Flavor Physics III

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Outline

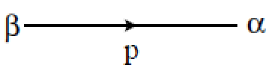
- Flavor Physics and the Standard Model
- Discrete Symmetry and CKM matrix
- Renormalization and Muon g-2
  - Renormalization and Dimensional Regularization
  - Muon g-2
- RG and Effective Field Theory
- CP Violation and BSM Flavor Physics

Renormalization: QED

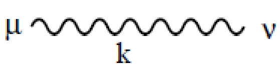
- The QED Lagrangian

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial \cdot A)^2 + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi$$

- The free propagators



$$\left(\frac{i}{p\!\!\!/ - m + i\varepsilon}\right)_{\beta\alpha} \equiv S^0_{F\beta\alpha}(p)$$

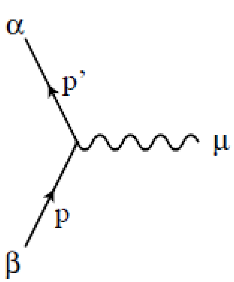


$$-i\left[\frac{g_{\mu\nu}}{k^2+i\varepsilon} + \frac{(\xi-1)}{1}\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\right]$$

$$= -i\left\{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2+i\varepsilon} + \xi\frac{k_\mu k_\nu}{k^4}\right\}$$

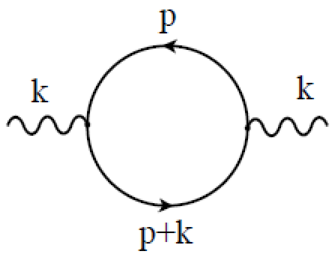
$$\equiv G^0_{F\mu\nu}(k)$$

- The vertex



$$+ie(\gamma_\mu)_{\beta\alpha} \qquad e = |e| > 0$$

Vacuum polarization



$$G^{(1)}_{\mu\nu}(k) \equiv G^{0}_{\mu\mu'}\; i\,\Pi_{\mu'\nu'}(k) G^{0}_{\nu'\nu}(k)$$

$$\begin{aligned}
i\,\Pi_{\mu\nu} &= -(+ie)^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu \frac{i}{\not{p} - m + i\varepsilon} \gamma_\nu \frac{i}{\not{p} + \not{k} - m + i\varepsilon}\right) \\
&= -e^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu(\not{p} + m)\gamma_\nu(\not{p} + \not{k} + m)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\
&= -4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\nu - g_{\mu\nu}(p^2 + p \cdot k - m^2)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)}
\end{aligned}$$

an extra minus sign for a fermion loop

- Simple counting indicates quadratic divergence for p→∞, but in fact the divergence is milder (logarithmically divergent)
- use dimensional regularization to regularize the infinities

Dimensional regularization

- the only known scheme that preserve: Lorentz invariance, gauge invariance, analytic structure of scattering amplitude, invariance under redefinitions of integration variable

- Analytic continuation from integer to noninteger dimensions

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d} \quad (d = 4 - \varepsilon)$$

- The degree of divergences of loop integrals can be reduced

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} (\text{logarithmical divergence}) \to \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^4} (\text{no UV divergence for } d < 4)$$

- Calculate a divergent integral in lower dimensions and analytically continue to d≥4

- The divergence at d=4 arises as a pole at ϵ

- In general the UV behavior becomes better for $\epsilon>0$, while the IR behavior becomes better for $\epsilon<0$

Dimensional regularization

- The Feynman rules are not changed except for the change of the coupling constants with an arbitrary mass scale μ which make the coupling constants dimensionless (e.g. in QCD)

$$S = \int d^d x \mathcal{L},$$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}G^{\mu\nu\,a}G^a_{\mu\nu} - g_s G^a_\mu \bar{\psi}\gamma^\mu T^a \psi,$$

$$[\psi] = \frac{d-1}{2}, \quad [G^a_\mu] = \frac{d-2}{2}, \quad [g_s] = d - \frac{d-2}{2} - 2 \times \frac{d-1}{2} = 2 - \frac{d}{2}.$$

$$\boxed{g_s \to g_s \mu^{(2-d/2)}}$$

- Relations in d dimensions

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = d = 4 - \varepsilon \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad Tr1 = 4$$

$$\gamma^\mu a\!\!\!/ \gamma_\mu = (2-d)a\!\!\!/ \qquad \gamma^\mu a\!\!\!/ b\!\!\!/ \gamma_\mu = (d-4)a\!\!\!/ b\!\!\!/ + 4a\cdot b \qquad \gamma^\mu a\!\!\!/ b\!\!\!/ c\!\!\!/ \gamma_\mu = (4-d)a\!\!\!/ b\!\!\!/ c\!\!\!/ - 2c\!\!\!/ b\!\!\!/ a\!\!\!/$$

- Scaleless integrals are zero: no available quantity with non-zero mass dim.

$$\int \frac{d^d k_E}{k_E^4} = \Omega_d \int_0^\Lambda dk_E\, k_E^{d-5} + \Omega_d \int_\Lambda^\infty dk_E\, k_E^{d-5} = \Omega_d \left(\ln\Lambda - \frac{1}{\varepsilon_{\text{IR}}} \right) + \Omega_d \left(\frac{1}{\varepsilon_{\text{UV}}} - \ln\Lambda \right)$$

Feynman parametrization

- Combine the products of the denominators of the propagators

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A+(B-A)x]^2} = \int_0^1 dx dy \delta(x+y-1) \frac{1}{[xA+yB]^2}$$

- differentiating by B

$$\frac{1}{AB^n} = \int_0^1 dx dy \delta(x+y-1) \frac{n y^{n-1}}{[xA+yB]^{n+1}}$$

- by induction

$$\frac{1}{ABC} = \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{[xA+yB+zC]^3}$$

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \delta(\sum x_i - 1) \, \frac{(n-1)!}{\left[x_1 A_1 + x_2 A_2 + \cdots x_n A_n\right]^n}$$

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^1 dx_1 \cdots dx_n \, \delta(\sum x_i - 1) \, \frac{\prod x_i^{m_i - 1}}{\left[\sum x_i A_i\right]^{\Sigma m_i}} \, \frac{\Gamma(m_1 + \cdots + m_n)}{\Gamma(m_1) \cdots \Gamma(m_n)}$$

Wick rotation

- Simplify the numerators

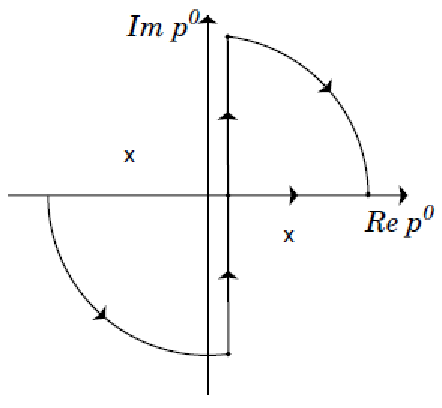
$$\int \frac{d^d p}{(2\pi)^d}\,\frac{p^\mu}{[p^2 - C + i\epsilon]^2} = 0$$

$$\int \frac{d^d p}{(2\pi)^d}\,\frac{p^\mu p^\nu}{[p^2 - C + i\epsilon]^2} = \frac{1}{d}\,g^{\mu\nu}\int \frac{d^d p}{(2\pi)^d}\,\frac{p^2}{[p^2 - C + i\epsilon]^2}$$

- Wick rotation (the deformation of the contour)

$$p^0 = \pm\left(\sqrt{|\vec{p}|^2 + C} - i\varepsilon\right)$$

We only have to calculate integrals of this form



$$\begin{aligned} I_{r,m} &= \int \frac{d^d p}{(2\pi)^d}\,\frac{(p^2)^r}{[p^2 - C + i\epsilon]^m} \\ &= \int \frac{d^{d-1} p}{(2\pi)^d}\int dp^0 \frac{(p^2)^r}{[p^2 - C + i\epsilon]^m} \end{aligned}$$

$$\oint dp_0 = \int_{-\infty}^{+\infty} dp_0 + \int_{+i\infty}^{-i\infty} dp_0 = 0$$

$$p^0 \to i p_E^0 \qquad ; \qquad \int_{-\infty}^{+\infty} \to i \int_{-\infty}^{+\infty} dp_E^0$$

$$p^2 = (p^0)^2 - |\vec{p}|^2 = -(p_E^0)^2 - |\vec{p}|^2 \equiv -p_E^2$$

$$I_{r,m} = i(-1)^{r-m}\int \frac{d^d p_E}{(2\pi)^d}\,\frac{p_E^{2r}}{[p_E^2 + C]^m}$$

we do not need the $i\epsilon$ anymore because the denominator is positive definite

Scalar integrals

- The Euclidean phase space

$$\int d^d k_E = \int d\bar{k}\, \bar{k}^{\,d-1}\, d\Omega_{d-1}$$

$$\bar{k} = \sqrt{(k_E^0)^2 + |\vec{k}|^2}$$

- The d-dimensional solid angle

$$k_E = \bar{k}(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \ldots, \sin\theta_1\cdots\sin\theta_{d-1})$$

$$\int d\Omega_{d-1} = \int_0^\pi \sin\theta_1^{d-2}\, d\theta_1 \cdots \int_0^{2\pi} d\theta_{d-1} = 2\,\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

$$\int_0^\pi \sin\theta^m\, d\theta = \sqrt{\pi}\,\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}$$

- The integration in $\bar{k}$

$$\int_0^\infty dx\, \frac{x^p}{(x^n + a^n)^q} = a^{p+1-nq}\,\frac{\Gamma\left(\frac{p+1}{n}\right)\Gamma\left(-\frac{p+1}{n}+q\right)}{n\Gamma(q)}$$

$$B(p+1, q+1) = \int_0^\infty \frac{u^p}{(1+u)^{p+q+2}}\, du$$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

- The scalar integrals

$$I_{r,m} = \int \frac{d^d k}{(2\pi)^d}\, \frac{k^{2r}}{[k^2 - C + i\epsilon]^m}$$

$n=2, q=m, p=2r+d-1$

$$= i(-1)^{r-m}\int \frac{d^d k_E}{(2\pi)^d}\,\frac{k_E^{2r}}{[k_E^2 + C]^m} = iC^{r-m+\frac{d}{2}}\,\frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}}\,\frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})}\,\frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)}$$

Back to vacuum polarization

$$
\begin{aligned}
i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\
&= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p,k)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)}
\end{aligned}
$$

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)$$

- Feynman parametrization $\displaystyle \frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}$

$$
\begin{aligned}
i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p,k)}{[x(p+k)^2 - xm^2 + (1-x)(p^2 - m^2) + i\varepsilon]^2} \\
&= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p,k)}{[p^2 + k\cdot px + xk^2 - m^2 + i\varepsilon]^2} \\
&= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p,k)}{[(p+kx)^2 + k^2x(1-x) - m^2 + i\varepsilon]^2}
\end{aligned}
$$

- Change of variables $p \to p - kx$

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p-kx,k)}{[p^2 - C + i\epsilon]^2} \qquad C = m^2 - k^2x(1-x)$$

$$N_{\mu\nu}(p-kx,k) = 2p_\mu p_\nu + 2x^2k_\mu k_\nu - 2xk_\mu k_\nu - g_{\mu\nu}\left(p^2 + x^2k^2 - xk^2 - m^2\right)$$

Vacuum polarization

$$\mathcal{N}_{\mu\nu} \equiv \mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p-kx,k)}{[p^2-C+i\epsilon]^2}$$

$$= \left(\frac{2}{d}-1\right) g_{\mu\nu}\mu^\epsilon I_{1,2} + \Big[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2\Big]\mu^\epsilon I_{0,2}$$

$$\mu^\epsilon I_{0,2} = \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}$$

$$= \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)$$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$$

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi$$

$$\mu^\epsilon I_{1,2} = -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} C\, \frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})} \frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2)}$$

$$= \frac{i}{16\pi^2}C\left(1+2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)$$

$$\frac{2}{d} - 1 = \frac{2}{4-\epsilon} - 1 = -\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2)$$

- Finally one obtains

$$\mathcal{N}_{\mu\nu} = \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)2x(1-x)$$

Vacuum polarization

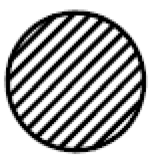
$$\Pi_{\mu\nu} = -4e^2 \frac{1}{16\pi^2} \left(g_{\mu\nu}k^2 - k_\mu k_\nu \right) \int_0^1 dx\ 2x(1-x) \left(\Delta_\epsilon - \ln \frac{C}{\mu^2} \right)$$

$$= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu \right) \Pi(k^2, \epsilon)$$

$$\Pi(k^2,\epsilon) \equiv \frac{2\alpha}{\pi} \int_0^1 dx\ x(1-x) \left[\Delta_\epsilon - \ln \frac{m^2 - x(1-x)k^2}{\mu^2} \right]$$

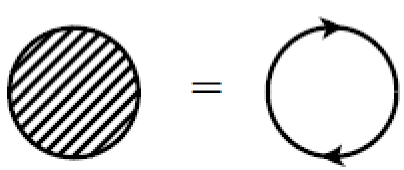
Π diverges as $\epsilon \to 0$

- The physical meaning of $\Pi_{\mu\nu}(k)$



$\equiv i\,\Pi_{\mu\nu}(k) =$ sum of all one-particle irreducible (*proper*) diagrams to *all* orders

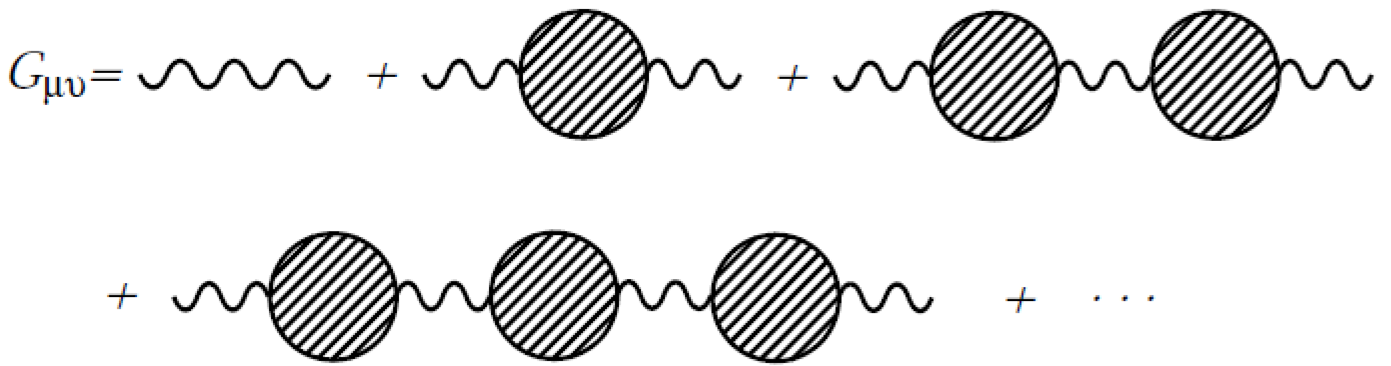
1PI diagrams



at lowest order

Photon propagator

- The full photon propagator is given by the series of



- Rewrite the free propagator $G^0_{\mu\nu}$

$$iG^0_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} = P^T_{\mu\nu}\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} \equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu}$$

$$P^T_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \qquad \begin{cases} k^\mu P^T_{\mu\nu} = 0 \\ P^{T\nu}_{\mu} P^T_{\nu\rho} = P^T_{\mu\rho} \end{cases}$$

Photon propagator

- The full propagator can be written in terms of transverse and longitudinal parts

$$G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu} \qquad\qquad G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu}$$

- The vacuum polarization tensor is transverse at the lowest order

$$i\,\Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu}\,\Pi(k)$$

valid to all orders

- This means that the longitudinal part is not renormalized (indep. of $\Pi_{\mu\nu}(k)$)

$$G^L_{\mu\nu} = G^{0L}_{\mu\nu}$$

- The transverse part is summed as a geometric series

$$\begin{aligned}
iG^T_{\mu\nu} &= P^T_{\mu\nu}\frac{1}{k^2} + P^T_{\mu\mu'}\frac{1}{k^2}(-i)k^2 P^{T\mu'\nu'}\Pi(k)(-i)P^T_{\nu'\nu}\frac{1}{k^2} \\
&\quad + P^T_{\mu\rho}\frac{1}{k^2}(-i)k^2 P^{T\rho\lambda}\,\Pi(k)(-i)P^T_{\lambda\tau}\frac{1}{k^2}(-i)k^2 P^{T\tau\sigma}\,\Pi(k)(-i)P^T_{\sigma\nu}\frac{1}{k^2} + \cdots \\
&= P^T_{\mu\nu}\frac{1}{k^2}\left[1 - \Pi(k) + \Pi^2(k^2) + \cdots\right] \\
&= P^T_{\mu\nu}\frac{1}{k^2[1+\Pi(k)]}
\end{aligned}$$

the function $\Pi(k)$ diverges

Counterterm

- The initial Lagrangian starts from the classical theory and in the quantum theory all physical parameters must be renormalized to cancel infinities

- The correct Lagrangian would be obtained by adding the corrections to the classical Lagrangian, order by order in perturbation theory → counterterms

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta\mathcal{L}$$

- The counterterms are defined from the normalization conditions that are imposed on the fields and other parameters of the theory

- It is convenient to keep the expressions as close as possible to the free field case

$$\lim_{k\to 0} k^2 iG^{RT}_{\mu\nu} = 1 \cdot P^T_{\mu\nu}$$

renormalized propagator

renormalization condition for photon propagator

in general, the renormalization conditions are arbitrary as long as the UV divergences cancel

- The counterterm lagrangian has to have the same form as the classical lagrangian in order to respect the symetries of the theory

$$\Delta\mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\delta Z_3\, F_{\mu\nu}F^{\mu\nu}$$

Renormalized photon propagator

- The Feynman rule for the counterterm

$$-i\,\delta Z_3 k^2\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)$$

Counter terms (~(Z-1)) are treated as interaction terms

- The loop correction + counterterm

$$i\Pi_{\mu\nu} \;=\; i\Pi^{loop}_{\mu\nu}-i\,\delta Z_3 k^2\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right) \;=\; -i\,\left(\Pi(k,\epsilon)+\delta Z_3\right)P^T_{\mu\nu}\,k^2$$

$$iG^T_{\mu\nu}=P^T_{\mu\nu}\frac{1}{k^2}\frac{1}{1+\Pi(k,\epsilon)+\delta Z_3}$$

$$\Delta_\epsilon=\frac{2}{\epsilon}-\gamma+\ln 4\pi$$

- The renormalization condition $\Pi(0,\epsilon)+\delta Z_3=0$

$$\delta Z_3 \;=\; -\Pi(0,\epsilon)=-\frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\left[\Delta_\epsilon-\ln\frac{m^2}{\mu^2}\right] \;=\; -\frac{\alpha}{3\pi}\left[\Delta_\epsilon-\ln\frac{m^2}{\mu^2}\right]$$

- The renormalized photon propagator

$$iG_{\mu\nu}(k)=\frac{P^T_{\mu\nu}}{k^2[1+\Pi(k,\epsilon)-\Pi(0,\epsilon)]}+i\,G^L_{\mu\nu}$$

$$\Pi(k,\epsilon)-\Pi(0,\epsilon)\equiv\Pi^R(k^2)$$

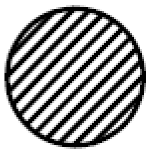
the photon mass is not renormalized because the pole is at $k^2=0$

Electron propagator

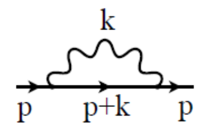
- The full propagators

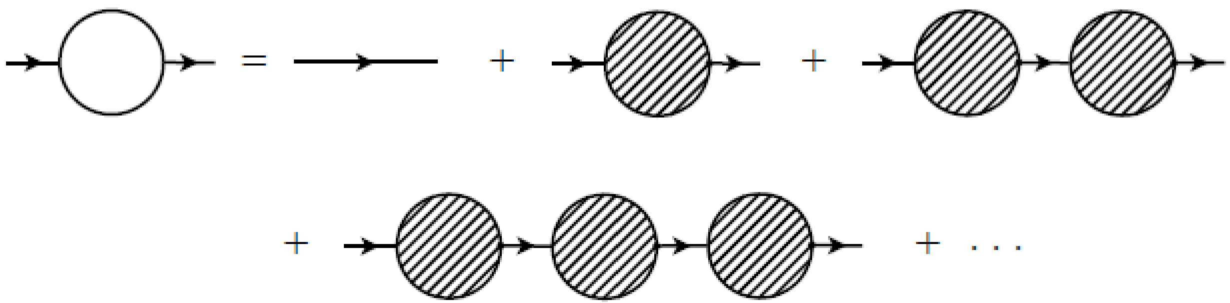
the free field propagator

$$S_0(p) = \frac{i}{\not{p} - m} \Longrightarrow S_0^{-1}(p) = -i(\not{p} - m)$$



$\equiv -i\,\Sigma(p)$ 1PI diagrams





$$S(p) \;=\; S^0(p) + S^0(p)\Big(-i\,\Sigma(p)\Big)S^0(p) + \cdots \;=\; S^0(p)\Big[1 - i\,\Sigma(p)S(p)\Big]$$

$S_0^{-1}(p)$ $S^{-1}(p)$ $S_0^{-1}(p)$ $S^{-1}(p)$

$$S^{-1}(p) \;=\; S_0^{-1}(p) + i\Sigma(p) \;=\; -i\Big[\not{p} - (m + \Sigma(p))\Big]$$

self-energy

Electron propagator

- counterterms (need to renormalize both electron field and mass)

$$\Delta\mathcal{L} = i\,\underbrace{(Z_2 - 1)}_{\delta Z_2}\,\overline{\psi}\gamma^\mu\partial_\mu\psi - (Z_2 - 1)\,m\,\overline{\psi}\psi + \quad \delta m\,\overline{\psi}\psi$$

- the self-energy (loop+counterterms)

$$-i\,\Sigma(p) = -i\,\Sigma^{loop}(p) + i\;(\not{p} - m)\,\delta Z_2 + i\,\delta m$$

- the on-shell renormalization scheme (used in QED)
  - the pole of the propagator = the physical mass

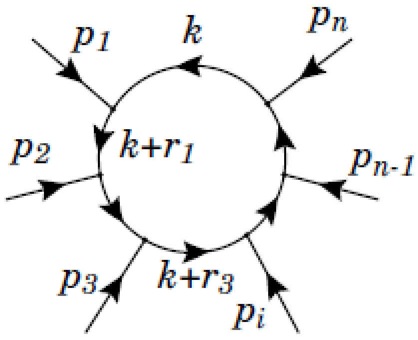
$$\Sigma(\not{p} = m) = 0 \quad \rightarrow \quad \delta m = \Sigma^{loop}(\not{p} = m)$$

  - the residue of the pole = the same value as the free propagator

$$\not{p} - (m + \Sigma(p)) \;=\; (\not{p} - m)\left(1 - \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=m}\right) + \mathcal{O}\left((\not{p} - m)^2\right)$$

$$\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m} = 0 \quad \rightarrow \quad \delta Z_2 = \left.\frac{\partial\Sigma^{loop}}{\partial\not{p}}\right|_{\not{p}=m}$$

Passarino-Veltman Integrals



$$T_n^{\mu_1 \cdots \mu_p} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d k \, \frac{k^{\mu_1} \ldots k^{\mu_p}}{D_0 D_1 D_2 \cdots D_{n-1}}$$

$$D_i = (k + r_i)^2 - m_i^2 + i\epsilon$$

$$r_j = \sum_{i=1}^{j} p_i \quad ; \quad j = 1, \ldots, n-1$$

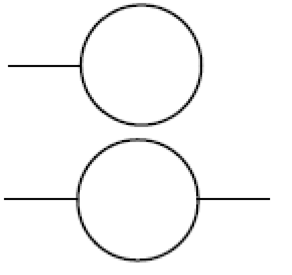
$$r_0 = \sum_{i=1}^{n} p_i = 0$$

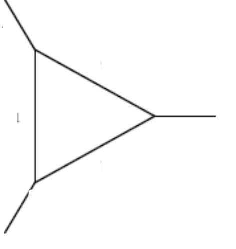
$$A_0(m_0^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \, \frac{1}{k^2 - m_0^2}$$

$$B_0(r_{10}^2, m_0^2, m_1^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \, \prod_{i=0}^{1} \frac{1}{[(k+r_i)^2 - m_i^2]}$$

$$C_0(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \, \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]}$$

$$D_0(r_{10}^2, r_{12}^2, r_{23}^2, r_{30}^2, r_{20}^2, r_{13}^2, m_0^2, \ldots, m_3^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \, \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]}$$





$$r_{ij}^2 = (r_i - r_j)^2 \quad ; \quad \forall \, i, j = (0, n-1)$$

$r_0 \;=\; 0$ so $r_{i0}^2 \;=\; r_i^2.$

Some formulas

$$A_0(m^2) = m^2\left(\Delta_\epsilon + 1 - \ln\frac{m^2}{\mu^2}\right)$$

$$B_0(p^2, m_0^2, m_1^2) = \Delta_\epsilon - \int_0^1 dx \ln\left[\frac{-x(1-x)p^2 + xm_1^2 + (1-x)m_0^2}{\mu^2}\right]$$

$$B_0(0, m_0^2, m_1^2) = \Delta_\epsilon + 1 - \frac{m_0^2 \ln m_0^2 - m_1^2 \ln m_1^2}{m_0^2 - m_1^2}$$

$$B_0(0, m_0^2, m_1^2) = \frac{A_0(m_0^2) - A_0(m_1^2)}{m_0^2 - m_1^2}$$

$$B_0(0, m^2, m^2) = \Delta_\epsilon - \ln\frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} - 1$$

$$B_0(m^2, 0, m^2) = \Delta_\epsilon + 2 - \ln\frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} + 1$$

$$B_0(0, 0, m^2) = \Delta_\epsilon + 1 - \ln\frac{m^2}{\mu^2}$$

$$B_0'(p^2, m_0^2, m_1^2) = -\int_0^1 dx\, \frac{x(1-x)}{-p^2x(1-x) + xm_1^2 + (1-x)m_0^2}$$

Feyncalc

Automatic installation

- Run the following instruction in a Kernel or Notebook session of Mathematica

```
Import["https://raw.githubusercontent.com/FeynCalc/feyncalc/master/install.m"]
InstallFeynCalc[]
```

If the above code fails with `URLFetch::invhttp: SSL connect error` (e.g. on Mathematica 9 under OS X), try

```
ImportString[URLFetch["https://raw.githubusercontent.com/FeynCalc/feyncalc/master/install.m"]]
InstallFeynCalc[]
```

Loading FeynCalc * To load FeynCalc 9 or newer run

```
<<FeynCalc`
```

Vacuum polarization

```
(* These are some shorthands for the FeynCalc notation *)

dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]
```

Vacuum polarization

```
(* Now write the numerator of the Feynman diagram. We define the
   constant

       C=alpha/(4 pi)
*)

num:= - C Tr[dm[mu] . (ds[q] + m) . dm[nu] .  (ds[q]+ds[k]+m)]
(* Tell FeynCalc to evaluate the integral in dimension D *)

SetOptions[OneLoop,Dimension->D]

(* Define the amplitude *)

amp:=num * FeynAmpDenominator[PropagatorDenominator[q+k,m], \
                              PropagatorDenominator[q,m]]

(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[q,amp]
ans=Simplify[res]
```

Vacuum polarization

```
                       2       2          2    2          2       2       2    2    2
Out[4]= (4  C  (k  + 6 m  B0[0, m , m ] - 3 (k  + 2 m ) B0[k , m , m ])
      2                                 2
   (k  g[mu, nu] - k[mu] k[nu])) / (9 k )
```

$$i\,\Pi_{\mu\nu}(k,\varepsilon) = -i\,k^2 P^T_{\mu\nu}\Pi(k,\varepsilon)$$

$$\Pi(k,\varepsilon) = \frac{\alpha}{4\pi}\left[-\frac{4}{9} - \frac{8}{3}\frac{m^2}{k^2}B_0(0,m^2,m^2) + \frac{4}{3}\left(1+\frac{2m^2}{k^2}\right)B_0(k^2,m^2,m^2)\right]$$

– In order to obtain the renormalized vacuum polarization, we need to calculate $\Pi(0,\varepsilon)$

$$\Pi(0,\varepsilon) = \frac{\alpha}{4\pi}\left[-\frac{4}{9} + \frac{4}{3}B_0(0,m^2,m^2) + \frac{8}{3}m^2 B_0'(0,m^2,m^2)\right]$$

$$B_0'(p^2,m_1^2,m_2^2) \equiv \frac{\partial}{\partial p^2}B_0(p^2,m_1^2,m_2^2)$$

$$B_0'(0,m^2,m^2) = \frac{1}{6m^2}$$

$$B_0(0,m^2,m^2) = \Delta_\varepsilon - \ln\frac{m^2}{\mu^2}$$

$$\Pi(0,\varepsilon) = -\delta Z_3 = \frac{\alpha}{4\pi}\left[\frac{4}{3}B_0(0,m^2,m^2)\right]$$

$$\Pi^R(k) = \frac{\alpha}{3\pi}\left[-\frac{1}{3} + \left(1+\frac{2m^2}{k^2}\right)\left(B_0(k^2,m^2,m^2) - B_0(0,m^2,m^2)\right)\right]$$

Anomalous magnetic moment of lepton

- Dirac equation of a point-like spin one-half particle with an external electromagnetic field $A_\mu(x)$

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\boldsymbol{\alpha}\cdot\left(-i\hbar\boldsymbol{\nabla} - \frac{e_\ell}{c}\boldsymbol{\mathcal{A}}\right) + \beta m_\ell c^2 + e_\ell\mathcal{A}_0\right]\psi$$

- In the non-relativistic limit, it reduces to the Pauli equation for the two-component spinor φ

$$i\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{(-i\hbar\boldsymbol{\nabla} - (e_\ell/c)\boldsymbol{\mathcal{A}})^2}{2m_\ell} - \frac{e_\ell\hbar}{2m_\ell c}\boldsymbol{\sigma}\cdot\mathbf{B} + e_\ell\mathcal{A}_0\right]\varphi$$

- A magnetic moment of the particle associated with its spin

$$\mathbf{M}_s = g_\ell\left(\frac{e_\ell}{2m_\ell c}\right)\mathbf{S}\,,\ \mathbf{S} = \hbar\,\frac{\boldsymbol{\sigma}}{2}$$

- A gyromagnetic ratio is predicted to be

$$g_\ell = 2$$

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- A gyromagnetic ratio is predicted to be

~~$g_\ell = 2$~~ $g_l \neq 2$

$$a_\mu \equiv \frac{g_\mu - 2}{2}$$ anomalous magnetic moment

The beginning of the g-2

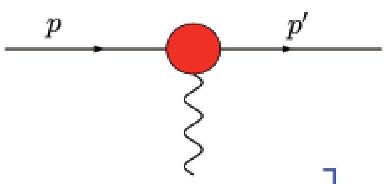
- **Kusch and Foley 1948:**

$$\mu_e^{\rm exp} = \frac{e\hbar}{2mc}\,(1.00119 \pm 0.00005)$$

- **Schwinger 1948 (triumph of QED!):**

$$\mu_e^{\rm th} = \frac{e\hbar}{2mc}\left(1 + \frac{\alpha}{2\pi}\right) = \frac{e\hbar}{2mc} \times 1.00116$$

- **Keep studying the lepton–γ vertex:**

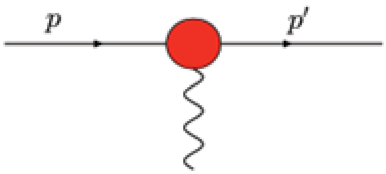


$$\bar{u}(p')\Gamma_\mu u(p) = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2) + \dots\right]u(p)$$

$$F_1(0) = 1 \qquad F_2(0) = a_l$$

A pure "quantum correction" effect!

(g-2) in the QFT



$$= \bar{u}(p')\Gamma^{\rho}(p',p)u(p)$$

$$k_\mu \equiv p'_\mu - p_\mu$$

- The most general form following Lorentz invariance, Dirac eq., etc.

$$\Gamma^{\rho}(p',p) = F_1(k^2)\gamma^{\rho} + \frac{i}{2m_\ell}F_2(k^2)\sigma^{\rho\nu}k_\nu - F_3(k^2)\gamma_5\sigma^{\rho\nu}k_\nu + F_4(k^2)[k^2\gamma^{\rho} - 2m_\ell k^{\rho}]\gamma_5$$

$F_1(k^2)$: the Dirac form factor — normalized as $F_1(0)=1$

$F_2(k^2)$: the Pauli form factor

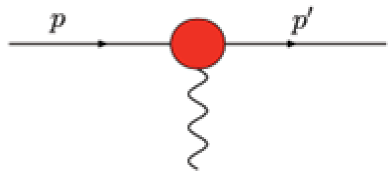
$F_3(k^2)$: the EDM form factor — P and T violation

$F_4(k^2)$: the anapole moment — P violation

- At the tree level in the SM

$$F_1^{\text{tree}}(k^2) = 1\,,\ F_i^{\text{tree}}(k^2) = 0,\ i = 2,\ 3,\ 4$$

(g-2) in the QFT



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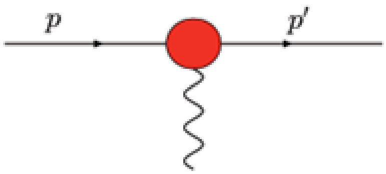
(F_3, F_4:) vanishing in QED and QCD

- At the tree level in the SM

$$F_1^{\text{tree}}(k^2) = 1\,,\; F_i^{\text{tree}}(k^2) = 0,\; i = 2,\,3,\,4$$

(g-2) in the QFT



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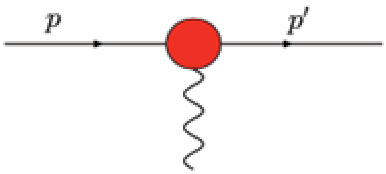
$F_4(k^2)$: the anapole moment — P violation

$a_\ell = \frac{1}{2}(g_\ell - 2) = F_2(0)$ anomalous magnetic moment

$d_\ell = e_\ell F_3(0)$ electric dipole moment

$F_4(0)$ anapole moment

(g-2) in the QFT



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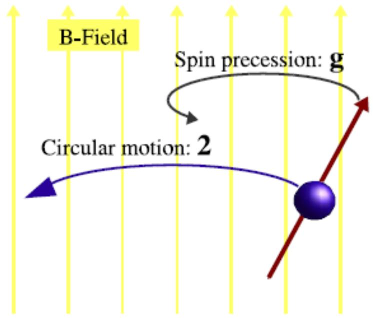
- project out form factors

$$F_i(k^2) = \mathrm{tr}\left[\Lambda_i^\rho(p',p)(\not{p}' + m_\ell)\Gamma_\rho(p',p)(\not{p} + m_\ell)\right]$$

$$\begin{aligned}
\Lambda_1^\rho(p',p) &= \frac{1}{4}\frac{1}{k^2 - 4m_\ell^2}\gamma^\rho + \frac{3m_\ell}{2}\frac{1}{(k^2-4m_\ell^2)^2}(p'+p)^\rho \\
\Lambda_2^\rho(p',p) &= -\frac{m_\ell^2}{k^2}\frac{1}{k^2-4m_\ell^2}\gamma^\rho - \frac{m_\ell}{k^2}\frac{k^2+2m_\ell^2}{(k^2-4m_\ell^2)^2}(p'+p)^\rho \\
\Lambda_3^\rho(p',p) &= -\frac{i}{2k^2}\frac{1}{k^2-4m_\ell^2}\gamma_5(p'+p)^\rho \\
\Lambda_4^\rho(p',p) &= -\frac{1}{4k^2}\frac{1}{k^2-4m_\ell^2}\gamma_5\gamma^\rho .
\end{aligned}$$

Muon magnetic moment: Exp

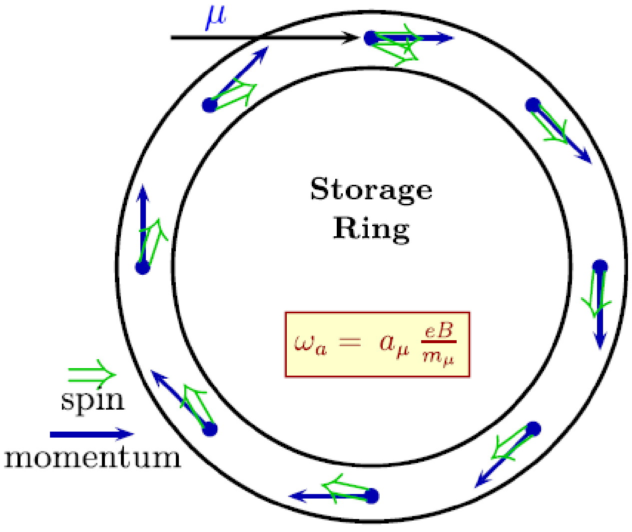


$$H_{\text{magnetic}} = -2(1 + a_\mu)\frac{e}{2m_\mu}\vec{B}\cdot\vec{S}$$

$$\omega_c = -\frac{qB}{m\gamma}$$ cyclotron precession

$$\omega_s = -\frac{gqB}{2m} - (1-\gamma)\frac{qB}{m\gamma}$$

spin precession (Larmor)



$$\omega_a = \omega_s - \omega_c = -\left(\frac{g-2}{2}\right)\frac{qB}{m} = -a_\mu \frac{qB}{m}$$

Muon g-2: the QED contribution

$a_\mu^{QED} = (1/2)(\alpha/\pi)$ Schwinger 1948

$+ \; 0.765857426\,(16)\,(\alpha/\pi)^2$

Sommerfield; Petermann; Suura&Wichmann '57; Elend '66; MP '04

$+ \; 24.05050988\,(28)\,(\alpha/\pi)^3$

Remiddi, Laporta, Barbieri … ; Czarnecki, Skrzypek; MP '04;
Friot, Greynat & de Rafael '05, Mohr, Taylor & Newell 2012

$+ \; 130.8773\,(61)\,(\alpha/\pi)^4$

Kinoshita & Lindquist '81, … , Kinoshita & Nio '04, '05;
Aoyama, Hayakawa,Kinoshita & Nio, 2007, Kinoshita et al. 2012 & 2015;
Lee, Marquard, Smirnov², Steinhauser 2013 (electron loops, analytic),
Kurz, Liu, Marquard, Steinhauser 2013 (τ loops, analytic);
Steinhauser et al. 2015 & 2016 (all electron & τ loops, analytic).

$+ \; 752.85\,(93)\,(\alpha/\pi)^5$ **COMPLETED!**

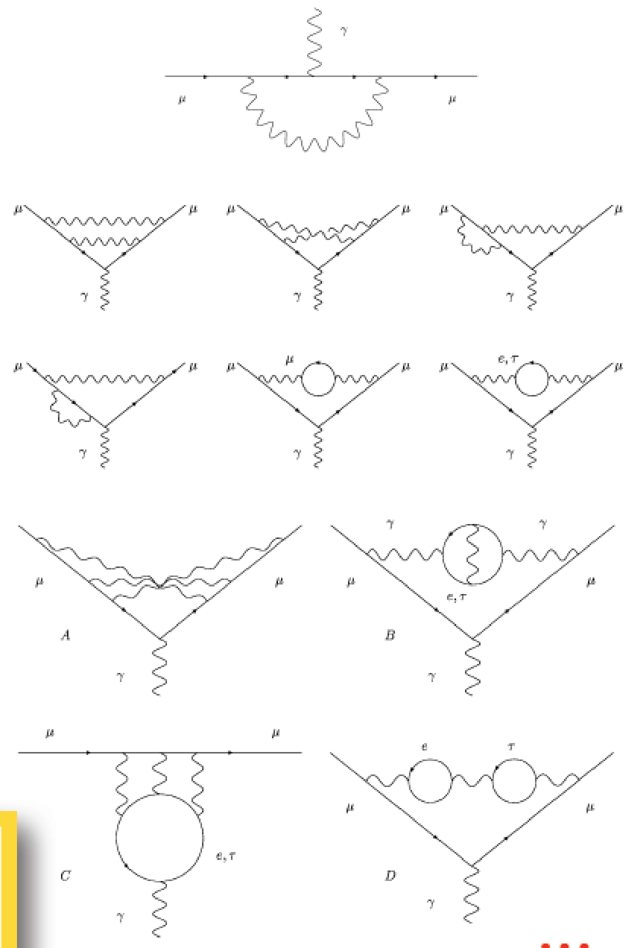
Kinoshita et al. '90, Yelkhovsky, Milstein, Starshenko, Laporta,…
Aoyama, Hayakawa, Kinoshita, Nio 2012 & 2015

Adding up, we get:

$a_\mu^{QED} = 116584718.941\,(21)(77) \times 10^{-11}$

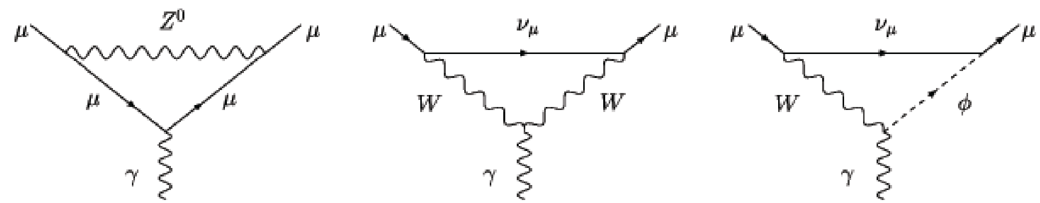
from coeffs, mainly from 4-loop unc ↲ ↳ from $\delta\alpha$(Rb)

with $\alpha = 1/137.035999049(90)$ [0.66 ppb]



Muon g-2: the EW contribution

- ## One-loop term:



$$a_\mu^{\rm EW}(1\text{-loop}) = \frac{5 G_\mu m_\mu^2}{24\sqrt{2}\pi^2}\left[1 + \frac{1}{5}\left(1 - 4\sin^2\theta_W\right)^2 + O\left(\frac{m_\mu^2}{M_{Z,W,H}^2}\right)\right] \approx 195\times 10^{-11}$$

1972: Jackiv, Weinberg; Bars, Yoshimura; Altarelli, Cabibbo, Maiani; Bardeen, Gastmans, Lautrup; Fujikawa, Lee, Sanda;

Studenikin et al. '80s

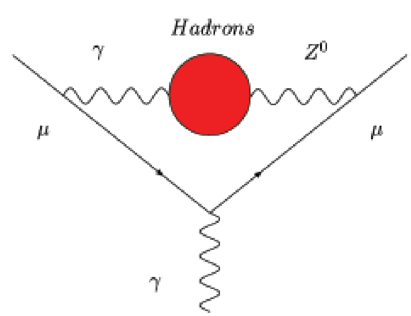
- ## One-loop plus higher-order terms:

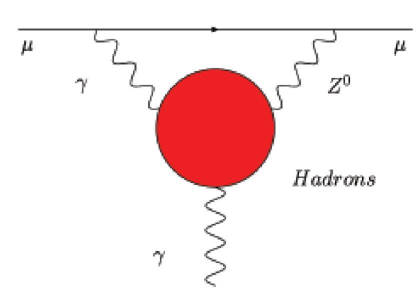
$a_\mu^{\rm EW} = 153.6\ (1) \times 10^{-11}$

with M_{Higgs} = 125.6 (1.5) GeV

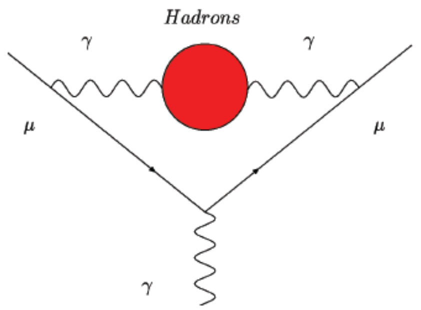
Hadronic loop uncertainties and 3-loop nonleading logs.

Kukhto et al. '92; Czarnecki, Krause, Marciano '95; Knecht, Peris, Perrottet, de Rafael '02; Czarnecki, Marciano and Vainshtein '02; Degrassi and Giudice '98; Heinemeyer, Stockinger, Weiglein '04; Gribouk and Czarnecki '05; Vainshtein '03; Gnendiger, Stockinger, Stockinger-Kim 2013.



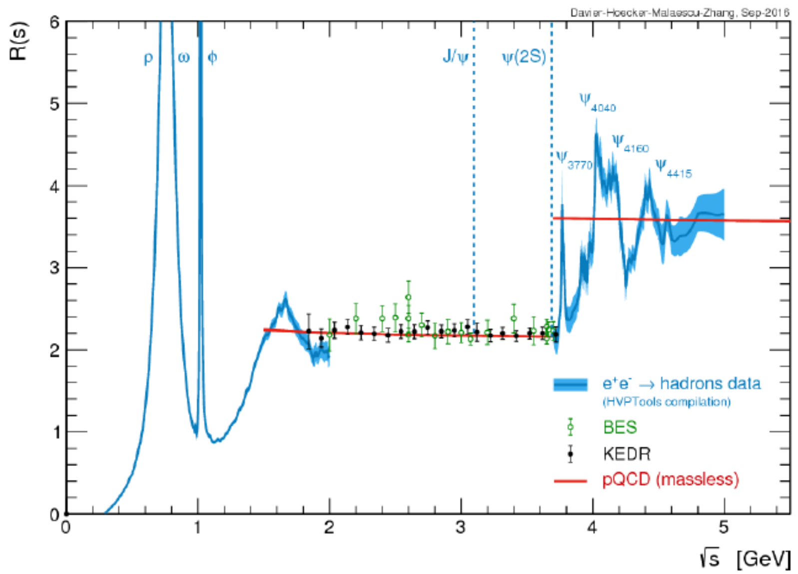


Muon g-2: the HVP contribution



$$R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

DHMZ , TAU 2016, arXiv:1612.02743



$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} \mathrm{d}s \frac{R(s)K(s)}{s^2} \qquad R \equiv \frac{\sigma_{\mathrm{total}}(e^+e^- \to \mathrm{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)(s/m^2)}$$

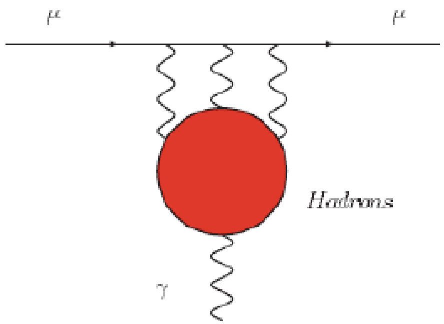
$a_\mu^{HLO} = 6870\ (42)_{tot} \times 10^{-11}$ — F. Jegerlehner, arXiv:1511.04473 (includes BESIII 2π)

$= 6928\ (33)_{tot} \times 10^{-11}$ — Davier et al, Tau2016, Beijing, Sep 2016, Preliminary

$= 6949\ (37)_{exp}\ (21)_{rad} \times 10^{-11}$ — Hagiwara et al, JPG 38 (2011) 085003

Muon g-2: the LBL contribution

- **HNLO: Light-by-light contribution**



Unlike the HLO term, the hadronic l-b-l term relies at present on theoretical approaches.

This term had a troubled life! Latest values:

$a_\mu^{HNLO}(lbl) = +80\ (40) \times 10^{-11}$	Knecht & Nyffeler '02
$a_\mu^{HNLO}(lbl) = +136\ (25) \times 10^{-11}$	Melnikov & Vainshtein '03
$a_\mu^{HNLO}(lbl) = +105\ (26) \times 10^{-11}$	Prades, de Rafael, Vainshtein '09
$a_\mu^{HNLO}(lbl) = +102\ (39) \times 10^{-11}$	Jegerlehner, arXiv:1511.04473

Results based also on Hayakawa, Kinoshita '98 & '02; Bijnens, Pallante, Prades '96 & '02

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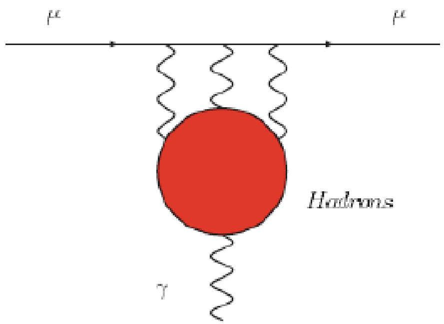

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Contribution	Result in 10^{-10} units	
QED(leptons)	11658471.885 ± 0.004	
HVP(leading order)	690.8 ± 4.7	
HVP(NLO)	-9.93 ± 0.07	
HVP(NNLO)	1.22 ± 0.01	
~~HLBL (+NLO)~~	~~11.7 ± 4.0~~	
EW	15.4 ± 0.1	
Total	~~11659179.1 ± 6.2~~ →	11659167.4 ± 4.7

NO HLBL

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 41.7(7.9) \times 10^{-10} \Rightarrow 5.3\,\sigma$$ (2σ effect)

Muon g-2: Exp. Vs Theory

- The E821 experiment at BNL

$$a_\mu^{\exp} = 11\,659\,209.1(6.3) \times 10^{-10}$$

- In the SM

Contribution	Result in 10^{-10} units	
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HVP(leading order)	690.8 ± 4.7	Davier *et al* 2011
HVP(NLO)	-9.93 ± 0.07	Hagiwara *et al* 2009
HVP(NNLO)	1.22 ± 0.01	Kurz *et al* 2014
HLBL (+NLO)*	11.7 ± 4.0	Jegerlehner, Nyffeler 2009
EW	15.4 ± 0.1	Czarnecki 2003, Gnendinger 2013
Total	11659179.1 ± 6.2	

* NLO: Colangelo et al 2014

$$a_\mu^{\exp} - a_\mu^{\mathrm{SM}} = 28.0(8.8) \times 10^{-10} \Rightarrow 3.2\,\sigma$$

Muon g-2: uncertainty budget in the SM

Table 1: Summary of the Standard-Model contributions to the muon anomaly. Two values are quoted because of the two recent evaluations of the lowest-order hadronic vacuum polarization.

	Value ($\times 10^{-11}$) units	
QED ($\gamma + \ell$)	$116\,584\,718.951 \pm 0.009 \pm 0.019 \pm 0.007 \pm 0.077_\alpha$	
HVP(lo) [20]	$6\,923 \pm 42$	0.6%, from e+e- exp.
HVP(lo) [21]	$6\,949 \pm 43$	
HVP(ho) [21]	-98.4 ± 0.7	
HLbL	105 ± 26	25%, from hadronic models
EW	154 ± 1	
Total SM [20]	$116\,591\,802 \pm 42_{\text{H-LO}} \pm 26_{\text{H-HO}} \pm 2_{\text{other}}\ (\pm 49_{\text{tot}})$	
Total SM [21]	$116\,591\,828 \pm 43_{\text{H-LO}} \pm 26_{\text{H-HO}} \pm 2_{\text{other}}\ (\pm 50_{\text{tot}})$	

Experimental uncertainty: 63×10^{-11} now,
goal: 17×10^{-11}.

Blum et al., arXiv:1311.2198v1 [hep-ph]

Electron g-2

- **The 2008 measurement of the electron g-2 is:**

$$a_e^{EXP} = 11596521807.3\ (2.8) \times 10^{-13}$$

Hanneke, Fogwell, Gabrielse
PRL100 (2008) 120801

- **Using α = 1/137.035 999 049 (90) from h/M measurement of ^{87}Rb (2011), the SM prediction for the electron g-2 is**

$$a_e^{SM} = 115\,965\,218\,16.5\ (0.2)\ (0.2)\ (0.2)\ (7.6) \times 10^{-13}$$

δC_4^{qed} δC_5^{qed} δa_e^{had} from $\delta\alpha$

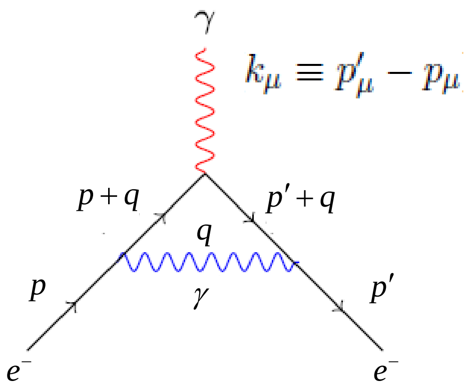
- **The EXP-SM difference is (note the negative sign):**

$$\Delta a_e = a_e^{EXP} - a_e^{SM} = -9.2\ (8.1) \times 10^{-13}$$

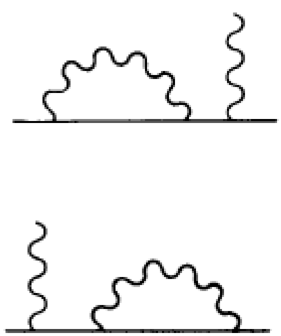
The SM is in very good agreement with experiment (1σ).

g-2 at one loop

- one loop diagrams



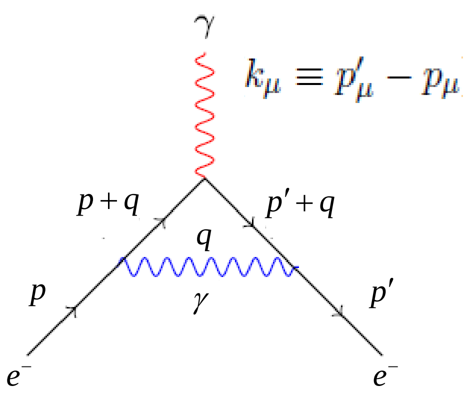
The (g-2) term does not diverge because there is no counter term at the tree level



these diagrams are proportional to γ^ρ which does not contribute to (g-2)

g-2 at one loop

- one loop diagram



$$\Gamma^\rho(p',p)|_{1\ \text{loop}} = (-ie)^2 \int \frac{d^4q}{(2\pi)^4}\gamma^\mu \frac{i}{\not p' + \not q - m_e}\gamma^\rho \frac{i}{\not p + \not q - m_e}\gamma^\nu \times \frac{(-i)}{q^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right].$$

- ξ dependent terms affect $F_1(0)$

$$(1-\xi)\bar{u}(p')\,\not q \frac{i}{\not p' + \not q - m_e}\gamma^\rho \frac{i}{\not p + \not q - m_e}\,\not q u(p) = (1-\xi)\bar{u}(p') i\gamma_\rho i u(p)$$

$$i\frac{\not p + \not q + m_e}{(p+q)^2 - m_e^2}\not q u(p) = i\frac{-\not q \not p + 2p\cdot q + q^2 + m_e \not q}{q^2 + 2p\cdot q}u(p) = iu(p)$$

g-2 at one loop

$$F_i(k^2) = \mathrm{tr}\left[\Lambda_i^\rho(p',p)(\not{p}' + m_\ell)\Gamma_\rho(p',p)(\not{p} + m_\ell)\right]$$

- After evaluating the trace for i=2, one finds

$$F_2(p',p) = -ie^2\left(-\frac{m_e}{k^2(k^2 - rm_e^2)^2}\right)\int \frac{d^4q}{(2\pi)^4}\frac{\mathcal{N}}{q^2[(p'+q)^2 - m_e^2][(p+q)^2 - m_e^2]}$$

$$\mathcal{N} = \mathrm{Tr}[\overline{\Lambda}^\rho(\not{p}' + m_e)\overline{\Gamma}_\rho(\not{p} + m_e)]$$

$$\overline{\Lambda}^\rho = m_e(k^2 - 4m_e^2)\gamma^\rho + (k^2 + 2m_e^2)(p' + p)^\rho$$

$$\overline{\Gamma}_\rho = \gamma^\mu(\not{p}' + \not{q} + m_e)\gamma_\rho(\not{p} + \not{q} + m_e)\gamma_\mu$$

- Using the standard Feynman parametrization

$$\frac{1}{q^2[(p'+q)^2 - m_e^2][(p+q)^2 - m_e^2]} = \int_0^1 dx \int_0^{1-x} dy \frac{2}{[(q + xp' + yp)^2 - (xp' + yp)^2]^3}$$

- Change of variables

$$q + xp' + yp \equiv q'$$

g-2 at one loop

- Then one obtains

$$F_2(k^2) = i4\pi\alpha \frac{2m_e}{k^2(k^2-4m_e^2)^2} \int dx \int dy \int \frac{d^4q'}{(2\pi)^4} \frac{\mathcal{N}(q')}{(q'^2-\Delta)^3}$$

- Remove the linear terms of q' and average over q^2 terms

$$\overline{\mathcal{N}}(q') = \frac{1}{2}(\mathcal{N}(q') + \mathcal{N}(q' \to -q'))$$

$$(q' \cdot k)^2 = k^2 {q'}^2/4, \quad (q' \cdot p)^2 = p^2 {q'}^2/4, \quad (q' \cdot k)(q' \cdot p) = k \cdot p {q'}^2/4$$

$$k.p = -k^2/2, \quad k.p' = k^2/2, \quad p.p' = (-k^2 + 2m_e^2)/2, \quad p^2 = {p'}^2 = m_e^2$$

- decouple the x,y integration

$$(x, y) \to (s, t)$$

$$x = st, \quad y = s(1-t)$$

$$\int_0^1 dx \int_0^{1-x} dy = \int_0^1 s ds \int_0^1 dt$$

g-2 at one loop

- Then one obtains

$$\overline{\mathcal{N}} = 4m_e k^2(k^2 - 4m_e^2)^2 s(1-s)$$

$$F_2(k^2) = i4\pi\alpha \frac{2m_e}{k^2(k^2-4m_e^2)^2} 4m_e k^2(k^2-4m_e^2)^2 \times \int_0^1 ds s^2(1-s) \int_0^1 dt \int \frac{d^4q'}{(2\pi)^4} \frac{1}{(q'^2-\Delta)^3}$$

- q' integration $\int \frac{d^4q'}{(2\pi)^4} \frac{1}{(q'^2-\Delta)^3} = I_{0,3} = -\frac{i}{2(4\pi)^2}\frac{1}{\Delta}$

- $F_2(k^2)$ is given by

$$F_2(k^2) = 4\pi\alpha\, 8m_e^2 \frac{1}{2(4\pi)^2} \int_0^1 ds s^2(1-s) \int_0^1 dt \frac{1}{\Delta}$$

$$\Delta = s^2 m_e^2 - s^2 t(1-t)k^2$$

- In the limit of $k^2 \to 0$

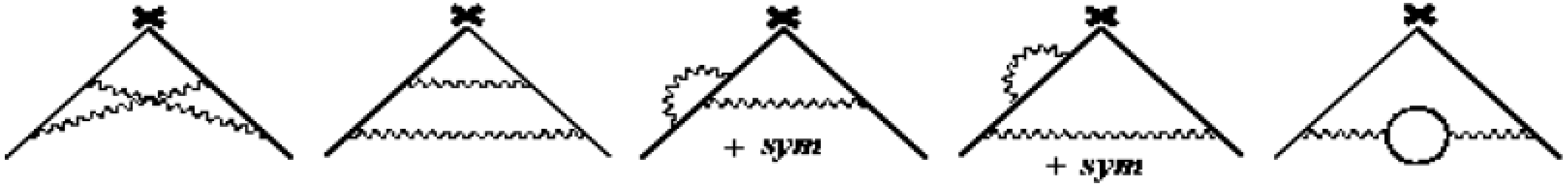
$$\Delta = s^2 m_e^2 - s^2 t(1-t)k^2 \longrightarrow s^2 m_e^2$$

$$F_2(0) = \frac{\alpha}{\pi} m_e^2 \int_0^1 ds s^2(1-s) \int_0^1 dt \frac{1}{s^2 m_e^2} = \frac{\alpha}{2\pi}$$

Schiwinger term

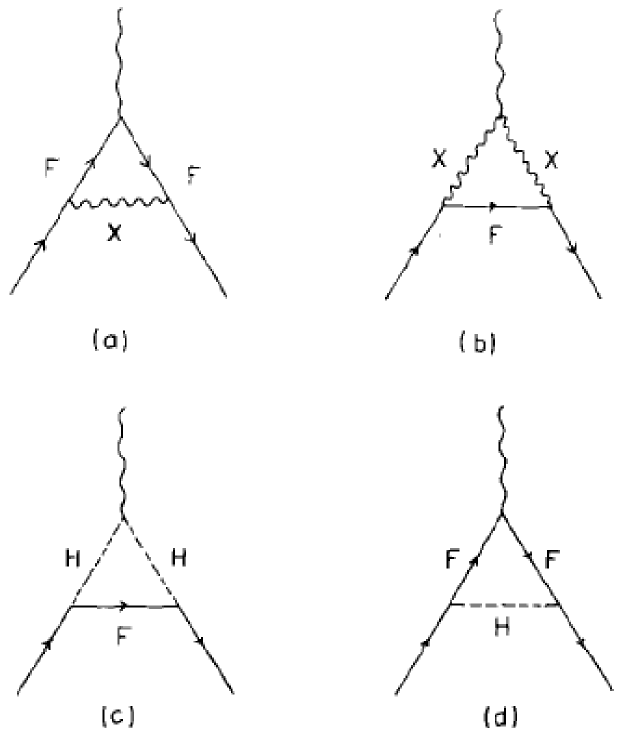
g-2 at two loops

- two-loop result



$$A_2 = \frac{197}{144} + \left(\frac{1}{2} - 3\ln 2\right)\zeta(2) + \frac{3}{4}\zeta(3)$$
$$= -0.328\,478\,965\ldots$$

Muon g-2 and new physics



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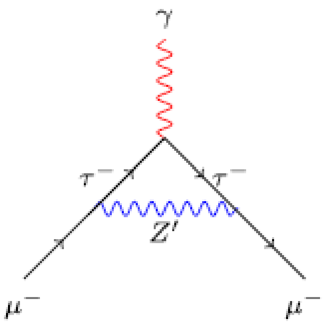
$$[a_\mu]_a = \frac{-q_F m_\mu^2}{4\pi^2} \int_0^1 dx \left[C_V^2 \left\{ (x - x^2)\left(x + \frac{2m_F}{m_\mu} - 2\right) - \frac{1}{2M_X^2}(x^3(m_F - m_\mu)^2 \right.\right.$$
$$\left.\left. + x^2(m_F^2 - m_\mu^2)\left(1 - \frac{m_F}{m_\mu}\right)\right\} + C_A^2 \{m_F \to -m_F\} \right]$$
$$\times \{m_\mu^2 x^2 + M_X^2(1 - x) + x(m_F^2 - m_\mu^2)\}^{-1}$$

$$[a_\mu]_b = \frac{q_X m_\mu^2}{8\pi^2} \int_0^1 dx \left[C_V^2 \left\{ \frac{4m_F}{m_\mu} x^2 - 2x^2(1 + x) + \frac{m_\mu^2}{M_X^2}\left[-x^2(x - 1) \right.\right.\right.$$
$$\left.\left. - \frac{m_F}{m_\mu}(-2x^3 + 3x^2 - x) - \frac{m_F^2}{m_\mu^2}(2x - 3x^2 + x^3) + \frac{m_F^3}{m_\mu^3}(x - x^2)\right]\right\}$$
$$\left. + C_A^2 \{m_F \to -m_F\} \right] \{[m_\mu^2 x^2 + (M_X^2 - m_\mu^2)x + m_F^2(1 - x)]\}^{-1}$$

$$[a_\mu]_c = \frac{-q_H m_\mu^2}{8\pi^2} \int_0^1 dx \frac{\left[C_S^2 \left\{ x^3 - x^2 + \frac{m_F}{m_\mu}(x^2 - x) \right\} + C_P^2 \{m_F \to -m_F\} \right]}{m_\mu^2 x^2 + (m_H^2 - m_\mu^2)x + m_F^2(1 - x)}$$

$$[a_\mu]_d = \frac{-q_F m_\mu^2}{8\pi^2} \int_0^1 dx \frac{\left[C_S^2 \left\{ x^2 - x^3 + \frac{m_F}{m_\mu} x^2 \right\} + C_P^2 \{m_F \to -m_F\} \right]}{m_\mu^2 x^2 + (m_F^2 - m_\mu^2)x + m_H^2(1 - x)}$$

Muon g-2 and new physics



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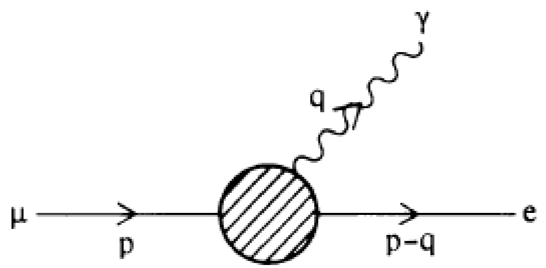
$$\mathcal{L}_{Z'} = g'_L(\bar{\mu}\gamma^\alpha P_L\tau + \bar{\nu}_\mu\gamma^\alpha P_L\nu_\tau)Z'_\alpha + g'_R(\bar{\mu}\gamma^\alpha P_R\tau)Z'_\alpha + \mathrm{H.c.}$$

$$a_\mu = \frac{m_\mu^2}{4\pi^2}\int_0^1 dx\left[C_V^2\left\{(x-x^2)\left(x+\frac{2m_\tau}{m_\mu}-2\right) - \frac{x^2}{2m_{Z'}^2}(m_\tau-m_\mu)^2\left(x-\frac{m_\tau}{m_\mu}-1\right)\right\} + C_A^2\left\{m_\tau \to -m_\tau\right\}\right] \times \left[m_\mu^2 x^2 + m_{Z'}^2(1-x) + x(m_\tau^2 - m_\mu^2)\right]^{-1}$$

$\mu \to e\gamma$

- In the SM, $\mu \to e\gamma$ is forbidden, but the neutrino oscillation may induce the flavor changing process



- The would-be-Goldstone boson contribution is not negligible (R_ξ gauge)
- The amplitude can be written as

$$T(\mu \to e\gamma) = \varepsilon^\lambda \langle e | J_\lambda^{em} | \mu \rangle$$

$$\langle e | J_\lambda^{em} | \mu \rangle = \bar{u}_e(p-q)[iq^\nu \sigma_{\lambda\nu}(A + B\gamma_5) + \gamma_\lambda(C + D\gamma_5) + q_\lambda(E + F\gamma_5)]u_\mu(p)$$

- From the charge conservation $\partial^\lambda J_\lambda^{em} = 0$

$$-m_e(C + D\gamma_5) + m_\mu(C - D\gamma_5) + q^2(E + F\gamma_5) = 0$$

$C = D = 0$ for an on-shell photon

$\mu \to e\gamma$

- Only the magnetic transition appears

$$T(\mu \to e\gamma) = \varepsilon^{\lambda}\bar{u}_e(p-q)[iq^{\nu}\sigma_{\lambda\nu}(A + B\gamma_5)]u_{\mu}(p)$$

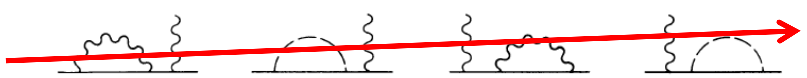
  – dimension-5 operators
  – no counterterm to absorb infinities → must be finite

- Strictly forbidden in the SM, but if there is mixing in the lepton sector like in the quark sector, this flavor-changing process can be generated

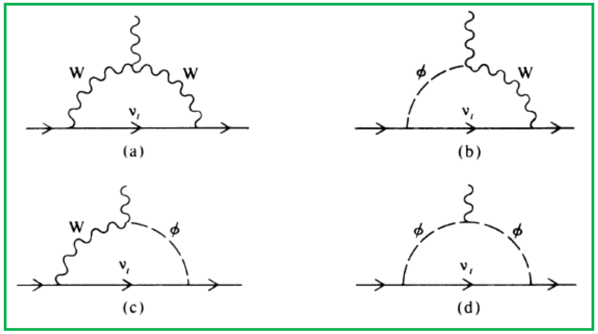
- Assume that neutrinos have Dirac masses and mixing between them

$$\nu_\alpha = \sum_i U_{\alpha i}\nu_i \qquad \alpha = e, \mu, \tau;\ i = 1, 2, 3$$

- keep the terms which produce the magnetic transition



$\propto \quad \bar{u}_e\gamma^{\lambda}u_{\mu}$



$\mu \rightarrow e\gamma$

- adopt the massless electron $m_e = 0$,

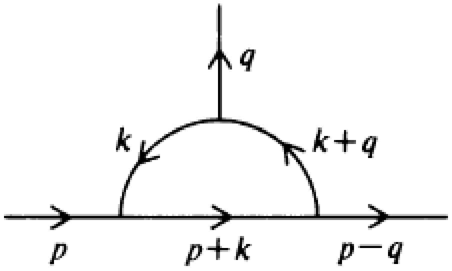
$$A = B$$

  – because the final electron is left-handed (couples to W)

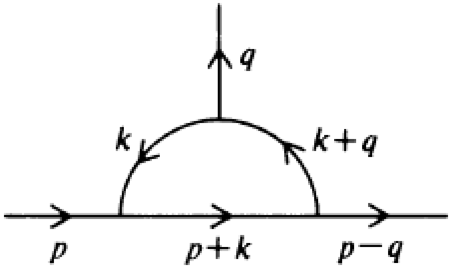
$$T = A\bar{u}_e(p-q)(1+\gamma_5)i\sigma_{\lambda\nu}q^\nu\varepsilon^\lambda u_\mu(p)$$
$$= A\bar{u}_e(p-q)(1+\gamma_5)(2p\cdot\varepsilon - m_\mu\gamma\cdot\varepsilon)u_\mu(p)$$

  – keep only the p·ϵ term

- The momentum assignment



$\mu \to e\gamma$



- diagram (a)

$$T_i(a) = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\bar{\mathrm{u}}_e(p-q) \left(\frac{\mathrm{i}g}{2\sqrt{2}} \right) U^*_{ei} \gamma_\mu (1-\gamma_5) \frac{\mathrm{i}}{\not{p} + \not{k} - m_i} \left(\frac{\mathrm{i}g}{2\sqrt{2}} \right) \right.$$

$$\left. \times\, U_{\mu i} \gamma_\nu (1-\gamma_5) \mathrm{u}_\mu(p) \right] [\mathrm{i}\Delta^{\nu\beta}(k)][\mathrm{i}\Delta^{\mu\alpha}(k+q)](-\mathrm{i}e)\Gamma_{\gamma\alpha\beta}\varepsilon^\gamma$$

  – the W boson-photon vertex for incoming momenta

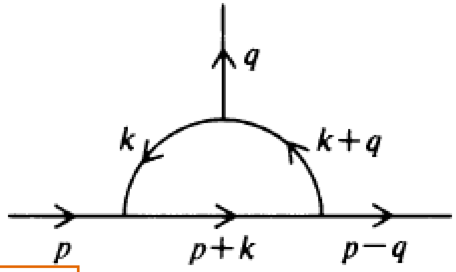
$$\Gamma_{\gamma\alpha\beta}(k_1, k_2, k_3) = [(k_3 - k_1)_\alpha g_{\gamma\beta} + (k_2 - k_3)_\gamma g_{\alpha\beta} + (k_1 - k_2)_\beta g_{\gamma\alpha}]$$

$$\varepsilon^\gamma \Gamma_{\gamma\alpha\beta}(-q, k+q, -k) \equiv \Gamma_{\alpha\beta} = [(2k \cdot \varepsilon) g_{\alpha\beta} - (k+2q)_\beta \varepsilon_\alpha - (k-q)_\alpha \varepsilon_\beta]$$

  – the W boson propagator

$$\Delta_{\mu\nu}(k) = -[g_{\mu\nu} - (1-\xi)k_\mu k_\nu/(k^2 - \xi M^2)]/(k^2 - M^2)$$

$\mu \to e\gamma$



– sum over three intermediate mass eigenstates

GIM $\Sigma_i U^*_{ei} U_{\mu i} = 0$

$$\sum_i \left\{ \frac{U^*_{ei} U_{\mu i}}{(p+k)^2 - m_i^2} \right\} = \sum_i U^*_{ei} U_{\mu i} \left\{ \frac{1}{(p+k)^2} + \frac{m_i^2}{[(p+k)^2]^2} + \dots \right\}$$

$$= \sum_i \frac{U^*_{ei} U_{\mu i} m_i^2}{[(p+k)^2]^2} + \dots$$

– then the amplitude becomes

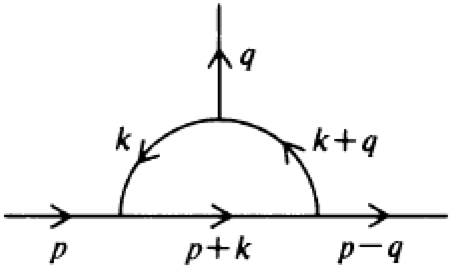
$$T(a) = \sum_i T_i(a) = \mathrm{i}c \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{R}{[(p+k)^2]^2}$$

$$c = \frac{g^2 e}{4} \sum_i U^*_{ei} U_{\mu i} m_i^2$$

$$R = \Delta^{\nu\beta}(k) \Delta^{\mu\alpha}(k+q) N_{\mu\nu} \Gamma_{\alpha\beta}$$

$$N_{\mu\nu} = \bar{\mathrm{u}}_e(p-q)\gamma_\mu (\not{p} + \not{k}) \gamma_\nu (1-\gamma_5) \mathrm{u}_\mu(p)$$

$\mu \to e\gamma$



– W boson propagators

$$\Delta^{\mu\nu}(k) \equiv \Delta_1^{\mu\nu}(k) + \Delta_2^{\mu\nu}(k)$$

$$\Delta_1^{\mu\nu}(k) = -(g^{\mu\nu} - k^\mu k^\nu / M^2)/(k^2 - M^2)$$

$$\Delta_2^{\mu\nu}(k) = -(k^\mu k^\nu / M^2)/(k^2 - \xi M^2)$$

$$(k+q)^\alpha k^\beta \Gamma_{\alpha\beta} = 0$$

$$\Delta_2^{\nu\beta}(k)\Delta_2^{\mu\alpha}(k+q)\Gamma_{\alpha\beta} = 0$$

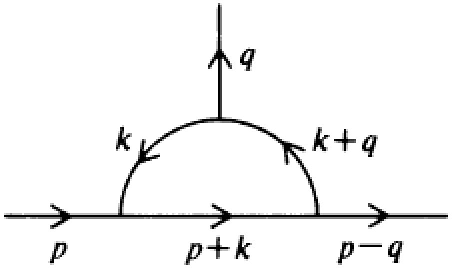
$$T(a) = \mathrm{i}c \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{[(k+p)^2]^2} \times \left\{ \frac{S_1 - S_2 - S_3}{(k^2 - M^2)[(k+q)^2 - M^2]} + \frac{S_2}{(k^2 - \xi M^2)[(k+q)^2 - \xi M^2]} + \frac{S_3}{(k^2 - M^2)[(k+q)^2 - \xi M^2]} \right\}$$

$$S_1 = \Gamma^{\mu\nu} N_{\mu\nu}$$

$$S_2 = (k^\lambda \Gamma^\mu_\lambda)(k^\nu N_{\mu\nu})/M^2$$

$$S_3 = [(k+q)^\lambda \Gamma^\mu_\lambda][(k+q)^\nu N_{\mu\nu}]/M^2$$

$\mu \to e\gamma$



– Feynman parametrization

$$T(a) = \mathrm{i}3!c \int \alpha_1 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \left\{ \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\frac{\tilde{S}_1 - \tilde{S}_2 - \tilde{S}_3}{(k^2 - a^2)^4} + \frac{\tilde{S}_2}{(k^2 - b^2)^4} + \frac{\tilde{S}_3}{(k^2 - d^2)^4} \right\} \right.$$

$$a^2 = (1 - \alpha_1) M^2 + \dots$$

$$b^2 = [(1 - \alpha_1 - \alpha_2)\xi + \alpha_2] M^2 + \dots$$

$$d^2 = [(1 - \alpha_1 - \alpha_2) + \alpha_2 \xi] M^2 + \dots .$$

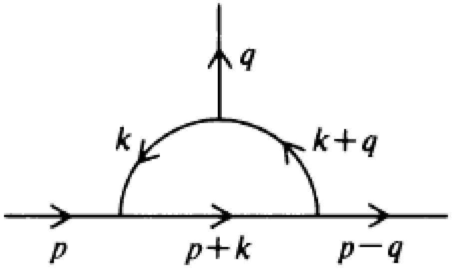
– Picking out only the p·ϵ term

$$S_1 \to \tilde{S}_1 = (p \cdot \varepsilon)[\bar{\mathrm{u}}_{\mathrm{e}}(1 + \gamma_5)\mathrm{u}_{\mu}] 2m_{\mu}[2(1 - \alpha_1)^2 + (2\alpha_1 - 1)\alpha_2]$$

$$S_2 \to \tilde{S}_2 = -k^2 (p \cdot \varepsilon)[\bar{\mathrm{u}}_{\mathrm{e}}(1 + \gamma_5)\mathrm{u}_{\mu}](m_{\mu}/M^2) \times \{(3\alpha_2 - 1) + [2\alpha_1^2 - \alpha_1 + \alpha_2(2\alpha_1 - 1/2)]\}$$

$$S_3 \to \tilde{S}_3 = -k^2 (p \cdot \varepsilon)[\bar{\mathrm{u}}_{\mathrm{e}}(1 + \gamma_5)\mathrm{u}_{\mu}](m_{\mu}/M^2)[2\alpha_1^2 + \alpha_1 + (2\alpha_1 - 1/2)\alpha_2]$$

$\mu \to e\gamma$



– Momentum integration

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - a^2)^4} = \frac{i}{96\pi^2 a^4}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 - a^2)^4} = \frac{-i}{48\pi^2} \frac{1}{a^2}$$

– After integrating α_i, one obtains

$$A(a) = \frac{c}{64\pi^2} \frac{m_\mu}{M^4} \left[1 - \frac{1}{3} \frac{\ln \xi}{\xi - 1} + \left(\frac{1}{\xi - 1} \right) \left(\frac{\xi \ln \xi}{\xi - 1} - 1 \right) \right]$$

- Similarly, one can calculate

$$A(b) = \frac{c}{64\pi^2} \frac{m_\mu}{M^4} \left[\frac{5}{6\xi} + \frac{4}{3} \frac{\ln \xi}{\xi - 1} - \frac{7}{3} \left(\frac{1}{\xi - 1} \right) \left(\frac{\xi \ln \xi}{\xi - 1} - 1 \right) \right]$$

$$A(c) = \frac{c}{64\pi^2} \frac{m_\mu}{M^4} \left[\frac{5}{6\xi} - \frac{\ln \xi}{\xi - 1} + \frac{1}{3} \left(\frac{1}{\xi - 1} \right) \left(\frac{\xi \ln \xi}{\xi - 1} - 1 \right) \right]$$

$$A(d) = \frac{-c}{32\pi^2} \left(\frac{m_\mu}{M^4} \right) \frac{5}{6\xi}$$

$\mu \to e\gamma$

$$f(\xi)=\frac{\ln \xi}{\xi -1}$$

$$g(\xi)=\left(\frac{1}{\xi -1}\right)\left(\frac{\xi \ln \xi}{\xi -1}-1\right)$$

- Summing the contributions of all diagrams

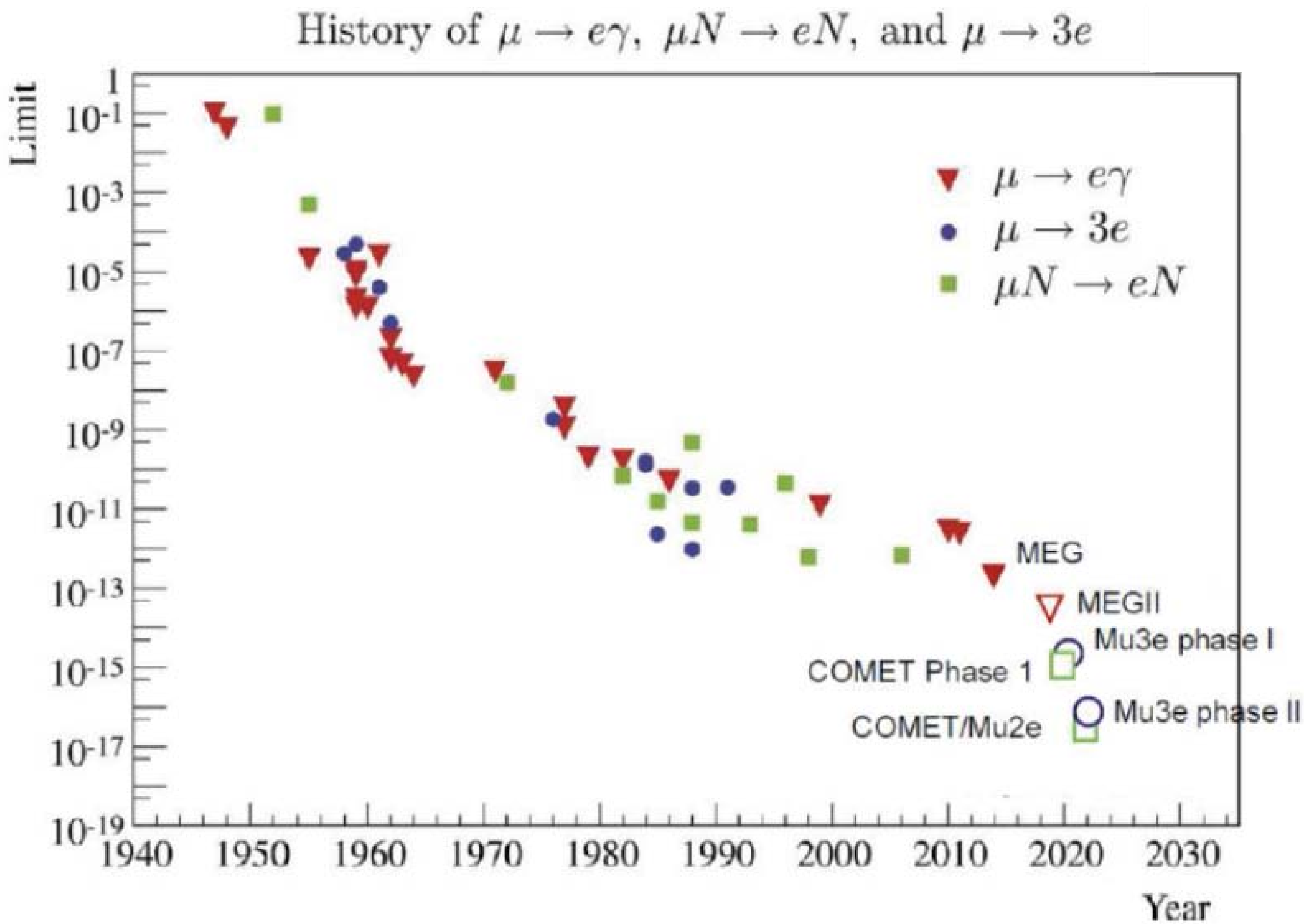
	R_ξ gauge	't Hooft gauge ($\xi = 1$)	unitary gauge ($\xi \to \infty$)
	$1-\frac{1}{3}f(\xi)+2g(\xi)$	$\frac{5}{3}$	1
	$\frac{5}{6\xi}+\frac{4}{3}f(\xi)-\frac{7}{3}g(\xi)$	1	0
	$\frac{5}{6\xi}-f(\xi)+\frac{1}{3}g(\xi)$	0	0
	$-\frac{5}{3\xi}$	$-\frac{5}{3}$	0

$$\Gamma(\mu \to e\gamma) = \frac{m_\mu^3}{8\pi}\left(|A|^2 + |B|^2\right)$$

$$A = B = e\,\frac{g^2}{8M^2}\,\frac{m_\mu}{32\pi^2}\,\delta_\nu$$

$$\delta_\nu = \sum_i U_{ei}^* U_{\mu i}(m_i^2/M^2)$$

Lepton flavor violation in μ



$Br_{sm}(\mu^+ \to e^+\gamma) < 10^{-54}$