Gravitational Compton scattering from the worldline formalism

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1. Worldline formalism in flat space-time, master formula for SQED
2. Worldline formalism in curved space, master formula for gravitational Compton scattering
3. Conclusions and Outlook

Worldline method: Quantum field theory results from quantization of **Quantum models.**

- Main tools in use:
  particle actions:

$$S[x,\psi,G]=\int_0^T d\tau\Big(\dot{x}^2+\psi\cdot\dot{\psi}+V(x,\dot{x},\psi,G)\Big)$$

$x\rightarrow\text{Bosonic}\qquad\psi\rightarrow\text{Fermionic}\qquad G\rightarrow\text{External}$

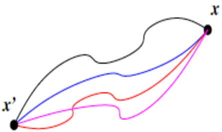
- Canonical quantization

- Path integral (integral over trajectories)

Purely Bosonic

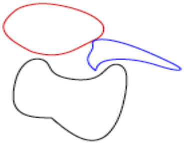
- Dirichlet boundary conditions (topology of a line)

$$\langle x|e^{-HT}|x'\rangle = \int_{x(0)=x'}^{x(T)=x} Dx(\tau)\, e^{-S[x,G]}$$



- Periodic boundary conditions (topology of a closed line)

$$Z(T) = \int_{x(0)=x(T)} Dx(\tau) e^{-S[x,G]}$$



Tools to compute:

- Green function (propagators)
- Effective action. i.e. functional generators of effective vertices using particle models.
  review by Schubert 2001

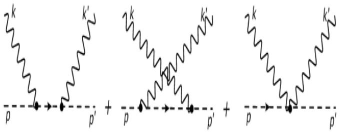
Why develop alternative (to 2nd qzn) tools?

- in some cases conventional QFT is not the most effective way to compute
- other methods have proved very succesful in the computation of S-matrix elements:
  - MHV amplitudes, holomorphic methods,... Bern, Kosower, Cachazo,.... mostly at tree level with massless particle

worldline formalism works well also with massive particle and at one loop

Advantages

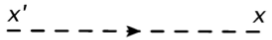
- no need to compute momentum integrals or Dirac traces explicitly
- efficient way to path order
- directly obtain off-shell Feynman amplitudes, rather than single Feynman diagrams
  Example: Compton scattering in scalar QED:



- gauge-invariance efficiently guaranteed

Worldline Green's function

$$\big(-\partial_\mu \partial^\mu + m^2\big)\Delta(x,x') = \delta(x-x') \quad \equiv$$



- Massive scalar field (Feynman) propagator

$$\Delta(x,x') = \langle \phi(x)\phi(x')\rangle = \int d^4p \, \frac{e^{-ip\cdot(x-x')}}{p^2+m^2}$$

- Schwinger representation

$$\begin{aligned}\langle \phi(x)\phi(x')\rangle &= \int_0^\infty dT \int d^4p \; e^{-ip\cdot(x-x')-T(p^2+m^2)} \\ &= \int_0^\infty dT \int d^4p \; \langle x|e^{-T(p^2+m^2)}|p\rangle\langle p|x'\rangle\end{aligned}$$

- Replacing p with $\mathbb{p}$ can integrate over p

$$\langle \phi(x)\phi(x')\rangle = \int_0^\infty dT \; e^{-Tm^2} \langle x|e^{-T\mathbb{p}^2}|x'\rangle \,, \quad \mathbb{H} = \mathbb{p}^2 = \delta_{\mu\nu}\mathbb{p}^\mu \mathbb{p}^\nu$$

- Path integral representation of transition element

$$\langle\phi(x)\phi(x')\rangle = \int_0^\infty dT\ e^{-Tm^2} \int_{x(0)=x'}^{x(T)=x} Dx\ e^{-S[x]}$$

$$S[x(\tau)] = \frac{1}{4T}\int_0^1 d\tau\ \delta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

$\langle\phi(x)\phi(x')\rangle \rightarrow$ Worldline representation for the free scalar field propagator $=$ path integral on the line

$S[x(\tau)] \rightarrow$ Worldline action

Tree-level scalar QED

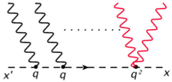
Coupling to external photons: replace $\mathbb{p}_\mu$ with $\Pi_\mu = \mathbb{p}_\mu + qA_\mu$

$$\langle \phi(x)\bar{\phi}(x')\rangle_A = \int_0^\infty dT\; e^{-Tm^2} \int_{x(0)=x'}^{x(T)=x} Dx\; e^{-S[x,A_\mu]}$$

$$S[x(\tau), A_\mu] = \int_0^1 d\tau\; \left(\frac{1}{4T}\delta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + iq\dot{x}^\mu A_\mu(x(\tau)) \right)$$

- treat A_μ perturbatively: the latter reproduces all tree-level diagrams of scalar QED with 2 scalars and n photons, the "n-propagator"

$$\langle \phi(x)\bar{\phi}(x')\rangle_A = \sum$$



- the WL linear coupling also reproduces the sea-gull coupling of QFT

Tree-level scalar QED

Recipe:

- Write potential as trivial background plus sum of photons

$$A_\mu(x(\tau)) = \sum_{i=1}^{n} \epsilon_{i,\mu} e^{ik_i\cdot x(\tau)}$$

- expand $e^{-iq\int \dot{x}\cdot A}$ and pick up terms linear in all polarizations: it involves a QM correlation function

$$\mathcal{A}[x,x',k_1,\epsilon_1;\cdots;k_n,\epsilon_n] = q^n \int_0^\infty dT e^{-Tm^2} \prod_{i=1}^{n} \int_0^1 d\tau_i$$

$$\times \int_{x(0)=x'}^{x(1)=x} Dx\ e^{-\frac{1}{4T}\int \dot{x}^2} e^{\sum_i \epsilon_i\cdot\dot{x}(\tau_i)+ik_i\cdot x(\tau_i)}\Big|_{m.l.\epsilon_i}$$

- split $x(\tau) = x_{bg}(\tau) + y(\tau)$ con $y(0) = y(1) = 0$
- then $\langle y(\tau)y(\tau')\rangle \sim \int_{y(0)=0}^{y(1)=0} Dy\ y(\tau)y(\tau')e^{-\frac{1}{4T}\int \dot{y}^2} = -2T\Delta(\tau,\tau')$

  $\Delta(\tau,\tau')$ particle Green's function

$$\mathcal{A}[x,x',k_1,\epsilon_1;\cdots;k_n,\epsilon_n] = q^n \int_0^\infty \frac{dT}{(4\pi T)^2} e^{-Tm^2-\frac{1}{4T}(x-x')^2} \prod_{i=1}^{n} \int_0^1 d\tau_i$$

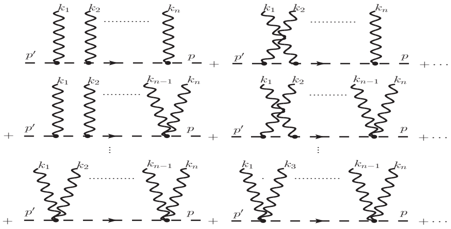
$$e^{\sum_i \left(ik_i\cdot x'+(i\tau_i k_i+\epsilon_i)\cdot(x-x')\right)} e^{T\sum_{i,j}\left(k_i\cdot k_j\Delta_{ij}-i2\epsilon_i\cdot k_j {}^\bullet\Delta_{ij}-\epsilon_i\cdot\epsilon_j {}^\bullet\Delta^\bullet_{ij}\right)}\Big|_{m.l.\epsilon_i}$$

To get a full momentum amplitude, use Fourier transform $\int dx dx' e^{i(p\cdot x+p'\cdot x')}$

$$\tilde{\mathcal{A}}[p,p',k_1,\epsilon_1;\cdots;k_n,\epsilon_n] = q^n \int_0^\infty dT\; e^{-T(m^2+p^2)} \prod_{i=1}^{n} \int_0^1 d\tau_i$$

$$e^{T(p-p')\cdot\sum_i(-\tau_i k_i + i\epsilon_i)} e^{T\sum_{i,j}\left(k_i\cdot k_j \Delta_{ij} - i2\epsilon_i\cdot k_j \dot{\Delta}_{ij} + \epsilon_i\cdot\epsilon_j \ddot{\Delta}_{ij}\right)}\Big|_{m.l.\epsilon_i}$$

where $\Delta_{ij} = \frac{1}{2}|\tau_i - \tau_j|, \quad \Rightarrow \quad \dot{\Delta}_{ij} = \frac{1}{2}\mathrm{sign}(\tau_i - \tau_j), \quad \ddot{\Delta}_{ij} = \delta(\tau_i - \tau_j)$



- integrals over T and τ_i are the Feynman parametrization of scalar free propagators
- on-shell the integrand is fully τ-translation invariant
- the external scalar lines aren't (yet) truncated
- see: N. Ahmadiniaz, F. Bastianelli and O. Corradini; arXiv:1508.05144
- N. Ahmadiniaz, A. Bashir and C. Schubert (to appear soon)

Worldline formalism in curved space

Nonlinear Sigma Model:

- Allows to compute amplitudes with gravitons.
- Scalar field line with external gravitons: described by scalar worldline actions in curved space.

$$\langle\phi(x)\phi(x')\rangle_g = \int_{x(0)=x'}^{x(1)=x} \mathcal{D}x \; e^{-S[x,g]}$$

$$S[x;g] = \frac{1}{4T}\int_0^1 d\tau g_{\mu\nu}(x(\tau))\dot{x}^\mu \dot{x}^\nu$$

- Einstein-invariant measure $\mathcal{D}x = \prod_\tau \sqrt{g(x(\tau))} d^4x(\tau)$
- can be reproduced via the path integral of auxiliary fields $\int \mathcal{D}x = \int DxDaDbDc \; e^{-\frac{1}{4T}\int g_{\mu\nu}(a^\mu a^\nu + b^\mu c^\nu)}$
- $a \rightarrow$ bosonic ghost field $\rightarrow \langle a^\mu(\tau_1)a^\nu(\tau_2)\rangle = 2\delta(\tau_1 - \tau_2)\delta^{\mu\nu}$
- $b, c \rightarrow$ fermionic ghost fields $\rightarrow \langle b^\mu(\tau_1)c^\nu(\tau_2)\rangle = -4\delta(\tau_1 - \tau_2)\delta^{\mu\nu}$.

Recipe:

- Write $g_{\mu\nu}$ as a sum of gravitons

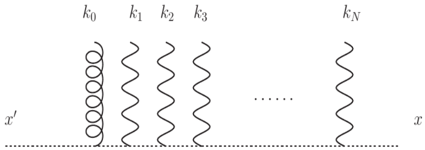
$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \kappa \sum_{l} \zeta_{\mu\nu,l}\ e^{ik_l \cdot x}$$

- get a vertex operator

$$V_h[k,\zeta] = \frac{1}{4T}\int_0^1 \zeta_{\mu\nu}(\dot{x}^{\mu}\dot{x}^{\mu} + a^{\mu}a^{\nu} + b^{\mu}c^{\nu})e^{ik\cdot x(\tau)}$$

- $\zeta_{\mu\nu} = \varepsilon_{0\mu}\varepsilon_{0\nu}$
- coupling is "abelian" $\rightarrow$ no path ordering.
- Vertex operator is quadratic.
- Single worldline diagrams are singular $\rightarrow$ need regularization.
- After regularization, the ghost field contributions will cancel all divergent terms.

Amplitude with two scalars, one-graviton and N-photons



Almost the same way that we found the master formula for N-photon one gets: for x-space

$$\mathcal{A}[x'; x; k_0, \varepsilon_0; k_1, \varepsilon_1; \cdots; k_N, \varepsilon_N] = (-\frac{1}{4}\kappa)(-ie)^N \int_0^\infty e^{-Tm^2} e^{-\frac{1}{4T}(x-x')^2} (4\pi T)^{-\frac{D}{2}}$$
$$\times \int_0^T \prod_{i=0}^N d\tau_i \, e^{\sum_{i=0}^N (\varepsilon_i \cdot \frac{(x-x')}{T} + ik_i \cdot (\frac{x-x'}{T})\tau_i + ik_i \cdot x')} \, e^{\sum_{i,j=0}^N [\Delta_{ij} k_i \cdot k_j - 2i {}^\bullet\Delta_{ij} \varepsilon_i \cdot k_j - {}^\bullet\Delta^\bullet_{ij} \varepsilon_i \cdot \varepsilon_j] - 2\delta(0)\varepsilon_0 \cdot \varepsilon_0}$$

which one must take terms which are **linear in all photon polarizations and multilinear in graviton's polarization**.

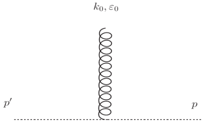
An in momentum space:

$$\mathcal{A}[p';p;k_0,\varepsilon_0;k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-\frac{1}{4}\kappa)(-ie)^N(2\pi)^D\delta^D\Big(p'+p+\sum_{i=0}^{N}k_i\Big)\int_0^\infty dT\, e^{-T(m^2+p^2)}$$

$$\times\int_0^T\prod_{i=0}^{N}d\tau_i\, e^{\sum_{i,j=0}^{N}\left[-2k_i\cdot p\tau_i+2i\varepsilon_i\cdot p+(\frac{|\tau_i-\tau_j|}{2}-\frac{\tau_i+\tau_j}{2})k_i\cdot k_j-i(\mathrm{sign}(\tau_i-\tau_j)-1)\varepsilon_i\cdot k_j+\delta(\tau_i-\tau_j)\varepsilon_i\cdot\varepsilon_j\right]-2\delta(0)\varepsilon_0\cdot\varepsilon_0}$$

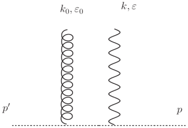
- Some especial cases:
  - $N=0$ →no photon → graviton coupling to scalar propagator!
  - $N=1$ → Compton scattering with one photon and one graviton!

$N=0$



$$\mathcal{A}[p;p';k_0,\varepsilon_0] = +(2\pi)^D\delta^D(p+p'+k_0)\frac{1}{(m^2+p^2)}\Big[\frac{\kappa}{2}\varepsilon_{0\mu\nu}p'^{\mu}p'^{\nu}\Big]\frac{1}{(m^2+p'^2)}$$

$N = 1$



$$\mathcal{A}[p; p'; k_0, \varepsilon_0; k_1, \varepsilon_1]\Big|_{\rm off-shell} = -\frac{\kappa e}{4}(2\pi)^D \delta^D(p' + p + k_0 + k_1)\frac{1}{(m^2 + p^2)(m^2 + p'^2)}$$

$$\times \Big\{ \Big[4(\varepsilon_0 \cdot p')^2(\varepsilon_1 \cdot p) + 4(\varepsilon_1 \cdot p)(\varepsilon_0 \cdot p')(\varepsilon_0 \cdot k_0) + 2(\varepsilon_0 \cdot p)(\varepsilon_0 \cdot k_1)(\varepsilon_1 \cdot k_1)$$

$$+2(\varepsilon_0 \cdot k_1)^2(\varepsilon_1 \cdot k_1) - 2(\varepsilon_1 \cdot k_1)(\varepsilon_0 \cdot p)(\varepsilon_0 \cdot p') + (\varepsilon_0 \cdot k_0)^2\left[(\varepsilon_1 \cdot p) + \frac{1}{2}(\varepsilon_1 \cdot k_1)\right]$$

$$+2(\varepsilon_0 \cdot k_1)(\varepsilon_0 \cdot k_0)(\varepsilon_1 \cdot k_1)\Big]\frac{1}{[m^2 + (p' + k_0)^2]}$$

$$+\Big[-4(\varepsilon_0 \cdot p)^2(\varepsilon_1 \cdot p') - 2(\varepsilon_0 \cdot p)^2(\varepsilon_1 \cdot k_1) - 4(\varepsilon_0 \cdot p)(\varepsilon_0 \cdot k_0)(\varepsilon_1 \cdot p')$$

$$+4(\varepsilon_1 \cdot k_0)(\varepsilon_0 \cdot k_0)^2 + 2(\varepsilon_0 \cdot k_0)^2(\varepsilon_1 \cdot p)$$

$$+\frac{1}{2}(\varepsilon_1 \cdot k_1)(\varepsilon_0 \cdot k_0)^2 - (\varepsilon_0 \cdot \varepsilon_0)[2\varepsilon_1 \cdot p' + \varepsilon_1 \cdot k_1]\Big]\frac{1}{[m^2 + (p + k_0)^2]}\Big\}$$

$$+\frac{2\kappa e}{4}(2\pi)^D \delta^D(p' + p + k_0 + k_1)(\varepsilon_0 \cdot \varepsilon_1)\varepsilon_0 \cdot (p - p')$$

$$\mathcal{A}[p;p';k_0,\varepsilon_0;k_1,\varepsilon_1]\Big|_{\mathrm{on-shell}} = 2\kappa e(2\pi)^D\delta^D\big(p+p'+k_0+k_1\big)$$

$$\times\Big[\frac{(\varepsilon_0\cdot p)^2(\varepsilon_1\cdot p')}{p'\cdot k_1} - \frac{(\varepsilon_1\cdot p)(\varepsilon_0\cdot p')^2}{p'\cdot k_0} - (\varepsilon_0\cdot\varepsilon_1)\varepsilon_0\cdot(p'-p)\Big]$$

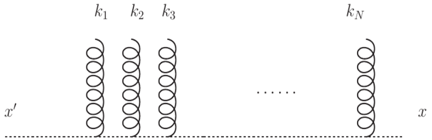
These results are in agreement with previous calculations (on-shell),

- B. R. Holstein, PRD, 74, 085002 (2006)
- S. Y. Chai, J. Lee, J. S. Shim, and H. S. Sang, PRD, 48, 769 (1993)

Amplitude with two scalars and N gravitons:

$$V_h[k,\zeta] = \frac{1}{4T}\int_0^1 \zeta_{\mu\nu}(\dot{x}^\mu \dot{x}^\mu + a^\mu a^\nu + b^\mu c^\nu)e^{ik\cdot x(\tau)}$$

$$\mathcal{A}(x,x';k_1,\zeta_1;...;k_N,\zeta_N) \sim \kappa^n \Big\langle V_h[k_1,\zeta_1]\cdots V_h[k_N,\zeta_N]\Big\rangle$$



Write $\zeta_{\mu\nu} =: \varepsilon_\mu \varepsilon'_\nu$ and $\epsilon_\mu := \varepsilon_\mu + \varepsilon'_\mu$

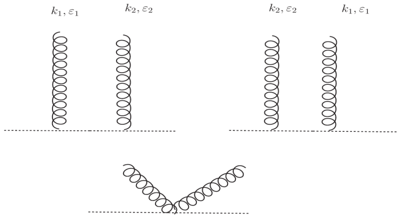
$$\zeta_{\mu\nu}(\dot{x}^\mu \dot{x}^\mu + a^\mu a^\nu + b^\mu c^\nu)e^{ik\cdot x} \cong e^{ik\cdot x + (\varepsilon+\varepsilon')\cdot(\dot{x}+a)+\varepsilon\cdot b+\varepsilon'\cdot c}\Big|_{\text{m.l., no mix}}$$

$$\left\langle V_h[k_1,\zeta_1]\cdots V_h[k_n,\zeta_n]\right\rangle = \prod_{l=1}^{n}\int_0^1 \frac{d\tau_i}{4T}\; e^{\sum_l \left[ik_l\cdot(x'+(x-x')\tau_l)+\epsilon_l\cdot(x-x')\right]}$$

$$e^{T\sum_{ll'}\left[k_l\cdot k_{l'}\Delta_{ll'}-2i\epsilon_l\cdot k_{l'}\,{}^{\bullet}\Delta_{ll'}-\epsilon_l\cdot\epsilon_{l'}({}^{\bullet}\Delta^{\bullet}_{ll'}+\Delta_{gh,ll'})\right]}\Big|_{\mathrm{m.l.}},$$

$$\mathcal{A}(x,x';k_1,\zeta_1;\ldots;k_N,\zeta_N)\sim\kappa^n\left\langle V_h[k_1,\zeta_1]\cdots V_h[k_N,\zeta_N]\right\rangle$$

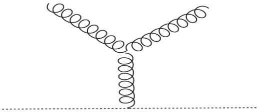
And from this master formula after some algebra one get the Gravitational Compton Scattering ($N=2$)



and its on-shell version

$$\mathcal{A}_{a,b}[p;p';k_1,\varepsilon_1;k_2,\varepsilon_2] = \frac{2\kappa^2}{(m^2+p'^2)(m^2+p^2)}(2\pi)^D\delta^D\Big(p+p'+k_1+k_2\Big) \times\left[\frac{(\varepsilon_1\cdot p')^2(\varepsilon_2\cdot p)^2}{k_2\cdot p}+\frac{(\varepsilon_1\cdot p)^2(\varepsilon_2\cdot p')^2}{k_1\cdot p}\right] \tag{1}$$

- We do not have the result for the seagull diagram yet.
- From our method we do not calculate the g-pole diagram, we are trying to find a way to include this diagram into our master formula.



- Worldline formalism efficient alternative to standard QFT
- Obtained several new applications of the method, at one-loop level tree level, for QED and QCD
- Fields with spin at tree-level: abelian and non-abelian spinning particles.
- Ball-Chiu vertices Ahmadiniaz and Schubert, 2013 and 2014
- Bound states. Done for scalar fields Bastianelli, Huet et al 2014
- KLT relations between graviton amplitudes and gauge amplitudes