Gauge Invariance

- Theories where the interaction is determined are called "gauge theories"

1) Gauge Invariance in Classical Electromagnetism

$$\vec{B} = \vec{\nabla} \times \vec{A} , \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad -(*)$$

- If the transformations $\begin{cases} \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} \chi \\ V \to V' = V - \frac{\partial \chi}{\partial t} \end{cases}$ are carried ou

  $(*)$ are unchanged.
- Combining $\vec{A}$ & V into a 4-vector, $A^\mu = (V ; \vec{A})$,

  the above transformations become $A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi$

2) Gauge Invariance in Quantum Theory

- Since observables depend on $|\psi|^2$, we can demand that the structure of the theory be invariant under

$$\psi \to \psi' = e^{-i\alpha} \psi$$

  $\to$ global gauge transformation.

  (it should be possible to choose the phase of ψ in an arbitrary way.)

- It should also be possible to choose the phase of ψ at each space-time point without affecting the theory

$$\psi(x) \to \psi'(x) = e^{-i\chi(x)} \psi(x)$$

$\Rightarrow$ local gauge transformatio

Consider a matter particle described by a wave ftn. ψ, which satisfi

$$-\frac{1}{2m}\nabla^2\psi(\vec{x},t) = i\frac{\partial\psi(\vec{x},t)}{\partial t} \quad \Rightarrow \text{Schrödinger eq.}$$

$\Rightarrow$ it is not invariant under the local gauge transformation

To satisfy the local gauge trasformation,

we can modify the Schröding eq. by including e-m field,

$$\Rightarrow \quad \frac{1}{2m}\left(-i\nabla + e\vec{A}\right)^2\psi = \left(i\frac{\partial}{\partial t} + eV\right)\psi$$

then, the eq. is invariant under $\begin{cases} \psi(\vec{x},t) \to \psi'(\vec{x},t) = e^{-i\chi(\vec{x},t)}\psi(\\ \vec{A} \to \vec{A}' = \vec{A} + \frac{1}{e}\vec{\nabla}\chi \\ V \to V' = V - \frac{1}{e}\frac{\partial\chi}{\partial t} \end{cases}$

$\Rightarrow$ we can reinterpret this result to say that the local phase invariance of the theory requires the presence of a field $A^\mu = (V$

" phase invariance of the theory for electrically charged partic requires that there must be a photon & an e-m interacti

" we can say that the effects of a local change in phase convention is equivalent to those of a new vector field.

(3) Covariant Derivatives

- By rewritting the above eqs. it is possible to put them in a nice form which makes their properties explicit & which is easily generalized to the gauge invariant theories

(ex. $\vec{D} = -\vec{\nabla} - ie\vec{A}$, $D^0 = \frac{\partial}{\partial t} - ieV$: $\frac{1}{2m}(i\vec{D})^2\psi = i$

- Define

· In general, for some particles with some other non-em charge, we can rewrite the trasformation as follows :

Suppose that the theory be invariant under a trasformation

$$\psi' = U\psi \qquad \text{for some } U$$

For the covariant derivative : we want to define $D^\mu = \partial^\mu - igA^\mu$

(※ A^μ : represents the interacting field that has to be added to keep the theory invariant under the trasformation) but we don't know how A^μ itself trasforms.

$$\Rightarrow \quad D^{\mu\prime}\psi' = (\partial^\mu - igA^{\mu\prime})U\psi \underset{\text{we require}}{\rightarrow} U(D^\mu\psi) = U(\partial^\mu - igA^\mu)$$

$$\Rightarrow \quad -igA^{\mu\prime}U = -\partial^\mu(U\psi) + U\partial^\mu\psi - igUA^\mu\psi$$

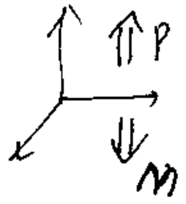
$$= -(\partial^\mu U)\psi - igUA^\mu\psi$$

$$\therefore) \quad A^{\mu\prime} = -\frac{i}{g}(\partial^\mu U)U^{-1} + UA^\mu U^{-1}$$

Non-Abelian Gauge Theories

- Similarity between n & p in strong interaction has led to the idea that we could think of n, p as 2 states of the same nucleon N.

- "strong isospin space" where the nucleon state points in some direc



To construct the theory that describes nucleon interac im is invariant under rotations in strong isospin spa

- Since there are 2 nucleon states it is like spin-up & spin-dow so we try putting p & n as states of a spin-like doublet

  $\rightarrow$ $SU(2)$ doublet : $N = \begin{pmatrix} p \\ n \end{pmatrix}$

Likewise, we can interprete the pion as an isospin-one stat

$$\pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \rightarrow \begin{cases} \pi^\pm = \frac{1}{\sqrt{2}}(\mp\pi_1 \pm i\pi_2) \\ \pi^0 = \pi_3 \end{cases}$$

- Let's consider the interactions of pion & nucleon.

  $\rightarrow$ the most general Lagrangian : $\mathcal{L}_{int} = g_{pn} p^\dagger n \pi^- + g_{np} n^\dagger p \pi^+ + g_{pp} p^\dagger p \pi^0$
  $+ g_{nn} n^\dagger n \pi^0$

  $\rightarrow$ but this form is not invariant under rotation in isospin spa unless certain relations hold among g's

$N \rightarrow N' = e^{i\frac{\vec{\tau}}{2}\cdot}$

- $\mathcal{L}_{int} = g (N^\dagger \vec{\tau} N) \cdot \vec{\pi}$ $\rightarrow$ invariant under rotations.

Now, let's combine the ideas of internal spaces & of phase invariance

For an SU(2) doublet, the phase transformation is given by

$$\begin{pmatrix} p' \\ n' \end{pmatrix} = e^{i\vec{\epsilon}\cdot\vec{\tau}/2} \begin{pmatrix} p \\ n \end{pmatrix} \qquad (\vec{\tau} : \text{pauli matrix})$$

※ the successive transformation matters since the rotations do not comm

$$[\tau_i, \tau_j] = 2i\,\epsilon_{ijk}\,\tau_k$$

Note that instead of $\chi(\vec{x},t)$, we have $\vec{\epsilon}\cdot\vec{\tau}$

To define D^μ, it is necessary for SU(2) case to introduce a set of 3 fields, each of which behaves as a 4-vector under Lorentz transformation, in order that we can write a term that transforms as ∂^μ does.

Define $\quad D^\mu = \partial^\mu - i g_2 \frac{\vec{\tau}}{2}\cdot\vec{W^\mu}$

$$D'^\mu\psi' = e^{i\vec{\epsilon}\cdot\vec{\tau}/2} D^\mu\psi \qquad \begin{pmatrix} \psi \to \psi' = e^{i\vec{\epsilon}\cdot\vec{\tau}/2}\psi \\ \vec{W}^\mu \to \vec{W}'^\mu = \vec{W}_i^\mu + \delta\vec{W}^\mu \end{pmatrix}$$

. As usual, let the transformation be infinitesimal so higher terms in ϵ can be neglected

$$D'^\mu\psi' \simeq (\partial^\mu - i g_2 \tau_i W_i'^\mu/2)(1 + i\epsilon_j\tau_j/2)\psi$$

$$= (\partial^\mu - i g_2 \tau_i W_i^\mu/2 - i g_2 \tau_i \delta W_i^\mu/2)(1 + i\epsilon_j\tau_j/2)\psi$$

$$= (\partial^\mu - i g_2 \tau_i W_i^\mu/2 - i g_2 \tau_i \delta W_i^\mu/2 + i\tau_j(\partial^\mu\epsilon_j)/2$$

$$+ g_2 \tau_i W_i^\mu \epsilon_j \tau_j/4)\psi + i\tau_j\epsilon_j(\partial^\mu\psi)/2$$

$$e^{i\vec{\epsilon}\cdot\vec{\tau}/2} D^\mu\psi \simeq (1 + i\epsilon_j\tau_j/2)(\partial^\mu - i g_2 \tau_i W_i^\mu/2)\psi$$

$$\tau_i \delta W_i^\mu = \frac{1}{g_2}(\partial^\mu \epsilon_i)\tau_i + \frac{i}{2}\epsilon_i W_j^\mu \underbrace{[\tau_i\tau_j - \tau_j\tau_i]}_{= 2i\epsilon_{ijk}\tau_k}$$

$$= \frac{1}{g_2}(\partial^\mu \epsilon_i)\tau_i - \underbrace{\epsilon_{ijk}\epsilon_i W_j^\mu \tau_k}_{\to\ \epsilon_{ijk}\epsilon_j W_k^\mu \tau_i}$$

$$\Rightarrow \quad \delta W_i^\mu = \frac{1}{g_2}\partial^\mu \epsilon_i - \epsilon_{ijk}\epsilon_j W_k$$

※ the 2nd term is familiar from classical mechanics as an example of how a vector transforms under rotations

- Let's consider how W transforms under an isospin rotation

$$\vec{W}' = e^{i\vec{\epsilon}\cdot\vec{T}}\vec{W}$$

where $\vec{T}$ is the appropriate representation of the rotation matric for spin-one : $(T_i)_{jk} = -i\epsilon_{ijk}$

- For an infinitesimal transf. $W_i' \simeq (1+i\epsilon_k T_k)_{ij} W_j \equiv W_i + \delta$

$$\rightarrow \delta W_i = i\epsilon_k(-i\epsilon_{kij})W_j$$

$$= \epsilon_{ijk}\epsilon_k W_j$$

- more generally, the covariant derivatives can be rewritten as

$$D^\mu = \partial^\mu - ig_2\vec{T}\cdot\vec{W}_\mu$$

For $SU(2)$ and spin-$\frac{1}{2}$ $\Rightarrow \vec{T} = \frac{\vec{\tau}}{2}$

For an $SU(n)$ with group generators $\vec{F}$,

$$D^\mu = \partial^\mu - ig_n\vec{F}\cdot\vec{G}^\mu, \quad [F_i, F_j] = i\epsilon_{ijk}F_k$$

Summary of free particle Lagrangians

a) Real spin -0 field of mass m (scalar or pseudoscalar)

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi\partial^\mu\phi) - m^2\phi^2\right]$$

$\rightarrow$ satisfies $\quad (\partial^\mu\partial_\mu + m^2)\phi = 0$

(b) complex scalar field of mass m

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)(\partial^\mu\phi_1) - m^2\phi_1^2\right] + \frac{1}{2}\left[(\partial_\mu\phi_2)(\partial^\mu\phi_2) - m^2\phi_2^2\right]$$

$$= (\partial^\mu\phi)^*(\partial_\mu\phi) - m^2\phi^*\phi \quad \left(\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\right)$$

(c) Spin $-\frac{1}{2}$ fermion field of mass m

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \xrightarrow{\text{eq. of motion}} (i\gamma^\mu\partial_\mu - m)\psi = 0 .$$

current

Any variations $\delta\phi$ and $\delta\phi^*$ leads to

$$\delta\mathcal{L} = \delta\phi\frac{\partial\mathcal{L}}{\partial\phi} + \left[\delta\left(\frac{\partial\phi}{\partial x_\mu}\right)\right]\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + (\phi\rightarrow\phi^*)$$

$$= \delta\phi\left\{\frac{\partial\mathcal{L}}{\partial\phi} - \partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right\} + (\phi\rightarrow\phi^*) + \partial_\mu\left\{\delta\phi\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} + (\phi\right.$$

$$\rightarrow \frac{\partial\mathcal{L}}{\partial\phi} - \partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} = 0 \quad : \quad \text{Euler-Lagrangian Eq.}$$

$$\rightarrow \quad \delta\mathcal{L} = \alpha\,\partial^\mu J_\mu = 0 \quad , \quad J_\mu = \left\{\delta\phi\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} + (\phi\rightarrow\phi^*)\right\} : \text{conse-} \\ \text{curr-}$$

2. The Standard Model Lagrangian

1) classical Electrodynamics

$$E^i = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = \partial^i A^0 - \partial^0 A^i \quad , \quad \vec{B}^i = \vec{\nabla}\times\vec{A}$$

It is convenient to define an antisymmetric tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \quad (F^{0i} = -E^i, \ F^{ij} = \epsilon^{ijk} B$$

$\rightarrow$ gauge invariant

The conventional Lagrangian for electromagnetism : $\mathcal{L} = \frac{1}{2}(E^2 - B$

Written in terms of $F^{\mu\nu}$, $\boxed{\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}$

2) Non-Abelian gauge fields

- What is the analogue of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$?
- Let's call it $W^i_{\mu\nu}$. is it $\partial_\mu W^i_\nu - \partial_\nu W^i_\mu$?
- check if it is gauge invariant

$$\delta(\partial_\mu W^i_\nu - \partial_\nu W^i_\mu) = \partial_\mu\left(\frac{1}{g_2}\partial_\nu \epsilon_i - \epsilon_{ijk}\epsilon_j W^k_\nu\right)$$
$$- \partial_\nu\left(\frac{1}{g_2}\partial_\mu \epsilon_i - \epsilon_{ijk}\epsilon_j W^k_\mu\right)$$

$$= -\epsilon_{ijk}\epsilon_j(\partial_\mu W^k_\nu - \partial_\nu W^k_\mu) - \underbrace{\epsilon_{ijk}(\partial_\mu \epsilon_j W^k_\nu - \partial_\nu \epsilon_j W^k_\mu)}_{\text{unwanted term}}$$

- To cancel the "unwanted term", let's observe that

$$g_2 \epsilon_{ijk} W^j_\mu W^k_\nu \Rightarrow \delta(g_2 \epsilon_{ijk} W^j_\mu W^k_\nu)$$

$$= g_2 \epsilon_{ijk}\left[\left(\frac{1}{g_2}\partial_\mu \epsilon_j - \epsilon_{jlm}\epsilon_l A^m_\mu\right)A^k_\nu + \epsilon_{ijk} A^j_\mu\left(\frac{1}{g_2}\partial_\nu \epsilon_k - \epsilon_{klm}\epsilon_l A\right.\right.$$

$$= \underline{\epsilon_{ijk}(\partial_\mu \epsilon_j A^k_\nu + \partial_\nu \epsilon_k A^j_\mu)} - \underline{g_2 \epsilon_{ijk}[\epsilon_{jlm}\epsilon_l A^m_\mu A^k_\nu + \epsilon_{klm}\epsilon_l A^j_\mu A}$$

$\epsilon_{ijk} [\epsilon_{jlm} \epsilon_l A_\mu^m A_\nu^k + \epsilon_{klm} \epsilon_l A_\mu^j A_\nu^m]$

$= \epsilon_l [\epsilon_{ijk} \epsilon_{jlm} A_\mu^m A_\nu^k + \underbrace{\epsilon_{ijk} \epsilon_{klm} A_\mu^j A_\nu^m}]$

$\Rightarrow j \to m, \; m \to k, \; k \to j$

$= \epsilon_l [\epsilon_{ijk} \epsilon_{jlm} A_\mu^m A_\nu^k + \epsilon_{imj} \epsilon_{jlk} A_\mu^m A_\nu^k]$

$= \epsilon_l [\epsilon_{ijk} \epsilon_{jlm} + \epsilon_{imj} \epsilon_{jlk}] A_\mu^m A_\nu^k$

※ $\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$

So, $\epsilon_{ijk} \epsilon_{jlm} + \epsilon_{imj} \epsilon_{jlk} = - \epsilon_{jik} \epsilon_{jlm} + \epsilon_{jim} \epsilon_{jlk}$

$= -(\delta_{il} \delta_{km} - \delta_{im} \delta_{kl}) + (\delta_{il} \delta_{mk} - \delta_{ik} \delta_{ml})$

$= \delta_{im} \delta_{kl} - \delta_{ik} \delta_{ml} = \epsilon_{jil} \epsilon_{jmk}$

$= \epsilon_l \epsilon_{jil} \epsilon_{jmk} A_\mu^m A_\nu^k \quad (l \to j, \; j \to k, \; m \to l, \; k \to m)$

$= \epsilon_{ijk} \epsilon_j \epsilon_{klm} A_\mu^l A_\nu^m$

Therefore we should modify $\partial_\mu W_\nu^i - \partial_\nu W_\mu^i$ to

$$W_{\mu\nu}^i \equiv \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g_2 \epsilon_{ijk} W_\mu^j W_\nu^k$$

Then. $\delta(W_{\mu\nu}^i) = - \epsilon_{ijk} \epsilon_j W_{\mu\nu}^k$

· The Lagragian for non-abelian gauge field : $\boxed{-\frac{1}{4} W_{\mu\nu}^i W^{\mu\nu}_i}$

In order to describe the particles and interactions know today, 3 internal symmetries are needed.

All experiments so far are consistent with the fact that the 3 sym
are necessary and sufficient to describe the interactions of the kno

3) The Quark & Lepton Lagrangian

- The full Lagrangian should arise by taking the free particle Lagrangian and replacing the ordinary derivative by the covariant derivative.
- It also has a part, $\mathcal{L}_{gauge}$, for the kinetic energies of gau
- Let's assign electroweak SU(2) states to the particle
- The remarkable thing is that L-H & R-H states transform differently under SU(2). R-H fermions are SU(2) singlet, while L-H

ex) $L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$, e^-_R, $Q_{L\alpha} = \begin{pmatrix} u_\alpha \\ d_\alpha \end{pmatrix}_L$, $d_{R\alpha}$

($\dot{x}$. $e_R^- = P_R \psi_{e^-} = \frac{1}{2}(1+\gamma_5)\psi_{e^-}$, $e_L^- = P_L \psi_{e^-} = \frac{1}{2}(1-\gamma_5)\psi_{e^-}$, γ^5

- The parity violation of electroweak interaction energes, which reflect that L-H & R-H fermions were put in different SU(2) mult

- $\bar{\psi} i \gamma^\mu \partial_\mu \psi \longrightarrow \bar{\psi} i \gamma^\mu D_\mu \psi$

$$D_\mu = \partial_\mu - i g_1 \frac{Y}{2} B_\mu - i g_2 \frac{\tau^i}{2} W_\mu^i - i g_3 \frac{\lambda^a}{2} G_\mu^a$$

ex) $\mathcal{L} = i \bar{e}_R \gamma^\mu \partial_\mu e_R + i \bar{e}_L \gamma^\mu \partial_\mu e_L + i \bar{\nu}_L \gamma^\mu \partial_\mu \nu_L$

$= i \bar{e}_R \gamma^\mu \partial_\mu e_R + i \bar{L}_L \gamma^\mu \partial_\mu L_L$

$\Rightarrow$ "gauging" $\partial_\mu \longrightarrow D_\mu$:

(i) U(1) terms : $-\mathcal{L}_f(U(1)) = \bar{L} i \gamma^\mu \left(i g_1 \frac{Y_L}{2} B_\mu\right) L + \bar{e}_R i \gamma^\mu \left(i g_1 \frac{Y_R}{2}\right.$

$$= -\frac{g_1}{2}\left(Y_L (\bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L) + Y_R \bar{e}_R \gamma^\mu\right.$$

(ii) SU(2) terms :

$-\mathcal{L}_f(SU(2)) = \bar{L} i \gamma^\mu \left[i g_2 \frac{\tau^i}{2} W_\mu^i\right] L \Rightarrow$ $= -\sqrt{2} W_\mu^+$

) Neutral Current

- Since we know how electrons & neutrinos interact, let's make $-\mathcal{L}_f$ (U(1), SU(2)) consistent with experiment.

→ We know that ν_L does not have an electromagnetic interac

from $-\mathcal{L}_f$, $\quad \left(-\frac{g_1}{2} Y_L B_\mu - \frac{g_2}{2} W_\mu^\circ\right) \bar{\nu}_L \gamma^\mu \nu_L$

such combination of B_μ & W_μ° should not be e-m field A_μ, rather it should be orthogonal to A_μ.

We call the above combination of B_μ & W_μ°, Z_μ

i.e) $\quad Z_\mu \propto g_1 Y_L B_\mu + g_2 W_\mu^\circ$

then, the orthogonal field to Z_μ is proportional to

$$g_2 B_\mu - g_1 Y_L W_\mu^\circ \propto A_\mu$$

After normalizing A_μ & Z_μ,

$$A_\mu = \frac{g_2 B_\mu - g_1 Y_L W_\mu^\circ}{\sqrt{g_2^2 + g_1^2 Y_L^2}}, \qquad Z_\mu = \frac{g_1 Y_L B_\mu + g_2 W_\mu^\circ}{\sqrt{g_2^2 + g_1^2 Y_L^2}}$$

$$\Rightarrow B_\mu = \frac{g_2 A_\mu + g_1 Y_L Z_\mu}{\sqrt{g_2^2 + g_1^2 Y_L^2}}, \qquad W_\mu^\circ = \frac{-g_1 Y_L A_\mu + g_2 Z_\mu}{\sqrt{g_2^2 + g_1^2 Y_L^2}}$$

Therefore, $\left(-\frac{g_1}{2} Y_L B_\mu - \frac{g_2}{2} W_\mu^\circ\right) \bar{\nu}_L \gamma^\mu \nu_L = -\frac{\sqrt{g_2^2 + g_1^2 Y_L^2}}{2} Z_\mu \bar{\nu}_L \gamma^\mu \nu_L$

$$= -\frac{\sqrt{g_2^2 + g_1^2}}{2} Z_\mu \bar{\nu}_L \gamma^\mu \nu_L \quad (Y_L = -1$$

For electrons : $\bar{e}_L \gamma^\mu e_L \left[-\frac{g_1}{2} Y_L B_\mu + \frac{g_2}{2} W_\mu^\circ\right] + \bar{e}_R \gamma^\mu e_R \left[-\frac{g_1}{2} Y_R B_\mu\right]$

$$= -A_\mu \left\{ \bar{e}_L \gamma^\mu e_L \left(\frac{g_1 g_2 Y_L}{\sqrt{g_2^2 + g_1^2 Y_L^2}}\right) + \bar{e}_R \gamma^\mu e_R \left(\frac{g_1 g_2 Y_R}{2\sqrt{g_2^2 + g_1^2 Y_L^2}}\right) \right\}$$

Then, $e \equiv \dfrac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}$, in the suggestive form, $\sin\theta_W = \dfrac{g_1}{\sqrt{g_1^2+g_2^2}}$, $\cos\theta_W = \dfrac{g}{\sqrt{g_1^2}}$

$$\Rightarrow \begin{cases} g_2 = \dfrac{e}{\sin\theta_W} \\ g_1 = \dfrac{e}{\cos\theta_W} \end{cases} \rightarrow \boxed{\sqrt{g_1^2+g_2^2} = \dfrac{e}{\cos\theta_W \sin\theta_W}}$$ (θ_W : Weinberg a

We call the interactions with Z "neutral current"

The couplings in the neutral currents :

$$\begin{cases} -\dfrac{\sqrt{g_1^2+g_2^2}}{2} = -\dfrac{g_2}{2\cos\theta_W} = -\dfrac{e}{2\cos\theta_W\sin\theta_W} & (\text{for } \nu_L) \\ \dfrac{g_1^2 Y_L^2 - g_2^2}{2\sqrt{g_2^2+g_1^2 Y_L^2}} = \dfrac{e}{\cos\theta_W \sin\theta_W}\left(-\dfrac{1}{2}+\sin^2\theta_W\right) & (\text{for } e_L) \\ \dfrac{g_1^2 Y_L Y_R}{2\sqrt{g_2^2+g_1^2 Y_L^2}} = \dfrac{e}{\cos\theta_W \sin\theta_W}\left(-\sin^2\theta_W\right) & \end{cases}$$

$$\Rightarrow \boxed{\dfrac{e}{\cos\theta_W\sin\theta_W}\left(T_3^f - Q_f \sin^2\theta_W\right)}$$

Charged Current

- off-diagonal part of $-\mathcal{L}_f(SU(2))$: $\dfrac{g_2}{\sqrt{2}}\left(\bar{\nu}_L \gamma^\mu e_L W_\mu^+ + \bar{e}_L \gamma^\mu \nu_L W_\mu^-\right)$
- $\bar{\nu}_L \gamma^\mu e_L = \frac{1}{2}\bar{\nu}\gamma^\mu(1-\gamma_5)e \Rightarrow$ V-A charged current interacti
- $\dfrac{(g_2\sqrt{2})^2}{4\pi} = \dfrac{(e^2/4\pi)}{2\sin^2\theta_W} \simeq \dfrac{2}{137}$
- Bring together in a place the Lagrangian for the interactions of fermions with gauge bosons :

$\mathcal{L} = \sum \quad e Q_f (\bar{f}\gamma^\mu f) A^\mu$

To realize the mechanism developed to construct a meaningful gauge theory which can include the possibility that particles have masses, we assume that the universe is filled with a spin-zero field, called a Higgs field.

→ a doublet in SU(2) & carries non-zero U(1) hypercharge

gauge bosons and fermions can interact with this field, and in its presence they no longer appear to have zero mass.

A crucial ingredient is that states with one or more Higgs are not orthogonal to vacuum (ground state) even though these states carry non-zero SU(2) & U(1) quantum numbers.

→ that means SU(2) & U(1) quantum numbers of vacuum are non-zero, so SU(2) & U(1) symmetries are effectively broken

When a sym. is broken this way, (i.e) sym. is valid for Lagrangian but for the ground state of the system. ⇒ called SSB.

Masses & the Higgs Mechanism

· Spontaneous Symmetry Breaking

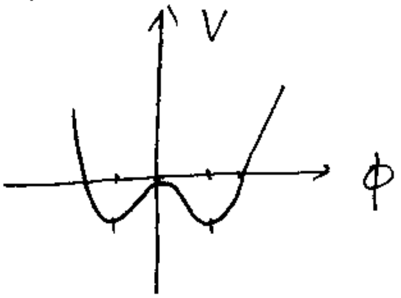
et's examine a very simple case

$$\mathcal{L} = T - V = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \left(\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4\right)$$

- we require $\lambda > 0$ in order that the potential be bounded from below as $\phi \to \infty$
- Invariant under $\phi \to -\phi$
- For $\mu^2 > 0$, vacuum corresponds to $\phi = 0$
- For $\mu^2 < 0$, $\phi = 0$ is not a minimum

$$\frac{\partial V}{\partial \phi} = 0 \Rightarrow \phi(\mu^2 + \lambda\phi^2) = 0$$

$$\phi = \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv \langle 0|\phi|0\rangle = v$$: vacuum expectation valu



※ It is obvious that vacuum is not inva under $\phi \to -\phi$.
$\Rightarrow$ spontaneous sym. b

- To determine the particle spectrum, we must study the th in the region of the minimum (quantum fluctuation around

$$\phi(x) = v + \eta(x)$$ (we are expanding around η

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta\,\partial^\mu\eta) - \left\{\frac{1}{2}\mu^2(v^2 + 2v\eta + \eta^2) + \frac{1}{4}\lambda(v^4 + 4v^3\eta + 6v^2\eta^2 + 4\right.$$

$$= \frac{1}{2}(\partial_\mu\eta\,\partial^\mu\eta) - \left\{\frac{v^2}{2}\left(\mu^2 + \frac{1}{2}\lambda v^2\right) + \eta v(\mu^2 + \lambda v^2) + \frac{\eta^2}{2}(\mu^2 + 3\lambda v^2)\right.$$

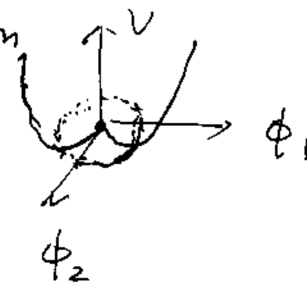
) A global symmetry – complex scalar field

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \longrightarrow \text{complex scalar}$$

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

· Invariant under a global gauge transformation

$$\phi \to \phi' = e^{i\chi}\phi$$



· Written in terms of the real components,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2$$

· for $\mu^2 < 0$, the minimum is along a circle of radiu

$$\phi_1^2 + \phi_2^2 = \frac{-\mu^2}{\lambda} \equiv v^2 \qquad \underline{O(2)} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

· we could choose any point on the circle, but to procee
we have to choose some point, which will break the sy

· we pick the point $\phi_1 = v$, $\phi_2 = 0$

$$\phi = \frac{1}{\sqrt{2}}(v + \eta(x) + i\rho(x))$$

$$\to \mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\partial_\mu \eta)^2 + \underbrace{\mu^2 \eta^2}_{\text{mass term of } \eta} - \lambda v(\eta\rho^2 + \eta^3) - \frac{\lambda}{2}\eta^2\rho^2 - \frac{\lambda}{4}\eta^4 - \frac{\lambda}{4}$$

we see no mass term of ρ $\longrightarrow$ Goldstone boson

※ Goldstone theorem : whenever a continuous global sym. is spontaneously broken, the spectrum will contain

3) The Abelian Higgs Mechanism

· $\mathcal{L} = (D_\mu \phi^*)(D^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$(D^\mu = \partial^\mu - igA^\mu)$$

· gauge transformation $\begin{cases} A_\mu \rightarrow A_\mu' = A_\mu - \frac{1}{g} \partial_\mu \chi(x) \\ \phi(x) \rightarrow \phi'(x) = e^{i\chi(x)} \phi(x) \end{cases}$

· In general, $\phi(x)$ could be written in the form

$$\phi(x) = \eta(x) e^{-i\rho(x)}$$

· Choosing $\phi_1 = v,\ \phi_2 = 0$, $\quad \phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x)) e^{-i\rho(x)}$

Using the gauge transformation, we can choose to write

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))$$

$\Rightarrow \mathcal{L} = \frac{1}{2}\left[(\partial^\mu + igA^\mu)(v+\eta)\right]\left[(\partial^\mu - igA^\mu)(v+\eta)\right] - \frac{1}{2}\mu^2(v+\eta)$

$\quad - \frac{1}{4}(v+\eta)^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$\quad = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \underline{\frac{1}{2} g^2 v^2 A_\mu A^\mu} - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4$

$\quad + g^2 v \eta A^\mu A_\mu + \frac{1}{2} g^2 \eta^2 A^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$\rightarrow$ we see that $\left(\begin{array}{l} \text{no massless boson} \\ \eta \text{ gets a mass } m_\eta^2 = 2\lambda v^2 \\ A_\mu \text{ get } \quad \text{"} \quad M_A = gv \end{array} \right.$

Note)) Before performing gauge transformation, a term $A_\mu \partial^\mu \rho$ appears.

· when such cross terms appear, one can go to eigenstates by diagon

The Higgs Mechanism in the Standard Model

- The Higgs field is assigned to an SU(2) doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \qquad \begin{cases} \phi^+ = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi^0 = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \end{cases}$$

→ electric charge assignment of ϕ corresponds to putting $Y_H = 1$

$$(\because\ Q = T_3 + \frac{Y_H}{2})$$

- only ϕ^0 get a VEV (if ϕ^+ get a VEV, Q is not conserved)

- Sym. breaking : SU(2) is broken because $\phi(x)$ is a double[t] and only one comp. gets a VEV.

  $U(1)_Y$ is broken because $Y_H \neq 0$.

  However, $Q\phi^0 = (T_3 + \frac{Y}{2})\phi_0 = 0$, so ϕ_0 is invariant under a transformation $\phi_0 \longrightarrow \phi_0' = e^{i\alpha(x)Q}\phi_0 = \phi_0$

  So, the vacuum is invariant under a particular $U'(1)$ whose generators are a particular linear combinations of the generato[rs] of the original SU(2) & U(1)

  ⇒ this $U'(1)$ is the U(1) of EM & the gauge boson that rema[ins] massless is the photon.

- $$\partial_\mu \longrightarrow D_\mu = \partial_\mu - ig_1 \frac{Y}{2} B_\mu - ig_2 \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu$$

$$\mathcal{L} = (D_\mu\phi)^\dagger (D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 + \cdots$$

$$\phi^\dagger\phi = (\phi^{+*}\ \phi^{0*})\begin{pmatrix}\phi^+ \\ \cdots\end{pmatrix}$$

For $\mu^2 < 0$, $V(\phi)$ has a minimum at

$$\phi^\dagger\phi = \frac{-\mu^2}{2\lambda} \equiv \frac{v^2}{2}$$

we choose, $\phi_1 = \phi_2 = \phi_4 = 0$, $\phi_3 = v$.

$$\langle\phi\rangle = \phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}$$

Expanding around the vacuum, $\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}$

(※ it's possible to rotate arbitrary $\phi(x)$ into the above
by making a gauge transformation, $\phi \to \phi' = e^{i\vec{\tau}\cdot\frac{\vec{\theta}}{v}}\phi$
→ 3 fields "gauging away" → become the longitudinal
parts of $W^\pm$ & Z^0)

$$D_\mu\phi = \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\partial_\mu H(x)\end{pmatrix} - \left[\frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\begin{pmatrix}W_\mu^3 & W_\mu^1 - iW_\mu^2\\ W_\mu^1+iW_\mu^2 & -W_\mu^3\end{pmatrix}\right]\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(v+H(x))\end{pmatrix}$$

$$= -\frac{i}{2}\begin{pmatrix}\frac{1}{\sqrt{2}}(v+H(x))\, g_2(W_\mu^1 - iW_\mu^2)\\ i\sqrt{2}\,\partial_\mu H(x) - (g_2 W_\mu^3 \quad g_1 B_\mu)\frac{1}{\sqrt{2}}(v+H(x))\end{pmatrix}$$

$$D_\mu\phi^\dagger D^\mu\phi = \frac{1}{2}(\partial_\mu H(x))^2 + \frac{1}{8}\left|\begin{pmatrix}g_1 B_\mu + g_2 W_\mu^3 & g_2(W_\mu^1 - iW_\mu^2)\\ g_2(W_\mu^1+iW_\mu^2) & g_1 B_\mu - g_2 W_\mu^3\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\right.$$

$$= \frac{1}{2}(\partial_\mu H(x))^2 + \frac{1}{8}v^2 g_2^2\left((W_\mu^1)^2 + (W_\mu^2)^2\right) + \frac{1}{8}v^2(g_1 B_\mu - g_2 W_\mu^3)$$
$+$

Define

$$Z_\mu = \frac{g_2 W_\mu^3 - g_1 B_\mu}{\sqrt{g_1^2+g_2^2}} \equiv \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu$$

5) Fermion Masses

- For $L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$ & $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$,

an SU(2) invariant is $\bar{L}\phi = \bar{\nu}_{eL}\phi^+ + \overline{e_L^-}\phi^0$

Multiplying by the singlet e_R^- does not change SU(2) sym

$$\rightarrow \quad \mathcal{L} = \underbrace{g_e}_{\text{arbitrary coupling}}(\bar{L}\phi e_R^- + \phi^+ \overline{e_R^-} L)$$

$$\Rightarrow \text{SSB}: \quad \phi \rightarrow \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

$$\Rightarrow \mathcal{L} = \underbrace{\frac{g_e v}{\sqrt{2}}\left(\overline{e_L^-}e_R^- + \overline{e_R^-}e_L^-\right)}_{\text{fermion mass terms.}} + \frac{g_e}{\sqrt{2}}\left(\overline{e_L^-}e_R^- + \overline{e_R^-}e_L^-\right)H(x)$$

$$m_e = \frac{g_e}{\sqrt{2}}v$$

- No mass term occurred for neutrino because of no ν_R in SM

- L has $Y_w = -1$, e_R^- has $Y_w = -2$

- For quarks, $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$

it is well known that if $\begin{pmatrix} a \\ b \end{pmatrix}$ is an SU(2) double,

$$\text{so is} \quad -i\tau_2\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b^* \\ a^* \end{pmatrix}$$

Cabibbo Theory & GIM mechanism

Befor gauge theory has been established for the description of weak interactions, physicists used Fermi's theory which is presented by current-current interactions

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} J^\lambda(x)\, J_\lambda^\dagger(x)$$

$$J^\lambda = j_e^\lambda(x) + j_\mu^\lambda(x) + a\, J_0^\lambda(x) + b\, J_1^\lambda(x)$$

$$\begin{cases} j_e^\lambda(x) = \bar{e}\gamma^\lambda(1+\gamma_5)\nu_e \\ j_\mu^\lambda(x) = \bar{\mu}\gamma^\lambda(1+\gamma_5)\nu_\mu \end{cases} \qquad \begin{cases} J_0^\lambda(x) = \bar{n}\gamma^\lambda(1+\gamma_5)p \\ J_1^\lambda(x) = \bar{\Lambda}\gamma^\lambda(1+\gamma_5)p \end{cases}$$

In quark model, $\begin{cases} J_0^\lambda(x) = \bar{d}\gamma^\lambda(1+\gamma_5)u \\ J_1^\lambda(x) = \bar{s}\gamma^\lambda(1+\gamma_5)u . \end{cases}$ $\left(\begin{array}{l} J^\lambda_{(ep)} = \bar{\nu}_{\mu\,(e)}\gamma^\lambda(1+\gamma_5) \\ J_\lambda^{(1)} + iJ_\lambda^{(2)} = (\Delta S = 0, \\ J_\lambda^{(4)} + iJ_\lambda^{(5)} = (\Delta S = \phi, \Delta \end{array} \right.$

- It has been known that strange particles (Kaons, hyperons...) slowly decay.

In early 60's, there is a puzzle why rates for decays with $\Delta S = 0 >$

(i.e) $BR(K^- \to \mu\bar{\nu}_\mu) \cong 63.5\%$, $BR(\pi^- \to \mu\bar{\nu}_\mu) \cong 100\%$.

Emperically, $|G^\mu|^2 \simeq |G^\beta|^2 + |G^\Lambda|^2$

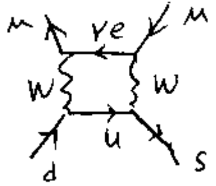
- In 1963, Cabibbo postulated that the quarks that participate in the weak interaction are a mixture of the quarks that participate in the strong interaction.

$$d' = d\cos\theta_c + s\sin\theta_c$$

$$\Rightarrow \begin{cases} d \to u + W^- \propto \cos\theta_c \\ s \to u + W^- \propto \sin\theta_c \end{cases} \Rightarrow \text{experimentally}, \ \theta_c \sim 12^\circ$$

Experimentally, $\frac{BR(K^0\to\mu^+\mu^-)}{BR(K^+\to\mu^+\nu_\mu)} = \frac{7\times10^{-9}}{0.64} \simeq 10^{-8}$

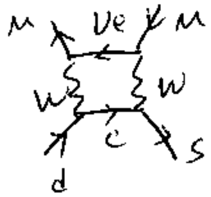
- Cabibbo theory shows



$\propto \cos\theta_c \sin\theta_c$: not suppressed.

- In 1970, GIM proposed the existence of a fourth quark (c) whose couplings to s and d carry factors of $\cos\theta_c$ & $-\sin\theta_c$

- Then there exists the following contribution to $K^0\to\mu^+\mu^-$



$\propto -\cos\theta_c \sin\theta_c \Rightarrow$ cancel the above contribution.

- Instead of the physical quark d and s, the "correct" states to use in the weak interactions are d' & s' given by

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}\begin{pmatrix} d \\ s \end{pmatrix}$$

- The W's couple to the "Cabibbo-rotated" states $\begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix}$

- Extension to 3 generations $\Rightarrow$ KM matrix

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} 3\times3 \text{ unitary} \\ \text{matrix} \end{pmatrix}\begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \end{pmatrix}\begin{pmatrix} d \\ s \end{pmatrix}$$

◎ Yukawa Interactions of Higgs fields with quarks

$$\mathcal{L}_Y = (g_d)_{ij}\, \overline{Q}_{Li} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_{Rj} + (g_u)_{ij}\, \overline{Q}_{Li} \begin{pmatrix} -\phi^{0*} \\ \phi^- \end{pmatrix} u_{Rj} + \text{h.c.}$$

- once ϕ^0 gets a VEV, fermion masses arise
- Then, the mass matrices for up-type & down-type quarks are proportional to $(g_d)_{ij}$, $(g_u)_{ij}$.
- Since the Yukawa couplings are arbitrary, so are the mass matrice
- We can diagonalize the mass matrices with the help of two unitary matrices $U_{uL}, U_{uR}, U_{DL}, U_{DR}$

$$\Rightarrow U_{uL} M_u U_{uR}^\dagger = M_u^{diag}, \quad U_{DL} M_D U_{DR}^\dagger = M_D^{diag}$$

$$\begin{pmatrix} u^m \\ c^m \\ t^m \end{pmatrix}_{L(R)} = U_{uL(R)} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L(R)}$$ (trasforming L(R) quarks into their mass eigens

- U_L & U_R can be found by diagonalizing $M_u M_u^\dagger$ & $M_u^\dagger M_u$

$$\Rightarrow (M_u^{diag})^2 = U_{uR} M_u^\dagger M_u U_{uR}^\dagger = U_{uL} M_u M_u^\dagger U_{uL}^\dagger$$

- For charged currents :

$$(\overline{u}_L, \overline{c}_L, \overline{t}_L)\, \gamma_\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = (\overline{u}_L^m, \overline{c}_L^m, \overline{t}_L^m)\, U_{uL} \gamma_\mu U_{DL}^\dagger \begin{pmatrix} d_L^m \\ s_L^m \\ b_L^m \end{pmatrix}$$

$$\equiv (\overline{u}_L^m, \overline{c}_L^m, \overline{t}_L^m)\, \gamma_\mu V \begin{pmatrix} d_L^m \\ s_L^m \\ b_L^m \end{pmatrix}$$

$$V \doteq U_{uL} U_{DL}^\dagger \Rightarrow \boxed{\text{CKM matrix}}$$

" Determining the number of independent physical parameters in V "

i) A general $n \times n$ complex matrix contains $2n^2$ real parameters

ii) Unitarity implies $\sum_j V_{ij} V_{jk}^* = \delta_{ik}$ yielding

$$\begin{cases} n \text{ constraints for } i=k \\ 2 \cdot \frac{1}{2} \cdot n \cdot (n-1) = n^2 - n \quad \text{for } i \neq k \end{cases}$$

→ So, a unitary $n \times n$ matrix contains n^2 independent real parameters.

iii) The phases of quark fields can be rotated freely

$$U_i^m \to e^{i\phi_i^U} U_i^m, \quad D_j^m \to e^{i\phi_j^D} D_j^m$$

leading to $$V \to \begin{pmatrix} e^{-i\phi_1^U} & & 0 \\ & \ddots & \\ 0 & & e^{-i\phi_n^U} \end{pmatrix} V \begin{pmatrix} e^{i\phi_1^D} & & 0 \\ & \ddots & \\ 0 & & e^{i\phi_n^D} \end{pmatrix}$$

Since the overall phase is irrelevant, $2n-1$ relative phases can be removed from V. Accordingly, V contains $(n-1)^2$ indep't p

iv) A general orthogonal $n \times n$ matrix is constructed from $\frac{1}{2}n(n-$ quantities describing the indep't rotation angles :

$$N_{angle} = \frac{1}{2} n(n-1)$$

So, the # of indep't phases in V : $(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-$

(ex for 3 generations, $n=3$, ⟹ $N_{angle} = 3$, $N_{phase} = 1$

$$V = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$