Model-independent approaches in the analysis of Dark Matter direct detection data

Stefano Scopel

IBS Dark World 2017, Daejeon, October 30, 2017
Plan of the talk

1) Short review of model-independent techniques to analyze WIMP direct detection data
2) Determination of WIMP velocity distribution $f(v)$ from data vs. determination of observables for the most general $f(v) \rightarrow$ observables seen as generalized moments of $f(v)$
WIMP direct searches: spin-independent interaction+Maxwellian distribution

Will the race discover DM before eventually reaching the irreducible background of solar and atmospheric neutrinos???

(from Y. Suzuki talk @IDM 2016, July 2016)
First DM results from XENON1T

(complete LUX exposure reached in 34.2 live days)

TON-size era for WIMP direct detection has begun

Aprile et al, 1705.06655
N.B.: theoretical predictions for the WIMP direct detection rate depend on two main ingredients:

1) a scaling law for the cross section, in order to compare experiments using different targets

   *Traditionally spin-independent cross section (proportional to (atomic mass number)^2) or spin-dependent cross section (proportional to the product S_{WIMP} \cdot S_{nucleus}) is assumed*

2) a model for the velocity distribution of WIMPs

   *Traditionally a Maxwellian distribution is assumed*
Getting an updated mass-cross section plot has never been easier!

DMTOOLS @ UCB

Sensitivity Plots for Direct Detection of WIMP Dark Matter

...An interactive plotter for experimental and theoretical data

Instructions & Tips

- Please select the data and the format for the plot you require using the form below. When you have finished the selection press "Generate Plot" once and then wait for the new results page to appear. The new page can take up to 30 sec to appear depending on the server load.
- Please note that you can use the browser "Back" button to return to this page, and your data/colour selections from your previous plot will be remembered.
- These data sets have not necessarily been endorsed by the Collaborations concerned. However, they do correspond to material that has been published in journals, to the arXiv, or given at conference talks.
- We are currently asking for any collaborations involved in dark matter searches to submit new data, or to correct any mistakes. If you would like to submit a data set please use the format described here.

A NOTE ON DATA COMPARISON

- We have used the prevailing convention of assuming that the Local Halo Density of Dark Matter is 0.3 GeV/cm³, and that the characteristic Halo Velocity (v²/2, rmax) is 220-240 km/s.
- The Spin-Independent Cross-section data is normalised to single nucleon.
- The Spin-Dependent Cross-section data is not YET shown in a format which allows fair comparison. (Dan Fox, Sildsfield, et al are developing a formalism to allow comparison of different target nuclei on a single graph.)
- Indirect detection, collider, and neutrino-mass independent limits have not yet been included.
- It is now possible to select the units of the y-axis. If Javascript is enabled on your browser, changing the selected plot units will update the limit options on this form. If Javascript is disabled, the plot will still generate correctly but the limits on this form must be entered in the default units (cm²).

We would like to thank a number of members of the community for the encouragement that they provided during the preparation of this site.

There is a backup cgi server that can be executed from the form of this address if this version appears offline.

Recent/Notable Data

- SL-Exp: Spin Independent - Experimental Limits
- SL-Th: Spin Independent - Theory
- SL-Pro: Spin Independent - Projected Limit
- SD-Exp: Spin Dependent - Experimental Limits
- SD-Th: Spin Dependent - Theory
- SD-Pro: Spin Dependent - Projected Limit

Historical Data

- SL-Exp: Spin Independent - Experimental Limits
- SL-Th: Spin Independent - Theory
- SL-Pro: Spin Independent - Projected Limit
- SD-Exp: Spin Dependent - Experimental Limits
- SD-Th: Spin Dependent - Theory
- SD-Pro: Spin Dependent - Projected Limit

(http://cedar.berkeley.edu/plotter/)

...at least for the most common assumptions: spin-independent, spin-dependent interaction+ Maxwellian
Indeed, spin-independent and spin-dependent cross sections are predicted for the neutralino in supersymmetry and numerical simulations of galaxy formation support the choice of a Maxwellian for the velocity distributions.

However a bottom-up approach would also be desirable, especially if no hints come from high-energy physics about the fundamental properties of the WIMP particle. Indeed two questions arise:

- what is the most general class of scaling laws for a WIMP-nucleus cross section?
- the detailed merger history of the Milky Way is not known, allowing for the possibility of the presence of sizeable non–thermal components for which the density, direction and speed of WIMPs are hard to predict, especially in the high velocity tail of the distribution: do we need to assume a Maxwellian velocity distribution?

Recently both aspects have been addressed
Compatibility among different experiments (ex. DAMA/Libra vs. CoGeNT) can be verified without assuming any model for the halo.

Write expected WIMP rate as:

\[ \frac{dR}{dE_R} = \frac{\rho_X \sigma_n}{2m_X \mu_{nX}^2} \frac{C_T}{f_n^2} F^2(E_R) \epsilon(E_R) g(v_{\text{min}}, t) \]

\( F^2(E_R) \) is the form factor, and the function:

\[ g(v_{\text{min}}, t) = \int_{v_{\text{min}}}^{\infty} \frac{f_{\text{local}}(\vec{v}, t)}{v} d^3v \]

contains all the dependence on the halo model with:

\[ v_{\text{min}} = \sqrt{\frac{m_N E_R}{2 \mu^2}} \]

So there is a one-to-one correspondence between the recoil energy \( E_R \) and \( v_{\text{min}} \) → map the event rate expected in different experiments into the same intervals in \( v_{\text{min}} \) (P.J. Fox, J. Liu, N. Weiner, PRD83,103514 (2011))

In this way the dependence on the galactic model cancels out in the ratio of the expected count rates of the two experiments because they depend on the same integrals of \( f_{\text{local}}(v) \)
halo-independent analysis for elastic scattering
Del Nobile, Gelmini, Gondolo, Huh, arXiv:1405.5582

\[ m_{\text{WIMP}} = 7 \text{ GeV} \]

\[ m_{\text{WIMP}} = 9 \text{ GeV} \]

\[
R_{[E'_1, E'_2]}^{\text{SI}}(t) = \int_0^\infty dv_{\text{min}} \tilde{\eta}(v_{\text{min}}, t) R_{[E'_1, E'_2]}^{\text{SI}}(v_{\text{min}})
\]

\[
\tilde{\eta}(v_{\text{min}}, t) \equiv \frac{\rho \sigma_p}{m} \int_{v \geq v_{\text{min}}} d^3v \frac{f(v, t)}{v}
\]

\[
\tilde{\eta}(v_{\text{min}}, t) \approx \tilde{\eta}^0(v_{\text{min}}) + \tilde{\eta}^1(v_{\text{min}}) \cos[\omega(t - t_0)]
\]

N.B.: only halo dependence factorized. Results depend on assumptions on other quantities such as quenching factors, \( L_{\text{eff}}, Q_y \) etc.
The usual “halo-independent” approach to analyze yearly modulation data: factorize a modulated halo function \( \tilde{\eta}_1 \) with the only constraint \( \tilde{\eta}_1 < \tilde{\eta}_0 \).

(In the case of a Maxwellian typically \( \tilde{\eta}_1 / \tilde{\eta}_0 \leq 0.07 \))

Standard lore: cannot predict \( \tilde{\eta}_1 / \tilde{\eta}_0 \) without a model for the velocity distribution. Is it really so? More on that later.
Summarizing, the minimal requirements for halo functions $\eta_{0,1}$ are:

\[
\tilde{\eta}_0(v_{\text{min},2}) \leq \tilde{\eta}_0(v_{\text{min},1}) \quad \text{if } v_{\text{min},2} > v_{\text{min},1} \\
\tilde{\eta}_1 \leq \tilde{\eta}_0 \quad \text{at the same } v_{\text{min}} \\
\tilde{\eta}_0(v_{\text{min}} \geq v_{\text{esc}}) = 0.
\]

(decreasing function)
(modulated part<100%)
(no bound WIMPs<escape velocity)
Several epicycles added to the usual scenario:
- Non-standard coupling
- Inelastic scattering
- Isospin violation
- ....
Inelastic Dark Matter

Two mass eigenstates $\chi$ and $\chi'$ very close in mass: $m_\chi - m_{\chi'} \equiv \delta$ with $\chi + N \rightarrow \chi + N$ forbidden

"Endothermic" scattering ($\delta > 0$)

"Exothermic" scattering ($\delta < 0$)

Kinetic energy needed to "overcome" step $\rightarrow$ rate no longer exponentially decaying with energy, maximum at finite energy $E_*$

$\chi$ is metastable, $\delta$ energy deposited independently on initial kinetic energy (even for WIMPs at rest)
Inelastic DM and the halo-independent approach: recoil energy $E_{ee}$ is no longer monotonically growing with $v_{\text{min}}$ (energy $E^*$ corresponds to minimal $v_{\text{min}}$)

$$v_{\text{min}} = \frac{1}{\sqrt{2m_N E_R}} \left( \frac{m_N E_R}{\mu} + \delta \right) = a \sqrt{E_r} + \frac{b}{\sqrt{E_R}}$$

N.B. for $\delta>0$ WIMPs need a minimal absolute incoming speed $v_*$ to upscatter to the heavier state $\rightarrow$ vanishing rate if $v_*>v_{\text{esc}}$ (escape velocity)

Need to rebin the data in such a way that the relation between $v_{\text{min}}$ and $E_R$ is invertible in each bin (easy: just ensure that for all target nuclei $E^*$ corresponds to one of the bin boundaries)

S. Scopel and K.H. Yoon, JCAP1408, 060 (2014)
isospin violation (more properly: isovector interaction)

\[ R = \sigma_p \sum_i \eta_i \frac{\mu_{A_i}^2}{\mu_p^2} I_{A_i} \left[ Z + (A_i - Z) \frac{f_n}{f_p} \right]^2 \]  

(sum over isotopes)

Cancellation between \( f_p \) (WIMP-proton coupling) and \( f_n \) (WIMP-nucleon coupling) when \( f_n/f_p \sim -Z/(A-Z) \) can suppress the scattering cross section on a specific target (i.e. \( f_n/f_p \sim -0.79 \) for Germanium)

Minimal “degrading factors”, i.e. maximal factors by which the reciprocal scaling law between two elements can be reduced (limited by multiple isotopes, one choice of \( f_n/f_p \) ratio cannot fit all)

<table>
<thead>
<tr>
<th>Element</th>
<th>Xe</th>
<th>Ge</th>
<th>Si</th>
<th>Ca</th>
<th>W</th>
<th>Ne</th>
<th>C</th>
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<tr>
<td>Xe (54, *)</td>
<td>1.00</td>
<td>8.79</td>
<td>149.55</td>
<td>138.21</td>
<td>10.91</td>
<td>34.31</td>
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<td>1.06</td>
<td>757.44</td>
<td>1.06</td>
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<tr>
<td>Ca (20, *)</td>
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<td>31.53</td>
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<td>4.09</td>
<td>782.49</td>
<td>1.10</td>
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<tr>
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<td>4.09</td>
<td>726.09</td>
<td>1.00</td>
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<tr>
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<td>1.07</td>
<td>4.09</td>
<td>789.59</td>
<td>1.12</td>
<td>1.00</td>
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<td>5.51</td>
<td>127.04</td>
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<td>20.68</td>
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<td>127.61</td>
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<td>1.08</td>
<td>1.03</td>
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(J.L.Feng, J.Kumar, D.Marfatia and D.Sanford, Phys.Lett.B703, 124 (2011), 1102.4331)
On the most general WIMP-nucleus cross section (i.e. beyond “spin-dependent” and “spin” independent)
Most general approach: consider ALL possible NR couplings, including those depending on velocity and momentum

\[ \mathcal{H} = \sum_i \left( c_i^0 + c_i^1 \tau_3 \right) \mathcal{O}_i \]

\( \tau_3 = \) nuclear isospin operator, i.e.

\[
\begin{align*}
  c_i^p &= \frac{(c_i^0 + c_i^1)}{2} \quad \text{(proton)} \\
  c_i^n &= \frac{(c_i^0 - c_i^1)}{2} \quad \text{(neutron)}
\end{align*}
\]

(if \( c_i^p = c_i^n \Rightarrow c_i^1 = 0 \))

N.R. operators \( \mathcal{O}_i \) guaranteed to be Hermitian if built out of the following four 3-vectors:

\[
\begin{align*}
  i \frac{\vec{q}}{m_N}, & \quad \vec{v}^\perp, & \quad \vec{S}_X, & \quad \vec{S}_N
\end{align*}
\]

with:

\[
\begin{align*}
  \vec{v}^\perp &= \vec{v} + \frac{\vec{q}}{2\mu_N} \\
  \vec{v} &\equiv \vec{v}_{\chi,\text{in}} - \vec{v}_{N,\text{in}}
\end{align*}
\]

\[ \vec{v}^\perp \cdot \vec{q} = 0 \]

Additional operators that do not arise for traditional spin≤1 mediators:

\[ \mathcal{O}_{12} = \vec{S}_X \cdot (\vec{S}_N \times \vec{v}^\perp), \]
\[ \mathcal{O}_{13} = i(\vec{S}_X \cdot \vec{v}^\perp)(\vec{S}_N \cdot \frac{\vec{q}}{m_N}), \]
\[ \mathcal{O}_{14} = i \left( \vec{S}_X \cdot \frac{\vec{q}}{m_N} \right)(\vec{S}_N \cdot \vec{v}^\perp), \]
\[ \mathcal{O}_{15} = -\left( \vec{S}_X \cdot \frac{\vec{q}}{m_N} \right) \left[ (\vec{S}_N \times \vec{v}^\perp) \cdot \frac{\vec{q}}{m_N} \right] \]
\[ \mathcal{O}_{16} = -\left[ (\vec{S}_X \times \vec{v}^\perp) \cdot \frac{\vec{q}}{m_N} \right] \left( \vec{S}_N \cdot \frac{\vec{q}}{m_N} \right) \]
In the expected rate WIMP physics (encoded in the R functions that depend on the $c_i$ couplings) and the nuclear physics (contained in 8 (6+2) response functions $W$ factorize in a simple way:

\[ \frac{dR}{dE_R} = \sum_T \frac{dR_T}{dE_R} = \sum_T \xi_T \frac{\rho_{\chi}}{2\pi m_{\chi}} \int \frac{f(\vec{v} + \vec{v}_e(t))}{v} P_{\text{tot}}(v^2, q^2) d^3v \]

\[ P_{\text{tot}}(v^2, q^2) = \frac{4\pi}{2j_N + 1} \sum_{\tau=0,1} \sum_{\tau'=0,1} \left[ R_{M}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{M}^{\tau\tau'}(y) + \right. \]

\[ + R_{\Sigma}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{\Sigma}^{\tau\tau'}(y) + R_{\Phi}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{\Phi}^{\tau\tau'}(y) \]

\[ + \frac{q^2}{m_N^2} \left[ R_{M}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{M}^{\tau\tau'}(y) + R_{\Sigma}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{\Sigma}^{\tau\tau'}(y) \right. \]

\[ + R_{\Phi}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{\Phi}^{\tau\tau'}(y) + R_{\Delta}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{\Delta}^{\tau\tau'}(y) \]

\[ + \left. R_{\Delta\Sigma}^{\tau\tau'}(v_{\chi T}^{1/2}, \frac{q^2}{m_N^2}) W_{\Delta\Sigma}^{\tau\tau'}(y) \right] \}

N.B.: besides usual spin-independent and spin-dependent terms new contributions arise, with explicit dependences on the transferred momentum $q$ and the WIMP incoming velocity.

WIMPs response functions

\[
\begin{align*}
R_M^{\tau\tau'} (\frac{v_T^2}{m_N^2}, \frac{q^2}{m_N^2}) &= c_1^{\tau} c_1^{\tau'} + \frac{j_x(j_x+1)}{3} \left[ \frac{q^2}{m_N^2} v_T^{\perp 2} c_5^{\tau} c_5^{\tau'} + v_T^{\perp 2} c_8^{\tau} c_8^{\tau'} + \frac{q^2}{m_N^2} c_1^{\tau} c_1^{\tau'} \right] \\
R_\Phi^{\tau\tau'} (\frac{v_T^2}{m_N^2}, \frac{q^2}{m_N^2}) &= \left[ \frac{q^2}{4m_N^2} c_3^{\tau} c_3^{\tau'} + \frac{j_x(j_x+1)}{12} \left( c_{12}^{\tau} - \frac{q^2}{m_N^2} c_{15}^{\tau'} \right) \left( c_{12}^{\tau'} - \frac{q^2}{m_N^2} c_{15}^{\tau} \right) \right] \frac{q^2}{m_N^2} \\
R_\Phi''^{\tau\tau'} (\frac{v_T^2}{m_N^2}, \frac{q^2}{m_N^2}) &= c_3^{\tau} c_3^{\tau'} + \frac{j_x(j_x+1)}{3} \left( c_{12}^{\tau} c_{12}^{\tau'} + \frac{q^2}{m_N^2} c_{13}^{\tau} c_{13}^{\tau'} \right) \frac{q^2}{m_N^2} \\
R_{\Sigma}^{\tau\tau'} (\frac{v_T^2}{m_N^2}, \frac{q^2}{m_N^2}) &= \frac{q^2}{4m_N^2} c_1^{\tau} c_1^{\tau'} + \frac{j_x(j_x+1)}{12} \left[ c_4^{\tau} c_4^{\tau'} + \frac{q^2}{m_N^2} (c_6^{\tau} c_6^{\tau'} + c_6^{\tau} c_4^{\tau'}) + \frac{q^2}{m_N^2} c_6^{\tau} c_6^{\tau'} + v_T^{\perp 2} c_{12}^{\tau} c_{12}^{\tau'} + \frac{q^2}{m_N^2} v_T^{\perp 2} c_{13}^{\tau} c_{13}^{\tau'} \right] \\
R_{\Delta}^{\tau\tau'} (\frac{v_T^2}{m_N^2}, \frac{q^2}{m_N^2}) &= \frac{1}{8} \left[ \frac{q^2}{m_N^2} v_T^{\perp 2} c_3^{\tau} c_3^{\tau'} + v_T^{\perp 2} c_7^{\tau} c_7^{\tau'} \right] + \frac{j_x(j_x+1)}{12} \left[ c_4^{\tau} c_4^{\tau'} + \frac{q^2}{m_N^2} c_9^{\tau} c_9^{\tau'} + \frac{v_T^{\perp 2}}{2} \left( c_{12}^{\tau} - \frac{q^2}{m_N^2} c_{15}^{\tau'} \right) \left( c_{12}^{\tau'} - \frac{q^2}{m_N^2} c_{15}^{\tau} \right) + \frac{q^2}{2m_N^2} v_T^{\perp 2} c_{14}^{\tau} c_{14}^{\tau'} \right] \\
R_{\Delta\Sigma}^{\tau\tau'} (\frac{v_T^2}{m_N^2}, \frac{q^2}{m_N^2}) &= \frac{j_x(j_x+1)}{3} \left( \frac{q^2}{m_N^2} c_5^{\tau} c_5^{\tau'} + c_8^{\tau} c_8^{\tau'} \right) \frac{q^2}{m_N^2} \\
R_{\Delta\Sigma'}^{\tau\tau'} (\frac{v_T^2}{m_N^2}, \frac{q^2}{m_N^2}) &= \frac{j_x(j_x+1)}{3} \left( c_5^{\tau} c_4^{\tau'} - c_8^{\tau} c_9^{\tau'} \right) \frac{q^2}{m_N^2}.
\end{align*}
\]

General form:

\[
R_{k}^{\tau\tau'} = R_{0k}^{\tau\tau'} + R_{1k}^{\tau\tau'} \frac{(v_T^2)^2}{c^2} = R_{0k}^{\tau\tau'} + R_{1k}^{\tau\tau'} \frac{v_T^2 - v_{\text{min}}^2}{c^2}
\]
Nuclear response functions

Assuming one-body dark matter-nucleon interactions, the Hamiltonian density for dark matter-nucleus interactions is:

\[ \mathcal{H}_{ET}(\vec{x}) = \sum_{i=1}^{A} l_0(i) \delta(\vec{x} - \vec{x}_i) + \sum_{i=1}^{A} l_0^A(i) \frac{1}{2M} \left[ -\frac{1}{i} \vec{\nabla}_i \cdot \vec{\sigma}(i) \delta(\vec{x} - \vec{x}_i) + \delta(\vec{x} - \vec{x}_i) \vec{\sigma}(i) \cdot \frac{1}{i} \vec{\nabla}_i \right] \]

\[ + \sum_{i=1}^{A} \vec{l}_5(i) \cdot \vec{\sigma}(i) \delta(\vec{x} - \vec{x}_i) + \sum_{i=1}^{A} \vec{l}_M(i) \cdot \frac{1}{2M} \left[ -\frac{1}{i} \vec{\nabla}_i \delta(\vec{x} - \vec{x}_i) + \delta(\vec{x} - \vec{x}_i) \frac{1}{i} \vec{\nabla}_i \right] \]

\[ + \sum_{i=1}^{A} \vec{l}_E(i) \cdot \frac{1}{2M} \left[ \vec{\nabla}_i \times \vec{\sigma}(i) \delta(\vec{x} - \vec{x}_i) + \delta(\vec{x} - \vec{x}_i) \vec{\sigma}(i) \times \vec{\nabla}_i \right] \]

So the WIMP-nucleus Hamiltonian has the general form:

\[ \int d\vec{x} \ e^{-i\vec{q} \cdot \vec{x}} \left[ l_0 \langle J_i M_i | \hat{\rho}(\vec{x}) | J_i M_i \rangle - \vec{q} \cdot \langle J_i M_i | \hat{j}(\vec{x}) | J_i M_i \rangle \right] \]

With:

\[ e^{i\vec{q} \cdot \vec{x}_i} = \sum_{J=0}^{\infty} \sqrt{4\pi} \ [J] i^J j_J(q \vec{x}_i) Y_{J0}(\Omega_{x_i}) \]

\[ \hat{e}_\lambda e^{i\vec{q} \cdot \vec{x}_i} = \begin{cases} \sum_{J=0}^{\infty} \sqrt{4\pi} \ [J] i^{J-1} \frac{\vec{\nabla}_i}{q} j_J(q \vec{x}_i) Y_{J0}(\Omega_{x_i}), & \lambda = 0 \\ \sum_{J=1}^{\infty} \sqrt{2\pi} \ [J] i^{J-2} \left[ \lambda j_J(q \vec{x}_i) \hat{Y}_{J1}(\Omega_{x_i}) + \frac{\vec{\nabla}_i}{q} \times j_J(q \vec{x}_i) \hat{Y}_{J1}(\Omega_{x_i}) \right], & \lambda = \pm 1 \end{cases} \]

which depends on the expectations of six distinct nuclear response functions, defined as:

\[ M_{JM}(q\bar{x}) \]
\[ \Delta_{JM}(q\bar{x}) = \frac{1}{q} \nabla \cdot \nabla \nabla M_{JM}(q\bar{x}) \]
\[ \Sigma'_{JM}(q\bar{x}) = -i \left\{ \frac{1}{q} \nabla \times M_{JM}(q\bar{x}) \right\} \cdot \vec{\sigma} = [J]^{-1} \left\{ -\sqrt{J} M_{JJ+1}(q\bar{x}) + \sqrt{J+1} M_{JJ-1}(q\bar{x}) \right\} \cdot \vec{\sigma} \]
\[ \Sigma''_{JM}(q\bar{x}) = \left\{ \frac{1}{q} \nabla M_{JM}(q\bar{x}) \right\} \cdot \vec{\sigma} = [J]^{-1} \left\{ \sqrt{J+1} M_{JJ+1}(q\bar{x}) + \sqrt{J} M_{JJ-1}(q\bar{x}) \right\} \cdot \vec{\sigma} \]
\[ \Phi'_{JM}(q\bar{x}) = \left( \frac{1}{q} \nabla \times M_{JM}(q\bar{x}) \right) \cdot \left( \vec{\sigma} \times \frac{1}{q} \nabla \right) + \frac{1}{2} M_{JJ}(q\bar{x}) \cdot \vec{\sigma} \]
\[ \Phi''_{JM}(q\bar{x}) = i \left( \frac{1}{q} \nabla M_{JM}(q\bar{x}) \right) \cdot \left( \vec{\sigma} \times \frac{1}{q} \nabla \right) \]

with \( M_{JM} = j_J Y_{JM} \) Bessel spherical harmonics and \( M_{M JL} = j_J Y_{JM} \) vector spherical harmonics.

- \( \mathbf{M} \) = vector-charge (scalar, \textbf{usual spin-independent part}, non-vanishing for all nuclei)
- \( \Phi'' \) = vector-longitudinal, related to spin-orbit coupling \( \vec{\sigma} \cdot \mathbf{l} \) (also spin-independent, non-vanishing for all nuclei)
- \( \Sigma' \) and \( \Sigma'' \) = associated to longitudinal and transverse components of nuclear spin, \textbf{their sum is the usual spin-dependent interaction}, require nuclear spin \( j>0 \)
- \( \Delta \) = associated to the orbital angular momentum operator \( \mathbf{l} \), also requires \( j>0 \)
- \( \Phi' \) = related to a vector-longitudinal operator that transforms as a tensor under rotations, requires \( j>1/2 \)

Squaring the amplitude get the following nuclear response functions:

\[
W_{O}^{\tau \tau'}(y) \equiv \sum_{J=0,2,\ldots}^{\infty} \langle j_{N} \| O_{J;\tau}(q) \| j_{N} \rangle \langle j_{N} \| O_{J;\tau'}(q) \| j_{N} \rangle \text{ for } O = M, \Phi'',
\]

\[
W_{O}^{\tau \tau'}(y) \equiv \sum_{J=1,3,\ldots}^{\infty} \langle j_{N} \| O_{J;\tau}(q) \| j_{N} \rangle \langle j_{N} \| O_{J;\tau'}(q) \| j_{N} \rangle \text{ for } O = \Sigma'', \Sigma', \Delta,
\]

\[
W_{\Phi'}^{\tau \tau'}(y) = \sum_{J=2,4,\ldots}^{\infty} \langle j_{N} \| \Phi'_{J;\tau}(q) \| j_{N} \rangle \langle j_{N} \| \Phi'_{J;\tau'}(q) \| j_{N} \rangle,
\]

\[
W_{\Phi''_{M}}^{\tau \tau'}(y) = \sum_{J=0,2,\ldots}^{\infty} \langle j_{N} \| \Phi''_{J;\tau}(q) \| j_{N} \rangle \langle j_{N} \| M_{J;\tau'}(q) \| j_{N} \rangle,
\]

\[
W_{\Delta_{\Sigma'}}^{\tau \tau'}(y) = \sum_{J=1,3,\ldots}^{\infty} \langle j_{N} \| \Delta_{J;\tau}(q) \| j_{N} \rangle \langle j_{N} \| \Sigma'_{J;\tau'}(q) \| j_{N} \rangle.
\]

(Interference terms)

These 8 (6+2 interferences) W nuclear response functions have been calculated for most nuclei using a numerical (truncated) harmonic potential shell model (Fitzpatrick et al., JCAP 1302 1302(2013), Catena and Schwabe, JCAP 1504 no. 04, 042 (2015)) with oscillator parameter:

\[
b[\text{fm}] = \sqrt{41.467/(45A^{-1/3} - 25A^{-2/3})} \quad y = (qb/2)^2
\]
P.S.: for a WIMP with spin>1/2 additional operators arise. They can be obtained as the spherical components of generic powers of $q, v_\perp, S_\chi$ and $S_N$. However, in the limit of one-nucleon interaction and including only terms linear in $v_\perp$ the number of operators remains finite for any WIMP spin. Moreover no additional nuclear response functions are needed.

Operators are numbered from 1 to 27, but $O_2$ depends on $v^2$ and $O_{16}$ can be expressed as a linear combination of $O_{12}$ and $O_{15} \rightarrow 27-2=25$ operators.

$\sigma=S_N$, $S=S_\chi$
General expressions for expected rates

N.B. : need to introduce 3 different energies:

\( E_R = \text{nuclear recoil energy}; \)
\( E_{ee} = \text{electron-equivalent energy}, \)
\( E' = \text{visible energy}. \)

The expected rate in a direct detection experiment for the energy bin \( E' _1 < E' < E' _2 \) is given by:

\[
R_{[E' _1, E' _2]} = MT \int_{E' _1}^{E' _2} \frac{dR}{dE'} dE'
\]

\[
\frac{dR}{dE'} = \int_0^{\infty} \frac{dR}{dE_{ee}} G(E', E_{ee}) \epsilon(E') dE_{ee}
\]

\[
E_{ee} = q(E_R) E_R,
\]

\( q(E_R) \) = quenching factor (fraction of recoil energy into ionization/scintillation),
\( \epsilon \) = acceptance/cuts efficiency, \( G(E', E_{ee}) \) = energy resolution, and

\[
\frac{dR}{dE_R} = N_T \int_{v_{min}}^{v_{esc}} \frac{\rho_\chi}{m_\chi} v \frac{d\sigma}{dE_R} f(v) dv
\]
The differential cross section is given by:

$$\frac{d\sigma}{dE_R} = \frac{1}{10^6} \frac{2m_N}{4\pi} \frac{c^2}{v^2} \left[ \frac{1}{2j_X + 1} \frac{1}{2j_N + 1} \sum_{\text{spin}} |M|^2 \right]$$

with:

$$\frac{1}{2j_X + 1} \frac{1}{2j_N + 1} \sum_{\text{spin}} |M|^2 = \frac{4\pi}{2j_N + 1} \sum_{\tau\tau'} \sum_l R_{l l}^{\tau\tau'} W_{l l}^{\tau\tau'}$$

WIMP response function nuclear response function

where the R’s contains a sum over all the couplings of the non-relativistic effective lagrangian

$$\mathcal{H} = \sum_{\alpha=n,p} \sum_{i=1}^{27} c_i^{\alpha} O_i^{\alpha}, \quad c_2 = c_{16} = 0$$

$$c_0^i = \frac{c_i^p + c_i^n}{2}, \quad c_1^i = \frac{c_i^p - c_i^n}{2},$$

while the W’s are the corresponding 8 nuclear response functions (including interferences)
To make contact to the previous formalism and in case of a single coupling $c_i \neq 0$ it is possible to factorize the conventional WIMP-proton cross section:

$$\sigma_p = (c_i^p)^2 \mu_x^2 \frac{\chi N}{\pi}$$

and get for the rate the expression:

$$\int_0^\infty dE_R \mathcal{R}(E_R) \{\tilde{\eta}[v_{\text{min}}(E_R)]c^2\} = \int_0^\infty d\nu_{\text{min}} \mathcal{R}(\nu_{\text{min}}) \{\tilde{\eta}(\nu_{\text{min}})c^2\}$$

$$\mathcal{R}(\nu_{\text{min}}) = \mathcal{R}(E_R) \frac{dE_R}{d\nu_{\text{min}}}$$

$$\tilde{\eta}(\nu_{\text{min}}) = \frac{\rho_x}{m_x} \sigma_p \eta(\nu_{\text{min}})$$

“generalized halo function”

$$\eta(\nu_{\text{min}}) \equiv \int_{\nu_{\text{min}}}^{\infty} \frac{f(v)}{v} \, dv$$

“halo function”
all the experimental quantities (energy resolution, efficiency, quenching, etc) are contained in the response function \( R \):

\[
\mathcal{R}(E_R) = N_T M T \frac{1}{10^6} \frac{m_N}{2\mu^2_N} \int_{E_1'}^{E_2'} dE' \epsilon(E') G[E', q(E_R)E_R] \frac{1}{2j_x + 1} \frac{1}{2j_N + 1} \sum_{\text{spin}} \left| \mathcal{M} \left( c_i^p = 1, r = \frac{c_i^n}{c_i^p} \right) \right|^2
\]

that depends also on the ratio \( r=c_n/c_p \). The isospin rotation can be easily done since the response function for a generic value of \( r=c_n/c_p \) is a linear combination of the three response functions for \( r=-1,0,1 \):

\[
\mathcal{R}(E_R, r) = \frac{r(r+1)}{2} \mathcal{R}(E_R, r = 1) + \frac{r(r-1)}{2} \mathcal{R}(E_R, r = -1) + (1 - r^2) \mathcal{R}(E_R, r = 0)
\]
N.B. the previous derivation requires no explicit velocity dependence in the cross section. However the same form is retained also in the case of an explicit velocity dependence in the cross section.

Take out the velocity integral, and write the expected number of events as:

\[ N(t)[E_1',E_2'] = \int_0^\infty \mathcal{H}[E_1',E_2'](v) f(v, t) \, dv \]

where the response function contains all the dependences on the cross section and the experimental quantities. By setting:

\[ f(v, t) \equiv -v \frac{\partial \eta(v, t)}{\partial v} \]

integrating by parts and incorporating as usual the point-like cross section and the local density in the definition of the halo function leads to:

\[ N[E_1',E_2'](v)(t) = \int_0^\infty \mathcal{R}[E_1',E_2'](v) \tilde{\eta}(v, t) \, dv \]

This expression looks pretty much the same as the previous one (with \( v_{\text{min}} \to v \)) but is valid in principle for any velocity dependence in the cross section.
So two alternative expressions for the expected rate:

\[
N_{[E_1', E_2']} (v) (t) = \int_0^\infty R_{[E_1', E_2']} (v) \tilde{n}(v, t) \, dv
\]

\[
N (t)_{[E_1', E_2']} = \int_0^\infty H_{[E_1', E_2']} (v) f (v, t) \, dv
\]

N.B: mathematically, expected rates can be seen a “generalized moments” of the distribution \( f(v) \), where “ordinary” moments are integrals

\[
\int v^n f (v) \, dv
\]

The two different response functions are related by:

\[
R_{[E_1', E_2']} (v) = \frac{\partial}{\partial v} \left[ v H_{[E_1', E_2']} (v) \right], \quad H_{[E_1', E_2']} (v) = \frac{1}{v} \int_0^v R_{[E_1', E_2']} (v') \, dv'
\]
If a large number of evaluations of the response function is needed (as in the case of a Monte Carlo simulation) it is convenient to tabulate the integrated response function:

\[ \bar{\mathcal{R}}(E_R) \equiv \int_0^{E_R} \mu_{\chi N}^2 \mathcal{R}(E') \, dE' \]

that does not depend on the WIMP mass or mass splitting, and calculate:

\[ \mathcal{H}(v) = \frac{1}{v} \frac{1}{\mu_{\chi N}^2} \bar{\mathcal{R}}[E_R(v)] \]

by interpolation.

Another approach is to parameterize the halo functions \( \eta \) (both for the average rate and for its modulated part) in terms of a sum of a “large-enough” number \( N \) of streams:

\[ \tilde{\eta}_{0,1}(v) = \sum_{k=1}^{N} \delta \bar{\eta}_{0,1}^k \theta(v_k - v) \]
in this case the rate can still be expressed in terms of interpolations of the $\bar{R}$ function:

$$ R = \int_0^{v_{esc}} d\nu \mathcal{R}[E_R(\nu)] \tilde{\eta}(\nu) = \sum_{k=1}^{N} \delta \tilde{\eta}_{0,1}^k \int_0^{v_{esc}} d\nu \mathcal{R}[E_R(\nu)] $$

$$ = \frac{1}{\mu^2 \chi \eta N} \sum_{k=1}^{N} \delta \tilde{\eta}_{0,1}^k \left[ \mathcal{R}(E_{max}(v_k, m_\chi, \delta)) - \mathcal{R}(E_{min}(v_k, m_\chi, \delta)) \right] $$

where the above expression includes the possibility of inelastic scattering
Schematic behaviors of the $\bar{R}$ function

- Differential rate, no energy resolution
- Differential rate, energy resolution
- Rate in an energy bin, no energy resolution
- Rate in an energy bin, energy resolution
Can tabulate full calculation of $\bar{R}$ response function for each:

1) Experiment
2) Energy bin/energy threshold/energy value
3) Isospin value ($c_n/c_p = -1,0,1$)
4) Nuclear target (including all stable isotopes)
5) Effective coupling

$c_4$ (spin-dependent interaction)

S.S., J.H.Yoon, S.H. Kang, in progress
Several attempts in the literature to determine the velocity distribution from the data, sampling $f(v)$ in a finite number of velocity bins $N_{\text{streams}}$ and using a maximum-likelihood method

In this case $f(v) \rightarrow \lambda_i \delta(v-v_i)$ with $\Sigma_i \lambda_i = 1$ with $i=1,N_{\text{streams}}$ and the likelihood $L(S^k)$ depends on the direct detection theoretical predictions $S^k$ with $k=1,N_{\exp}$ and $S^k$ in the form:

$$S^k = \sum_{i=1}^{N} \lambda_i H^k_i$$

with $H^k_i$ fixed constants determined by experimental properties (response functions).

The problem of this approach is that in general $N_{\text{streams}} \ll N_{\exp}$ (actually, the idea it to take $N_{\text{streams}} \rightarrow \infty$) so fixing $N_{\exp}$ experimental values leaves a large degeneracy in the $\lambda$’s.
The WIMP velocity distribution \( f(v) \) can also be inferred from estimations of the DM density profile in our Galaxy \( \rho_{DM}(r) \) (either through measurements of the galactic rotational velocity or by simulations of Galaxy formation. Both quantities depend on the same 6-dimensional distribution \( F \):

\[
F(\vec{r}, \vec{v})
\]

In particular:

\[
f(\vec{v}) = F(\vec{r}_0, \vec{v})
\]

\( (r_0=\text{Earth's position}) \)

while:

\[
\rho_{DM}(\vec{r}) = \int d^3v F(\vec{r}, \vec{v})
\]

**ADD INFORMATION: SIMMETRIES**

For instance, assuming spherical symmetry by Jean's theorem \( F \) can only depend on \( v \) through the only integral of motion, the total mechanical energy \( E=T+V \):

\[
F(\vec{r}, \vec{v}) = F(E)
\]

This allows to get Eddington's equation, i.e. a determination of \( F \) as a functional of the DM density profile
Eddington’s equation

\[ F'(\epsilon) = \frac{1}{\sqrt{8\pi}^2} = \frac{d}{d\epsilon} \int_{0}^{\epsilon} \frac{d\rho}{d\psi} \sqrt{\epsilon - \psi} \]

\[ \epsilon = \psi - \frac{1}{2}v^2 \]

= mechanical energy per unit mass

input: density profile
output: velocity distribution function

Consistency check: given a density profile \( \rho(r) \) the necessary condition \( F(\epsilon) > 0 \) is not guaranteed (not every density profile is actually consistent to a steady-state solution)
Alternative approach are being attempted: for instance, can get \( f(v) \) from a Bayesian analysis → break the degeneracy by choosing a prior distribution for \( f(v) \) and get a well defined posterior. Of course quantifying our prior knowledge of \( f(v) \) is a sticky issue...

Can assume that our prior knowledge of \( f(v) \) favours a (truncated) maxwellian \( m(v) \). Find the distribution \( f(v) \) that maximizes the relative entropy

\[
S[f, m] = - \int f(v) \ln \left( \frac{f(v)}{m(v)} \right) dv
\]

subject to the experimental constraints:

\[
\int f(v) w_i(v) dv = \mu_i
\]

Using Lagrange multipliers \( f \) turns out to be:

\[
f(v) \propto m(v) \exp \left( \sum_i \lambda_i w_i(v) \right)
\]

A. Fowlie, JCAP10(2017)002
In this way the prior for $f$ is:

$$p(f \mid \text{prior knowledge}) \propto e^{\beta S[f,m]}$$

where $\beta$ is a regularization parameter that describes the strength of our prior information:

$\beta=0$ : no prior information
Minimal entropy

$\beta=\infty$ : strong belief in a Maxwellian and large entropy ($f=m$ irrespective of the data)

Crucially, for $\beta \neq 0$ the posterior probability is a strictly convex function $\rightarrow$ unique minimum $\rightarrow$ degeneracy broken

**Example: fit to the DAMA modulation effect:**

Continuous “morphing” driven by $\beta$ from a minimal-entropy solution (two streams at $v \approx 250$ km/s and $v \approx 400$ km/s) to a maximal-entropy solution (Maxwellian)

(isotropic $f(v)$ assumed)
The bottom line: getting $f(v)$ from experimental data is an “ill-posed” or under-constrained problem (mathematically, it boils down to determining $f(v)$ from its “generalized moments”): need additional assumptions to break the degeneracy

An alternative approach: give up the determination of $f(v)$ and instead integrate out $f(v)$ as a *nuisance* parameter and look for the **maximal range** of any other observable calculated in terms of $f(v)$ (P. Gondolo, S. Scopel, JCAP 1709 (2017) no.09, 032 arXiv:1703.08942).

is it possible? (in principle $f(v)$ is a function of an infinite number of nuisance parameters (i.e. the superposition of an infinite number of streams):

$$f(\vec{u}) = \int d^3u' f(\vec{u}') \delta(\vec{u} - \vec{u}')$$
Consider the case of a finite number of streams $N_{\text{streams}}$

Fixing $N_{\text{exp}}$ observations with $N_{\text{streams}}$ unknown $\lambda$’s corresponds to solving the linear system:

$$\begin{align*}
\sum_{i=1}^{N_{\text{streams}}} H_i^1 \lambda_i &= N_{\text{exp}}^1 \\
\vdots \\
\sum_{i=1}^{N_{\text{streams}}} H_i^{N_{\text{exp}}} \lambda_i &= N_{\text{exp}}^{N_{\text{exp}}} \\
\sum_{i=1}^{N_{\text{streams}}} \lambda_i &= 1 \\
\lambda_i &> 0
\end{align*}$$

The system is unconstrained ($N_{\text{exp}} < N_{\text{streams}}$) so it singles out a polyhedron (i.e. the subset of a hyperplane when $0<\lambda_i<1$) of dimension $N_{\text{streams}} - N_{\text{exp}} - 1$ in the $N_{\text{streams}}$-dimensional space of the $\lambda$’s.

The vertices of the polyhedron are found by fixing to zero $N_{\text{streams}} - N_{\text{exp}} - 1$ of the $\lambda$'s and solving the remaining system of $N_{\text{exp}} + 1$ unknowns and $N_{\text{exp}} + 1$ constraints.

An intuitive theorem: given a linear function $W = \sum_i \lambda_i W_i$ defined on a polyhedron its maximum and minimum must correspond to one of the vertices.

So to find the extrema of $W$ need only to check the vertices – in practice this means to always consider $N_{\text{exp}} + 1$ $\lambda$'s at a time (setting the other $\lambda$'s to zero).

The problem is $N_{\text{exp}} + 1$ dimensional, not $N_{\text{streams}}$ dimensional, and since $N_{\text{exp}} \ll N_{\text{streams}}$ this implies a huge simplification!
This theorem is also valid in the space of continuous functions:

- given the $N + 1$ known functions $g^i(x) \ (i = 1, \ldots, N)$ and $h(x)$ and the unknown function $f(x)$, all defined in the same domain, the $N$ constraints:

\[ I^i_g = \int_0^\infty g^i(x) f(x) \, dx, \quad i = 1, \ldots, N, \]

imply that the extreme values of the integral:

\[ I_h = \int_0^\infty h(x) f(x) \, dx, \]

are obtained by expressing the unknown function $f(x)$ in terms of the $N$ parametrizations:

\[ f_n(x) = \sum_{j=1}^n \lambda_j \delta(x - x_j), \quad n = 1, \ldots, N, \]

with $\sum_{i=1}^n \lambda_i = 1$ and $n = 1, \ldots, N$.

In the continuum case the theorem reduces an extremization problem in infinite dimensions (the moment set) into an extremization problem in a finite number of dimensions (the space of extreme distributions, which has dimension at most 
\[(1+d)N,\]
where \(N = N+1\) is the number of moment conditions and \(d\) is the dimensionality of the velocity space).

In practice, this means that, at fixed \( n \), the maximal range of the \( I_g \) integral is swept by the \( \lambda_j, x_j \) parameters that satisfy the \( n \) constraints with \( f_n(x) \) given by the superposition of \( n \) streams, i.e. the system of \( n + 1 \) linear equations:

\[
\sum_{j=1}^{n} \lambda_j g^i(x - x_j) = I_f^i, \quad i = 1, \ldots, n
\]

\[
\sum_{k=1}^{n} \lambda_k = 1, \quad \lambda_k > 0.
\]

The full range of \( I_g \) is then obtained by combining the \( N \) intervals at fixed \( n \).

**Direct application to the analysis of direct detection data:** given \( n \) experimental measurements any other quantity of the form

\[
A = \int_0^{\infty} A(v) f(v) \, dv
\]

can be bracketed for any \( f(v) \).
Recap:

• Given \( n \) independent direct detection measurements a parameterization of the velocity distribution in terms of \( n \) streams, combined with analogous parameterizations for \( n-1, n-2, \ldots, 1 \) brackets any observable of the form 
  \[
  A = \int_0^{\infty} A(v) f(v) \, dv
  \]
  where only \( A(v) \) is known.

Where do we get from here?
Halo-independent yearly-modulated fractions

Due to the rotation of the Earth around the Sun the signal in a direct detection experiment depends on time. Assuming that the only time dependence is due to the boost from the Galactic to the Lab rest frame:

\[
S(t)_{[E'_1, E'_2]} = \int \mathcal{H}_{[E'_1, E'_2]}(v) f(v, t) \, dv =
\]

\[
S(t)_{0,[E'_1, E'_2]} + S_{m,[E'_1, E'_2]} \cos \left[ \frac{2\pi}{365 \text{ days}} (t - t_0) \right] =
\]

\[
\int \mathcal{R}_{[E'_1, E'_2]}(v) \tilde{\eta}(v, t) \, dv =
\]

\[
\int \mathcal{R}_{[E'_1, E'_2]}(v) \left\{ \tilde{\eta}_0(v) + \tilde{\eta}_1(v) \cos \left[ \frac{2\pi}{365 \text{ days}} (t - t_0) \right] \right\} \, dv
\]

Standard lore: need to know explicitly \( f(v) \) to get the modulated fraction \( \tilde{\eta}_1(v)/\tilde{\eta}_0(v) \)

(ex: <10% for a Maxwellian)

The problem: how to estimate the time dependence (yearly modulation) of the signal from the time dependence of an unknown quantity \([f(v)]\)?

Very simple solution: a change of variable

\[ f(\vec{v}, t) = f_{\text{gal}} (\vec{u} = \vec{v} + \vec{v}_\odot + \vec{v}_\oplus(t)) \]

\[ S_{[E'_1, E'_2]}(t) = \int \mathcal{H}_{[E'_1, E'_2]} (\vec{u} - \vec{v}_\odot - \vec{v}_\oplus(t)) \ f_{\text{gal}}(\vec{u}) \ d^3u \]

N.B. : the time dependence is now only in the response function

The unmodulated and modulated parts are obtained via a Fourier time analysis:

\[ S_{0,[E'_1, E'_2]} = \frac{1}{T} \int_0^T dt, S_{[E'_1, E'_2]}(t) \]

\[ S_{m,[E'_1, E'_2]} = \frac{1}{T} \int_0^T dt, \cos \left[ \frac{2\pi}{365} (t - t_0) \right] S_{[E'_1, E'_2]}(t) \]

N.B. present detectors are not sensitive to directionality (they only measure the nuclear recoil energy, that depends on the WIMP's speed in the lab rest frame) → isotropic response functions \( H(v) \) in the lab rest frame

However, when making the change of variable from \( v \) (lab frame) to \( u \) (galactic rest frame) the detector's response functions \( H(\vec{u}) \) are non longer necessarily isotropic

In the following we will assume that \( f(u) \) is isotropic → in that case can average the response functions over angles:

\[
\mathcal{H}[E'_1, E'_2](u) \equiv \frac{1}{4\pi} \int d\Omega \, \mathcal{H}[E'_1, E'_2](\vec{u})
\]
In this case the case:

\[
S_{0, [E'_1, E'_2]} = \int \mathcal{H}_{0, [E'_1, E'_2]}(u) f_{\text{gal}}(u) \, du
\]

\[
S_{m, [E'_1, E'_2]} = \int \mathcal{H}_{m, [E'_1, E'_2]}(u) f_{\text{gal}}(u) \, du
\]

with:

\[
\mathcal{H}_{0, \text{gal}}(u) = \frac{1}{4\pi} \int d\Omega_u \frac{1}{T} \int_0^T dt \mathcal{H}(|\vec{u} - \vec{v}|)
\]

\[
\mathcal{H}_{m, \text{gal}}(u) = \frac{1}{4\pi} \int d\Omega_u \frac{1}{T} \int_0^T dt \cos \left[ \frac{2\pi}{365} (t - t_0) \right] \mathcal{H}(|\vec{u} - \vec{v}|)
\]

N.B. The modulated amplitude depends on the cosine transform of the response function, which is completely known → modulation as a property of the detector

Angle-averaged modulated response functions in DAMA (galactic rest frame)

Angle-averaged unmodulated response functions in DAMA (galactic rest frame)

$S_0$ and $S_m$ are both given by the integral of a known response function times the same unknown $f(u) \rightarrow$ use theorem on extreme distributions to profile out the unmodulated amplitudes in DAMA starting from measured modulated amplitudes.

\[-2 \ln \mathcal{L} = \sum_{k=1}^{12} \left( \frac{S_m^k - S_{m,exp}^k}{\sigma_k} \right)^2\]

\[f_{gal}(u) = \sum_{k=1}^{N} \lambda_k \delta(u - u_k), \quad N = 1, 12\]

N.B.: The velocity distribution is a nuisance parameter, manageable because we need it only on the boundary, where the dimensionality is reduced.

According to the previous theorem for any choice of the quantities $S_{m}^{k}$ $(k=1,\ldots,N)$ any other quantity of the form $A = \int_{0}^{\infty} A(v) f(v) dv$ can be bracketed for any $f(v)$. Actually, fixing $S_{m}^{k}$ fixes also the Lagrangian $L$, so this is also true for a fixed value of $L$. 

If we can bracket the full range of $A$ at fixed $L$ by turning the plot 90 degrees we can get the profile-likelihood of $A$

The $n$-sigma range of $A$ is obtained by taking the points with $L-L_{\text{min}}<n^2$

N.B. the new physics is contained in the cross section $\sigma$, which is a normalizing factor (together with the WIMP local density $\rho_\chi$) in the response functions. In particular, in the lab rest frame:

$$
\mathcal{H}_i(v) = \frac{N_T}{M_{\text{det}} \Delta E} \frac{\rho_\chi}{m_\chi} \sigma \chi T \frac{\mathcal{H}_i(v)}{
u E_R^{\max}(v)}
$$

where $\nu E_R^{\max}(v)$ is the energy resolution.

Since, for an extreme distribution:

$$
S_{0,[E'_1,E'_2]} = \sum \lambda_k \mathcal{H}_{0,[E'_1,E'_2]}(u_k)
$$

$$
S_{m,[E'_1,E'_2]} = \sum \lambda_k \mathcal{H}_{m,[E'_1,E'_2]}(u_k)
$$

a convenient way is to normalize the streams to $\sigma$ (at fixed local DM density):

$$
\sigma = \sum_k \lambda_k
$$

Problem: how do we sample the parameter space with $N=1,2,3,4...,N$ streams?

According to the theorem the full range of $S_{0,[E'_1,E'_2]}$ is spanned by using:

$$S_{0,[E'_1,E'_2]} = \sum_{k=1}^{m} \lambda_k \mathcal{H}_{0,[E'_1,E'_2]}(u_k) \quad m = 1, 2, ..., N$$

Need to do that numerically.

Suitable for a Markov Chain sampling. Two advantages:

- the sampling is driven by the Likelihood itself, don’t waste time in low-probability regions
- Perfect for profiling, highest density of points where $-2 \ln L$ is minimal

Can use a Markov–Chain Monte Carlo code* to generate large sets $\{v\}$ of $v_k$ velocities and $\{\lambda\}$ of $\lambda_k$ coefficients for $1 \leq k \leq m$ and $1 \leq m \leq N+1=4$ to calculate both $-2 \ln L$ and $\langle \tilde{\eta}(v_{\min}) \rangle$

12 DAMA measurements corresponding 1-sigma intervals from profile likelihood

\[ \chi^2 \]

\[ m_\chi = 5 \text{ GeV} \]

5x10^6 points Markov chain, 250 independent walkers Metropolis-Hastings sampler

$m_\chi = 10 \text{ GeV}$

12 DAMA measurements

corresponding 1-sigma intervals from profile likelihood

$5 \times 10^6$ points Markov chain, 250 independent walkers Metropolis-Hastings sampler

5x10^6 points Markov chain, 250 independent walkers Metropolis-Hastings sampler

\[ m_\chi = 15 \text{ GeV} \]

Disentangled from the background for any isotropic velocity distribution

<table>
<thead>
<tr>
<th>$E_i$ [keVee]</th>
<th>$S_{m,i}$</th>
<th>$S_{0,i}$, $m_\chi = 5$ GeV</th>
<th>$S_{0,i}$, $m_\chi = 10$ GeV</th>
<th>$S_{0,i}$, $m_\chi = 15$ GeV</th>
<th>$B_i + S_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0–2.5</td>
<td>0.0161(39)</td>
<td>0.513$^{+0.051}_{-0.050}$</td>
<td>0.333$^{+0.046}_{-0.045}$</td>
<td>0.227$^{+0.35}_{-0.027}$</td>
<td>1.029</td>
</tr>
<tr>
<td>2.5–3.0</td>
<td>0.0260(44)</td>
<td>0.311$^{+0.031}_{-0.030}$</td>
<td>0.239$^{+0.039}_{-0.037}$</td>
<td>0.165$^{+0.33}_{-0.024}$</td>
<td>1.228</td>
</tr>
<tr>
<td>3.0–3.5</td>
<td>0.0219(44)</td>
<td>0.165$^{+0.016}_{-0.016}$</td>
<td>0.160$^{+0.032}_{-0.030}$</td>
<td>0.112$^{+0.30}_{-0.020}$</td>
<td>1.294</td>
</tr>
<tr>
<td>3.5–4.0</td>
<td>0.0084(40)</td>
<td>0.0791$^{+0.0078}_{-0.0078}$</td>
<td>0.103$^{+0.027}_{-0.025}$</td>
<td>0.073$^{+0.27}_{-0.017}$</td>
<td>1.140</td>
</tr>
<tr>
<td>4.0–4.5</td>
<td>0.0107(36)</td>
<td>0.0352$^{+0.0035}_{-0.0036}$</td>
<td>0.066$^{+0.022}_{-0.021}$</td>
<td>0.047$^{+0.24}_{-0.014}$</td>
<td>0.956</td>
</tr>
<tr>
<td>4.5–5.0</td>
<td>0.0054(32)</td>
<td>0.0148$^{+0.0015}_{-0.0016}$</td>
<td>0.042$^{+0.019}_{-0.016}$</td>
<td>0.030$^{+0.22}_{-0.011}$</td>
<td>0.853</td>
</tr>
<tr>
<td>5.0–5.5</td>
<td>0.0089(32)</td>
<td>0.00600$^{+0.0059}_{-0.0065}$</td>
<td>0.026$^{+0.016}_{-0.012}$</td>
<td>0.018$^{+0.20}_{-0.0084}$</td>
<td>0.868</td>
</tr>
<tr>
<td>5.5–6.0</td>
<td>0.0039(31)</td>
<td>0.00236$^{+0.00023}_{-0.00026}$</td>
<td>0.0156$^{+0.013}_{-0.0084}$</td>
<td>0.011$^{+0.18}_{-0.0059}$</td>
<td>0.853</td>
</tr>
<tr>
<td>6.0–6.5</td>
<td>0.00018(308)</td>
<td>$9.04^{+0.89}_{-1.03} \times 10^{-4}$</td>
<td>0.0090$^{+0.011}_{-0.0054}$</td>
<td>0.0064$^{+0.16}_{-0.0038}$</td>
<td>0.868</td>
</tr>
<tr>
<td>6.5–7.0</td>
<td>0.00018(281)</td>
<td>3.41$^{+0.34}_{-0.40} \times 10^{-4}$</td>
<td>0.0050$^{+0.0094}_{-0.0033}$</td>
<td>0.0035$^{+0.15}_{-0.0023}$</td>
<td>0.860</td>
</tr>
<tr>
<td>7.0–7.5</td>
<td>0.0015(28)</td>
<td>1.27$^{+0.12}_{-0.15} \times 10^{-4}$</td>
<td>0.0026$^{+0.0073}_{-0.0018}$</td>
<td>0.0019$^{+0.13}_{-0.0013}$</td>
<td>0.860</td>
</tr>
<tr>
<td>7.5–8.0</td>
<td>-0.0013(29)</td>
<td>4.67$^{+0.46}_{-0.56} \times 10^{-5}$</td>
<td>0.0013$^{+0.0054}_{-0.0010}$</td>
<td>$0.95^{+123}_{-0.71} \times 10^{-3}$</td>
<td>0.890</td>
</tr>
</tbody>
</table>

Example of correlation ellipsoid among different $S_{0,i}$

\[ -2\Delta \ln \mathcal{L}_p(\{S_{0,i}\}) \leq 1 \]

\[ -2\Delta \ln \mathcal{L}_p(\{S_{0,i}\}) \leq 3 \]

$m_\chi = 10 \text{ GeV}$
N.B. Non-isotropic distributions can easily predict larger modulation fractions (up to 100 %). However, also the space of isotropic $f(u)$ contains large modulation solutions, which however are disfavored by the data.

1-sigma ranges for modulation fractions:
$m_\chi=5$: $0.03 < S_m/S_0 < 0.13$; $m_\chi=10$: $0.05 < S_m/S_0 < 0.13$; $m_\chi=20$: $0.07 < S_m/S_0 < 0.19$

A few comments:

- Cannot adopt this procedure to get a best-likelihood determination of $f(v)$ - the streams parameterization of $f(v)$ is only valid on the boundaries – unless assume minimal-entropy solution à-la Fowlie (regularization parameter $\beta=0$)
- Mathematically and statistically sound procedure, with many potential applications.
- Straightforward generalization to non-isotropic velocity distribution – no conceptual problems, but numerically more challenging.
This procedure leads also to a determination of the velocity distribution, but only to the solution with minimal entropy. Indeed:

1) in our Markov Chain the absolute minimum of the chi square corresponds to 2 streams

2) Excellent agreement with Minimal Entropy result from Fowlie (JCAP(2017)002) \( \rightarrow \) our result corresponds to \( \beta = 0 \) (so, indeed, when \( \beta \) is fixed the \( f(v) \) is determined)
The next step: 3-d response functions for non-isotropic velocity distributions

\[ \mathcal{H}_{0,i}^{\text{gal}}(u) = \frac{1}{T} \int_0^T dt \mathcal{H}_i(|u - v_\odot - v_\odot(t)|), \]

\[ \mathcal{H}_{Y,i}^{\text{gal}}(u) = \frac{2}{T} \int_0^T dt \cos[\omega(t-t_0)] \mathcal{H}_i(|u - v_\odot - v_\odot(t)|). \]

\[ \mathcal{H}_{Z,i}^{\text{gal}}(u) = \frac{2}{T} \int_0^T dt \sin[\omega(t-t_0)] \mathcal{H}_i(|u - v_\odot - v_\odot(t)|). \]

In all plots: \(|u|=300\text{ km/s}\)
Also the halo-function $\eta$ can be treated as a nuisance parameter

\[
R = \int_0^{v_{esc}} d\nu \mathcal{R}[E_R(\nu)] \tilde{\eta}(\nu) = \sum_{k=1}^{N} \delta \tilde{\eta}_{0,1}^k \int_0^{v_{esc}} d\nu \mathcal{R}[E_R(\nu)] \\
= \frac{1}{\mu_{\chi N}^2} \sum_{k=1}^{N} \delta \tilde{\eta}_{0,1}^k \left[ \mathcal{R}(E_{\max}(v_k, m_\chi, \delta)) - \mathcal{R}(E_{\min}(v_k, m_\chi, \delta)) \right]
\]

nuisance parameters

One explicit example: a frequentist analysis of pSIDM (proton-philic Spin Inelastic Dark Matter)
One of the most popular scenarios for WIMP-nucleus scattering is a spin-dependent interaction where the WIMP particle is a $\chi$ fermion (either Dirac or Majorana) that recoils through its coupling to the spin of nucleons N=p,n: 

$$L_{int} \propto \vec{S}_\chi \cdot \vec{S}_N = c^p \vec{S}_\chi \cdot \vec{S}_p + c^n \vec{S}_\chi \cdot \vec{S}_n$$

(for instance, predicted by supersymmetry when the WIMP is a neutralino that couples to quarks via Z-boson or squark exchange)
A few facts of life:
Nuclear spin is mostly carried by odd-numbered nucleons. Even-even isotopes carry no spin.
• the DAMA effect is measured with Sodium Iodide. Both Na and I have spin **carried by an unpaired proton**

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Spin</th>
<th>Z (# of protons)</th>
<th>A-Z (# of neutrons)</th>
<th>Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{23}\text{Na}$</td>
<td>3/2</td>
<td>11</td>
<td>12</td>
<td>100 %</td>
</tr>
<tr>
<td>$^{127}\text{I}$</td>
<td>5/2</td>
<td>53</td>
<td>74</td>
<td>100 %</td>
</tr>
</tbody>
</table>

Germanium experiments carry only a very small amount of $^{73}\text{Ge}$, the only isotope with spin, **carried by an unpaired neutron**

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Spin</th>
<th>Z (# of protons)</th>
<th>A-Z (# of neutrons)</th>
<th>Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{73}\text{Ge}$</td>
<td>9/2</td>
<td>32</td>
<td>41</td>
<td>7.7 %</td>
</tr>
</tbody>
</table>

Xenon experiment contain two isotopes with spin, **both carried mostly by an unpaired neutron**

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Spin</th>
<th>Z (# of protons)</th>
<th>A-Z (# of neutrons)</th>
<th>Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{129}\text{Xe}$</td>
<td>½</td>
<td>54</td>
<td>75</td>
<td>26%</td>
</tr>
<tr>
<td>$^{131}\text{Xe}$</td>
<td>3/2</td>
<td>54</td>
<td>77</td>
<td>21%</td>
</tr>
</tbody>
</table>

→several authors have considered the possibility that $c_n<<c_p$: in this case the WIMP particle is seen by DAMA but does not scatter on xenon and germanium detectors
However another class of Dark Matter experiments (superheated droplet detector and bubble chambers) **all use nuclear targets with an unpaired proton:**

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Target</th>
<th>Type</th>
<th>Energy thresholds (keV)</th>
<th>Exposition (kg day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIMPLE</td>
<td>C$_2$Cl F$_5$</td>
<td>superheated droplets</td>
<td>7.8</td>
<td>6.71</td>
</tr>
<tr>
<td>COUPP</td>
<td>C F$_3$ I</td>
<td>bubble chamber</td>
<td>7.8, 11, 15.5</td>
<td>55.8, 70, 311.7</td>
</tr>
<tr>
<td>PICASSO</td>
<td>C$_3$F$_8$</td>
<td>bubble chamber</td>
<td>1.7, 2.9, 4.1, 5.8, 6.9, 16.3, 39, 55</td>
<td>114</td>
</tr>
<tr>
<td>PICO-2L</td>
<td>C$_3$F$_8$</td>
<td>bubble chamber</td>
<td>3.2, 4.4, 6.1, 8.1</td>
<td>74.8, 16.8, 82.2, 37.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Spin</th>
<th>Z (# of protons)</th>
<th>A-Z (# of neutrons)</th>
<th>Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{19}$F</td>
<td>1/2</td>
<td>9</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>$^{35}$Cl</td>
<td>3/2</td>
<td>17</td>
<td>18</td>
<td>75.77 %</td>
</tr>
<tr>
<td>$^{37}$Cl</td>
<td>3/2</td>
<td>17</td>
<td>20</td>
<td>24.23 %</td>
</tr>
<tr>
<td>$^{127}$I</td>
<td>5/2</td>
<td>53</td>
<td>74</td>
<td>100</td>
</tr>
</tbody>
</table>

These experiments are sensitive to $c_p$, so for $c_n << c_p$ spin-dependent scatterings on Fluorine have been shown to lead to tension with the DAMA (C. Amole et al., (PICO Coll.) PLB711, 153(2012), E. Del Nobile, G.B. Gelmini, A. Georgescu and J.H. Huh, 1502.07682)

N.B. All only sensitive to the energy threshold, which for bubble and droplets nucleation is controlled by the pressure of the liquid
Evading Fluorine constraints for a WIMP with spin-dependent coupling to protons: inelastic scattering (proton-philic Spin-dependent IDM, pSIDM)

\[ v_{\text{min}} = \frac{1}{\sqrt{2m_N E_R}} \left| \frac{m_N E_R}{\mu_{\chi N}} + \delta \right| \]

\[ v_{\text{min}} > v_{\text{min}}^* \]

\[ v_{\text{min}}^* = \sqrt{\frac{2\delta}{\mu_{\chi N}}} \]

\[ A_{\text{Sodium}} = 23 \quad A_{\text{Fluorine}} = 19 \]

\[ m_{\text{Sodium}} > m_{\text{Fluorine}} \quad \mu_{\chi N}^{\text{Sodium}} > \mu_{\chi N}^{\text{Sodium}} \]

\[ v_{\text{min}}^*_{\text{Sodium}} < v_{\text{min}}^*_{\text{Fluorine}} \]

what if \( v_{\text{min}}^*_{\text{Sodium}} < v_{\text{esc}} < v_{\text{min}}^*_{\text{Fluorine}} \)?  

(N.B. \( v_{\text{esc}} \) in lab frame)
m_\chi = 11.4 \text{ GeV} \quad \delta = 23.7 \text{ keV}

depending on m_\chi and \delta, can drive Fluorine (and Iodine in COUPP) beyond v_\text{esc} while Sodium remains below \rightarrow no constraint on DAMA from droplet detectors and bubble chambers.
very tuned region. but this is just kinematics
taking $v_{\text{esc}} = v_{\text{DAMA}}^{\text{MAX}}(m_\chi, \delta)$ the kinematic region enlarges considerably
Construct a likelihood function fixing \( c_i = c_4 \) (spin-dependent interaction).
Including the DAMA modulation amplitudes and all constraints
In the likelihood both the non-modulated halo function \( \eta_0 \) and the modulated one \( \eta_1 \) are parameterized as:

\[
\tilde{\eta}_{0,1}(v) = \sum_{k=1}^{N} \delta \tilde{\eta}_{0,1}^k \theta(v_k - v)
\]

with the constraints:

\[
\tilde{\eta}_0(v_{\text{min},2}) \leq \tilde{\eta}_0(v_{\text{min},1}) \quad \text{if } v_{\text{min},2} > v_{\text{min},1}
\]

\[
\tilde{\eta}_1 \leq \tilde{\eta}_0 \quad \text{at the same } v_{\text{min}}
\]

\[
\tilde{\eta}_0(v_{\text{min}} \geq v_{\text{esc}}) = 0.
\]

Also backgrounds \( b_i \) are taken as free parameters with the only requirement of positivity.
Then using:

\[
\mathcal{L}(m_\chi, \delta, r = c_4^n / c_4^p, \delta \tilde{\eta}_0^k, \delta \tilde{\eta}_1^k, b_i)
\]

sample the parameter space with a Markov-Chain Montecarlo to profile the WIMP parameters \( m_\chi, \delta, r = c_4^n / c_4^p \)

S.S., J.H.Yoon, S.H. Kang, in progress
Contour plot for absolute minimum of chi square (integrating over # of streams)

Conclusions

• A model independent analysis of direct detection data implies:
  1) using non-relativistic effective theory which introduces new response functions with explicit dependence on the transferred momentum and the WIMP incoming velocity
  2) factorizing the halo-function dependence
  3) allowing for inelastic scattering
  4) allowing for isovector couplings
• In this way a much wider parameter space opens up.

Theoretical predictions for WIMP signals depend on integrals of the velocity distribution f(v) (generalized moments) → cannot be directly inverted using exp data to determine the velocity distribution (degenerate problem, integration implies information loss)

An alternative approach: treat f(v) (or the halo function η) as a nuisance parameter and profile the other WIMP parameters (couplings, mass, etc) or other observables.