Introduction to SUSY

Ref) Wess & Bagger

Intriligator & Seiberg 9509066

Motivation

i) gauge hierarchy problem ← tomorrow

ii) understanding of the strong coupling behavior of the field theory

relativistic field theory — the extent of spacetime symmetry

← Poincaré symmetry ⇒ conserved quantities

energy-momentum 4-vector P_μ

Lorentz generators $M_{\mu\nu}$

boost + rotation

+ internal symmetry

conserved charges

e.g.) $S = -\int d^4x \; \partial_\mu \phi^+ \partial^\mu \phi + m^2 \phi^+ \phi + \lambda |\phi^+ \phi|^2$

Poincaré + $\phi \to e^{i\alpha}\phi$ $U(1)$

Noether current

$j_\mu = i(\phi^+ \partial_\mu \phi - \partial_\mu \phi^+ \phi)$ $\quad \partial_\mu j^\mu = 0$

" $i\, \phi^+ \overleftrightarrow{\partial_\mu} \phi$

$Q = \int d^3x\, j^0$

If we have N ϕ's

$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_N \end{pmatrix}$

we can have $U(N)$ symmetry

$\phi^+\phi$ invariant

and we have N^2 conserved charges

$Q_A = i\int d^3x\, \phi^+ X_A \overleftrightarrow{\partial_0} \phi$

X_A is a basis of hermitian $N \times N$ matrices

$1 \sim N^2$

One can have enlarged sym

If we add higher spin fields in the free field limit

Thus for one free real scalar ϕ and one free real vector A_μ

of the same mass m

$J_{\mu\nu} = \phi \overleftrightarrow{\partial_\mu} A_\nu$ obeys the conservation law

$\partial^\mu J_{\mu\nu} = \phi \Box A_\nu - (\Box\phi) A_\nu = (m^2 - m^2)\phi A_\nu = 0$

One can introduce higher spin fields and obtain more conserved currents for the free fields.

One may wonder if this can hold for interacting cases

No! Coleman, Mandula theorem

For $d > 2$ relativistic quantum field theory

with a discrete spectrum of massive one-particle states

with some nonzero scattering amplitudes

The only tensorial conserved charges (not Lorentz scalars)

energy-mom 4-vector P_μ & $M_{\mu\nu}$

$\downarrow$

span Poincare algebra

All other conserved charges Lorentz scalars

$\downarrow$

internal symmetry

(totally massless case: $\mathcal{P} \to$ conformal algebra)

Suppose $\exists$ a tensorial conserved charge $Q_{\alpha\beta}$ ← assume traceless

assume a scalar particle of mass m carrying $Q_{\alpha\beta}$

Let $|p\rangle$ be the corresponding one-particle state $|p\rangle$

with $p^2 = -m^2$

the expectation value $\langle p| Q_{\alpha\beta} |p\rangle = C(p_\alpha p_\beta - \frac{1}{d}\eta_{\alpha\beta} p^2)$
$\neq 0$

Suppose $p_1 + p_2 \to p_1' + p_2'$

due to conserved $Q_{\alpha\beta}$

$p_{1\alpha} p_{1\beta} + \frac{1}{d}\eta_{\alpha\beta} m^2 + p_{2\alpha} p_{2\beta} + \frac{1}{d}\eta_{\alpha\beta} m^2 = (1 \to 1') + (2 \to 2')$

This equation implies that

the scattering must be in the forward or backward direction

($\alpha \approx$, $\beta \approx$ $E_1^2 + G_2^2 = G_1'^2 + G_2'^2$)

while no scattering for the other directions

2d - exception

→ incompatible with analyticity of S-matrix

→ impossibility of nontrivial Sym that connect particles of different spins

& spin difference is integer

Connect ev integer spin - half-integer spin particle?

possible → SuSy

Have to deal with fermions in 4d

Choose $\eta_{nm} \sim (-1\ 1\ 1\ 1)$

explain the notation of Wess & Bagger

When one writes the Dirac equation

$$(i \gamma^m \partial_m + m)\psi = 0$$

$$\{\gamma^m, \gamma^n\} = -2\eta^{nm}$$

there are several convenient choices.

Wess & Bagger use the Weyl basis

with $\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar\sigma^m & 0 \end{pmatrix}$

$$\sigma^m = (-1, \vec\sigma)$$
$$\bar\sigma^m = (-1, -\vec\sigma)$$

one can check $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

4-comp spinor $\psi = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}$

Introduce the undotted spinor & dotted spinor

χ_α $\bar\psi^{\dot\alpha}$ are Weyl spinor

($m=0$ → we can have the Dirac equation with 2x2 gamma matrices) the associated group theory

Lorentz group $SO(1,3)$

↕

matrix A satisfying $\eta = A^T \eta A$ $\eta = (-1, 1, 1, 1)$

the transf. $(-1\,1\,1\,1)$ invariant with $\det A = +1$

$-x_0^2 + \vec{x}^2 = -(x_0')^2 + \vec{x}'^2$

Lie algebra of $SO(1,3)$

↕ infinitesimal transf.

↕ three rotation generators J_i, three boost generators K_i

If we define $A = \frac{J + iK}{2}$ $B = \frac{J - iK}{2}$

then A, B satisfy the Lie algebra of $SU(2)$

$[A_i, A_j] = i\epsilon_{ijk} A_k$ $[B_i, B_j] = i\epsilon_{ijk} B_k$ $[A_i, B_j] = 0$

cf) $SO(4) \simeq SU(2) \times SU(2)$

we need complexification

↓

$SO(1,3) \rightarrow SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$ $(\epsilon^{\mu\nu}) - A^{\mu}$

Since we introduce i component in the def

↓

2×2 matrix of $\det \pm 1$

the representation of $SL(2\mathbb{C})$ essentially follows from the repre of $SU(2)$ → angular momentum algebra spin j repre

$M \in SL(2\mathbb{C})$,

M, M^*, $(M^*)^{-1}$, M^{-1} all $SL(2, \mathbb{C})$

two-component spinors with upper & lower dotted/undotted spinors transf

$(\frac{1}{2}, 0)$ $\psi'_\alpha = M_\alpha{}^\beta \psi_\beta$ $\bar{\psi}'_{\dot\alpha} = M^*{}_{\dot\alpha}{}^{\dot\beta} \bar{\psi}_{\dot\beta}$ → $(0, \frac{1}{2})$ repre

$\exp i\vec{\theta}\cdot\vec{\tau}$

$\exp -i\vec{\theta}\cdot\vec{\tau}^*$

$\psi'^\alpha = (M^{-1})_\beta{}^\alpha \psi^\beta$ $\bar{\psi}'^{\dot\alpha} = (M^{*-1})_{\dot\beta}{}^{\dot\alpha} \bar{\psi}^{\dot\beta}$

Hermitian

↕ the index α can be lowered and raised by $\epsilon_{\alpha\beta}$ ↕ 2nd rk tensor

$\tau = \tau^\dagger$

$\tau^* = (\tau^\dagger)^*$

$= \tau^T$

$\psi^\alpha \psi_\alpha$ ← invariant

↕ transf by $M_\alpha{}^\beta$ — transf by its inverse

Connection bet $SL(2\mathbb{C})$ and Lorentz gp is established

using $\sigma^m = (\sigma^0, \sigma^i) = (-1, \vec{\sigma})$

$$P = P_m \sigma^m = \begin{pmatrix} -P_0 + P_3 & P_1 - i P_2 \\ P_1 + i P_2 & -P_0 - P_3 \end{pmatrix}$$

Any hermitian matrix P can be expanded with σ^m with real coeffs P_m

For any hermitian matrix P
we can always obtain another by
$$P' = M P M^{+}$$
Both P & P' have expansions in σ
$$\sigma^m P_m' = M \sigma^m P_m M^{+}$$
Since M is unimodular $\quad \det \sigma^m P_m' = \det M \sigma^m P_m M^{+}$

$$P_0'^2 - \vec{P}'^2 = P_0^2 - \vec{p}^2$$

We define $\varepsilon^{\alpha\beta}$ & $\varepsilon_{\alpha\beta}$ s.t. $\varepsilon_{\alpha\beta} \varepsilon^{\beta\gamma} = \delta_\alpha^\gamma$

$$\varepsilon^{12} = \varepsilon_{21} = +1 \qquad \varepsilon_{12} = \varepsilon^{21} = -1$$

ε is ~~$SL(2\mathbb{C})$ invariant~~
2nd rank tensor of $SL(2\mathbb{C})$

$$\varepsilon'_{\alpha\beta} = M_\alpha{}^\gamma M_\beta{}^\delta \varepsilon_{\gamma\delta}$$

$$\underset{\substack{\| \\ -1}}{\varepsilon_{12}} = M_1{}^\gamma M_2{}^\delta \varepsilon_{\gamma\delta} = M_1{}^1 M_2{}^2 \varepsilon_{12} + M_1{}^2 M_2{}^1 \varepsilon_{21}$$
$$= -\det M = -1$$

Using ε we can form $SL(2\mathbb{C})$ invariant expression

$\psi^\alpha \psi_\alpha \equiv \varepsilon_{\alpha\beta} \psi^\alpha \psi^\beta$ or ~~raise~~ raise and lower the indices

Wess & Bagger spinor convention is

$$\psi\chi = \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}$$

For $\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \quad \gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \qquad \bar{\sigma}^{m\,\dot{\alpha}\alpha} \chi_\alpha$

$$\sigma^m{}_{\alpha\dot{\alpha}} \qquad \bar{\sigma}^{m\,\dot{\alpha}\alpha} \qquad \gamma^m \quad m\psi$$

$$\bar{\sigma}^{m\,\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\alpha\beta} \sigma^m{}_{\beta\dot{\beta}}$$

For the gamma matrix

$$(\sigma^m \bar{\sigma}^n + \sigma^m \bar{\sigma}^n)_\alpha{}^\beta = -2\eta^{mn}\delta_\alpha{}^\beta$$

$$(\bar{\sigma}^m \sigma^n + \bar{\sigma}^n \sigma^m)^{\dot\alpha}{}_{\dot\beta} = -2\eta^{mn}\delta^{\dot\alpha}{}_{\dot\beta}$$

SUSY algebra and its multiplet structure $\quad \bar{\psi}\gamma^m\psi$

$$\boxed{\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma_{\alpha\dot\beta}{}^m P_m}$$

natural $\quad (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}) \simeq$ ~~vector~~ $(\tfrac{1}{2}, \tfrac{1}{2})$

$= $ vector

$\{Q_\alpha, Q_\beta\} = 0 \quad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \quad [P_m, Q_\alpha] = 0$

$[P_m, P_n] = 0$

Lorentz indices

multiplet structure

Massive states $\quad p^2 = -M^2$

Choose the frame $\quad P_m = (-M, 0, 0, 0)$

$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2M\,\delta_{\alpha\dot\beta}$

fermion harmonic oscillator

st. $\quad \{a_\alpha, a_\beta^\dagger\} = \delta_{\alpha\beta} \quad \alpha, \beta = 1, 2$

State $\quad |0\rangle, \quad a_\alpha^\dagger|0\rangle \quad a_\alpha^\dagger a_\beta^\dagger|0\rangle$

defined to be $a_\alpha|0\rangle = 0$

stability group

If $|0\rangle$ is spin 0 under $SU(3)$ rotation in the rest frame

Spin 0 $\quad 2$

Spin 1/2 $\quad 2$

If $|0\rangle$ is spin 1/2 $\quad \tfrac{1}{2} \oplus \tfrac{1}{2} = 1 \oplus 0$

$\lvert 0\rangle$	$a_\alpha^\dagger\lvert 0\rangle$	$a_\alpha^\dagger a_\beta^\dagger \lvert 0\rangle$
1/2	1, 0	1/2

Spin	
0	1
1/2	2
1	1

not interested in spin > 1

super gravity

Massless states $\quad p^2 = 0 \qquad P_m = (-E,\ \underbrace{0,\ 0}_{SO(2)},\ E)$

$$\{ Q_\alpha\ \bar{Q}_\beta \} = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$\{ a_1,\ a_1^\dagger \} = 1$ $\quad$ Single harmonic oscillator

$|0\rangle,\quad a_1^\dagger|0\rangle$

If helicity of $|0\rangle$ is 0

$\downarrow$

quantum # of $SO(2)$ rotation

spin along P_3 $\qquad$ $+1$ $\qquad$ -1

spin 0 $\quad 1$

spin $\frac{1}{2}$ $\quad 1$

If helicity of $|0\rangle$ is $\frac{1}{2}$ $\qquad$ spin $\frac{1}{2}$ $\quad 1$

spin 1 $\quad 1$

under CPT $\qquad$ helicity $\to$ $-$helicity

parity $\quad$ momentum odd $\quad$ angular momentum even

Combined with CPT conjugate

0	2	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1
$-\frac{1}{2}$	1	$-\frac{1}{2}$	1
		-1	1

$\downarrow$ chiral multiplet $\qquad$ $\downarrow$ vector multiplet $\quad (A_\mu,\ \lambda)$

$(\phi,\ \psi)$

ϕ complex scalar, ψ are Dirac spinor

We can have massive and massless chiral multiplets

and for the gauge interaction we need massless vector multiplets

Formulate the SUSY algebra in terms of fields not restricted by any mass shell condition

Introducing anti commuting parameters $\xi^\alpha, \bar{\xi}_{\dot\alpha}$

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = [P^m, \xi^\alpha] = 0$$

SUSY algebra can be written in terms of commutator

$$[\xi Q, \bar{\xi}\bar{Q}] = 2\,\xi\sigma^m\bar{\xi}\,P_m$$

$$[\xi Q, \xi Q] = 0 \qquad [P^m, \xi Q] = 0$$

$$(\xi Q = \xi^\alpha Q_\alpha \qquad \bar{\xi}\bar{Q} = \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha})$$

Define $\quad \delta_\xi A = (\xi Q + \bar{\xi}\bar{Q}) \times A$

SUSY transf $\quad \delta_\xi \psi = (\xi Q + \bar{\xi}\bar{Q}) \times \psi$

This should satisfy the SUSY algebra

$$(\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)A = 2(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})P_m A$$

$$= -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\partial_m A$$

SUSY transf. maps tensor fields into spinor fields & vice versa

Q has mass dimension $1/2 \sim \sqrt{P}$

dim $l \rightarrow l + 1/2$

One can check

$$\delta_\xi A = \sqrt{2}\,\xi\psi$$

$$\delta_\xi\psi = i\sqrt{2}\,\sigma^m\bar{\xi}\,\partial_m A + \sqrt{2}\,\xi F$$

$$\delta_\xi F = i\sqrt{2}\,\bar{\xi}\bar{\sigma}^m\partial_m\psi$$

satisfies the SUSY algebra

If we consider a Lagrangian

$$\mathcal{L}_0 = i\,\partial_m\bar{\psi}\bar{\sigma}^m\psi + A^*\Box A + F^*F$$

At the Lagrangian F is just an auxiliary field

and one can eliminate it by using the eqn of motion

$F = 0$

One can simply has

$$\mathcal{L} = i\partial_m\bar{\psi}\bar{\sigma}^m\psi + A^*\Box A$$

and $S = \int d^4x\,\mathcal{L}$ is invariant under

$$\delta_\xi A = \sqrt{2}\xi\psi \qquad \delta_\xi\psi = i\sqrt{2}\sigma^m\bar{\xi}\partial_m A$$

However using this transf law

the susy algebra closes only if one uses the eqn of motion of A and ψ

We said susy holds on-shell

In order to have susy without the on-shell condition we have to introduce the auxiliary field F → off-shell

For $N=1$ susy in 4D we know how to realize susy off-shell

For an extended susy and supergravity this is not easy task

Typically one has on-shell susy

In general checking the susy is painful exercise

and we need formulation where susy is manifest

And this is superfield formalism

Here components are obtained by power series expansion in θ

susy algebra ~ Lie algebra with anticommuting parameter

→ define a group element

$$G(x,\theta,\bar{\theta}) = e^{i(-x^m P_m + \theta Q + \bar{\theta}\bar{Q})}$$

multiplication of group elts

using $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$

$$G(a,\xi,\bar{\xi})\,G(x^m,\theta,\bar{\theta}) = G(x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta},\ \theta+\xi,\ \bar{\theta}+\bar{\xi})$$

The multiplication of group elements induce a motion in a parameter space

$$g(\xi,\bar{\xi}) : (x^m,\theta,\bar{\theta}) \to (x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta},\ \theta+\xi,\ \bar{\theta}+\bar{\xi})$$

↓

this motion may be generalized by diff op Q and $\bar{Q}$

$$\xi Q + \bar{\xi}\bar{Q} = \xi^\alpha \left(\frac{\partial}{\partial \theta^\alpha} - i \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial_m \right) + \bar{\xi}_{\dot{\alpha}} \left(\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i \theta^\alpha \sigma_{\alpha\dot{\beta}}^m \varepsilon^{\dot{\beta}\dot{\alpha}} \partial_m \right)$$

From this

$$\{ Q_\alpha , \bar{Q}_{\dot{\alpha}} \} = 2 i \sigma_{\alpha\dot{\alpha}}^m \partial_m$$

$$\{ Q_\alpha \; Q_\beta \} = 0$$

($P_m = -i \partial_m$ product of successive group elements

→ order of the multiplication reverses)

† Left multiplication

We can also consider right multiplication

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{\dot{\alpha}} \partial_m$$

$$\bar{D}_{\dot{\alpha}} = - \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \theta^\alpha \sigma_{\alpha\dot{\alpha}}^m \partial_m$$

$$\{ D_\alpha , \bar{D}_{\dot{\alpha}} \} = -2 i \sigma_{\alpha\dot{\alpha}}^m P_m \qquad D \; Q \text{ anticommute}$$

superspace $(x \; \theta \; \bar{\theta})$

superfields = functions of superspace

→ power series in $\theta \; \bar{\theta}$

$$F(x,\theta,\bar\theta) = f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^n\bar\theta V_n(x) + \cdots + \theta\theta\bar\theta\bar\theta\, d(x)$$

The trans law for superfields

$$\delta_\xi F(x,\theta,\bar\theta) = \delta_\xi f(x) + \theta\delta_\xi\phi(x) + \bar\theta\delta_\xi\bar\chi(x) + \cdots$$

$$= (\xi Q + \bar\xi\bar Q) F \quad \leftarrow \text{just compute this}$$

SUSY transf — diff op. we defined before

linear combs and products are of an superfields

Since Q & $\bar Q$ are linear diff op

→ Superfields form linear rep of SUSY algebra

reps are highly reducible

can ~~impose~~ eliminate extra comp fields by imposing covari constraints s.t. $\bar D F = 0$, $F = F^\dagger$

$\bar D F = 0$ → chiral scalar superfields → chiral mults

$F = F^\dagger$ → vector superfields

Chiral superfields

$\bar D_{\dot\alpha}\Phi = 0$ → chiral multiplets

easy to solve in terms of $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$

Since $\bar D_{\dot\alpha}(x^\mu + i\theta\sigma^\mu\bar\theta) = 0$ and $\bar D_{\dot\alpha}\theta = 0$

$$\to \Phi = A(y) + \sqrt2\,\theta\psi(y) + \theta\theta F(y)$$

$$= A(x) + i\theta\sigma^m\bar\theta\,\partial_m A(x) + \tfrac14\theta\theta\bar\theta\bar\theta\,\Box A(x) + \sqrt2\,\theta\psi(x) - \tfrac{i}{\sqrt2}\theta\theta\,\partial_m\psi(x)\sigma^m\bar\theta + \theta\theta F(x)$$

$\Phi^\dagger$ satisfies $D_\alpha\Phi^\dagger = 0$

← ~~natural~~ natural function of $y^{\dagger\mu} = x^\mu - i\theta\sigma^\mu\bar\theta$

easy to verify the transf law for A ψ F are exactly those for the comp mults

$\Phi = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$

$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta$

$\delta_\xi\Phi = (\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\bar{Q}^{\dot\alpha})\Phi = (\xi^\alpha Q_\alpha - \bar{\xi}^{\dot\alpha}\bar{Q}_{\dot\alpha})\Phi$

$\epsilon_{\alpha\beta}\theta^\alpha\theta^\beta$

$= \left(\xi^\alpha\frac{\partial}{\partial\theta^\alpha} - \bar{\xi}^{\dot\alpha}\left(-\frac{\partial}{\partial\bar\theta} + 2i\,\theta^\alpha\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial y^m}\right)\right)\left(A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\right)$

$= \sqrt{2}\,\xi^\alpha\psi_\alpha(y) + \dots$

$\delta_\xi A = \sqrt{2}\xi\psi$

The highest comp of Φ or $\Phi^\dagger$ are F & F*

all higher powers in $\theta\bar\theta$ → space time deriv

$\delta_\xi F$ → spacetime deriv

products of chiral super fields are $\Phi_i\Phi_j\Phi_k$ chiral super fields

$\Phi_i\Phi_j = A_i(y)A_j(y) + \sqrt{2}\theta(\psi_i A_j(y) + A_i\psi_j(y))$

$+\theta\theta(A_iF_j + A_jF_i - \psi_i\psi_j(y))$

products of $\Phi^\dagger\Phi$ not a chiral super f

$\Phi_i^\dagger\Phi_j = A_i^*(x)A_j(x) + \dots$

$+\theta\theta\bar\theta\bar\theta\,(F_i^*F_j + \frac{1}{4}A_i^*\Box A_j + \frac{1}{4}\Box A_i^*A_j - \frac{1}{2}\partial A_i^*\partial A_j$

$+\frac{i}{2}\partial\bar\psi\bar\sigma\psi_j - \frac{i}{2}\bar\psi_i\bar\sigma\partial\psi_j]$

$\theta\theta\bar\theta\bar\theta$ comp transf into a spacetime deriv

Sperl susy renorm Lagr. builts chiral superfield

$\mathcal{L} = \Phi_i^\dagger\Phi_i\big|_{\theta\theta\bar\theta\bar\theta} + \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3 + g\Phi\big|_{\theta\theta}$ + h.c

+ Grassman integral $\int d\theta\,1 = 0$

$\int d\theta\,\theta = 1$

$\mathcal{L} = i\partial\bar\psi_i\bar\sigma\psi_i + A_i^*\Box A + F_i^*F_i$

$+ m_{ij}(A_iF_j - \frac{1}{2}\psi_i\psi_j)$ + h.c.

$+ g_{ijk}(A_iA_jF_k - \psi_i\psi_jA_k) + \lambda_iF_i$ $\int\theta\theta\,R$

One can solve the auxiliary fields F_i using $\frac{\partial \mathcal{L}}{\partial F_i^*} = 0$

this introduce the potential

$$V(A_i, A_j^*) = F_k^* F_k \geq 0$$

one can define the R-charge

R-symmetry

$$R\, \Phi(\theta, x) = e^{in\alpha}\, \Phi(e^{-i\alpha}\theta, x) \qquad \theta \text{ has R-charge } -1$$

$$R: A \to e^{in\alpha} A$$

$$\psi \to e^{i(n-1)\alpha}\psi$$

$$F \to e^{i(n-2)\alpha} F$$

in general one can have

$$\int d\theta\, d\theta\, W(\phi)$$

ϕ chiral fields   W has R charge +2

F superpotential   a F-term

Non-renormalization theorem

$\Leftarrow$ holomorphicity

W is holomorphic in chiral super fields

depend on Φ not on $\Phi^\dagger$

Regard the coupling constant λ in tree level W

as background fields

Then the renormalized effective superpotential is constrained by

1. Symmetry $\leftarrow$ By assigning transf laws both to the fields and to the coupling constants, the theory has a large symmetry

   The effective Lagrangian should be invariant under it

2. Holomorphy : W_{eff} is indep of $\lambda_i^\dagger$ $\leftarrow$ holomorphic in fields in coupling constants

3. various limits : W_{eff} can be analyzed approxi at weak coupling:

e.g.) Wess–Zumino Model

$$W_{tree} = m\phi^2 + g\phi^3$$

We will use two $U(1)$ symmetries

	$U(1)$	$\times$ $U(1)_R$
ϕ	1	1
m	-2	0
λ	-3	-1

W R-charge 2

$$W_{eff} = m\phi^2 f\left(\frac{\lambda\phi}{m}\right)$$

↑ charge neutral combination

$$W_{eff} = m\phi^2 \sum a_n \left(\frac{\lambda\phi}{m}\right)^n = \sum a_n \lambda^n \phi^{n+2} \frac{1}{m^{n-1}}$$

Consider the limit $\lambda \to 0$ and $m \to 0$ with arbitrary $\frac{\lambda}{m}$

We must have $W_{eff} \to W_{tree}$

$\left(f\left(t = \frac{\lambda\phi}{m}\right) \to 1+t \right.$ in this limit

Since t is arbitrary $\quad f(t) = 1+t$ for all t

Thus $W_{eff} = m\phi^2 + g\phi^3 = W_{tree}$

superpotential is not renormalized $\leftarrow$ this is an exact statement

Vector superfield

$$V = V^\dagger$$

$$V(x, \theta, \bar\theta) = \cdots - \theta\sigma^m\bar\theta V_m(x) + \cdots$$

↓ real

$$\Phi + \Phi^\dagger = i\theta\sigma^m\bar\theta\, \partial_m (A - A^*)$$

↓

SUSY generalization of a gauge transf.

$$V \to V + \Phi + \Phi^\dagger$$

Using this one can choose a suitable gauge st.

$$V = -\theta\sigma^m\bar\theta V_m(x) + i\theta\theta\,\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\,\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x)$$

↑ gaugino

SUSY gen. of YM potential

susy gen of field strength ?

$\lambda_\alpha \ \bar{\lambda}_{\dot{\alpha}}$ lowest dim gauge invari comp in V

↓

lowest-dim comp fields in

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\, D_\alpha V \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}} V$$

$$W_\alpha = -i\lambda_\alpha(y) + [\delta_\alpha{}^\beta D(y) - \frac{i}{2}(\sigma^m\bar{\sigma}^n)_\alpha{}^\beta(\partial_m V_n - \partial_n V_m)(y)]\theta_\beta$$
$$+ \theta\theta\,\sigma^m_{\alpha\dot{\alpha}}\,\partial_m\bar{\lambda}^{\dot{\alpha}}(y)$$

the gauge invariant Lagrangian is

$$\mathcal{L} = \frac{1}{4}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}})$$

$$\int d^4x\,\mathcal{L} = \int d^4x\,(\tfrac{1}{2}D^2 - \tfrac{1}{4}V^{mn}V_{mn} - i\lambda\sigma^m\partial_m\bar{\lambda})$$

gauge boson   gaugino

gauge interaction

$$\Phi' = e^{-it\lambda}\Phi$$

λ should be promoted to a full chiral matter

$$\Phi' = e^{-it\Lambda}\Phi \qquad \bar{D}_{\dot{\alpha}}\Lambda = 0$$

kinetic term $\quad \Phi^\dagger\Phi' = \Phi^\dagger\Phi\, e^{it(\Lambda^\dagger - \Lambda)}$

we need the vector superfield and

$$V' = V + i(\Lambda - \Lambda^\dagger)$$

$$\rightarrow \quad \mathcal{L} = 1/4\,(\overbrace{W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}}^{①}) + \overbrace{\Phi_i^\dagger e^{tV}\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}}}^{②}$$

$$+ \frac{1}{2}(m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k)|_{\theta\theta} + h.c$$

↓

susy QED

and suitable non-abelianization can be done

① ∈ ~~usual gauge kinetic~~ term

$$\frac{1}{4} \quad \frac{1}{2}D^2 - \frac{1}{4}V^{mn}V_{mn} - i\lambda\sigma^m\partial_m\bar{\lambda}$$

auxiliary field   gauge boson   gaugino

② ∈ gauge invariant interactions of scalars

Scalar potential

$$\frac{1}{2}D^2 + \frac{e}{2}D\,(A_+^\dagger A_+ - A_-^\dagger A_-)$$

$\rightarrow$

interaction term bet scalar & gauge boson

$$\frac{e}{2}D\,(|A_+|^2 - |A_-|^2) \qquad - \frac{1}{4}t^2 v_n v^n\,(|A_+|^2 + |A_-|^2)$$

charge +1 charge −1

gives the potential $(|A_+|^2 - |A_-|^2)^2$

$$|A_+| = |A_-|$$

$$A_+ = |a|\,e^{i\theta}$$

$$A_- = |a|\,e^{+i\theta'}$$

using gauge transf ~~$A_+ \to A_+ e^{i\Lambda}$~~

$$A_+ \to A_+ e^{i\alpha} \qquad A_- \to A_- e^{-i\alpha}$$

one can set $A_+ = A_- = a$ for any complex a

$\rightarrow$ degenerate vacua

$a \neq 0$ the phase group is broken by the susy Higgs mechanism

The gauge susy field gets mass $|a|$ by eating one chiral susy field

the other one is left massless

gauge invariant description $X = Q\tilde{Q}$

$$Q = \tilde{Q} = a \qquad X = a^2$$

Classically no potential for X — the field whose vev parametrizes the degen vacua.

"Modulus"

Classical vacuum degen is accidental and can be lifted quantum mechanically

∃ Non abelian gen of the above vector superfield:

Classical, SUSY th with gauge gp G

matter superfields Φ_f in repe $R(f)$ of G

the tree level potential term is

$$V = \sum_a \left(\sum_f \phi_f^+ T_f^a \phi_f \right)^2 \qquad \leftarrow \text{gen of } \left(|A_1|^2 - |A_{-1}|^2 \right)^2$$

T_f^a are the G-generators in repe $R(f)$

e.g., SUSY QCD

$SU(N_c)$ gauge theory with N_f quark flavors Q^λ in fundamental repe

$\tilde{Q}_{\tilde{\lambda}}$ in anti fundamental

$\lambda, \tilde{\lambda} = 1$ to N_f

In the absence of mass terms $\left(\leftarrow \right.$

$\int d^2\theta \; m_\lambda Q^\lambda \tilde{Q}_{\tilde{\lambda}}$

classical moduli space of vacua can be described by

$$Q = \tilde{Q} = \begin{array}{c} \\ \downarrow \\ N_f \end{array} \overset{\longrightarrow N_c}{\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_{N_f} & \end{pmatrix}} \qquad N_f < N_c$$

with a_i arbitrary

$$Q = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_{N_c} \end{pmatrix} \qquad \tilde{Q} = \begin{pmatrix} \tilde{a}_1 & & \\ & \tilde{a}_2 & \\ & & \tilde{a}_{N_c} \end{pmatrix}$$

$|a_\lambda|^2 - |\tilde{a}_\lambda|^2 =$ indep of λ

$(\sigma^a, \sigma^b) =$

σ^b

$[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}] = i \frac{\sigma^c}{2}$

$[-\frac{\sigma^{a*}}{2}, -\frac{\sigma^{b*}}{2}] = -i \frac{\sigma^{c*}}{2}$

$SU(2)$ $\quad N_f = 3 \quad T_f^a = \sigma^a \leftarrow$ Pauli matrices

$\swarrow Q^\lambda \quad \tilde{Q}^\lambda \quad (\sigma^a)^T$

$\searrow -(\sigma^a)^*$

$$Q^1 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \quad Q^2 = \begin{pmatrix} 0 \\ a_2 \end{pmatrix} \quad Q^3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \tilde{Q}_1 = \begin{pmatrix} \tilde{a}_1 \\ 0 \end{pmatrix} \quad \tilde{Q}_2 = \begin{pmatrix} 0 \\ \tilde{a}_2 \end{pmatrix}$$

$$V = \left(|a_1|^2 - |\tilde{a}_1| - |a_2|^2 + |\tilde{a}_2|^2 \right)^2$$

$N_f < N_c$ gauge invariant description of the classical moduli space is mesons $M^i_{\tilde{j}} = Q^i \tilde{Q}_{\tilde{j}}$ $(q\bar{q})$

$N_f \geq N_c$ can also form baryons, totally antisymmetric of baryon qqq color singlet

$B^{A_1 \dots A_{N_c}} = Q^{A_1} \dots Q^{A_{N_c}}$

$\tilde{B}_{\tilde{A}_1 \dots \tilde{A}_{N_c}} = \tilde{Q}_{\tilde{A}_1} \dots \tilde{Q}_{\tilde{A}_{N_c}}$

up to global sym transf

$$M = \begin{pmatrix} a_1 \tilde{a}_1 & & \\ & a_2 \tilde{a}_2 & \\ & & \ddots \\ & & & a_{N_c} \tilde{a}_{N_c} \end{pmatrix}$$

$B^{1 \dots N_c} = a_1 a_2 \dots a_{N_c}$

$\tilde{B}_{1 \dots N_c} = \tilde{a}_1 \tilde{a}_2 \dots \tilde{a}_{N_c}$

rk M is at most N_c

If the rk M is equal to N_c

B & $\tilde{B}$ have rk 1 and product of their eigenvalues is

$\det M = B\tilde{B}$ the same as the product of non-zero eigenvalue of M

I) $N_f < N_c$ No vacuum

determine the form of superpotential by the symmetries

At the classical level we have

$SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B \times U(1)_R$

where the quarks transf

$Q \; (N_f, 1, 1, 1, \frac{N_f - N_c}{N_f})$

$\tilde{Q} \; (1, \bar{N}_f, 1, -1, \frac{N_f - N_c}{N_f})$

$U(1)_R$ R-sym (gauginos $+1$, squark of Q $\tilde{Q}$ $\overbrace{R(Q)}$ the charge fermion $R(Q)-1$)

$\to$ charges were chosen s.t. only the $\underline{U(1)_A \text{ is anomalous}}$

$\exists$ a unique superpotential compatible with these sym

$$W_{eff} = \left(\frac{\Lambda^{3N_c - N_f}}{\det \tilde{Q} Q} \right)^{\frac{1}{N_c - N_f}} \quad -(1)$$

quark has d(1)

mass dim (3)

$\uparrow$ R charge of W is 2

(2N_f)

every scale factor is inserted by dimensional analysis

(m_{Q})

$$\Lambda^{3N_c - N_f} \qquad \Lambda^{\frac{3N_c - N_f}{N_c - N_f}} - \frac{2N_f}{N_c - N_f}$$

$\parallel (\Lambda^3) \parallel$

$N_f \geq N_c$: the superpotential does not exist

$\left(\frac{1}{\tilde{Q}Q}\right)^{(\tilde{Q}Q)}$

$N_f = N_c$ the exponent diverges

$N_f > N_c$ $\det \tilde{Q} Q$ vanishes

$\downarrow$

vacuum degen cannot be lifted $\propto \left(\frac{1}{\det \tilde{Q} Q} \right)$

on the other hand $N_c > N_f \Rightarrow$ no vacuum for finite moduli

all vacua are lifted

Consider give Q_{N_f} $\tilde{Q}_{N_f}$ a large mass " $+ \tilde{Q}_{N_f} Q_{N_f}$

by adding $W_{tree} = m M^{N_f}_{\tilde{N}_f}$

$$W_{eff} = \left(\frac{\Lambda^{3N_c - N_f}}{\det \tilde{Q} Q} \right)^{\frac{1}{N_c - N_f}} + m M^{N_f}_{\tilde{N}_f}$$

one can check that in the large m limit

this flows nice to W_{eff} for $(N_c, N_f - 1)$ } consistency check

It's known that $N_c = N_f - 1$ the superpotential is generated due to one instanton effect

Consider $W = (1) + \mathrm{Tr}\, m M$

" Any masses for all quarks

For simplicity choose $N_c = N_f + 1$

$$W = \frac{\Lambda^{3N_c - N_f}}{\det M} + (m M_1 + \cdots + m M_{N_f})$$

$\det M = M_1 \cdots M_{N_f}$

$$\frac{\partial W}{\partial M_1} = 0 \rightarrow -\frac{1}{M_1^2}\frac{\Lambda^{3N_c-N_f}}{M_2 \cdots M_{N_f}} + m = 0$$

$$\rightarrow \frac{1}{M_1}\frac{\Lambda^{3N_c-N_f}}{\det M} = m$$

$$\rightarrow M_1 = \frac{1}{m}(m\,\Lambda^{3N_c-N_f})^{\frac{1}{N_c}} \rightarrow M^i_j = (\det m\, \Lambda^{3N_c-N_f})^{1/N_c}\left(\frac{1}{m}\right)^i_j \quad (1)$$

$M_1 = M_2 = \cdots = M_{N_f}$

$\downarrow$

N_c - vacua

For large mass m, $\rightarrow$ pure $SU(N_c)$ SYM

$\downarrow$

discrete N_c-vacua with mass gap /

- $N_f = N_c$ quantum moduli space with confinement
chiral sym. breaking

Classically $\det M - B\tilde{B} = 0$

this space has a singular submanifold $B = \tilde{B} = 0$, $\mathrm{rk} M \leq N_c - 2$

where $d(\det M - B\tilde{B}) = 0$

the singularities reflect that there will be additional massless d.o.f. ($\in$ gluons)

The quantum moduli space

$$\det M - B\tilde{B} = \Lambda^{2N_c} \quad (2)$$

If we add $W = m\,\mathrm{Tr}\, M$

from (1) we have $\det\langle M\rangle = \Lambda^{2N_c}$ indep. of m for $N_f = N_c$

Agrees with (2) and $\langle B\tilde{B}\rangle = 0$

no singularities on the quantum moduli space

all the classical singularities are smoothed as

cf $XY = \mu$

no singularities

the only massless d.o.f are moduli

no photons → confinement just see M. B $\tilde{B}$

massless

$M = B = \tilde{B} = 0$ is not a part of the moduli space

so chiral $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$ should break

e.g. $M^i_{\tilde{j}} = \Lambda^2 \delta^i_{\tilde{j}}$ $B = \tilde{B} = 0$

$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \to SU(N_f)_V \times U(1)_B \times U(1)_R$

$M = 0$ $B = -\tilde{B} = \Lambda^{N_c}$

$\to SU(N_f)_L \times SU(N_f)_R \times U(1)_R$

no vacuum with the full chiral sym

$N_f = N_c + 1$ ∵ confinement without chiral sym breaking

$$W = \underline{M^i_{\tilde{j}} B_n \tilde{B}^{\tilde{j}} - \det M}$$

dim $M - B\tilde{B} =$

check + W-tree $= m M^{N_c+1}_{N_c+1}$

are consistent $N_f = N_c$ case

quantum moduli space is the same as the classical moduli space

but at the origin we have massless mesons and baryons

→ confinement without chiral symmetry breaking