Flavor Physics IV

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Outline

- Flavor Physics and the Standard Model
- Discrete Symmetry and CKM matrix
- Renormalization and Muon g-2
- RG and Effective Field Theory
  - Renormalization Group Equation
  - Effective Field Theory
- CP Violation and BSM Flavor Physics

Quantum Chromodynamics (QCD)

- The Lagrangian density of QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2$$

gauge fixing

$$+ \bar{q}(i\not{\partial} - m_q)q + \chi^{a*}\partial^\mu\partial_\mu\chi^a$$

ghost

$$- \frac{g}{2}f^{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A^{b\mu}A^{c\nu} - \frac{g^2}{4}f^{abe}f^{cde}A_\mu^a A_\nu^b A^{c\mu}A^{d\nu}$$

$$+ g\bar{q}_i T_{ij}^a \gamma^\mu q_j A_\mu^a + g f^{abc}(\partial^\mu \chi^{a*})\chi^b A_\mu^c$$

- Use dimensional regularization by continuing to d=4-2ϵ

- To eliminate the divergences one has to introduce renormalized fields

$$A_{0\mu}^a = Z_3^{1/2} A_\mu^a \qquad q_0 = Z_q^{1/2} q \qquad \chi_0^a = \tilde{Z}_3^{1/2}\chi^a$$

$$g_0 = Z_g g \mu^\varepsilon \qquad \xi_0 = Z_3 \xi \qquad m_0 = Z_m m$$

"0" indicates unrenormalized quantities that are independent of the scale μ

do not need the gauge parameter renormalization if we are dealing with the gauge independent quantities

Renormalization scheme

- Counter terms are added to implement renormalization

$$\mathcal{L}_F = \bar{q}_0 i \not{\partial} q_0 - m_0 \bar{q}_0 q_0 \equiv \bar{q} i \not{\partial} q - m\bar{q}q + (Z_q - 1)\bar{q} i \not{\partial} q - (Z_q Z_m - 1) m \bar{q} q$$

Only renormalized quantities appear in the Lagrangian

Counter terms (~(Z-1)) are treated as interaction terms

$$i\delta_{\alpha\beta}[(Z_q - 1)\not{p} - (Z_q Z_m - 1)m]$$

- The constant Z_i are determined such that the counter terms cancel divergences in the Green functions

- Arbitrary in subtracting finite parts → renormalization scheme depen.

- The Minimal Subtraction (MS) scheme: subtract only divergences

$$Z_i = \frac{\alpha_s}{4\pi}\frac{a_{1i}}{\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{a_{2i}}{\varepsilon^2} + \frac{b_{2i}}{\varepsilon}\right) + \mathcal{O}(\alpha_s^3)$$

The renormalization constants do not have any explicit μ-dependence and depends on μ only through g_s

- The modified Minimal Subtraction ($\overline{\text{MS}}$) scheme: $\mu_{\overline{MS}} = \mu e^{\gamma_E/2}(4\pi)^{-1/2}$

$\ln 4\pi - \gamma_E$

$$\frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\left(\frac{\mu^2_{\overline{MS}}}{-p^2}\right)^{\varepsilon} = \frac{1}{16\pi^2}[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{-p^2} + \mathcal{O}(\varepsilon)]$$

MS-bar scheme

- Comparison of MS and MS-bar scheme

$$\{\mathrm{MS} \to \overline{\mathrm{MS}}\} \quad \equiv \quad \{\mu \to \mu_{\overline{MS}}\}$$
$$\{Z_i^{MS} \to Z_i^{\overline{MS}}\} \quad \equiv \quad \{\alpha_s^{MS} \to \alpha_s^{\overline{MS}}\}$$

- drop the MS-bar superscript for simplicity
- In the MS-bar scheme the renormalization constants are given by

$$Z_q = 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$
$$Z_m = 1 - \frac{\alpha_s}{4\pi} 3C_F \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$
$$Z_3 = 1 - \frac{\alpha_s}{4\pi} \left[\frac{2}{3} f - \frac{5}{3} N\right] \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$

$N = 3$ in QCD

$$Z_g = 1 - \frac{\alpha_s}{4\pi} \left[\frac{11}{6} N - \frac{2}{6} f\right] \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$

- In this scheme, the mass m(μ) and the coupling $\alpha(\mu)$ depend on the renormalization scheme while in the on-shell scheme the physical mass and coupling which are measured in the experiment are defined

Renormalization scheme

Neubert, EFT in Particle Physics and Cosmology

- in the on-shell renormalization scheme an observable is expresses as

$$O = O(\alpha, m, \ln s/m^2, \ldots)$$

Thomson limit ($q^2=0$) → α; physical pole mass → m; dependence on kinematic variables → $\ln s/m^2$

on-shell renormalization scheme might not be well defined in QCD

- in the $\overline{\text{MS}}$ scheme

$$O = O(\underbrace{\alpha(\mu), m(\mu)}_{\mu\text{-dependent}}, \ln s/\mu^2, \ldots)$$

- both results are equivalent and μ independent
- sometimes on-shell renormalization is inconvenient since it leaves large log $\ln s/m^2$ (for $s \gg m^2$), which might spoil the perturbative expansion
- choose $\mu^2 \approx s$, and expand in terms of $\alpha(\mu)$ and $m(\mu)$
- μ-independence implies

$$\mu \frac{d}{d\mu} O = \mu \frac{d\alpha(\mu)}{d\mu}\frac{\partial O}{\partial \alpha} + \mu \frac{dm(\mu)}{d\mu}\frac{\partial O}{\partial m} + \frac{\partial O}{\partial \ln \mu} = 0$$

Callan-Symanzik equation

- large logarithms in $\alpha(\mu)$ and $m(\mu)$ need to be resummed

Renormalization group equation

- The renormalized coupling constant g_s and mass m depends on an arbitrary mass scale μ

$$g_{0,s} = Z_g g_s \mu^{\varepsilon} \quad\Rightarrow\quad \frac{dg(\mu)}{d\ln\mu} = \beta(g(\mu), \varepsilon)$$

$$g \equiv g_s$$

$$m_0 = Z_m m \quad\Rightarrow\quad \frac{dm(\mu)}{d\ln\mu} = -\gamma_m(g(\mu))m(\mu)$$

$$\beta(g,\epsilon) = \frac{d}{d\ln\mu}\frac{g_0}{Z_g\mu^{\epsilon}} = \frac{g_0}{Z_g}\frac{d\mu}{d\ln\mu}\frac{d\mu^{-\epsilon}}{d\mu} - \frac{g_0}{\mu^{\epsilon}}\frac{1}{Z_g^2}\frac{dZ_g}{d\ln\mu} = -\epsilon g + \beta(g)$$

– in 4 dimensions, the beta function $\beta(g) = -g\frac{1}{Z_g}\frac{dZ_g}{d\ln\mu}$

– the anomalous dimension of the mass operator

$$\gamma_m(g) = \frac{1}{Z_m}\frac{dZ_m}{d\ln\mu}$$

– the beta function and anomalous dimension govern the μ-dependence of g(μ) and m(μ)

Renormalization group equation

- We can write $Z_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_{i,k}(g)$
- Rewrite $\beta(g,\varepsilon) = -\varepsilon g - g f(g)$

$$f(g) = \frac{\mu}{Z_g}\frac{dZ_g}{d\mu}$$

$$f(g) Z_g = \frac{dg}{d\ln\mu}\frac{dZ_g}{dg}$$

$$f(g)\left(1 + \frac{Z_{g,1}}{\varepsilon} + \frac{Z_{g,2}}{\varepsilon^2} + \ldots\right) = \frac{1}{\varepsilon}\beta(g,\varepsilon)\left(\frac{dZ_{g,1}}{dg} + \frac{1}{\varepsilon}\frac{dZ_{g,2}}{dg} + \ldots\right)$$

$$f(g) = -g\frac{dZ_{g,1}}{dg}$$

$$\Rightarrow \quad \beta(g) = -g f(g) = 2g^3\frac{dZ_{g,1}(g)}{dg^2}$$

- Similarly, one obtains $\gamma_m(g) = -2g^2\frac{dZ_{m,1}(g)}{dg^2}$

β(g) and γm(g) can be directly obtained from the $1/\epsilon$ parts of the renormalization constants

$$\alpha_s = \frac{g_s^2}{4\pi}$$

$$Z_g = 1 - \frac{\alpha_s}{4\pi}\left[\frac{11}{6}N - \frac{2}{6}f\right]\frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$

$$Z_m = 1 - \frac{\alpha_s}{4\pi}3C_F\frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$

$$\Rightarrow \quad \beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}N - \frac{2}{3}f\right]$$

$$\gamma_m(g) = \frac{g^2}{16\pi^2}6C_F$$

Running coupling constant

- We can obtain the running coupling constant

$$\frac{dg(\mu)}{d\ln\mu} = \beta(g(\mu), \varepsilon) \qquad \beta(g) = -\beta_0 \frac{g^3}{16\pi^2} \qquad \beta_0 = \frac{11N - 2f}{3}$$

$$\frac{dg(\mu)}{d\ln\mu} = \frac{dg}{dg^2}\frac{dg^2}{d\ln\mu} = \frac{1}{2g}4\pi\frac{d\alpha_s}{d\ln\mu} = \beta(g(\mu), \varepsilon) \simeq -\beta_0 \frac{g^3}{16\pi^2}$$

$$\Rightarrow \quad \boxed{\frac{d\alpha_s}{d\ln\mu} = -2\beta_0 \frac{\alpha_s^2}{4\pi}}$$

- integrate $-\frac{d\alpha_s}{\alpha_s^2} = \frac{\beta_0}{4\pi} d\ln\mu^2$

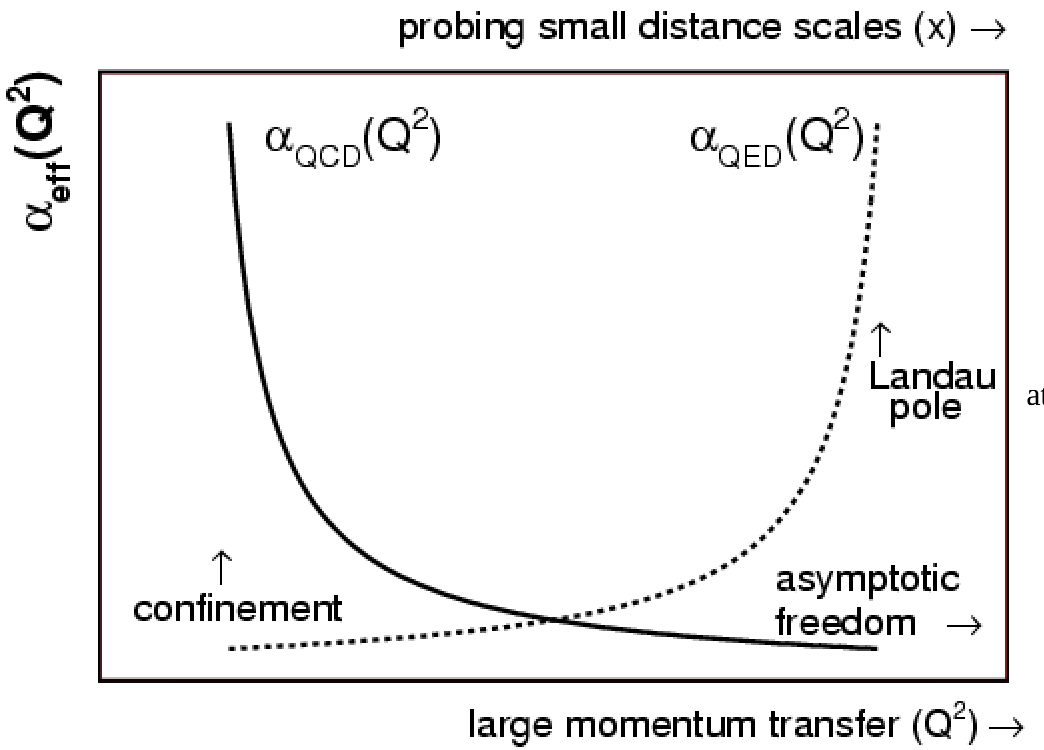
$$-\int_{\alpha_s(Q)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\alpha_s^2} = \frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(Q)} = \frac{\beta_0}{4\pi}\ln\frac{\mu^2}{Q^2}$$

$$\boxed{\alpha_s(\mu) = \frac{\alpha_s(Q)}{1 + \frac{\beta_0}{4\pi}\alpha_s(Q)\ln\frac{\mu^2}{Q^2}}}$$

Q: a reference scale

- running mass $\boxed{m(\mu) = m(\mu_0)\exp\left[-\int_{g(\mu_0)}^{g(\mu)} dg' \frac{\gamma_m(g')}{\beta(g')}\right]}$

Running coupling constant



at $\mu \approx me^{3\pi/2\alpha} \approx 10^{277}\,\text{GeV}$

QCD

$$\alpha_s(\mu) = \frac{\alpha_s(q)}{1 + \frac{\beta_0}{4\pi}\alpha_s(q)\ln\frac{\mu^2}{q^2}}$$

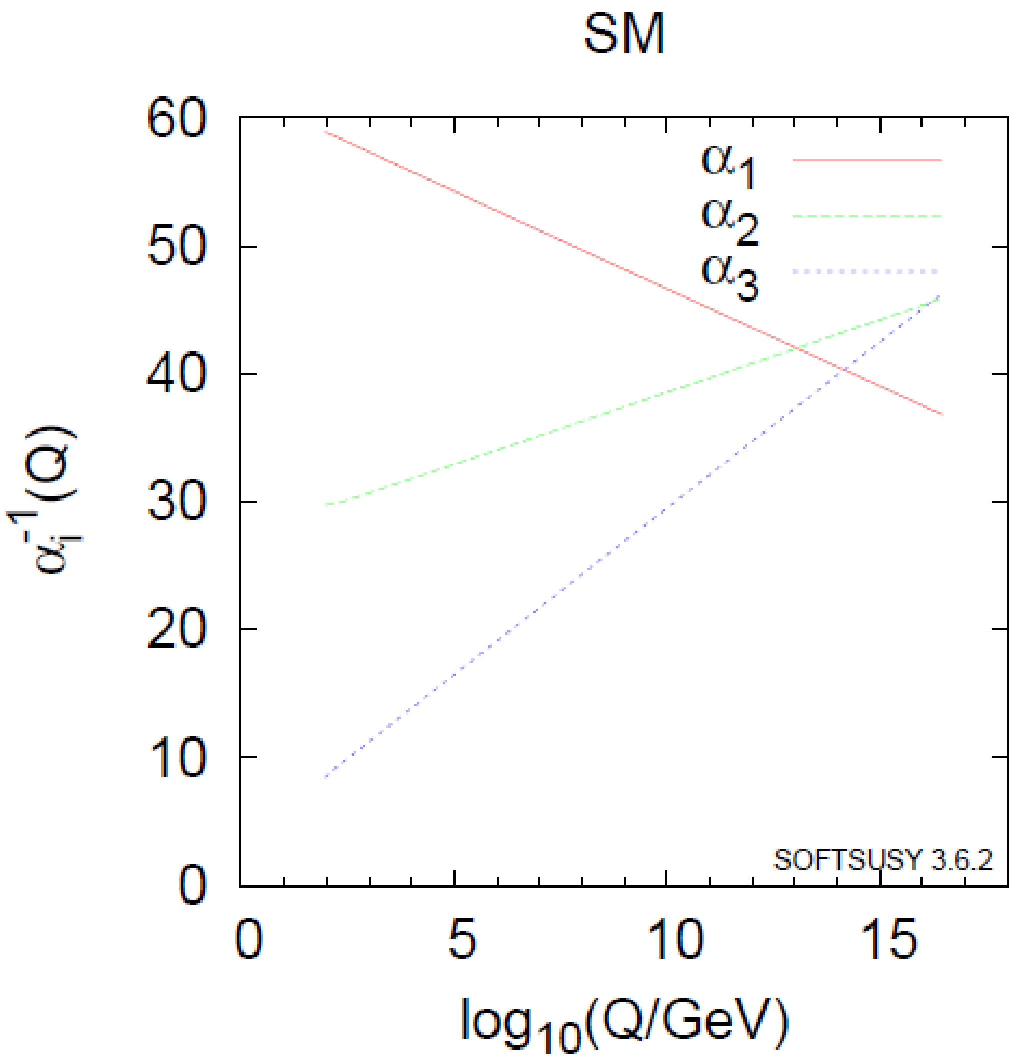
If $n_f \leq 16$ $\beta_0 = \frac{11N}{3} - \frac{2}{3}n_f > 0$

QED

$$\alpha(\mu) = \frac{\alpha}{1 + \frac{\beta_0}{4\pi}\alpha\ln\frac{\mu^2}{m^2}}$$

$\beta_0 = -\frac{4}{3} < 0$

Evolution of coupling constants



Quark mass

- The quarks (except the top quark) are confined in hadrons and their masses have to be inferred indirectly from properties of hadrons
  - The lattice QCD provides the most reliable determination for light quarks (u,d,s).
  - The masses of c and b can be determined by charmonium and upsilon spectroscopy, in conjunction with lattice calculation
  - The mass of the top quark is precisely determined by the top quark pair production at the LHC by comparing it with various Monte Carlo simulations

- The measured mass for a scale can be extrapolated to another scale by a renormalization group evolution coming from QCD loops
  - the pole mass: the physical mass which appears as the pole in the propagator
  - the running mass: defined in the MS-bar scheme and includes corrections from QCD loops

$$m_Q = \overline{m}_Q(\overline{m}_Q)\left\{1 + \frac{4\overline{\alpha}_s(\overline{m}_Q)}{3\pi} + \left[-1.0414\sum_k\left(1 - \frac{4}{3}\frac{\overline{m}_{Q_k}}{\overline{m}_Q}\right) + 13.4434\right]\left[\frac{\overline{\alpha}_s(\overline{m}_Q)}{\pi}\right]^2 + \left[0.6527N_L^2 - 26.655N_L + 190.595\right]\left[\frac{\overline{\alpha}_s(\overline{m}_Q)}{\pi}\right]^3\right\}$$

Quark mass

assuming MSSM

$m_i \diagdown \mu$	$m_c(m_c)$	2 GeV	$m_b(m_b)$	$m_t(m_t)$	1 TeV	$\Lambda_{\rm GUT}^{\tan\beta=10}$	$\Lambda_{\rm GUT}^{\tan\beta=50}$
m_u(MeV)	2.57	**2.2**	1.86	1.22	1.10	0.49	0.48
m_d(MeV)	5.85	**5.0**	4.22	2.76	2.50	0.70	0.51
m_s(MeV)	111	**95**	80	52	47	13	10
m_c(GeV)	**1.25**	1.07	0.901	0.590	0.532	0.236	0.237
m_b(GeV)	5.99	5.05	**4.20**	2.75	2.43	0.79	0.61
m_t(GeV)	*384.8*	*318.4*	*259.8*	**162.9**	150.7	92.2	94.7
m_e(MeV)	0.4955	~	0.4931	0.4853	0.4959	0.2838	0.206
m_μ(MeV)	104.474	~	103.995	102.467	104.688	59.903	43.502
m_τ(MeV)	1774.90	~	1767.08	1742.15	1779.74	1021.95	773.44

– The bold characters are inputs

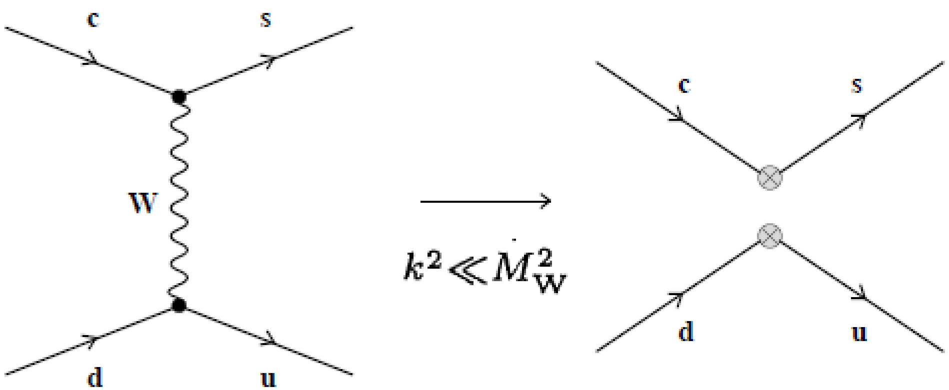
– The light quark masses decrease by about a factor of two form 2 GeV to 1 TeV

– The top quark mass below m_t is unphysical because the top quark will be integrated out at the scale

– The running effect in the lepton mass is very small $m_\ell(\mu) = M_\ell\left[1 - \frac{\alpha}{\pi}\left\{1 + \frac{3}{2}\ln\frac{\mu}{m_\ell(\mu)}\right\}\right]$

Effective field theory

- At low energies, the exchange of heavy virtual particles (M≫E) leads to quasi-local effective interactions



$$(\bar{s}c)_{V-A} \equiv \bar{s}\gamma_\mu(1-\gamma_5)c$$

$$A = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \frac{M_W^2}{k^2 - M_W^2} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A} + \mathcal{O}(\frac{k^2}{M_W^2})$$

- The same amplitude can be obtained from the effective Hamiltonian

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A}$$

- Operator product expansion (OPE)

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \rightarrow \sum_n C_{12}{}^n(x)\mathcal{O}_n(0)$$

Effective field theory

- Basic idea of effective field theory
  - choose cutoff Λ ~ typically the mass scale of heavy fields
  - heavy fields are integrated out at the scale
  - expand non-local operators in terms of $1/\Lambda$
  - OPE series is equivalent to the original theory if summed to all orders
  - truncate the series neglecting higher-order contributions

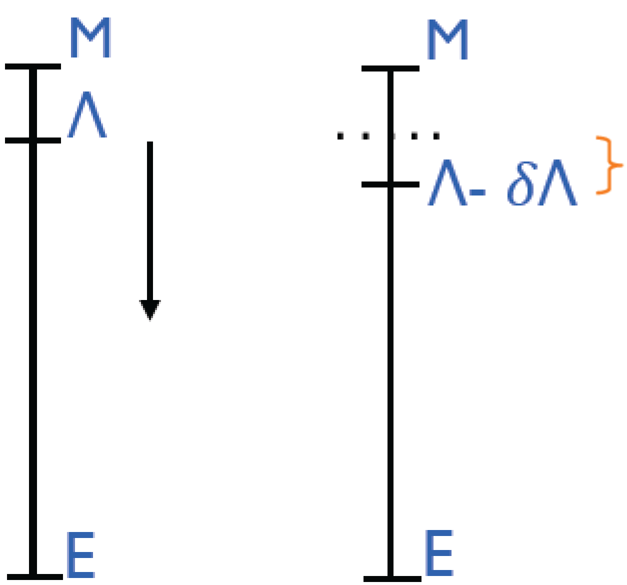
Effective field theory

- separation of short- and long-distance physics

$$\langle f|\mathcal{H}_{\text{eff}}|i\rangle = \sum_k \frac{1}{\Lambda^k} \sum_i c_{k,i}(\Lambda/\mu)\ \langle f|\mathcal{O}_{k,i}|i\rangle\Big|_{\mu}$$

– the Wilson coefficients $c_{k,i}$ are determined by matching the effective theory to the full theory at the scale μ=Λ

– lowering the cutoff Λ



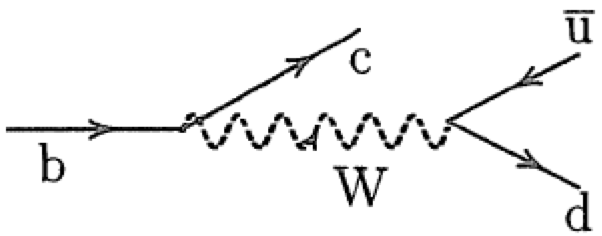
integrate out high frequency of light particle
(as opposed to integrating out heavy particle)

– the operators remain the same

– lowering Λ is absorbed into change in the coefficients

– the effective Hamiltonian at the desired scale E by RG running of the coefficients

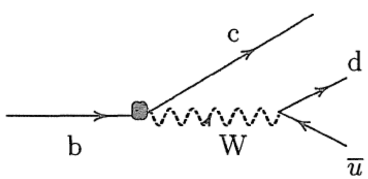
$b \to c d \overline{u}$



$$i \left(\frac{-ig}{2\sqrt{2}} \right)^2 V_{cb} V_{ud}^* \frac{-i}{k^2 - M_W^2} (H_\mu^{bc}) (H_{ud}^{\mu\dagger}) \underset{k^2 \ll M_W^2}{\longrightarrow} \frac{G_F}{\sqrt{2}} V_{cb} V_{ud}^* \mathcal{O}_A$$

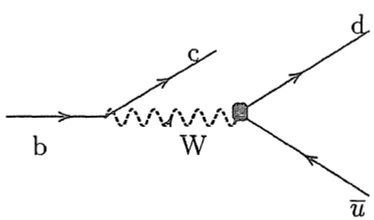
$$\mathcal{O}_A = (H_\mu^{bc}) (H_{ud}^{\mu\dagger}) = [\overline{c}\, \gamma_\mu (1 - \gamma_5)\, b]\, [\overline{d}\, \gamma^\mu (1 - \gamma_5)\, u]$$

- QCD corrections (for massless c,u,d)



5 diagrams for
virtual+real corrections

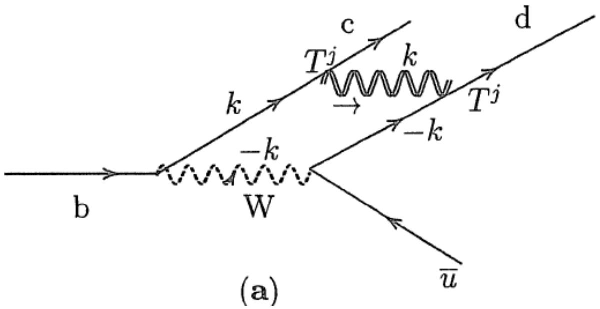
$$\frac{-2\alpha_s}{3\pi} \left(\pi^2 - \frac{25}{4} \right) + \frac{\alpha_s}{\pi} = \frac{-2\alpha_s}{3\pi} \left(\pi^2 - \frac{31}{4} \right) = -1.41 \frac{\alpha_s}{\pi}$$

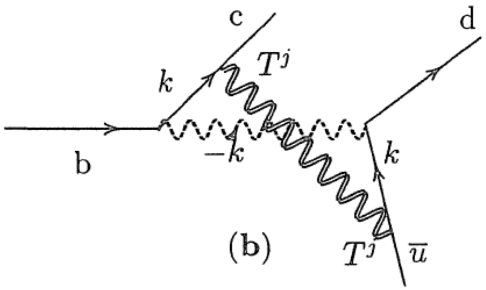


5 diagrams for
virtual+real corrections

$$\frac{+\alpha_s}{\pi}$$

$b \to c d \bar{u}$





+ 2 diagrams

$$I_{c\leftrightarrow d} = i\left(\frac{-ig}{2\sqrt{2}}\right)^2 (-ig_s)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \frac{-i}{k^2 - M_W^2} \left[\bar{d}\, T^j\, \gamma^\lambda \frac{i(-\not{k} + m_d)}{k^2 - m_d^2} \gamma^\mu (1-\gamma_5)\, u\right]$$

$$\times \left[\bar{c}\, T^j\, \gamma_\lambda \frac{i(\not{k} + m_c)}{k^2 - m_c^2} \gamma_\mu (1-\gamma_5)\, b\right]$$

– the external momenta are taken to be zero, but k^2 in the W propagator cannot be ignored

– neglect m_d and CKM factors are suppressed

$$T^j = \frac{1}{2}\lambda^j$$

$$g^2 = \frac{8M_W^2 G_F}{\sqrt{2}}$$

$$I_{c\leftrightarrow d} = \frac{-i\, G_F\, g_s^2}{\sqrt{2}} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{M_W^2}{k^2 - M_W^2} \frac{1}{k^2 - m_c^2} \times \frac{1}{4}$$

$$\times \sum_{j=1}^{8} [\bar{d}\ T^j\ \gamma^\lambda \gamma^\rho \gamma^\mu (1-\gamma_5)\, u]\ [\bar{c}\ T^j\ \gamma_\lambda \gamma_\rho \gamma_\mu (1-\gamma_5)\, b]$$

the m_c term in the numerator vanish because the numerator is proportional to k

Leading Logarithmic Approximation

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{M_W^2}{k^2-M_W^2}\frac{1}{k^2-m_c^2} = 2M_W^2\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-xM_W^2-ym_c^2]^3}$$
$$= M_W^2\int_0^1 dx\int_0^{1-x} dy\left(-\frac{i}{16\pi^2}\right)\frac{1}{xM_W^2+ym_c^2}$$
$$= \left(-\frac{iM_W^2}{16\pi^2}\right)\int_0^1 ds\int_0^1 dt\frac{1}{t(M_W^2-m_c^2)+m_c^2}$$
$$= -\frac{i}{16\pi^2}\frac{M_W^2}{M_W^2-m_c^2}\log\frac{M_W^2}{m_c^2} \simeq -\frac{i}{16\pi^2}\log\frac{M_W^2}{m_c^2}$$

large logarithm

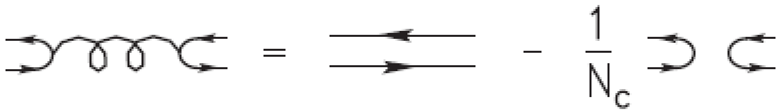
- if external momenta are not ignored, mc must be replaced by external momenta which are dominated by the b quark mass
- the perturbative correction ~ $\alpha_s \log(M_W^2/m_Q^2)$ ~ O(1)
- one cannot ignore higher-order corrections $[\alpha_s \log(M_W^2/m_Q^2)]^n$
- these α_s^n corrections correspond to multiple hard gluons exchanged between the two vertices
- sum over the leading logarithmic terms to all orders of α_s by using the renormalization group equation

⇒ leading logarithmic approximation

$b \to c d \bar{u}$

- by making use of the relations

$$T^a_{\alpha\beta} T^a_{\gamma\rho} = -\frac{1}{2N}\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{2}\delta_{\alpha\delta}\delta_{\gamma\beta}$$



$$[\bar{d}\gamma^\lambda\gamma^\rho\gamma^\mu(1-\gamma_5)u][\bar{c}\gamma_\lambda\gamma_\rho\gamma_\mu(1-\gamma_5)b] = 16[\bar{d}\gamma^\rho(1-\gamma_5)u][\bar{c}\gamma_\rho(1-\gamma_5)b]$$

$$\begin{aligned}
&[\bar{d}_\alpha T^j_{\alpha\beta}\gamma^\lambda\gamma^\rho\gamma^\mu(1-\gamma_5)u_\beta][\bar{c}_\gamma T^j_{\gamma\delta}\gamma_\lambda\gamma_\rho\gamma_\mu(1-\gamma_5)b_\delta] \\
&= -\frac{1}{2N}[\bar{d}_\alpha\gamma^\lambda\gamma^\rho\gamma^\mu(1-\gamma_5)u_\alpha][\bar{c}_\beta\gamma_\lambda\gamma_\rho\gamma_\mu(1-\gamma_5)b_\beta] + \frac{1}{2}[\bar{d}_\alpha\gamma^\lambda\gamma^\rho\gamma^\mu(1-\gamma_5)u_\beta][\bar{c}_\beta\gamma_\lambda\gamma_\rho\gamma_\mu(1-\gamma_5)b_\alpha] \\
&= 4\left(-\frac{2}{N}[\bar{d}_\alpha\gamma^\mu(1-\gamma_5)u_\alpha][\bar{c}_\beta\gamma_\mu(1-\gamma_5)b_\beta] + 2[\bar{d}_\alpha\gamma^\mu(1-\gamma_5)u_\beta][\bar{c}_\beta\gamma_\mu(1-\gamma_5)b_\alpha]\right) \\
&= 4\left(-\frac{2}{N}\mathcal{O}_A + 2\mathcal{O}_B\right)
\end{aligned}$$

$$\mathcal{O}_A = [\bar{d}\gamma^\mu(1-\gamma_5)u][\bar{c}\gamma_\mu(1-\gamma_5)b]$$

$$\mathcal{O}_B = [\bar{d}\gamma^\mu(1-\gamma_5)b][\bar{c}\gamma_\mu(1-\gamma_5)u]$$

Fierz rearrangement
and an extra (-) sign
for fermion exchange

$b \to c d \bar{u}$

$$I_{\mathrm{c}\leftrightarrow\mathrm{d}} = \frac{G_{\mathrm{F}}}{\sqrt{2}} \left(\frac{\alpha_{\mathrm{s}}}{4\pi} \log \frac{M_{\mathrm{W}}^2}{M^2} \right) \left(\frac{2}{3} \mathcal{O}_{\mathrm{A}} - 2\mathcal{O}_{\mathrm{B}} \right)$$

$$I_{\mathrm{c}\leftrightarrow\bar{\mathrm{u}}} = -\tfrac{1}{4}\, I_{\mathrm{c}\leftrightarrow\mathrm{d}}$$

$(+\not{k} \leftrightarrow -\not{k})$ in the propagator

and the order of gamma matrices

$$I_{\mathrm{c}\leftrightarrow\mathrm{d}} + I_{\mathrm{c}\leftrightarrow\bar{\mathrm{u}}} = \frac{3}{4} I_{\mathrm{c}\leftrightarrow\mathrm{d}} = \frac{G_{\mathrm{F}}}{\sqrt{2}} \left(\frac{\alpha_{\mathrm{s}}}{4\pi} \log \frac{M_{\mathrm{W}}^2}{M^2} \right) \left(\frac{1}{2} \mathcal{O}_{\mathrm{A}} - \frac{3}{2} \mathcal{O}_{\mathrm{B}} \right)$$

$$I_{\mathrm{b}\leftrightarrow\bar{\mathrm{u}}} = I_{\mathrm{c}\leftrightarrow\mathrm{d}}$$

$$I_{\mathrm{b}\leftrightarrow\mathrm{d}} = I_{\mathrm{c}\leftrightarrow\bar{\mathrm{u}}}$$

- the total contribution from gluon exchanges between two lines

Bare Electroweak		Renormalization by QCD
$\frac{G_{\mathrm{F}}}{\sqrt{2}} \mathcal{O}_{\mathrm{A}}$	$\Longrightarrow$	$\frac{G_{\mathrm{F}}}{\sqrt{2}} \left(\frac{\alpha_{\mathrm{s}}}{4\pi} \log \frac{M_{\mathrm{W}}^2}{M^2} \right) [\mathcal{O}_{\mathrm{A}} - 3\mathcal{O}_{\mathrm{B}}]$

a new operator O_B appears

a logarithmic enhancement

$$\mathcal{O}_{\mathrm{A}} \Longrightarrow c_{\mathrm{A}} \mathcal{O}_{\mathrm{A}} = \left(1 + \frac{\alpha_{\mathrm{s}}}{4\pi} \log \frac{M_{\mathrm{W}}^2}{M^2} \right) \mathcal{O}_{\mathrm{A}} \qquad 0 \Longrightarrow c_{\mathrm{B}} \mathcal{O}_{\mathrm{B}} = -3 \left(\frac{\alpha_{\mathrm{s}}}{4\pi} \log \frac{M_{\mathrm{W}}^2}{M^2} \right) \mathcal{O}_{\mathrm{B}}$$

Renormalization group improvement

- the QCD effects on the two operators

$$\mathcal{O}_{\mathrm{A}} \Longrightarrow \mathcal{O}_{\mathrm{A}} + \left[\frac{\alpha_s}{2\pi} \log\left(\frac{M_{\mathrm{W}}}{\mu}\right)\right] [\mathcal{O}_{\mathrm{A}} - 3\,\mathcal{O}_{\mathrm{B}}]$$

$$\mathcal{O}_{\mathrm{B}} \Longrightarrow \mathcal{O}_{\mathrm{B}} + \left[\frac{\alpha_s}{2\pi} \log\left(\frac{M_{\mathrm{W}}}{\mu}\right)\right] [\mathcal{O}_{\mathrm{B}} - 3\,\mathcal{O}_{\mathrm{A}}]$$

there is always a mixing between two operators

- unmixed (bare) operators

$$\mathcal{O}_{\pm} = \tfrac{1}{2}\,[\mathcal{O}_{\mathrm{A}} \pm \mathcal{O}_{\mathrm{B}}]$$

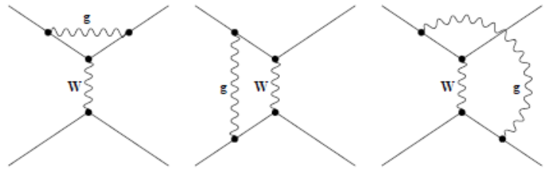
- the bare operators are multiplicatively renormalized by QCD

$$\mathcal{O}_{\pm} \Longrightarrow c_{\pm}\,\mathcal{O}_{\pm} \qquad c_{\pm} = 1 + d_{\pm}\left[\frac{\alpha_s}{\pi} \log\left(\frac{M_{\mathrm{W}}}{\mu}\right)\right] \qquad \begin{aligned} d_+ &= -1 \\ d_- &= +2 \end{aligned}$$

- need to sum over all $\sum_n [\alpha_s \log(M_{\mathrm{W}}/\mu)]^n$ terms with the renormailzation group equation

Full theory vs effective theory

- the full amplitude



$$S_1 \equiv \mathcal{O}_{\rm B} = (\bar{c}_\alpha b_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A}$$

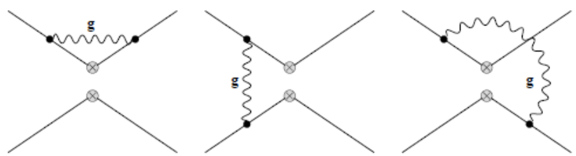
$$S_2 \equiv \mathcal{O}_{\rm A} = (\bar{c}_\alpha b_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}$$

$$m_i = 0,\ p^2 < 0$$

$$A_{full} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \Big[\left(1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) \right) S_2 + \frac{3}{N} \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{-p^2} S_2 - 3 \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{-p^2} S_1 \Big]$$

– all external quark lines are massless and carry off-shell momentum p

- the unrenormalized current-current matrix element



$$\langle Q_1 \rangle^{(0)} = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) \right) S_1 + \frac{3}{N} \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_1 - 3 \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_2$$

$$\langle Q_2 \rangle^{(0)} = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) \right) S_2 + \frac{3}{N} \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_2 - 3 \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_1$$

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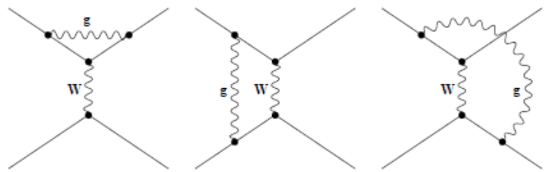

$S_1 \equiv \langle Q_1 \rangle_{tree} = (\bar{s}_\alpha c_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A}$

$S_2 \equiv \langle Q_2 \rangle_{tree} = (\bar{s}_\alpha c_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}$

$m_i = 0,\ p^2 < 0$

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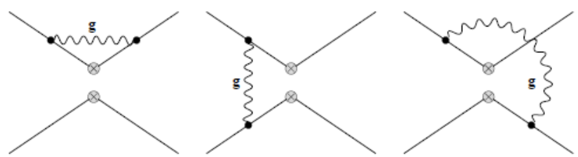

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cancel by the quark field renormalization

$$-\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{k^2 - m_c^2} = \frac{-i}{16\pi^2} \frac{\Gamma(2 - \frac{n}{2})}{\mu^{4-n}}$$

$M_W \to \infty$

Operator Renormalization

- the operator renormalization is necessary to remove additional divergences

$$\langle Q_i \rangle^{(0)} = Z_q^{-2} Z_{ij} \langle Q_j \rangle$$

$$\hat{Z} = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\begin{pmatrix} 3/N & -3 \\ -3 & 3/N \end{pmatrix}$$

a 2×2 matrix $\hat{Z}$

- the renormalized matrix element

$$\langle Q_1 \rangle = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2}\right) S_1 + \frac{3}{N}\frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} S_1 - 3\frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} S_2$$
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- the matching of the full theory onto the effective theory

$$A_{full} = A_{eff} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_1 \langle Q_1 \rangle + C_2 \langle Q_2 \rangle)$$

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$$\langle Q_2\rangle = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \ln\frac{\mu^2}{-p^2}\right) S_2 + \frac{3}{N}\frac{\alpha_s}{4\pi} \ln\frac{\mu^2}{-p^2} S_2 - 3\frac{\alpha_s}{4\pi} \ln\frac{\mu^2}{-p^2} S_1$$

- the matching of the full theory onto the effective theory

$$A_{full} = A_{eff} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_1\langle Q_1\rangle + C_2\langle Q_2\rangle)$$

$$C_1(\mu) = -3\frac{\alpha_s}{4\pi} \ln\frac{M_W^2}{\mu^2} \qquad C_2(\mu) = 1 + \frac{3}{N}\frac{\alpha_s}{4\pi} \ln\frac{M_W^2}{\mu^2}$$

valid only for $\mu \sim O(M_W)$

For $\mu \ll M_W$, sum large logarithms to all orders

Renormalization Group Equation

- operator mixing and diagonalization

$$Q_\pm = \frac{Q_2 \pm Q_1}{2} \qquad C_\pm = C_2 \pm C_1$$

$$Q_\pm^{(0)} = Z_\pm Q_\pm$$

unrenormalized renormalized

$$Z_\pm = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\left(\mp 3\frac{N \mp 1}{N}\right)$$

- in the new basis

$$A \equiv A_+ + A_- = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_+(\mu)\langle Q_+(\mu)\rangle + C_-(\mu)\langle Q_-(\mu)\rangle)$$

$$A_\pm = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud}\left[\left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right)S_\pm + \left(\frac{3}{N} \mp 3\right)\frac{\alpha_s}{4\pi}\ln\frac{M_W^2}{-p^2}S_\pm\right]$$

$$\langle Q_\pm(\mu)\rangle = \left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right)S_\pm + \left(\frac{3}{N} \mp 3\right)\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}S_\pm$$

$$C_\pm(\mu) = 1 + \left(\frac{3}{N} \mp 3\right)\frac{\alpha_s}{4\pi}\ln\frac{M_W^2}{\mu^2}$$

$$S_\pm = (S_2 \pm S_1)/2$$

Renormalization Group Equation

- unrenormalized Wilson coefficients $C_i^{(0)}$ do not depend on μ

$$C_\pm = Z_\pm C_\pm^{(0)} \qquad Q_\pm^{(0)} = Z_\pm Q_\pm$$

$$Z_\pm = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\left(\mp 3\frac{N \mp 1}{N}\right)$$

- the anomalous dimension $\gamma_\pm$ of the operator $Q_\pm$

$$\frac{dC_\pm(\mu)}{d\ln\mu} = \gamma_\pm(g) C_\pm(\mu)$$

$$\gamma_\pm(g) = \frac{1}{Z_\pm}\frac{dZ_\pm}{d\ln\mu}$$

- in the MS(-bar) scheme

$$Z_\pm = 1 + \sum_{k=1}^{\infty}\frac{1}{\varepsilon^k} Z_{\pm,k}(g) \;= 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\left(\mp 3\frac{N \mp 1}{N}\right)$$

$$\gamma_\pm(g) = -2g^2\frac{\partial Z_{\pm,1}(g)}{\partial g^2}$$

$$\gamma_\pm(\alpha_s) = \frac{\alpha_s}{4\pi}\gamma_\pm^{(0)} \qquad \gamma_\pm^{(0)} = \pm 6\frac{N \mp 1}{N}$$

Renormalization Group Equation

- solve the RG equation $\dfrac{dC_\pm(\mu)}{d\ln\mu} = \gamma_\pm(g)C_\pm(\mu)$

$$\frac{dC_\pm}{C_\pm} = \gamma_\pm \frac{d\ln\mu}{dg}dg = \frac{\gamma_\pm(g)}{\beta(g)}dg$$

$$C_\pm(\mu) = U_\pm(\mu,\mu_W)C_\pm(\mu_W)$$

$$\beta(g) = -\beta_0 g^3/16\pi^2$$

$$U_\pm(\mu,\mu_W) = \exp\left[\int_{g(\mu_W)}^{g(\mu)} dg' \frac{\gamma_\pm(g')}{\beta(g')}\right]$$

$$= \exp^{\int -\frac{\gamma_\pm^{(0)}}{\beta_0}\frac{1}{g'}dg'} = \left(\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)}\right)^{\frac{\gamma_\pm^{(0)}}{2\beta_0}}$$

$$C_\pm(\mu) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{\frac{\gamma_\pm^{(0)}}{2\beta_0}} C_\pm(M_W)$$

- at the LO $C_\pm(M_W) = 1$

$$C_\pm(\mu_b) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma_\pm^{(0)}}{2\beta_0}}$$

$$C_+(\mu_b) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{6}{23}} \qquad C_-(\mu_b) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{-12}{23}} \qquad f = 5$$

$$C_+(\mu_b) = 0.847 \qquad C_-(\mu_b) = 1.395$$

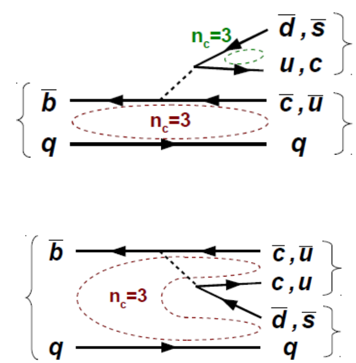
Tree and Penguin operators

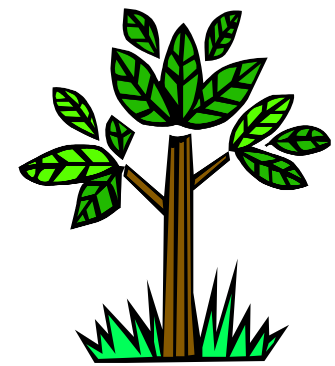
- tree operators

$$Q_1 = (\bar{b}_\alpha c_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A} \qquad Q_2 = (\bar{b}_\alpha c_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}$$

$$Q_1 = (\bar{s}_\alpha c_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A} \qquad Q_2 = (\bar{s}_\alpha c_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}$$

$$Q_1 = (\bar{s}_\alpha u_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A} \qquad Q_2 = (\bar{s}_\alpha u_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}$$





- penguin operators

$$Q_3 = (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}q)_{V-A} \qquad Q_4 = (\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_\beta q_\alpha)_{V-A}$$

$$Q_5 = (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}q)_{V+A} \qquad Q_6 = (\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_\beta q_\alpha)_{V+A}$$

$$Q_7 = \frac{3}{2}\,(\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} e_q\,(\bar{q}q)_{V+A}$$

$$Q_8 = \frac{3}{2}\,(\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} e_q(\bar{q}_\beta q_\alpha)_{V+A}$$

$$Q_9 = \frac{3}{2}\,(\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} e_q(\bar{q}q)_{V-A}$$

$$Q_{10} = \frac{3}{2}\,(\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} e_q\,(\bar{q}_\beta q_\alpha)_{V-A}$$