Modular Zeros

Michael Ratz

8/19/2026

FLASY 2026

Based on collaborations with:
Mu-Chun Chen, Xueqi Li, Xiang-Gan Liu, Harold Matias, Omar Medina, Hans-Peter Nilles, Shaheed Perez & Alex Stewart

Outline

Motivation

Zeros in Model Building

- Zeros: Forbidden Couplings
- Wanted: Zeros defying Traditional Picture

Zeros: Forbidden Couplings

☞ in (flavor) model building we want/need to forbid couplings, e.g.

- coefficient of $U^{\mathcal{C}}\,D^{\mathcal{C}}\,D^{\mathcal{C}}$ in supersymmetric theories

up quark superfield

down quark superfield

Zeros: Forbidden Couplings

☞ in (flavor) model building we want/need to forbid couplings, e.g.

- coefficient of $U^{\mathcal{C}}\,D^{\mathcal{C}}\,D^{\mathcal{C}}$ in supersymmetric theories
- texture zero: $M = \begin{pmatrix} \bullet & \bullet & \bullet \\ {\color{red}0} & \bullet & \bullet \\ {\color{red}0} & {\color{red}0} & \bullet \end{pmatrix}$

Zeros: Forbidden Couplings

- in (flavor) model building we want/need to forbid couplings, e.g.
  - coefficient of $U^\mathcal{C}\,D^\mathcal{C}\,D^\mathcal{C}$ in supersymmetric theories
  - texture zero: $M = \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix}$
- traditional way to achieve these "zeros": symmetries that forbid the corresponding term(s)

Zeros: Forbidden Couplings

- in (flavor) model building we want/need to forbid couplings, e.g.
  - coefficient of $U^\mathcal{C}\,D^\mathcal{C}\,D^\mathcal{C}$ in supersymmetric theories
  - texture zero: $M = \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix}$
- traditional way to achieve these "zeros": symmetries that forbid the corresponding term(s)
- in supersymmetric theories: holomorphic zeros

▸ details

Wanted: Zeros defying Traditional Picture

☞ Weinberg's mass matrix for u and c quarks Weinberg (1977)

$$\mathcal{M}_{\mathsf{Weinberg}} = \begin{pmatrix} 0 & \delta \\ \delta & \mu \end{pmatrix}$$

with $\delta \ll \mu$ cannot be a consequence of "normal" symmetries

Wanted: Zeros defying Traditional Picture

- Weinberg's mass matrix for u and c quarks — Weinberg (1977)

$$\mathcal{M}_{\text{Weinberg}} = \begin{pmatrix} 0 & \delta \\ \delta & \mu \end{pmatrix}$$

with $\delta \ll \mu$ cannot be a consequence of "normal" symmetries

- mysterious string selection rule that does not correspond to known symmetry such as "Rule 4" — Font, Ibáñez, Nilles & Quevedo (1988)

Wanted: Zeros defying Traditional Picture

- Weinberg's mass matrix for u and c quarks — Weinberg (1977)

$$\mathcal{M}_{\text{Weinberg}} = \begin{pmatrix} 0 & \delta \\ \delta & \mu \end{pmatrix}$$

with $\delta \ll \mu$ cannot be a consequence of "normal" symmetries

- mysterious string selection rule that does not correspond to known symmetry such as "Rule 4" — Font, Ibáñez, Nilles & Quevedo (1988)

this talk:

"modular zeros" can explain these features

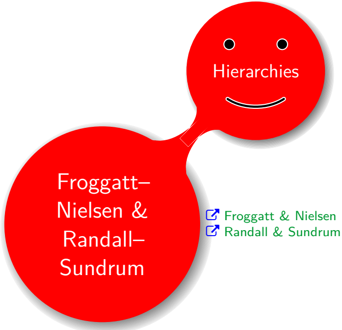
Modular Flavor Symmetries

- Modular Flavor Symmetries
- Modular Flavor Symmetries were hiding in plain Sight
- Interlude: Fermat's Last Theorem

Flavor symmetries

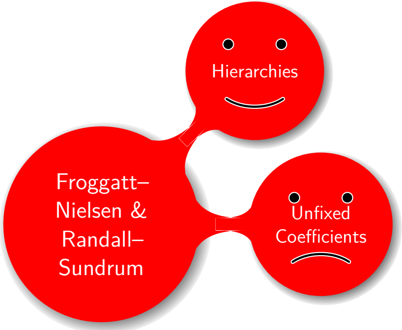
Froggatt & Nielsen (1979);...
Randall & Sundrum (1999);...

Flavor symmetries

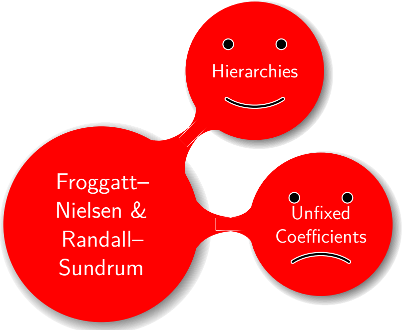


Froggatt & Nielsen (1979);. . .
Randall & Sundrum (1999);. . .

Flavor symmetries

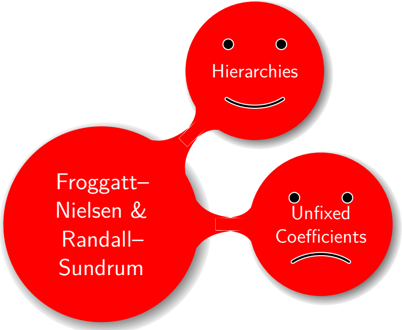


Flavor symmetries

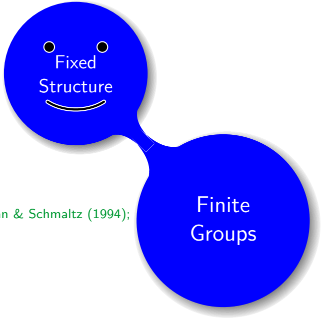


Kaplan & Schmaltz (1994);
...

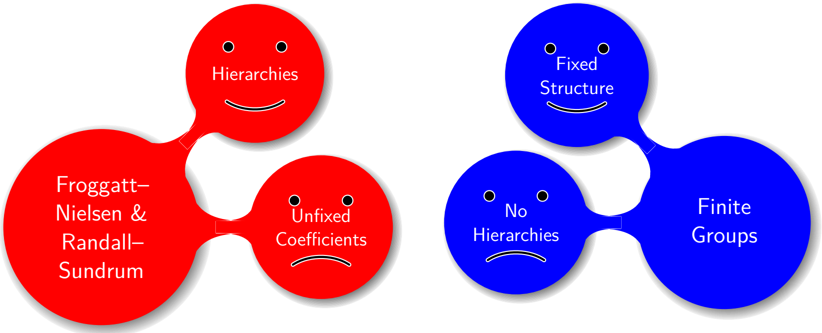
Flavor symmetries



Kaplan & Schmaltz (1994); ...



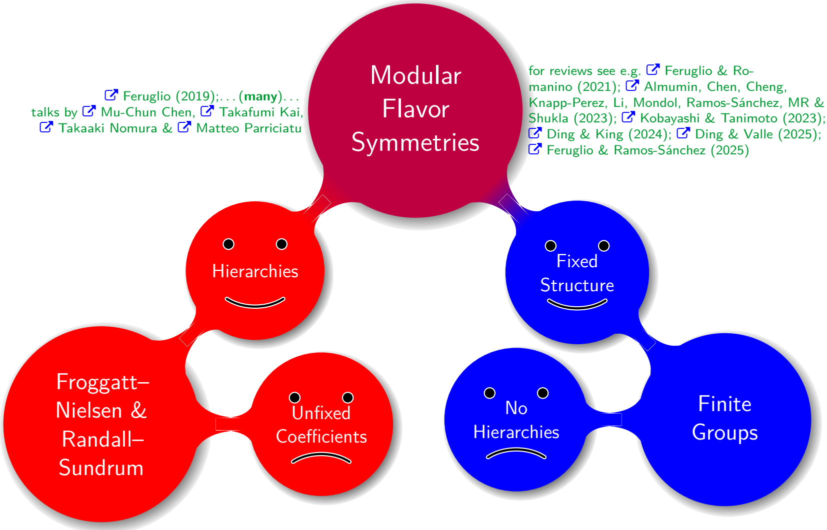
Flavor symmetries

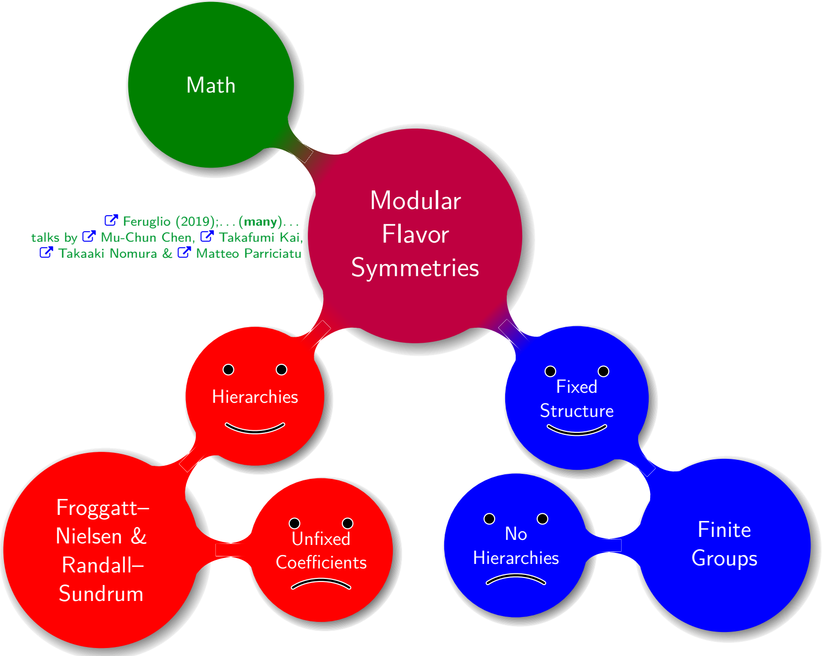


Flavor symmetries

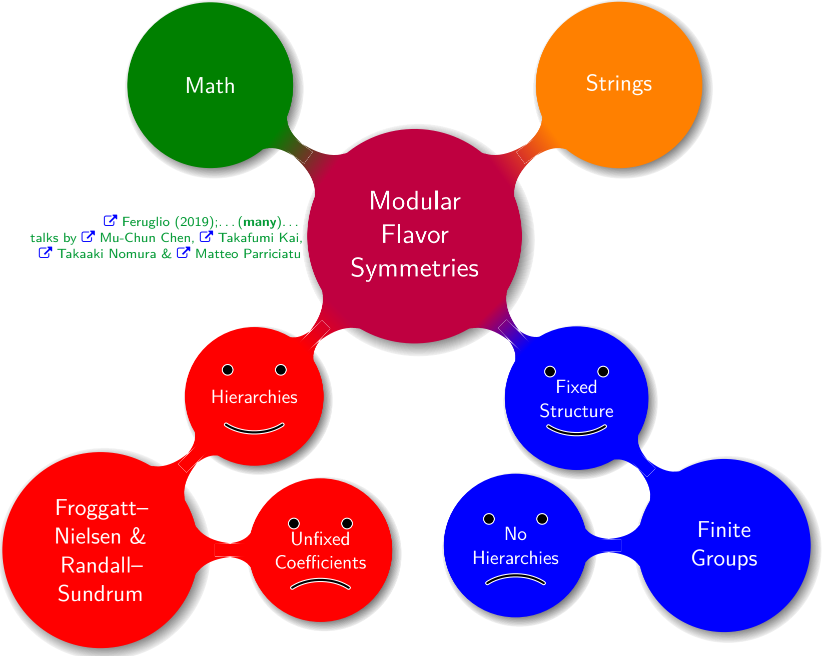
Feruglio (2019);…(**many**)…
talks by Mu-Chun Chen, Takafumi Kai, Takaaki Nomura & Matteo Parriciatu

for reviews see e.g. Feruglio & Romanino (2021); Almumin, Chen, Cheng, Knapp-Perez, Li, Mondol, Ramos-Sánchez, MR & Shukla (2023); Kobayashi & Tanimoto (2023); Ding & King (2024); Ding & Valle (2025); Feruglio & Ramos-Sánchez (2025)





Math
Modular Flavor Symmetries
Feruglio (2019);. . . (**many**). . .
talks by Mu-Chun Chen, Takafumi Kai,
Takaaki Nomura & Matteo Parriciatu
Hierarchies
Fixed Structure
Froggatt–Nielsen & Randall–Sundrum
Unfixed Coefficients
No Hierarchies
Finite Groups


Math
Strings
Modular Flavor Symmetries
Feruglio (2019);. . . (many). . .
talks by Mu-Chun Chen, Takafumi Kai,
Takaaki Nomura & Matteo Parriciatu
Hierarchies
Fixed Structure
Froggatt–Nielsen & Randall–Sundrum
Unfixed Coefficients
No Hierarchies
Finite Groups

Modular Flavor Symmetries were hiding in plain Sight

- world-sheet instantons & flavor — Dine, Seiberg, Wen & Witten (1986)

  "We cannot resist speculating on possible implications of these results for super-string phenomenology. A number of possibilities come to mind. The fact that some Yukawa couplings vanish to all orders of sigma model perturbation theory but may be generated by instantons suggests a possible origin for fermion mass hierarchies. Of course, for such a possibility to be realized, it will be necessary for e^{-r} *to be at least moderately small."*

Modular Flavor Symmetries were hiding in plain Sight

- world-sheet instantons & flavor — Dine, Seiberg, Wen & Witten (1986)
- (Yukawa) couplings *are* modular forms in heterotic orbifolds — Quevedo (1996)

This is nothing but one of the $SL(2,\mathbf{Z})_{T,U}$ transformation for toroidal orbifold compactifications ($a = b = d = 1, c = 0$ in eq. (10)). Therefore the only conditions these symmetries impose on W is that it should transform as a modular form of a given weight ($W \to (cT+d)^{-3}W$ for the simplest toroidal orbifolds with T the overall size of the compactification space)[36]. In fact, explicit calculations for specific orbifold models show that

$$W_{tree}(T, Q^I) = \chi_{IJK}(T)\, Q^I Q^J Q^K + \cdots \tag{19}$$

with $\boxed{\chi(T) \text{ a particular modular form of } SL(2,\mathbf{Z})}$ or any other duality group and the ellipsis represent higher powers of Q, exponentially suppressed. The identification of $\chi(T)$ with modular forms was a highly nontrivial check of the explicit orbifold calculations which were preformed in refs. [37] without any relation (nor knowledge) of the underlying duality symmetry $SL(2,\mathbf{Z})$. This kind of symmetry puts also strong constraints to the higher order, nonrenormalizable, corrections to W, since each matter field Q transforms in a particular way under that symmetry ($Q \to (cT+d)^n Q$ with n the modular weight of Q). There are also other $\boxed{\text{discrete symmetries}}$ as those defined by the $\boxed{\text{point group } \mathcal{P} \text{ and space group } \mathcal{S}}$ of an orbifold which have to be respected by the superpotential W. These 'selection rules' are very important to find vanishing couplings and uncover flat directions which can be used to break the original gauge symmetries and construct quasi-realistic models.

Modular Flavor Symmetries were hiding in plain Sight

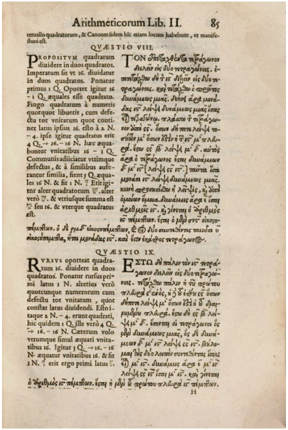
- world-sheet instantons & flavor — Dine, Seiberg, Wen & Witten (1986)
- (Yukawa) couplings *are* modular forms in heterotic orbifolds — Quevedo (1996)
- a nontrivial modular flavor symmetry (T') has been discovered early on
  Ferrara, Lüst, Shapere & Theisen (1989); Chun, Mas, Lauer & Nilles (1989);...
- vector-valued modular form couplings were obtained long ago using orbifold CFT
  Chun, Mas, Lauer & Nilles (1989); Lauer, Mas & Nilles (1991);...
- world-sheet instantons are leading-order terms in q-expansion of vector-valued modular forms (VVMFs)

Modular Flavor Symmetries were hiding in plain Sight

- ☞ world-sheet instantons & flavor — Dine, Seiberg, Wen & Witten (1986)
- ☞ (Yukawa) couplings *are* modular forms in heterotic orbifolds — Quevedo (1996)
- ☞ a nontrivial modular flavor symmetry (T') has been discovered early on
  Ferrara, Lüst, Shapere & Theisen (1989); Chun, Mas, Lauer & Nilles (1989);...
- ☞ vector-valued modular form couplings were obtained long ago using orbifold CFT
  Chun, Mas, Lauer & Nilles (1989); Lauer, Mas & Nilles (1991);...
- ☞ world-sheet instantons are leading-order terms in q-expansion of vector-valued modular forms (VVMFs)
- ☞ full picture requires an analysis of massive (and not so massive 😉) winding and KK modes
  Li, Liu, Nilles, MR & Stewart (2025) ▸ details
- ☞ modular symmetries are R-symmetries — Dixon, Kaplunovsky & Louis (1990);... ▸ details
- ➥ important implications for solutions to the strong $\mathcal{CP}$ problem ▸ details
  Liu, MR & Stewart (2026); Chen, Li, Liu, Matias, MR & Stewart (2026)

Interlude: Fermat's Last Theorem

en.wikipedia.org/wiki/Pierre_de_Fermat



Arithmeticorum Lib. II. 85

teruallo quadratorum, & Canones iidem hîc etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$. & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

ΤΟΝ ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο τετραγώνους. ἐπιτετάχθω δὴ τὸν ις διελεῖν εἰς δύο τετραγώνους. καὶ τετάχθω ὁ πρῶτος δυνάμεως μιᾶς. δεήσει ἄρα μονάδας ις λεῖψει δυνάμεως μιᾶς ἴσας εἶναι τετραγώνῳ. πλάσσω τὸν τετράγωνον ἀπὸ ςς. ὅσων δήποτε λεῖψει τοσούτων μο ὅσων ἐστὶν ἡ τῶν ις μο πλευρά. ἔστω ςς β λεῖψει μο δ. αὐτὸς ἄρα ὁ τετράγωνος ἔσται δυνάμεων δ μο ις [λεῖψει ςς ις.] ταῦτα ἴσα μονάσι ις λεῖψει δυνάμεως μιᾶς. κοινὴ προσκείσθω ἡ λεῖψις, καὶ ἀπὸ ὁμοίων ὅμοια. δυνάμεις ἄρα ε ἴσαι ἀριθμοῖς ις. καὶ γίνεται ὁ ἀριθμὸς ις πέμπτων. ἔσται ὁ μὲν σνς εἰκοστοπέμπτων. ὁ δὲ ρμδ εἰκοστοπέμπτων, καὶ οἱ δύο συντεθέντες ποιοῦσι υ εἰκοστόπεμπτα, ἤτοι μονάδας ις. καὶ ἔστιν ἑκάτερος τετράγωνος.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit ergo primi latus $\frac{16}{5}$.

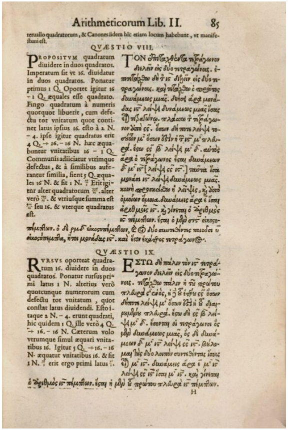
ΕΣΤΩ δὴ πάλιν τὸν ις τετράγωνον διελεῖν εἰς δύο τετραγώνους. τετάχθω πάλιν ἡ τοῦ πρώτου πλευρὰ ς ἑνός, ἡ δὲ τοῦ ἑτέρου ςς ὅσων δήποτε λεῖψει μο ὅσων ἐστὶν ἡ τοῦ διαιρουμένου πλευρά. ἔστω δὴ ςς β λεῖψει μο δ. ἔσονται οἱ τετράγωνοι ὃς μὲν δυνάμεως μιᾶς, ὃς δὲ δυνάμεων δ μο ις λεῖψει ςς ις. βούλομαι τοὺς δύο λοιπὸν συντεθέντας ἴσους εἶναι μο ις. δυνάμεις ἄρα ε μο ις λεῖψει ςς ις ἴσαι μο ις. καὶ γίνεται ὁ ἀριθμὸς ις πέμπτων. ἔσται ἡ μὲν τοῦ πρώτου πλευρὰ ις πέμπτων.

H

en.wikipedia.org/wiki/Fermat%27s_Last_Theorem

Interlude: Fermat's Last Theorem

☞ what does Fermat's Last Theorem say?



Arithmeticorum Lib. II. 85

teruallo quadratorum, & Canones iidem hîc etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. [illegible] Erit igitur alter quadratorum [illegible]. alter verò [illegible]. & vtriusque summa est [illegible] seu 16. & vterque quadratus est.

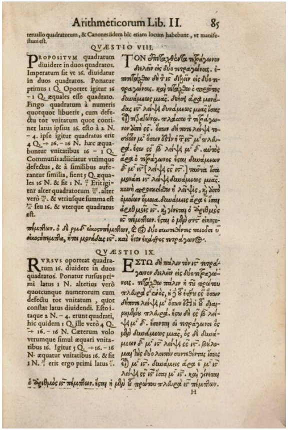
QVÆSTIO IX.

RVRSVS oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. [illegible] erit ergo primi latus [illegible].

H

Interlude: Fermat's Last Theorem

☞ what does Fermat's Last Theorem say?
$a^n + b^n = c^n$ has no integer solutions
for $n > 2$ Fermat (1637)



Arithmeticorum Lib. II. 85

teruallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. 16/5 Erit igitur alter quadratorum 256/25. alter verò 144/25. & vtriusque summa est 400/25 seu 16. & vterque quadratus est.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. 16/5 erit ergo primi latus 16/5.

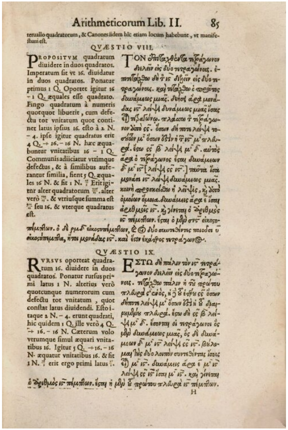
H

en.wikipedia.org/wiki/Fermat%27s_Last_Theorem

Interlude: Fermat's Last Theorem

- what does Fermat's Last Theorem say?
  $a^n + b^n = c^n$ has no integer solutions for $n > 2$ Fermat (1637)

- how was this theorem proven?



Arithmeticorum Lib. II. 85

teruallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

Propositvm quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$. Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$. & vtriusque summa est $\frac{400}{25}$. seu 16. & vterque quadratus est.

QVÆSTIO IX.

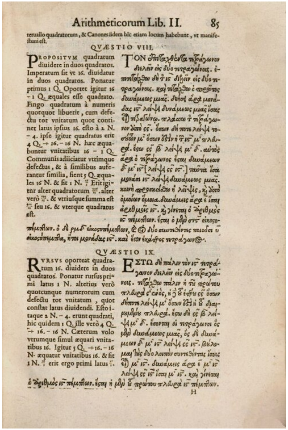
Rvrsvs oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit ergo primi latus $\frac{16}{5}$.

en.wikipedia.org/wiki/Fermat%27s_Last_Theorem

Interlude: Fermat's Last Theorem

- what does Fermat's Last Theorem say?
  $a^n + b^n = c^n$ has no integer solutions for $n > 2$ Fermat (1637)

- how was this theorem proven?
  - if there were solutions for $n > 2$ this would imply the existence of certain "weird" modular forms



Arithmeticorum Lib. II. 85

teruallo quadratorum, & Canones iidem hîc etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$. & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit ergo primi latus $\frac{16}{5}$.

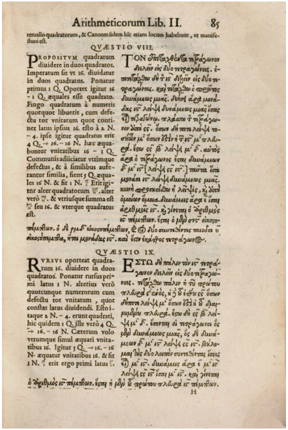
H

en.wikipedia.org/wiki/Fermat%27s_Last_Theorem

Interlude: Fermat's Last Theorem

- what does Fermat's Last Theorem say?
  $a^n + b^n = c^n$ has no integer solutions for $n > 2$ — Fermat (1637)
- how was this theorem proven?
  - if there were solutions for $n > 2$ this would imply the existence of certain "weird" modular forms
  - Taniyama–Shimura conjecture: these "weird" modular forms cannot exist — proved by Wiles (1994)

cf. public talk by Karl Rubin at `https://www.math.uci.edu/~krubin/lectures/psbreakfast.pdf`



Arithmeticorum Lib. II. 85

terualllo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$. & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

QVÆSTIO IX.

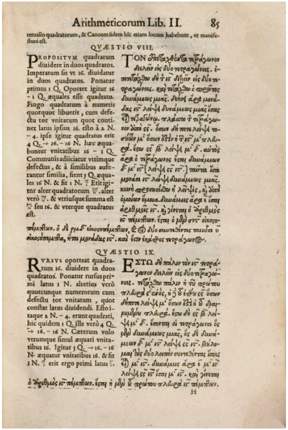
RVRSVS oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit ergo primi latus $\frac{16}{5}$.

Interlude: Fermat's Last Theorem

- what does Fermat's Last Theorem say?
  $a^n + b^n = c^n$ has no integer solutions for $n > 2$ — Fermat (1637)
- how was this theorem proven?
  - if there were solutions for $n > 2$ this would imply the existence of certain "weird" modular forms
  - Taniyama–Shimura conjecture: these "weird" modular forms cannot exist — proved by Wiles (1994)

this talk:

use gaps in modular forms to explain "mysterious" zeros in physics


Arithmeticorum Lib. II. 85

teruallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. 16/5. Erit igitur alter quadratorum 256/25. alter verò 144/25. & vtriusque summa est 400/25 seu 16. & vterque quadratus est.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16. diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. 16/5. erit ergo primi latus 16/5.

en.wikipedia.org/wiki/Fermat%27s_Last_Theorem

Modular Zeros

Liu & MR (2025)

- Gaps in Modular Forms
- Modular Spin
- A nontrivial modular Zero
- Stringy Zeros
- Weinberg's Mass Texture

Gaps in Modular Forms for $\Gamma'_3 \cong T'$

$$T' = \left\langle \mathsf{S}, \mathsf{T} \;\middle|\; \mathsf{S}^4 = (\mathsf{S}\,\mathsf{T})^3 = \mathsf{T}^3 = \mathsf{S}^2\,\mathsf{T}\,\mathsf{S}^{-2}\,\mathsf{T}^{-1} = \mathbb{1} \right\rangle$$

- irreps $\boldsymbol{s}$: 3-plet $\mathbf{3}$, 3 2-plets $\mathbf{2}$, $\mathbf{2'}$ & $\mathbf{2''}$, 3 1-plets $\mathbf{1}$, $\mathbf{1'}$ & $\mathbf{1''}$

$k_{\mathcal{Y}}$	VVMFs $\mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}$
1	$\mathcal{Y}_{\mathbf{2''}}^{(1)}$
2	$\mathcal{Y}_{\mathbf{3}}^{(2)}$
3	$\mathcal{Y}_{\mathbf{2'}}^{(3)}, \quad \mathcal{Y}_{\mathbf{2''}}^{(3)}$
4	$\mathcal{Y}_{\mathbf{1}}^{(4)}, \quad \mathcal{Y}_{\mathbf{1'}}^{(4)}, \quad \mathcal{Y}_{\mathbf{3}}^{(4)}$
5	$\mathcal{Y}_{\mathbf{2}}^{(5)}, \quad \mathcal{Y}_{\mathbf{2'}}^{(5)}, \quad \mathcal{Y}_{\mathbf{2''}}^{(5)}$
6	$\mathcal{Y}_{\mathbf{1}}^{(6)}, \quad \mathcal{Y}_{\mathbf{3}\mathrm{I}}^{(6)}, \quad \mathcal{Y}_{\mathbf{3}\mathrm{II}}^{(6)}$
$\vdots$	$\vdots$

$k \backslash \boldsymbol{s}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{2}$	$\mathbf{2'}$	$\mathbf{2''}$	$\mathbf{3}$
0	✓	∅	∅	∅	∅	∅	∅
1	∅	∅	∅	∅	∅	✓	∅
2	∅	∅	∅	∅	∅	∅	✓
3	∅	∅	∅	∅	✓	✓	∅
4	✓	✓	∅	∅	∅	∅	✓
5	∅	∅	∅	✓	✓	✓	∅
6	✓	∅	∅	∅	∅	∅	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

- evidently VVMFs do not exist for all combinations of $k_{\mathcal{Y}}$ and $\boldsymbol{s}$

Gaps in Modular Forms for $\Gamma_3' \cong T'$

$$T' = \left\langle \mathsf{S}, \mathsf{T} \;\middle|\; \mathsf{S}^4 = (\mathsf{S}\,\mathsf{T})^3 = \mathsf{T}^3 = \mathsf{S}^2\,\mathsf{T}\,\mathsf{S}^{-2}\,\mathsf{T}^{-1} = \mathbb{1} \right\rangle$$

- irreps $\boldsymbol{s}$: 3-plet $\mathbf{3}$, 3 2-plets $\mathbf{2}$, $\mathbf{2'}$ & $\mathbf{2''}$, 3 1-plets $\mathbf{1}$, $\mathbf{1'}$ & $\mathbf{1''}$

$k_{\mathcal{Y}}$	VVMFs $\mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}$
1	$\mathcal{Y}^{(1)}_{\mathbf{2''}}$
2	$\mathcal{Y}^{(2)}_{\mathbf{3}}$
3	$\mathcal{Y}^{(3)}_{\mathbf{2'}}$, $\mathcal{Y}^{(3)}_{\mathbf{2''}}$
4	$\mathcal{Y}^{(4)}_{\mathbf{1}}$, $\mathcal{Y}^{(4)}_{\mathbf{1'}}$, $\mathcal{Y}^{(4)}_{\mathbf{3}}$
5	$\mathcal{Y}^{(5)}_{\mathbf{2}}$, $\mathcal{Y}^{(5)}_{\mathbf{2'}}$, $\mathcal{Y}^{(5)}_{\mathbf{2''}}$
6	$\mathcal{Y}^{(6)}_{\mathbf{1}}$, $\mathcal{Y}^{(6)}_{\mathbf{3}\mathrm{I}}$, $\mathcal{Y}^{(6)}_{\mathbf{3}\mathrm{II}}$
$\vdots$	$\vdots$

k \ $\boldsymbol{s}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{2}$	$\mathbf{2'}$	$\mathbf{2''}$	$\mathbf{3}$
0	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	$\emptyset$
4	✓	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
5	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	✓	$\emptyset$
6	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

gaps = modular zeros

- evidently VVMFs do not exist for all combinations of $k_{\mathcal{Y}}$ and $\boldsymbol{s}$

  …note: these gaps were implicit in many models

Modular Spin

☞ central element of the modular group: $\mathsf{S}^2 = -\mathbb{1}$

$$\mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}(\varrho) \xmapsto{\mathsf{S}^2} \mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}(\varrho) = (-1)^{k_{\mathcal{Y}}}\, \rho_{\boldsymbol{s}}(\mathsf{S}^2)\, \mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}(\varrho)$$

Modular Spin

- central element of the modular group: $\mathsf{S}^2 = -\mathbb{1}$

$$\mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}(\varrho) \xmapsto{\mathsf{S}^2} \mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}(\varrho) = (-1)^{k_{\mathcal{Y}}}\, \rho_{\boldsymbol{s}}(\mathsf{S}^2)\, \mathcal{Y}_{\boldsymbol{s}}^{(k_{\mathcal{Y}})}(\varrho)$$

➥ $\rho_{\boldsymbol{s}}(\mathsf{S}^2) = (-1)^{-k_{\mathcal{Y}}}\, \mathbb{1}$

Modular Spin

- central element of the modular group: $\mathsf{S}^2 = -\mathbb{1}$

$$\mathcal{Y}_{\boldsymbol{s}}^{(k_\mathcal{Y})}(\varrho) \xrightarrow{\mathsf{S}^2} \mathcal{Y}_{\boldsymbol{s}}^{(k_\mathcal{Y})}(\varrho) = (-1)^{k_\mathcal{Y}}\,\rho_{\boldsymbol{s}}(\mathsf{S}^2)\,\mathcal{Y}_{\boldsymbol{s}}^{(k_\mathcal{Y})}(\varrho)$$

➥ $\rho_{\boldsymbol{s}}(\mathsf{S}^2) = (-1)^{-k_\mathcal{Y}}\,\mathbb{1}$

Liu & Ding (2022)

- weight-k VVMFs from contractions of k weight-1 "fundamental" doublets $\mathcal{Y}_{\mathbf{2''}}^{(1)}$

$$\mathcal{Y}_{\boldsymbol{s}}^{(k)} = \Big(\underbrace{\mathcal{Y}_{\mathbf{2''}}^{(1)} \times \cdots \times \mathcal{Y}_{\mathbf{2''}}^{(1)}}_{k\ \text{times}}\Big)_{\boldsymbol{s}}$$

contraction to irrep s

$k \backslash \boldsymbol{s}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{2}$	$\mathbf{2'}$	$\mathbf{2''}$	$\mathbf{3}$
0	✓	∅	∅	∅	∅	∅	∅
1	∅	∅	∅	∅	∅	✓	∅
2	∅	∅	∅	∅	∅	∅	✓
3	∅	∅	∅	∅	✓	✓	∅
4	✓	✓	∅	∅	∅	∅	✓
5	∅	∅	∅	✓	✓	✓	∅
6	✓	∅	∅	∅	∅	∅	✓
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Modular Spin

- central element of the modular group: $\mathsf{S}^2 = -\mathbb{1}$

$$\mathcal{Y}_{\boldsymbol{s}}^{(k_\mathcal{Y})}(\varrho) \xmapsto{\mathsf{S}^2} \mathcal{Y}_{\boldsymbol{s}}^{(k_\mathcal{Y})}(\varrho) = (-1)^{k_\mathcal{Y}}\, \rho_{\boldsymbol{s}}(\mathsf{S}^2)\, \mathcal{Y}_{\boldsymbol{s}}^{(k_\mathcal{Y})}(\varrho)$$

➥ $\rho_{\boldsymbol{s}}(\mathsf{S}^2) = (-1)^{-k_\mathcal{Y}}\, \mathbb{1}$

Liu & Ding (2022)

- weight-k VVMFs from contractions of k weight-1 "fundamental" doublets $\mathcal{Y}_{\mathbf{2''}}^{(1)}$

$$\mathcal{Y}_{\boldsymbol{s}}^{(k)} = \Big(\underbrace{\mathcal{Y}_{\mathbf{2''}}^{(1)} \times \cdots \times \mathcal{Y}_{\mathbf{2''}}^{(1)}}_{k \text{ times}}\Big)_{\boldsymbol{s}}$$

➥ SU(2)-like pattern

$$\dim \boldsymbol{s} = \begin{cases} \text{even} & \text{if } k \text{ is odd} \\ \text{odd} & \text{if } k \text{ is even} \end{cases}$$

odd (1, 1′, 1″ and 3) — even (2, 2′, 2″)

$k \backslash \boldsymbol{s}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{2}$	$\mathbf{2'}$	$\mathbf{2''}$	$\mathbf{3}$
0	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	$\emptyset$
4	✓	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
5	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	✓	$\emptyset$
6	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

A nontrivial modular Zero

- $\Gamma_3' \cong T'$ w/ Higgs doublets H_u & H_d carrying modular weight $k_H = -5/2$
- scenario A: Higgs doublets in the representation $\mathbf{1'}$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1'}} H_u\, H_d$

constant

$$\mathcal{Y}_{\mathbf{3}}^{(2)} := \begin{pmatrix} \mathcal{Y}_{\mathbf{3},1}^{(2)} \\ \mathcal{Y}_{\mathbf{3},2}^{(2)} \\ \mathcal{Y}_{\mathbf{3},3}^{(2)} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\, \left[\mathcal{Y}_{\mathbf{2''},1}^{(1)}\right]^2 \\ 2\, \mathcal{Y}_{\mathbf{2''},1}^{(1)}\, \mathcal{Y}_{\mathbf{2''},2}^{(1)} \\ \sqrt{2}\, \left[\mathcal{Y}_{\mathbf{2''},2}^{(1)}\right]^2 \end{pmatrix}$$

A nontrivial modular Zero

- ☞ $\Gamma_3' \cong T'$ w/ Higgs doublets H_u & H_d carrying modular weight $k_H = -5/2$
- ☞ scenario A: Higgs doublets in the representation $\mathbf{1'}$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1'}} H_u\, H_d$
- ☞ scenario B: Higgs doublets in the representation $\mathbf{1''}$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1''}} H_u\, H_d$

A nontrivial modular Zero

- $\Gamma_3' \cong T'$ w/ Higgs doublets H_u & H_d carrying modular weight $k_H = -5/2$
- scenario A: Higgs doublets in the representation $\mathbf{1}'$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)}\mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}'} H_u\, H_d$
- scenario B: Higgs doublets in the representation $\mathbf{1}''$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)}\mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}''} H_u\, H_d = 0$
- $\left(\mathcal{Y}_{\mathbf{3}}^{(2)}\mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}''}$ contraction vanishes

Feruglio (2019); Chen, Li, Liu, Medina & MR (2024)
talk by Mu-Chun Chen

$k \backslash s$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{2}$	$\mathbf{2}'$	$\mathbf{2}''$	$\mathbf{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	✓	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

A nontrivial modular Zero

- $\Gamma_3' \cong T'$ w/ Higgs doublets H_u & H_d carrying modular weight $k_H = -5/2$
- scenario A: Higgs doublets in the representation $\mathbf{1}'$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}'} H_u\, H_d$
- scenario B: Higgs doublets in the representation $\mathbf{1}''$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}''} H_u\, H_d = 0$
- $\left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}''}$ contraction vanishes

on same footing in T'

Feruglio (2019); Chen, Li, Liu, Medina & MR (2024)
talk by Mu-Chun Chen

$k \backslash s$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{2}$	$\mathbf{2}'$	$\mathbf{2}''$	$\mathbf{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	✓	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

"sporadic"

structural

- even though from a T' point of view $\mathbf{1}'$ and $\mathbf{1}''$ are on the same footing the VVMFs distinguish them maximally: one $\mathbf{1}''$ has a "gap" or "sporadic modular zero" at weight 4

A nontrivial modular Zero

- $\Gamma_3' \cong T'$ w/ Higgs doublets H_u & H_d carrying modular weight $k_H = -5/2$
- scenario A: Higgs doublets in the representation $\mathbf{1}'$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}'} H_u\, H_d$
- scenario B: Higgs doublets in the representation $\mathbf{1}''$
- ➥ μ term: $\mathscr{W}_\mu = \lambda \left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}''} H_u\, H_d = 0$
- $\left(\mathcal{Y}_{\mathbf{3}}^{(2)} \mathcal{Y}_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}''}$ contraction vanishes

on same footing in T'

Feruglio (2019); Chen, Li, Liu, Medina & MR (2024)
talk by Mu-Chun Chen

$k \backslash s$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{2}$	$\mathbf{2}'$	$\mathbf{2}''$	$\mathbf{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	✓	✓	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

"sporadic"

- even though from a T' point of view $\mathbf{1}'$ and $\mathbf{1}''$ are on the same footing the VVMFs distinguish them maximally: one $\mathbf{1}''$ has a "gap" or "sporadic modular zero" at weight 4

nontrivial (or "sporadic") modular zeros:

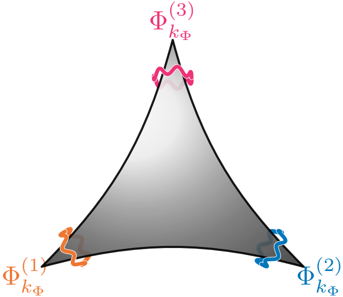
couplings are prohibited without an obvious reason

Stringy Zeros

☞ mysterious string selection rule in orbifolds that does not correspond to known symmetry Font, Ibáñez, Nilles & Quevedo (1988)

***Rule 4**: Couplings $\left(\Phi_{k_\Phi}^{(i)}\right)^{3n}$ of twisted fields $\Phi_{k_\Phi}^{(i)}$ at the same fixed point are only allowed if the sum of the number of derivatives Σ equals* $0 \bmod 6$.

orbifold looks like "Napjak Mandu" w/ twisted strings sitting at the corners



AI generated

Stringy Zeros

***Rule 4**: Couplings $\left(\Phi^{(i)}_{k_\Phi}\right)^{3n}$ of twisted fields $\Phi^{(i)}_{k_\Phi}$ at the same fixed point are only allowed if the sum of the number of derivatives Σ equals $0 \bmod 6$.*

☞ explicit expression ($N = 3n$ w/ $n \in \mathbb{N}$)

$$\Sigma = \begin{cases} N-3 & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ 2N-3 & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

☞ to have $\mathscr{W} \supset \mathcal{Y}_{k_\mathcal{Y}} \left(\Phi^{(i)}_{k_\Phi}\right)^{3n}$

$$k_\mathcal{Y} = \begin{cases} 2n-1 & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ 5n-1 & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

➥ “Modular Spin” implies constraint $\rho_\Phi^{3n}(\mathsf{S}^2)\,\rho_{\boldsymbol{s}}(\mathsf{S}^2) = (-1)^{3n}\,(-1)^{-k_\mathcal{Y}} \overset{!}{=} 1$

Modular Zeros

Stringy Zeros

***Rule 4**: Couplings $\left(\Phi^{(i)}_{k_\Phi}\right)^{3n}$ of twisted fields $\Phi^{(i)}_{k_\Phi}$ at the same fixed point are only allowed if the sum of the number of derivatives Σ equals* $0 \bmod 6$.

- explicit expression ($N = 3n$ w/ $n \in \mathbb{N}$)

$$\Sigma = \begin{cases} N-3 & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ 2N-3 & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

- to have $\mathscr{W} \supset \mathcal{Y}_{k_\mathcal{Y}} \left(\Phi^{(i)}_{k_\Phi}\right)^{3n}$

$$k_\mathcal{Y} = \begin{cases} 2n-1 & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ 5n-1 & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

➥ "Modular Spin" implies constraint $\rho^{3n}_\Phi(\mathsf{S}^2)\,\rho_{\boldsymbol{s}}(\mathsf{S}^2) = (-1)^{3n}\,(-1)^{-k_\mathcal{Y}} \overset{!}{=} 1$

$$\curvearrowright 1 \overset{!}{=} \begin{cases} (-1)^{n+1} & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ (-1)^{-2n+1} & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

Stringy Zeros (cont'd)

- constraint:

$$1 \stackrel{!}{=} \begin{cases} (-1)^{n+1} & \text{for couplings of non-oscillator states w/ } k_\Phi = -{}^2\!/_3 \\ (-1)^{-2n+1} & \text{for couplings of oscillator states w/ } k_\Phi = -{}^5\!/_3 \end{cases}$$

- non-oscillator states: only odd multiples of 3 are allowed
- oscillator states: $-2n+1$ is odd $\curvearrowright$ no couplings

Stringy Zeros (cont'd)

- constraint:

$$1 \stackrel{!}{=} \begin{cases} (-1)^{n+1} & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ (-1)^{-2n+1} & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

- non-oscillator states: only odd multiples of 3 are allowed
- oscillator states: $-2n+1$ is odd $\curvearrowright$ no couplings

Rule 4*: Couplings $\left(\Phi^{(i)}_{k_\Phi}\right)^{3n}$ of twisted fields $\Phi^{(i)}_{k_\Phi}$ at the same fixed point are only allowed if the sum of the number of derivatives Σ equals* $0 \bmod 6$.

$$\Sigma = \begin{cases} 3n-3 & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ 6n-3 & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

Stringy Zeros (cont'd)

- constraint:

$$1 \stackrel{!}{=} \begin{cases} (-1)^{n+1} & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ (-1)^{-2n+1} & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

- non-oscillator states: only odd multiples of 3 are allowed
- oscillator states: $-2n+1$ is odd $\curvearrowright$ no couplings

Rule 4: *Couplings* $\left(\Phi^{(i)}_{k_\Phi}\right)^{3n}$ *of twisted fields* $\Phi^{(i)}_{k_\Phi}$ *at the same fixed point are only allowed if the sum of the number of derivatives* Σ *equals* $0 \bmod 6$.

$$\Sigma = \begin{cases} 3n-3 & \text{for couplings of non-oscillator states w/ } k_\Phi = -2/3 \\ 6n-3 & \text{for couplings of oscillator states w/ } k_\Phi = -5/3 \end{cases}$$

Liu & MR (2025)
see also Knapp-Perez, Liu, Nilles & Ramos-Sanchez (2026)

bottom-line:

Rule 4 can be explained via modular zeros

Weinberg's Mass Texture

☞ Weinberg's mass matrix for u and c quarks Weinberg (1977)

$$\mathcal{M}_{\text{Weinberg}} = \begin{pmatrix} 0 & \delta \\ \delta & \mu \end{pmatrix}$$

with $\delta \ll \mu$ cannot be a consequence of "normal" symmetries

Weinberg's Mass Texture

- Weinberg's mass matrix for u and c quarks — Weinberg (1977)

$$\mathcal{M}_{\text{Weinberg}} = \begin{pmatrix} 0 & \delta \\ \delta & \mu \end{pmatrix}$$

with $\delta \ll \mu$ cannot be a consequence of "normal" symmetries

- zero may be explained as holomorphic zero but not the hierarchy
- e.g. pseudo-anomalous $\mathrm{U}(1)_{\mathrm{anom}}$ & flavon ϕ with charges

$$\left.\begin{aligned} q_{Q_\mathrm{L}} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ q_{\overline{d}_\mathrm{R}} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ q_\phi &= 1 \end{aligned}\right\} \curvearrowright \quad M_{\text{holomorphic zero}} \sim \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & \varepsilon^4 \end{pmatrix}$$

Weinberg's Mass Texture

- Weinberg's mass matrix for u and c quarks

Weinberg (1977)

$$\mathcal{M}_{\text{Weinberg}} = \begin{pmatrix} 0 & \delta \\ \delta & \mu \end{pmatrix}$$

with $\delta \ll \mu$ cannot be a consequence of "normal" symmetries

- zero may be explained as holomorphic zero but not the hierarchy
- e.g. pseudo-anomalous $\mathrm{U}(1)_{\mathrm{anom}}$ & flavon ϕ with charges

$$\left.\begin{aligned} q_{Q_\mathrm{L}} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ q_{\overline{d}_\mathrm{R}} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ q_\phi &= 1 \end{aligned}\right\} \curvearrowright \quad M_{\text{holomorphic zero}} \sim \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & \varepsilon^4 \end{pmatrix}$$

- zero can also emerge from non-invertible symmetries

talks by Junichiro Kawamura, Tatsuo Kobayashi, Hiroshi Okada & Morimitsu Tanimoto

Weinberg's Mass Texture with a Modular Zero

☞ assign both the quark doublet & d-type quarks the representations $(\mathbf{1''}, \mathbf{1})$ under a modular T' & weight $-5/2$

 Modular Zeros

Weinberg's Mass Texture with a Modular Zero

- ☞ assign both the quark doublet & d-type quarks the representations $(\mathbf{1''}, \mathbf{1})$ under a modular T' & weight $-5/2$

modular zero

- ➥ Weinberg's mass texture can be reproduced as ↗ Liu & MR (2025)

$$M_{\text{modular zero}} = \begin{pmatrix} 0 & \mathcal{Y}^{(4)}_{\mathbf{1}'} \\ \mathcal{Y}^{(4)}_{\mathbf{1}'} & \mathcal{Y}^{(4)}_{\mathbf{1}} \end{pmatrix}$$

- ☞ mass hierarchies

$$M_{\text{modular zero}} \xrightarrow{\operatorname{Im}\varrho > 1} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

$$\varepsilon := \mathrm{e}^{2\pi\,\mathrm{i}\,\varrho/3}$$

Weinberg's Mass Texture with a Modular Zero

☞ assign both the quark doublet & d-type quarks the representations $(\mathbf{1}'', \mathbf{1})$ under a modular T' & weight $-5/2$

➥ Weinberg's mass texture can be reproduced as Liu & MR (2025)

$$M_{\text{modular zero}} = \begin{pmatrix} 0 & \mathcal{Y}^{(4)}_{\mathbf{1}'} \\ \mathcal{Y}^{(4)}_{\mathbf{1}'} & \mathcal{Y}^{(4)}_{\mathbf{1}} \end{pmatrix}$$

☞ mass hierarchies

$$M_{\text{modular zero}} \xrightarrow{\operatorname{Im} \varrho > 1} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

➥ modular mass matrix can naturally explain the Cabibbo angle & the mass hierarchy of the lighter two generations, $\vartheta_{\mathrm{C}} \approx \sqrt{m_d/m_s} = \varepsilon \simeq 0.2$

Summary

&

Outlook

Summary

Modular Zeros

Trivial Zeros

e.g. $k \leq 0$

- similar to holomorphic zeros

Summary

Modular Zeros

Trivial Zeros

Structural Zeros

e.g. $k \leq 0$

- similar to holomorphic zeros

e.g. Modular Spin

- explains mysterious Rule 4 (stringy zeros)

Summary

Modular Zeros

- **Trivial Zeros**
- **Structural Zeros**
- **Sporadic Zeros**

Trivial Zeros

e.g. $k \leq 0$

- similar to holomorphic zeros

Structural Zeros

e.g. Modular Spin

- explains mysterious Rule 4 (stringy zeros)

Sporadic Zeros

no obvious pattern

- contradicts naive EFT rules
- Weinberg's mass texture
- leads to simple modular invariant holomorphic observables

talk by Mu-Chun Chen ⟷ modular invariant holomorphic observables

Outlook

- We have only started to use the analytic properties of modular forms to address the flavor puzzle
  - modular invariant holomorphic observables
  - (almost) immune to quantum corrections

  } talk by Mu-Chun Chen

- Modular Zeros may be also available in non-holomorphic scenarios


정말
정말 감사합니다
THANK YOU VERY MUCH

Holomorphic Zeros

Binetruy & Ramond (1995); Binétruy, Lavignac & Ramond (1996)

- string models often have a pseudo-anomalous $\mathrm{U}(1)$ factor: $\mathrm{U}(1)_{\mathrm{anom}}$

Fischler, Nilles, Polchinski, Raby & Susskind (1981)

- $\mathrm{U}(1)_{\mathrm{anom}}$ symmetry leads to 1-loop Fayet–Iliopoulos (FI) term

$$\mathscr{V}_D \supset \frac{1}{2}\left(g_{\mathrm{anom}}\sum_i q_{\mathrm{anom}}^{(i)}\left|\Phi^{(i)}\right|^2-\xi_{\mathrm{FI}}\right)^2$$

Leurer, Nir & Seiberg (1993); Leurer, Nir & Seiberg (1994)

- holomorphy & sign of the charge of the field(s) that cancel the FI term lead to restrictions on the couplings, known as "holomorphic zeros"
- if only fields with positive $\mathrm{U}(1)_{\mathrm{anom}}$ charges get VEVs, superpotential term with net positive $\mathrm{U}(1)_{\mathrm{anom}}$ charge are prohibited at the perturbative level

Holomorphic Zeros (cont'd)

- consider monomial of superfields

$$\mathscr{M} = \prod_i \Phi_i^{n_i}$$

- two cases:
  1. $q_\mathrm{anom}(\mathscr{M}) \le 0$: we can find an $n_+ \ge 0$ such that a superpotential term $\mathscr{M} \cdot \Phi_+^{n_+}$ is $\mathrm{U}(1)_\mathrm{anom}$ invariant
  2. $q_\mathrm{anom}(\mathscr{M}) > 0$: no such n_+ $\curvearrowright$ corresponding superpotential term is prohibited

Araki et al. (2008)

- holomorphic zeros are not expected to be exact

$$\mathscr{W} \supset B\,\mathrm{e}^{-b\,S}\,\mathscr{M}$$

dilaton

back to intro

back to modular zeros

Modular Symmetries are R-Symmetries

- combining supersymmetry with gravity yields a unique framework: supergravity
- Kähler potential of modulus τ is not modular invariant

$$K = -\ln(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}) \xmapsto{\gamma} -\ln\left(\frac{-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}}{|c\,\tau + d|^2}\right)$$
$$= -\ln(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}) + \ln|c\,\tau + d|^2$$

$$\tau \xmapsto{\gamma} \frac{a\,\tau + b}{c\,\tau + d}$$

Modular Symmetries are R-Symmetries

- combining supersymmetry with gravity yields a unique framework: supergravity
- Kähler potential of modulus τ is not modular invariant

$$\begin{aligned} K = -\ln(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}) \xmapsto{\gamma} & -\ln\left(\frac{-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}}{|c\,\tau + d|^2}\right) \\ = & -\ln(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}) + \ln|c\,\tau + d|^2 \end{aligned}$$

- supergravity Kähler function $G = \mathrm{e}^K \ln|\mathscr{W}|^2$ is modular invariant provided that
  $\mathscr{W} \xmapsto{\gamma} (c\,\tau + d)^{-1}\,\mathscr{W}$

Modular Symmetries are R-Symmetries

- combining supersymmetry with gravity yields a unique framework: supergravity
- Kähler potential of modulus τ is not modular invariant

$$K = -\ln\bigl(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}\bigr) \xmapsto{\gamma} -\ln\left(\frac{-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}}{|c\,\tau + d|^2}\right)$$
$$= -\ln\bigl(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}\bigr) + \ln|c\,\tau + d|^2$$

- supergravity Kähler function $G = \mathrm{e}^K \ln|\mathscr{W}|^2$ is modular invariant provided that
  $\mathscr{W} \xmapsto{\gamma} (c\,\tau + d)^{-1}\,\mathscr{W}$

➥ modular symmetries are R-symmetries

Dixon, Kaplunovsky & Louis (1990); . . . ; Liu, MR & Stewart (2026)

Modular Symmetries are R-Symmetries

- combining supersymmetry with gravity yields a unique framework: supergravity
- Kähler potential of modulus τ is not modular invariant

$$K = -\ln\bigl(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}\bigr) \xmapsto{\gamma} -\ln\left(\frac{-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}}{|c\,\tau + d|^2}\right)$$
$$= -\ln\bigl(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}\bigr) + \ln|c\,\tau + d|^2$$

- supergravity Kähler function $G = \mathrm{e}^K \ \ln|\mathscr{W}|^2$ is modular invariant provided that
  $\mathscr{W} \xmapsto{\gamma} (c\,\tau + d)^{-1}\,\mathscr{W}$
- ➥ modular symmetries are R-symmetries

Dixon, Kaplunovsky & Louis (1990); …; Liu, MR & Stewart (2026)

- (why) is this important?

Modular Symmetries are R-Symmetries

- combining supersymmetry with gravity yields a unique framework: supergravity
- Kähler potential of modulus τ is not modular invariant

$$K = -\ln(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}) \xmapsto{\gamma} -\ln\left(\frac{-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}}{|c\,\tau + d|^2}\right)$$
$$= -\ln(-\mathrm{i}\,\tau + \mathrm{i}\,\overline{\tau}) + \ln|c\,\tau + d|^2$$

- supergravity Kähler function $G = \mathrm{e}^K \ln|\mathscr{W}|^2$ is modular invariant provided that
  $\mathscr{W} \xmapsto{\gamma} (c\,\tau + d)^{-1}\,\mathscr{W}$

➥ modular symmetries are R-symmetries

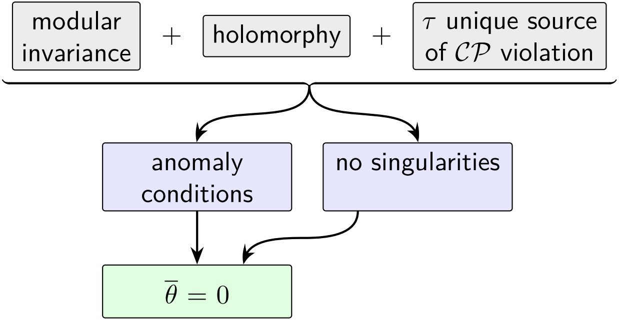
Dixon, Kaplunovsky & Louis (1990); …; Liu, MR & Stewart (2026)

- (why) is this important?
  one implication: duality anomaly coefficients are different

back

Solutions to the Strong $\mathcal{CP}$ Problem

- modular invariance has been proposed as solution to the strong $\mathcal{CP}$ problem Feruglio, Strumia & Titov (2023); Feruglio, Parriciatu, Strumia & Titov (2024); Petcov & Tanimoto (2024); Penedo & Petcov (2024); Feruglio, Marrone, Strumia & Titov (2025);...
- solutions were based on established duality anomaly coefficients Ibáñez & Lüst (1992);...



back

Solutions to the Strong $\mathcal{CP}$ Problem

- modular invariance has been proposed as solution to the strong $\mathcal{CP}$ problem
  Feruglio, Strumia & Titov (2023); Feruglio, Parriciatu, Strumia & Titov (2024); Petcov & Tanimoto (2024); Penedo & Petcov (2024); Feruglio, Marrone, Strumia & Titov (2025);...
- solutions were based on established duality anomaly coefficients
  Ibáñez & Lüst (1992);...

... which are not entirely correct   Liu, MR & Stewart (2026); Chen, Li, Liu, Matias, MR & Stewart (2026)

$$\underbrace{\boxed{\text{modular invariance}} + \boxed{\text{holomorphy}} + \boxed{\tau \text{ unique source of } \mathcal{CP} \text{ violation}}}$$

anomaly conditions → $\bar{\theta} = 0$

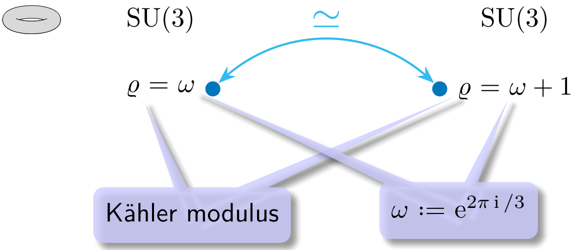
no singularities + supergravity → $\bar{\theta} = 0$

no singularities + supergravity → mass hierarchies

back

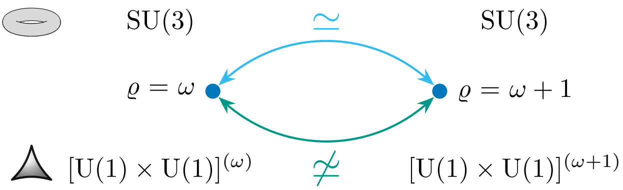
Modular Symmetries & Discrete Gauge Symmetries

- how to get modular flavor symmetries & VVMFs? Li, Liu, Nilles, MR & Stewart (2025)
- well known: compactifying heterotic string on $\mathfrak{su}(3)$ torus leads to $\mathrm{SU}(3) \times \mathrm{SL}(2,\mathbb{Z})$ at critical radius



Modular Symmetries & Discrete Gauge Symmetries

- how to get modular flavor symmetries & VVMFs? Li, Liu, Nilles, MR & Stewart (2025)
- well known: compactifying heterotic string on $\mathfrak{su}(3)$ torus leads to $\mathrm{SU}(3)\times\mathrm{SL}(2,\mathbb{Z})$ at critical radius

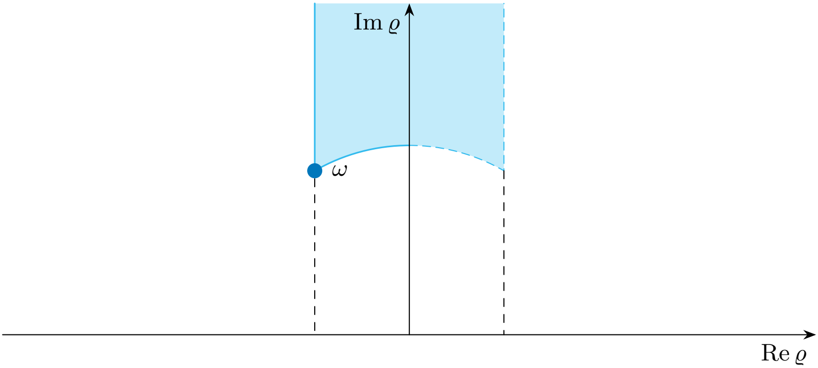


- orbifold: $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetries at critical radius Ibáñez, Lerche, Lüst & Theisen (1991)
- certain $\mathrm{SL}(2,\mathbb{Z})_\varrho$ transformations relate *physically different* $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetries Li, Liu, Nilles, MR & Stewart (2025)

Enhanced Gauge Symmetries & Fundamental Domains

back

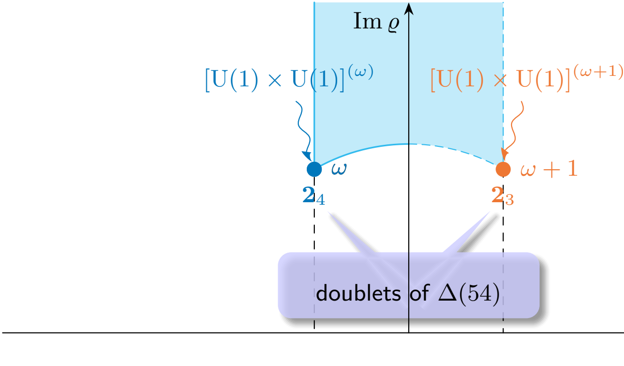
fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$



Enhanced Gauge Symmetries & Fundamental Domains

back

- it is well known that there are $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetries at ω and $\omega + 1$ Ibáñez, Lerche, Lüst & Theisen (1991)
- however, the $\mathrm{U}(1)$ symmetries at ω and $\omega + 1$ are physically inequivalent Li, Liu, Nilles, MR & Stewart (2025)
- ➥ fundamental domain of $\mathrm{SL}(2, \mathbb{Z})$ is *too small*



https://www.pinterest.com/

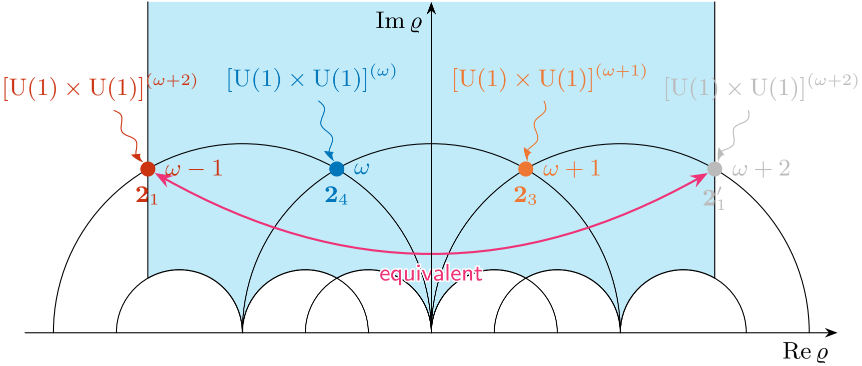
Enhanced Gauge Symmetries & Fundamental Domains

back

$\omega + 3n$ and ω are physically equivalent

$\curvearrowright$ need fundamental domain of $\Gamma(3)$

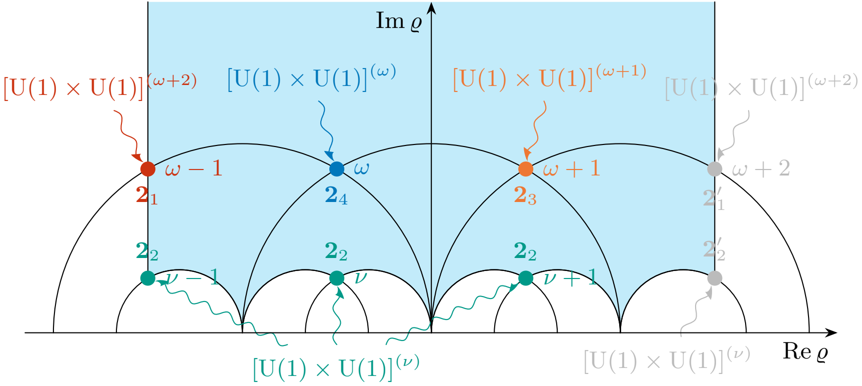
$$\Gamma(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_\varrho ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 3 \right\}$$



Enhanced Gauge Symmetries & Fundamental Domains

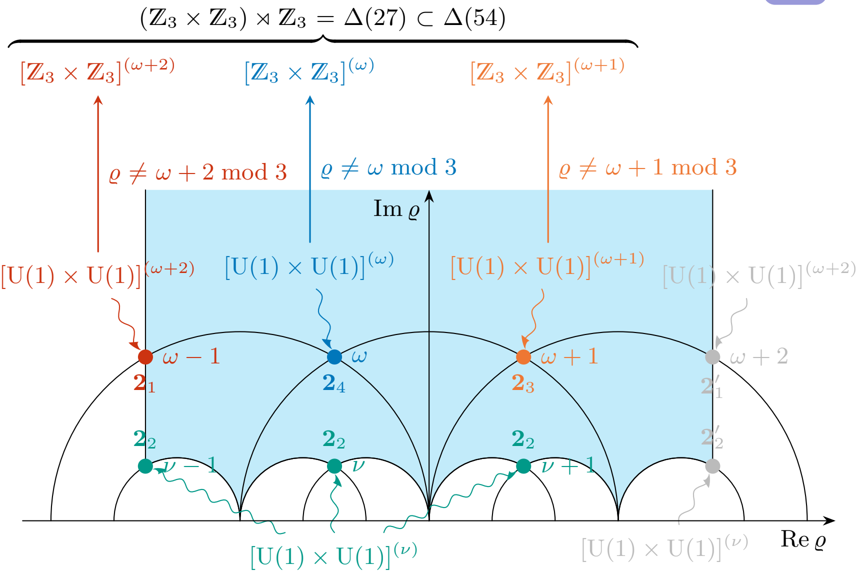
back

the critical points ν, $\nu+1$ & $\nu+2$ carry the 'same' $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\nu)}$ symmetries but can be shown to be physically distinct



Enhanced Gauge Symmetries & Fundamental Domains

back



References

References I

[1] Steven Weinberg. "The Problem of Mass". In: *Trans. New York Acad. Sci.* 38 (1977), pp. 185–201. DOI: 10.1111/j.2164-0947.1977.tb02958.x.

[2] A. Font et al. "On the Concept of Naturalness in String Theories". In: *Phys. Lett. B* 213 (1988), pp. 274–278. DOI: 10.1016/0370-2693(88)91760-1.

[3] C. D. Froggatt & Holger Bech Nielsen. "Hierarchy of Quark Masses, Cabibbo Angles and CP Violation". In: *Nucl. Phys. B* 147 (1979), pp. 277–298. DOI: 10.1016/0550-3213(79)90316-X.

[4] Lisa Randall & Raman Sundrum. "A Large mass hierarchy from a small extra dimension". In: *Phys. Rev. Lett.* 83 (1999), pp. 3370–3373. DOI: 10.1103/PhysRevLett.83.3370. arXiv: hep-ph/9905221.

[5] David B. Kaplan & Martin Schmaltz. "Flavor unification and discrete nonAbelian symmetries". In: *Phys. Rev. D* 49 (1994), pp. 3741–3750. DOI: 10.1103/PhysRevD.49.3741. arXiv: hep-ph/9311281.

References II

[6] Ferruccio Feruglio. "Are neutrino masses modular forms?" In: *From My Vast Repertoire ...: Guido Altarelli's Legacy*. Ed. by Aharon Levy, Stefano Forte & Giovanni Ridolfi. 2019, pp. 227–266. DOI: 10.1142/9789813238053_0012. arXiv: 1706.08749 [hep-ph].

[7] Ferruccio Feruglio & Andrea Romanino. "Lepton flavor symmetries". In: *Rev. Mod. Phys.* 93.1 (2021), p. 015007. DOI: 10.1103/RevModPhys.93.015007. arXiv: 1912.06028 [hep-ph].

[8] Yahya Almumin et al. "Neutrino Flavor Model Building and the Origins of Flavor and Violation". In: *Universe* 9.12 (2023), p. 512. DOI: 10.3390/universe9120512. arXiv: 2204.08668 [hep-ph].

[9] Tatsuo Kobayashi & Morimitsu Tanimoto. "Modular flavor symmetric models". In: July 2023. arXiv: 2307.03384 [hep-ph].

[10] Gui-Jun Ding & Stephen F. King. "Neutrino mass and mixing with modular symmetry". In: *Rept. Prog. Phys.* 87.8 (2024), p. 084201. DOI: 10.1088/1361-6633/ad52a3. arXiv: 2311.09282 [hep-ph].

References III

[11] Gui-Jun Ding & Jose W. F. Valle. "The symmetry approach to quark and lepton masses and mixing". In: *Phys. Rept.* 1109 (2025), pp. 1–105. DOI: 10.1016/j.physrep.2024.12.005. arXiv: 2402.16963 [hep-ph].

[12] Ferruccio Feruglio & Saul Ramos-Sánchez. "Quark and lepton masses". In: (June 2025). arXiv: 2506.20755 [hep-ph].

[13] Michael Dine et al. "Nonperturbative Effects on the String World Sheet". In: *Nucl. Phys. B* 278 (1986), pp. 769–789. DOI: 10.1016/0550-3213(86)90418-9.

[14] Fernando Quevedo. "Lectures on superstring phenomenology". In: (1996). [AIP Conf. Proc.359,202(1996)]. DOI: 10.1063/1.49735. arXiv: hep-th/9603074 [hep-th].

[15] S. Ferrara et al. "Modular Invariance in Supersymmetric Field Theories". In: *Phys. Lett. B* 225 (1989), p. 363. DOI: 10.1016/0370-2693(89)90583-2.

[16] E. J. Chun et al. "Duality and Landau–Ginzburg Models". In: *Phys. Lett. B* 233 (1989), pp. 141–146. DOI: 10.1016/0370-2693(89)90630-8.

References IV

[17] J. Lauer, J. Mas & Hans Peter Nilles. "Twisted sector representations of discrete background symmetries for two-dimensional orbifolds". In: *Nucl. Phys.* B351 (1991), pp. 353–424. DOI: 10.1016/0550-3213(91)90095-F.

[18] Xueqi Li et al. "Flavor symmetries and winding modes". In: *JHEP* 09 (2025), p. 026. DOI: 10.1007/JHEP09(2025)026. arXiv: 2506.12887 [hep-th].

[19] Lance J. Dixon, Vadim Kaplunovsky & Jan Louis. "On Effective Field Theories Describing (2,2) Vacua of the Heterotic String". In: *Nucl. Phys. B* 329 (1990), pp. 27–82. DOI: 10.1016/0550-3213(90)90057-K.

[20] Xiang-Gan Liu, Michael Ratz & Alexander Stewart. "Strong $\mathcal{CP}$ and Quark Mass Hierarchies from Modular Invariance". In: (Aug. 2026). arXiv: 2608.13796 [hep-ph].

[21] Mu-Chun Chen et al. "Chiral Modular Anomalies". In: (2026). in preparation.

References V

[22] Xiang-Gan Liu & Michael Ratz. "Modular Zeros". In: *Phys. Lett. B* 875 (2025), p. 140372. DOI: 10.1016/j.physletb.2026.140372. arXiv: 2512.21386 [hep-th].

[23] Xiang-Gan Liu & Gui-Jun Ding. "Modular flavor symmetry and vector-valued modular forms". In: *JHEP* 03 (2022), p. 123. DOI: 10.1007/JHEP03(2022)123. arXiv: 2112.14761 [hep-ph].

[24] Mu-Chun Chen et al. "Modular invariant holomorphic observables". In: *Phys. Lett. B* 852 (2024), p. 138600. DOI: 10.1016/j.physletb.2024.138600. arXiv: 2401.04738 [hep-ph].

[25] V. Knapp-Perez et al. "Demystifying stringy miracles with eclectic flavor symmetries". In: *Phys. Rev. D* 113.10 (2026), p. 106015. DOI: 10.1103/p6p2-46s1. arXiv: 2512.21382 [hep-th].

[26] Pierre Binetruy & Pierre Ramond. "Yukawa textures and anomalies". In: *Phys. Lett. B* 350 (1995), pp. 49–57. DOI: 10.1016/0370-2693(95)00297-X. arXiv: hep-ph/9412385.

References VI

[27] Pierre Binétruy, Stéphane Lavignac & Pierre Ramond. "Yukawa textures with an anomalous horizontal abelian symmetry". In: *Nucl. Phys.* B477 (1996), pp. 353–377. arXiv: hep-ph/9601243.

[28] W. Fischler et al. "Vanishing Renormalization of the D Term in Supersymmetric U(1) Theories". In: *Phys. Rev. Lett.* 47 (1981), p. 757. DOI: 10.1103/PhysRevLett.47.757.

[29] Miriam Leurer, Yosef Nir & Nathan Seiberg. "Mass matrix models". In: *Nucl. Phys. B* 398 (1993), pp. 319–342. DOI: 10.1016/0550-3213(93)90112-3. arXiv: hep-ph/9212278.

[30] Miriam Leurer, Yosef Nir & Nathan Seiberg. "Mass matrix models: The Sequel". In: *Nucl. Phys.* B420 (1994), pp. 468–504. arXiv: hep-ph/9310320.

[31] Takeshi Araki et al. "(Non-)Abelian discrete anomalies". In: *Nucl. Phys.* B805 (2008), pp. 124–147. DOI: 10.1016/j.nuclphysb.2008.07.005. arXiv: 0805.0207 [hep-th].

References VII

[32] Ferruccio Feruglio, Alessandro Strumia & Arsenii Titov. "Modular invariance and the QCD angle". In: *JHEP* 07 (2023), p. 027. DOI: 10.1007/JHEP07(2023)027. arXiv: 2305.08908 [hep-ph].

[33] Ferruccio Feruglio et al. "Solving the strong CP problem without axions". In: *JHEP* 08 (2024), p. 214. DOI: 10.1007/JHEP08(2024)214. arXiv: 2406.01689 [hep-ph].

[34] S. T. Petcov & M. Tanimoto. "A_4 modular invariance and the strong CP problem". In: *Eur. Phys. J. C* 84.9 (2024), p. 914. DOI: 10.1140/epjc/s10052-024-13272-w. arXiv: 2404.00858 [hep-ph].

[35] J. T. Penedo & S. T. Petcov. "Finite modular symmetries and the strong CP problem". In: *JHEP* 10 (2024), p. 172. DOI: 10.1007/JHEP10(2024)172. arXiv: 2404.08032 [hep-ph].

[36] Ferruccio Feruglio et al. "Solving the strong CP problem in string-inspired theories with modular invariance". In: *JHEP* 08 (2025), p. 076. DOI: 10.1007/JHEP08(2025)076. arXiv: 2505.20395 [hep-ph].

References VIII

[37] Luis E. Ibáñez & Dieter Lüst. "Duality anomaly cancellation, minimal string unification and the effective low-energy Lagrangian of 4-D strings". In: *Nucl. Phys. B* 382 (1992), pp. 305–361. DOI: 10.1016/0550-3213(92)90189-I. arXiv: hep-th/9202046.

[38] Luis E. Ibáñez et al. "Some Considerations About the Stringy Higgs Effect". In: *Nucl. Phys. B* 352 (1991), pp. 435–450. DOI: 10.1016/0550-3213(91)90450-C.