance (even τ-independent), RG invariant

    MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)
    MCC, S. Perez, M. Ratz (2026)

  - Near-critical behavior of mass matrices ⇒ Mass hierarchy at near critical points, determined by modular weights

    MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

- Other unique salient features

See talk by Michael Ratz

Predictive Power of Modular Symmetries

- Ingredients
  - Modular invariance
  - Holomorphicity
  - Finiteness
- However, typical observables are not holomorphic, e.g.

$$\mathscr{W} = \frac{\mathscr{M}(\tau)}{2}\Phi^2$$

$$\mathscr{K} = \frac{1}{(-i\tau + i\bar{\tau})^{k_\Phi}}\bar{\Phi}\Phi$$

$\Rightarrow$

$$\begin{aligned} m_{physical} &= m_{physical}(\bar{\tau}, \tau) \\ &= |\mathscr{M}(\tau)|(-i\tau + i\bar{\tau})^{k_\Phi} \end{aligned}$$

Predictive Power of Modular Symmetries

- Ingredients
  - Modular invariance
  - Holomorphicity
  - Finiteness
- However, typical observables are not holomorphic, e.g.

$$\mathscr{W} = \frac{\mathscr{M}(\tau)}{2}\Phi^2$$

$$\mathscr{K} = \frac{1}{(-i\tau + i\bar{\tau})^{k_\Phi}}\bar{\Phi}\Phi$$

$$\Rightarrow$$

$$m_{physical} = m_{physical}(\bar{\tau}, \tau)$$
$$= |\mathscr{M}(\tau)|(-i\tau + i\bar{\tau})^{k_\Phi}$$

Are there observables fulfilling the three properties?

Holomorphic Observables

- Typical model

$$\mathscr{W}_{\text{lepton}} = Y_e^{ij} L_i H_d E_j + \frac{1}{2}\kappa_{ij}(\tau) L_i H_u L_j H_u$$

In diagonal Y_e basis:

- Modular invariant holomorphic observables

$$I_{ij}(\tau) = \frac{\mathscr{M}_{ii}(\tau)\mathscr{M}_{jj}(\tau)}{(\mathscr{M}_{ij}(\tau))^2} = \frac{\kappa_{ii}\,\kappa_{jj}}{\kappa_{ij}^2} = \frac{m_{ii}(\tau,\bar{\tau})m_{jj}(\tau,\bar{\tau})}{(m_{ij}(\tau,\bar{\tau}))^2}$$

I_{ij} invariant under renormalization group

Chang, Kuo (2002)

Renormalization Group Invariants

Chang, Kuo (2002)

- In P-diagonal basis: $P = C_e Y_e^\dagger Y_e$

RG Running effects on neutrino oscillation: see talk by Chui-Fan Kong

$$\frac{d}{dt}\kappa = \tilde{P}\kappa\tilde{Q}^T + \tilde{Q}\kappa\tilde{P}^T + \tilde{\alpha}\kappa,$$

$$\tilde{P} = \mathrm{diag}\left(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3\right), \quad \tilde{Q} = \mathrm{diag}\left(\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3\right)$$

- At 1-loop: $\tilde{P} = \frac{1}{16\pi^2}P, \quad \tilde{Q} = I, \quad \tilde{\alpha} = \frac{1}{16\pi^2}\alpha$

$$\dot{\kappa}_{ij} = \kappa_{ij}\left(\tilde{P}_i\tilde{Q}_j + \tilde{P}_j\tilde{Q}_i + \tilde{\alpha}\right), \qquad \frac{d}{dt}I_{ij} = 2\left(\tilde{P}_i - \tilde{P}_j\right)\left(\tilde{Q}_i - \tilde{Q}_j\right)I_{ij}$$

$\Rightarrow I_{ij}$ is RG invariant

A Toy Modular A_4 Model

Feruglio (2017)

- Lepton sector of the (supersymmetric) standard model

	(E_1^c, E_2^c, E_3^c)	L	H_d	H_u	φ
$\mathrm{SU(2)_L \times U(1)}_Y$	$\mathbf{1}_1$	$\mathbf{2}_{-1/2}$	$\mathbf{2}_{-1/2}$	$\mathbf{2}_{1/2}$	$\mathbf{1}_0$
Γ_3	$(\mathbf{1}, \mathbf{1}', \mathbf{1}'')$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$
k	$(k_{E_1}, k_{E_2}, k_{E_3})$	k_L	k_d	k_u	k_φ

- Charged fermion masses obtained (with couplings to flavon) by adjusting 3 parameters

- Weinberg Operator: $$\mathscr{W}_\nu = \frac{1}{\Lambda}\left[(H_u \cdot L)\, Y\, (H_u \cdot L)\right]_{\mathbf{1}}$$

- Kähler potential of leptons: $$K_L = (-\mathrm{i}\tau + \mathrm{i}\overline{\tau})^{-1}\,(\overline{L}\,L)_{\mathbf{1}}$$

Invariants in Toy Modular A_4 Model

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

- Mass matrix in canonical basis:

$$m_\nu(\tau,\bar\tau) = (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau)\frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_2(\tau) & -Y_3(\tau) \\ -Y_2(\tau) & 2Y_3(\tau) & -Y_1(\tau) \\ -Y_3(\tau) & -Y_1(\tau) & 2Y_2(\tau) \end{pmatrix} =: (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau) v_u^2 \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{pmatrix}$$

- Invariants

$$I_{12}(\tau) = 4\frac{Y_1(\tau)\,Y_3(\tau)}{\big(Y_2(\tau)\big)^2}, \qquad I_{13}(\tau) = 4\frac{Y_1(\tau)\,Y_2(\tau)}{\big(Y_3(\tau)\big)^2}, \qquad I_{23}(\tau) = 4\frac{Y_2(\tau)\,Y_3(\tau)}{\big(Y_1(\tau)\big)^2}$$

- Algebraic constraint $$Y_2^2 + 2Y_1Y_3 = 0$$

- Thus

$$I_{12}(\tau) = -2, \qquad I_{13}(\tau) = -2\Big(1+\frac{1}{3}j_3(\tau)\Big)^3, \qquad I_{23}(\tau) = -\frac{32}{I_{13}(\tau)}$$

Invariants in Toy Modular A_4 Model

- Two interesting relations: **RG invariant, independent of τ**

$$I_{12}(\tau) = -2, \qquad I_{13}(\tau) I_{23}(\tau) = -32$$

- Invariants I_{ij} : functions of physical observables

$$\left(m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \alpha_{12}, \alpha_{23}\right)$$

$\Rightarrow$ **sum rules among physical observables:**

RG invariant, τ independent

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A_4 Model

- Invariants I_{ij} : functions of physical observables

$$\left(m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \alpha_{12}, \alpha_{23}\right)$$

$$I_{12} = -2$$

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2+\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)+\widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23}-\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}\right)-\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3s_{13}s_{23}\right]^2}$$

$$\widetilde{m}_1 := m_1\,\mathrm{e}^{\mathrm{i}\,\varphi_1}$$

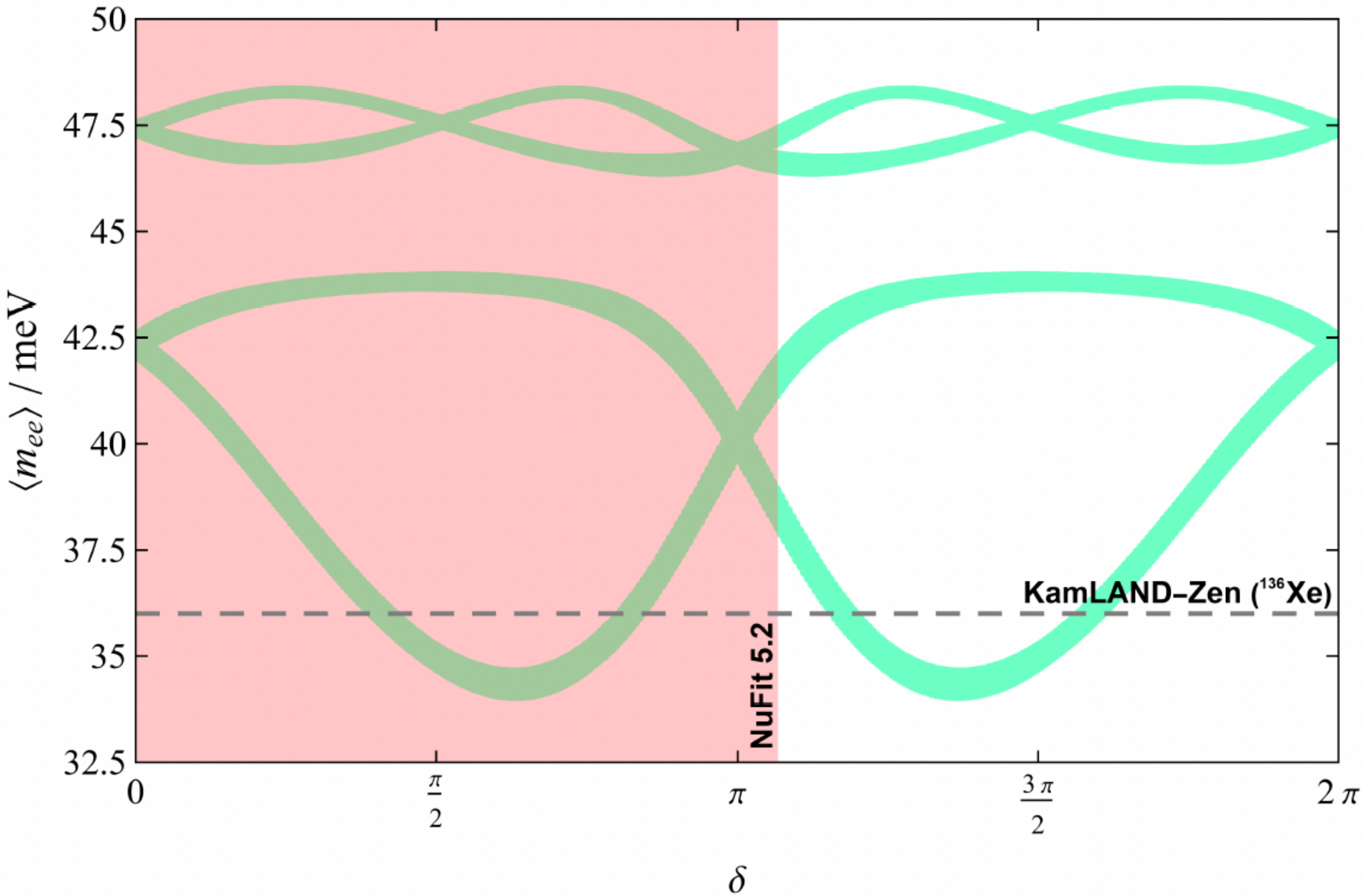
$$\widetilde{m}_2 := m_2\,\mathrm{e}^{\mathrm{i}\,\varphi_2}.$$

$$\begin{aligned}
a_0 &:= \left(\widetilde{m}_1c_{12}^2+\widetilde{m}_2s_{12}^2\right)c_{13}^2+\mathrm{e}^{2\,\mathrm{i}\,\delta}m_3s_{13}^2\;,\\
a_1 &:= \left[\left(\mathrm{e}^{2\,\mathrm{i}\,\delta}s_{12}^2-c_{12}^2s_{13}^2\right)\sin(2\theta_{23})-\mathrm{e}^{\mathrm{i}\,\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}\right]\;,\\
a_2 &:= \left[\mathrm{e}^{\mathrm{i}\,\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}+\left(\mathrm{e}^{2\,\mathrm{i}\,\delta}c_{12}^2-s_{12}^2s_{13}^2\right)\sin(2\theta_{23})\right]\;.
\end{aligned}$$

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A$_4$ Model

- Predictions from $I_{12} = -2$ invariant, Inverted Ordering



MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

Reality of I_{12} $\Rightarrow$

Constraints on CP phases:

Given δ $\Rightarrow$

Majorana phases α_{12}, α_{13}

Invariants in Toy Modular A_4 Model

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(e^{i\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(e^{i\delta}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2+e^{2i\delta}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(e^{i\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)+\widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23}-e^{i\delta}c_{12}c_{23}\right)-e^{2i\delta}m_3s_{13}s_{23}\right]^2}\,,$$

$$I_{13} = \frac{a_0\left[\widetilde{m}_1\left(c_{12}c_{23}s_{13}-e^{i\delta}s_{12}s_{23}\right)^2+\widetilde{m}_2\left(c_{23}s_{12}s_{13}+e^{i\delta}c_{12}s_{23}\right)^2+e^{2i\delta}m_3c_{13}^2c_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(c_{12}c_{23}s_{13}-e^{i\delta}s_{12}s_{23}\right)+\widetilde{m}_2s_{12}\left(c_{23}s_{12}s_{13}+e^{i\delta}c_{12}s_{23}\right)-e^{2i\delta}m_3c_{23}s_{13}\right]^2}\,,$$

$$\begin{aligned}I_{23} = {} & \left[e^{2i\delta}m_3c_{13}^2s_{23}^2+\widetilde{m}_1\left(e^{i\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(e^{i\delta}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2\right] \\ & \times\frac{4\left[e^{2i\delta}m_3c_{13}^2c_{23}^2+\widetilde{m}_2\left(c_{23}s_{12}s_{13}+e^{i\delta}c_{12}s_{23}\right)^2+\widetilde{m}_1\left(c_{12}c_{23}s_{13}-e^{i\delta}s_{12}s_{23}\right)^2\right]}{\left[\widetilde{m}_1a_1+\widetilde{m}_2a_2-e^{2i\delta}m_3\sin(2\theta_{23})c_{13}^2\right]^2}\,,\end{aligned}$$

$$\widetilde{m}_1 := m_1\, e^{i\varphi_1}$$

$$\widetilde{m}_2 := m_2\, e^{i\varphi_2}.$$

$$\begin{aligned}a_0 &:= \left(\widetilde{m}_1c_{12}^2+\widetilde{m}_2s_{12}^2\right)c_{13}^2+e^{2i\delta}m_3s_{13}^2\,,\\ a_1 &:= \left[\left(e^{2i\delta}s_{12}^2-c_{12}^2s_{13}^2\right)\sin(2\theta_{23})-e^{i\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}\right],\\ a_2 &:= \left[e^{i\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}+\left(e^{2i\delta}c_{12}^2-s_{12}^2s_{13}^2\right)\sin(2\theta_{23})\right].\end{aligned}$$

Invariants in Toy Modular A_4 Model

- No simultaneous solution for I_{ij} that is consistent with data
  - Agree with previous analysis by scanning parameter space (i.e. toy modular A4 model does not fit all data)
  - Here, arrived at conclusion without the need to scan

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A5 Model

- In a model based on modular A5:

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

$$I_{12} = \frac{2\sqrt{6}}{3}\frac{Y_1(\tau)Y_4(\tau)}{Y_5^2(\tau)}\,, \qquad I_{13} = \frac{2\sqrt{6}}{3}\frac{Y_1(\tau)Y_3(\tau)}{Y_2^2(\tau)}\,, \qquad I_{23} = 6\frac{Y_3(\tau)Y_4(\tau)}{Y_1^2(\tau)}$$

- Algebraic relations among the invariants

$$0 = 4 + 18I_{12} + 18I_{13} + 9I_{12}I_{13} + I_{12}I_{13}I_{23}\,,$$
$$0 = 8 + 12I_{12} - 108I_{12}^2 + 12I_{13} + 414I_{12}I_{13} + 108I_{12}^2I_{13} - 108I_{13}^2 + 108I_{12}I_{13}^2 + 81I_{12}^2I_{13}^2 - I_{12}^2I_{23} - I_{13}^2I_{23}\,.$$

- Exchange symmetry: $I_{12} \leftrightarrow I_{13} \;\Rightarrow\; \mu - \tau$ symmetry built in

Validity at High Orders?

- **SUSY setups:**
  - RG corrections vanish to all orders
- **Non-SUSY setups: SM**

MCC, S. Perez, M. Ratz (2026)

  - 2 loop:

$$\frac{\mathrm{d}I_{fg}}{\mathrm{d}t} = \frac{2\left(y_f^2 - y_g^2\right)^2}{(16\pi^2)^2} I_{fg}$$

  - Corrections:

$$\Delta I_{fg} \approx \frac{2\left(y_f^2 - y_g^2\right)^2}{(16\pi^2)^2} I_{fg} \Delta t$$

  - Example: running between $10^3 - 10^6$ GeV
    - Negligible

$$\boxed{\begin{aligned} \Delta I_{12} &\approx -1.2 \times 10^{-15} \\ \Delta I_{13} &\approx -1.3 \times 10^{-11} \\ \Delta I_{23} &\approx 2.6 \times 10^{-12} \end{aligned}}$$

Determinants & VVMFs

- Determinants of mass matrices are 1-dim VVMFs

$$\det M(\tau) \xmapsto{\gamma} (c\,\tau + d)^{\sum_i k_{\varphi_i^c} + k_{\varphi_i}} \left(\det \rho^{\mathcal{C}} \rho\right)^* \det M(\tau)$$

X.-G. Liu, G.-J. Ding (2022)

- 1-dim VVMFs form a finite-dimensional linear space generated by η^2, E_4, E_6

$$\mathcal{M}_k^{(1\mathrm{D})}(\mathrm{SL}(2,\mathbb{Z})) = \sum_{\substack{a+b+c=k \\ a,b,c\geq 0}} \alpha_{abc}\, \eta^{2c}\, {E_4}^a\, {E_6}^b$$

η : Dedekind η function

$E_{4,\,6}$: Eisenstein series

- We know a lot about the determinants from theory of modular forms

Location of Zeros

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

c.f. Feruglio, Gherardi, Romanino, Titov (2021); Novichkov, Penedo, Petcov (2021); Feruglio (2023)

Quark mass hierarchy, CPV: See talk by Matteo Parriciatu

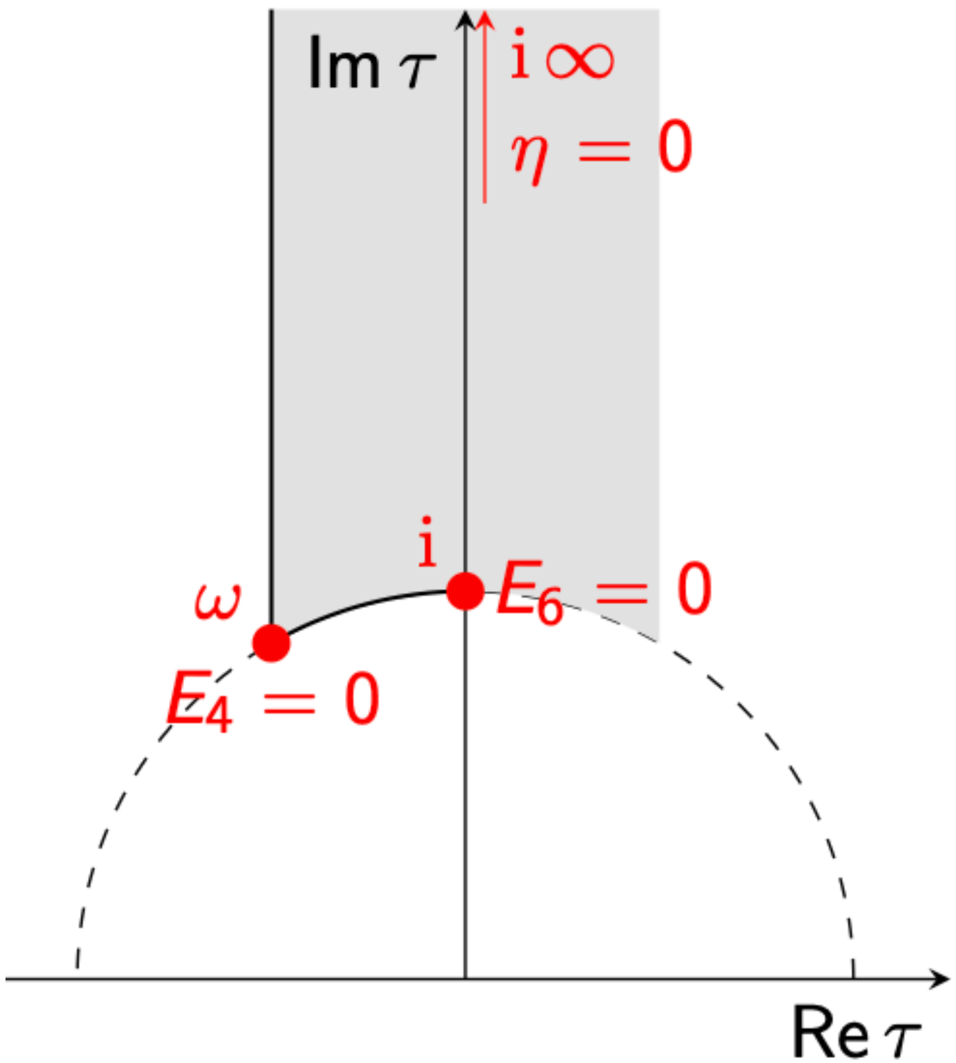
- Close to zeros of det M $\Rightarrow$ hierarchical masses

- Orders of zeros constrained by Valence Formula

Prop 2 in Bruinier, van der Geer, Harder, Zagier (2008)

$$\frac{1}{12}n_{\mathrm{i}\,\infty} + \frac{1}{2}n_{\mathrm{i}} + \frac{1}{3}n_{\omega} + \sum_{\substack{\tau_0 \neq \\ \mathrm{i},\omega,\mathrm{i}\,\infty}} n_{\tau_0} = \frac{k}{12}$$

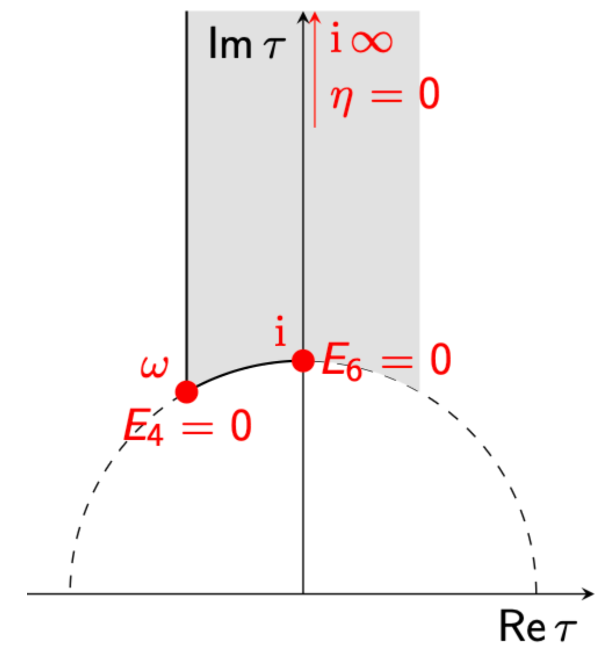
- Absence of cancellations of different terms for $k_{\det M} < 12$, the zeros can only be at $\tau = \omega,\, i,\, i\infty$



Determinants of Weights up to 6

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

$k_{\det M}$	Modular Form	Zeros	$(n_{\mathrm{i}\,\infty}, n_{\mathrm{i}}, n_{\omega})$	irrep
1	η^2	$\mathrm{i}\,\infty$	$(1,0,0)$	$\mathbf{1}_1$
2	η^4	$\mathrm{i}\,\infty$	$(2,0,0)$	$\mathbf{1}_2$
3	η^6	$\mathrm{i}\,\infty$	$(3,0,0)$	$\mathbf{1}_3$
4	E_4	ω	$(0,0,1)$	$\mathbf{1}_0$
	η^8	$\mathrm{i}\,\infty$	$(4,0,0)$	$\mathbf{1}_4$
5	$\eta^2 E_4$	ω and $\mathrm{i}\,\infty$	$(1,0,1)$	$\mathbf{1}_1$
	η^{10}	$\mathrm{i}\,\infty$	$(5,0,0)$	$\mathbf{1}_5$
6	E_6	i	$(0,1,0)$	$\mathbf{1}_0$
	$\eta^4 E_4$	ω and $\mathrm{i}\,\infty$	$(2,0,1)$	$\mathbf{1}_2$
	η^{12}	$\mathrm{i}\,\infty$	$(6,0,0)$	$\mathbf{1}_6$



$$\det M(\tau) \xmapsto{\gamma} (c\,\tau + d)^{\sum_i k_{\varphi_i^{\mathcal{C}}} + k_{\varphi_i}} \left(\det \rho^{\mathcal{C}} \rho\right)^{*} \det M(\tau)$$

$$\frac{1}{12} n_{\mathrm{i}\,\infty} + \frac{1}{2} n_{\mathrm{i}} + \frac{1}{3} n_{\omega} + \sum_{\substack{\tau_0 \neq \\ \mathrm{i},\omega,\mathrm{i}\,\infty}} n_{\tau_0} = \frac{k}{12}$$

Determinants of Weights 7–9

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

$k_{\det M}$	Modular Form	Zeros	$(n_{\mathrm{i}\infty}, n_{\mathrm{i}}, n_{\omega})$	irrep
7	$\eta^6 E_4$	ω and $\mathrm{i}\infty$	$(3,0,1)$	$\mathbf{1}_3$
	$\eta^2 E_6$	i and $\mathrm{i}\infty$	$(1,1,0)$	$\mathbf{1}_1$
	η^{14}	$\mathrm{i}\infty$	$(7,0,0)$	$\mathbf{1}_7$
8	E_4^2	ω	$(0,0,2)$	$\mathbf{1}_0$
	$\eta^4 E_6$	i and $\mathrm{i}\infty$	$(2,1,0)$	$\mathbf{1}_2$
	$\eta^8 E_4$	ω and $\mathrm{i}\infty$	$(4,0,1)$	$\mathbf{1}_4$
	η^{16}	$\mathrm{i}\infty$	$(8,0,0)$	$\mathbf{1}_8$
9	$\eta^{10} E_4$	ω and $\mathrm{i}\infty$	$(5,0,1)$	$\mathbf{1}_5$
	$\eta^6 E_6$	i and $\mathrm{i}\infty$	$(3,1,0)$	$\mathbf{1}_3$
	$\eta^2 E_4^2$	ω and $\mathrm{i}\infty$	$(1,0,2)$	$\mathbf{1}_1$
	η^{18}	$\mathrm{i}\infty$	$(9,0,0)$	$\mathbf{1}_9$

Determinants of Weights 10, 11

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

$k_{\det M}$	Modular Form	Zeros	$(n_{\mathrm{i}\infty}, n_{\mathrm{i}}, n_{\omega})$	irrep
10	$E_4 E_6$	ω and i	$(0,1,1)$	$\mathbf{1}_0$
	$\eta^4 E_4^2$	ω and $\mathrm{i}\infty$	$(2,0,2)$	$\mathbf{1}_2$
	$\eta^8 E_6$	i and $\mathrm{i}\infty$	$(4,1,0)$	$\mathbf{1}_4$
	$\eta^{12} E_4$	ω and $\mathrm{i}\infty$	$(6,0,1)$	$\mathbf{1}_6$
	η^{20}	$\mathrm{i}\infty$	$(10,0,0)$	$\mathbf{1}_{10}$
11	$\eta^2 E_4 E_6$	ω, i and $\mathrm{i}\infty$	$(1,1,1)$	$\mathbf{1}_1$
	$\eta^6 E_4^2$	ω and $\mathrm{i}\infty$	$(3,0,2)$	$\mathbf{1}_3$
	$\eta^{10} E_6$	i and $\mathrm{i}\infty$	$(5,1,0)$	$\mathbf{1}_5$
	$\eta^{14} E_4$	ω and $\mathrm{i}\infty$	$(7,0,1)$	$\mathbf{1}_7$
	η^{22}	$\mathrm{i}\infty$	$(11,0,0)$	$\mathbf{1}_{11}$

Orders of Zeros and Mass Hierarchy

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

- Convincing explanation of hierarchical masses by modular flavor symmetries ⇒ requires zeros of high orders
- At $\tau = i$ or ω : zeros at most orders 1 or 2
- At $\tau = i\infty$: only occur for nontrivial 1-plets: $1_{p>0}$
- Certain finite modular groups do not have appropriate $1_{p>0}$ -plets
- Perfect groups have no nontrivial 1-plets, e.g.
    - A_5 : hierarchy can only be obtained by balancing coefficients

MCC, Fallbacher, Ratz, Trautner, Vaudrevange (2015)

Orders of Zeros and Mass Hierarchy

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

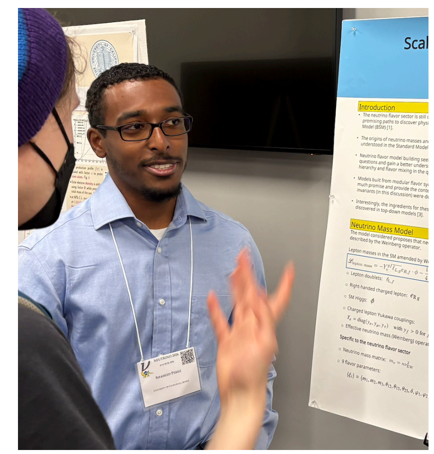
- Convincing explanation of hierarchical masses by modular flavor symmetries ⇒ requires zeros of high orders
- At $\tau = i$ or ω : zeros at most orders 1 or 2
- At $\tau = i\infty$: only occur for nontrivial 1-plets: $1_{p>0}$
- Certain finite modular groups do not have appropriate $1_{p>0}$-plets
- Perfect groups have no nontrivial 1-plets, e.g.
  - A_5 : hierarchy can only be obtained by balancing coefficients
- Bottom line: given finite modular group + modular weights of matter fields ⇒ can immediately tell whether mass hierarchy can emerge as properties of modular forms

Acknowledgements

Xueqi Li
(UCI Grad)

Xiang-Gan Liu
(UNAM PD/
former UCI
PD)

Omar Medina
(IFIC Valencia
Grad/former
UCI visiting
student)



Shaheed
Perez
(UCI Grad)

Michael Ratz
(UCI)

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz, PLB852 (2024) 138600

MCC, X.-Q. Li, X.-G. Liu, M. Ratz, JHEP10 (2025) 033

MCC, S. Perez, M. Ratz, Universe 12 (2026) 2, 46

Conclusions

- Fundamental origin of fermion mass & mixing patterns still unknown
- Uniqueness of Neutrino masses exciting opportunities to explore BSM Physics
- Modular Flavor Symmetries:
  - Significant reduction of the number of parameters
  - Two generic predictions:
    - Modular Invariant (and even τ-independent), RG Invariants:
      - Robust sum rules among physical observables
      - Independent of renormalization scale, model parameters
    - Near-critical behavior of mass matrices:
      - Mass hierarchy can arise due to properties of modular forms
      - Expansion parameter: deviation from fixed point
      - Classification of symmetries based on modular weights
      - Preference for $\tau = i\infty$