# Gravity on Equation of State

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- **Newtonian gravity:**
  **In preparation,** H.K., Gungwon Kang (KISTI),
- **General relativity:**
  **In Preparation,** H.K., Chueng Ji (NCSU).

## Motivations

Matters in a star

$p_{\rm poly} \equiv K\rho^{\Gamma}$ ?

Palatini-*f(R)* and Eddington-inspired Born-Infeld gravity are plagued by the surface singularity problem:(Sotiriou, Faraoni, 2010; Olomo, 2011).

Singularity disappears if one allow extremely strong gravity modifies the matter EoS (H.K., 2014)
→ Save the theory.

Q: It appears genuine that strong gravity affects on EoS.
Are there similar effects in GR and Newtonian?

# Equation of State in General Relativity (Ordinary Wisdom)

## General Covariance:

Freely falling frame = locally flat
EoS in freely falling frame = EoS in flat ST

## Scalar quantity

Density, pressure, temperature are scalar quantities.
Therefore, their values in other frames must be the same as those in the freely falling frame.

## Extremely strong curvature?

A statistical system has a size.
When curvature or gravity is extremely strong, is it possible to set up a locally flat coordinates for a given system?

→ We need to check it.

# Newtonian Stars and Equation of State

## Spherically symmetric Newtonian star:

$$dP(r) = -\rho(r) g(r) dr, \quad g(r) = \frac{GM(r)}{r^2} = \frac{4\pi G}{r^2} \int_0^r dr' r'^2 \rho(r') dr'$$

**A <u>relation btw pressure and density</u> is necessary.**

This is not appropriate to integrate the EoS.

## Equation of state 1:

$$PV = Nk_B T,$$ **(Ideal gas)**

## An additional constraint: Adiabaticity

$$dS = 0,$$

$$dT = -PdV/C_V, \leftarrow dU = TdS - PdV,$$

$$PdV + VdP = -\frac{Nk_B}{C_V} PdV \quad \Longrightarrow \quad \boxed{P = K\rho^\gamma;} \quad \gamma = \frac{C_V + Nk_B}{C_V},$$

## Equation of state 2:

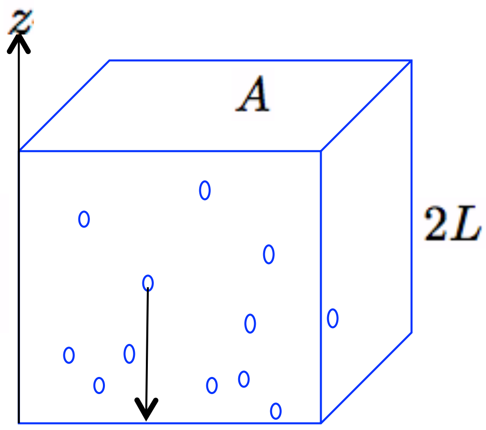
**With polytropic type EoS, one can integrate the EoM.**

# Generalization: Ideal Gas in Constant Gravity

## *N*-particle system in a box:

**One particle Hamitonian:**

$$H = \frac{1}{2}\mu_0 v^2 + \mu_0 g z, \qquad -L \le z \le L,$$




Landsberg, et. al. (1994).

## Statistics in a canonical ensemble:

$$\log Z_N = N \log Z_1 - \log N!\,,$$

$$\log Z_1 \equiv \log\left[\left(\frac{\mu_0}{h}\right)^3 \int_V d^3x \int d^3v\, e^{-\beta H}\right] = \log \frac{V}{(\hbar/(\mu_0 c))^3} + \frac{3}{2}\log\frac{k_B T}{2\pi\mu_0 c^2} + \log\frac{\sinh X}{X}.$$

$$V = 2L \times A,$$

## Order parameter for gravity:

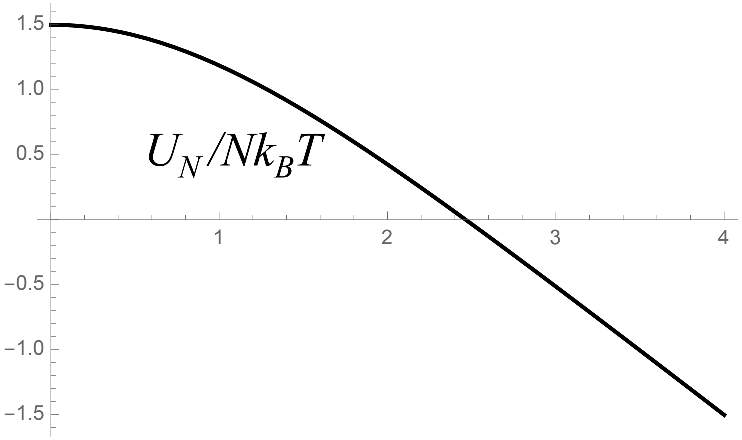
$$X \equiv \frac{M\mathcal{G}}{N k_B T} = \frac{MgL}{N k_B T} \ge 0; \qquad \mathcal{G} \equiv gL, \quad M = N\mu_0.$$

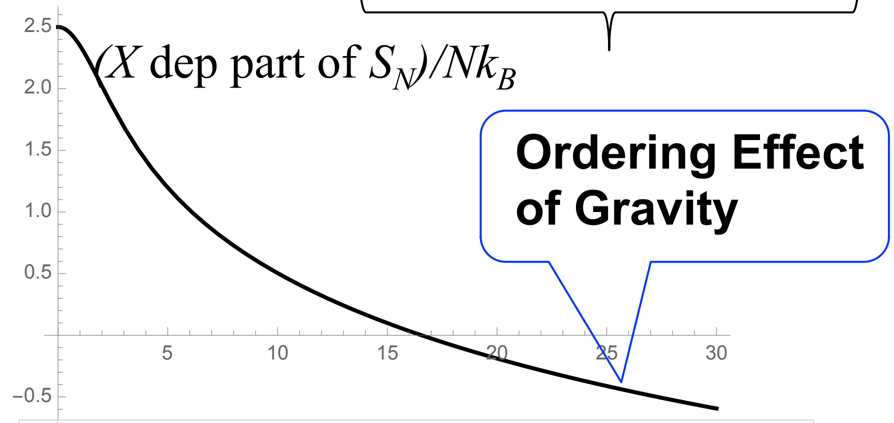
**Ratio btw the grav. Potential energy to the thermal kinetic energy.**

# Internal energy and entropy:

$$U_N(T,X) \equiv -\left[\frac{\partial \log Z_N}{\partial \beta}\right]_V = \left(\frac{5}{2} - X\coth(X)\right) N k_B T.$$

$$\frac{S_N}{N k_B} \equiv \frac{U_N}{N k_B T} + N^{-1}\log Z_N(X) = \frac{7}{2} + \log\frac{V/N}{(\hbar/\mu_0 c)^3} + \frac{3}{2}\log\frac{k_B T}{2\pi\mu_0 c^2} + \log\left(\frac{\sinh X}{X}\right) - X\coth X.$$



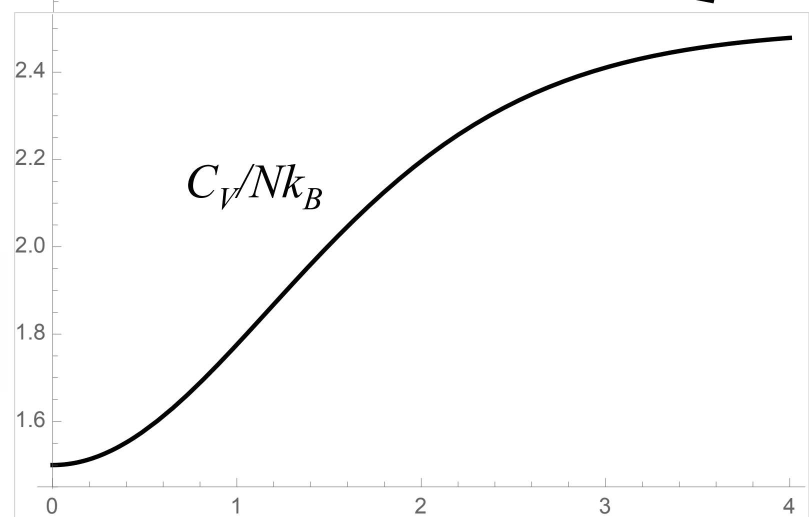




# Heat capacity:

$$C_V \equiv \frac{\partial U_N}{\partial T} = C_{V,0} + N k_B \left(1 - \frac{X^2}{\sinh^2 X}\right)$$

**Monatomic gas**

$$C_{V,0} = 3 N k_B / 2.$$



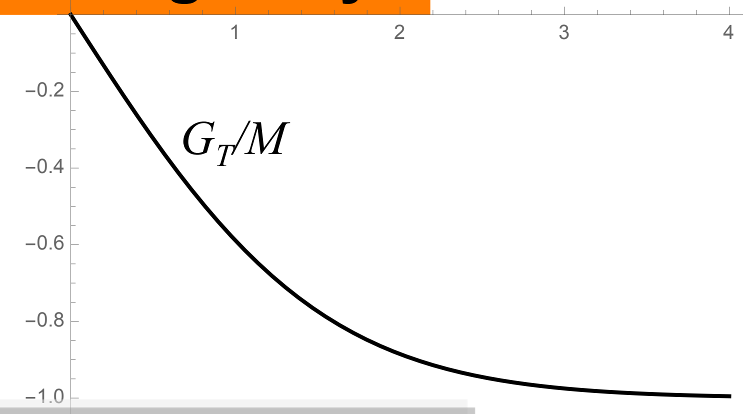
# Ideal Gas in Constant Gravity

## The total energy is dependent on the gravity:

$$\Omega = U_N - K = Nk_BT(1 - X\coth X),$$

**Heat capacity with $T$ fixed:**

$$\frac{G_T}{M} \equiv \frac{1}{M}\frac{\partial\Omega}{\partial\mathcal{G}} = \frac{X}{\sinh^2 X} - \coth X.$$




## Distribution of particles is position dependent:

$$n(z) \equiv \int d^3v\, n(z,v) = \frac{N}{V}\frac{X}{\sinh X}e^{-\beta\mu_0 gz}.$$

$$P(z) \equiv \int_{v_z>0} d^3v(2p_z)v_z n(z,v) = \bar{P}\frac{X}{\sinh X}e^{-\frac{\mu_0 gz}{k_BT}} = n(z)k_BT,$$

**However, local and the averaged values satisfy the ideal gas law.**

$$\bar{P} \equiv \frac{1}{2L}\int_{-L}^{L} P(z)dz = \frac{Nk_BT}{V}$$

# The new EoS in the **adiabatic** case

**First law:** $dU_N = k_B T dS - \bar{P} dV + \Omega (d \log \mathcal{G}).$

**The energy is dependent on both of the temperature and gravity:**

$$dU_N = C_V dT + G_T d\mathcal{G} = \frac{C_V}{N k_B}(V d\bar{P} + \bar{P} dV) + G_T d\mathcal{G},$$

**Adiabaticity:** $dS = 0,$

**We get,**

$$\frac{3}{2}\frac{d\bar{P}}{\bar{P}} + \frac{5}{2}\frac{dV}{V} = \left(1 - \frac{X^2}{\sinh^2 X}\right)\frac{dX}{X}.$$

**which can be integrated to give a *new* EoS,** $\rho \equiv M/V$

$$\bar{P} = K\rho^{5/3}\left(\frac{X e^{X \coth(X) - 1}}{\sinh X}\right)^{2/3} = \bar{K}(X)\rho^{5/3}; \qquad \bar{K} \equiv K\left(\frac{X e^{X \coth(X) - 1}}{\sinh X}\right)^{2/3}.$$

$X = \frac{M\mathcal{G}}{N k_B T} = \frac{M\mathcal{G}}{\bar{P}V} = \frac{\rho g L}{\bar{P}}$ **contains thermodynamic variables.**

**This gives difference from the old EoS.**

# The new EoS: Limiting behaviors

**Weak gravity limit:** $\bar{P} \approx K\rho^{5/3}\Big[1 + \frac{1}{9}\Big(\frac{\mathcal{G}}{K\rho^{2/3}}\Big)^2 + \cdots\Big].$

The correction is **second order**.
Therefore, one can ignore this correction in the small size limit of the system.

**Therefore, for most astrophysical systems, the gravity effects on EoS can be ignored.**

**Strong gravity (macroscopic system) limit:** $\bar{P} \approx K^{3/5}\left(\frac{2M\mathcal{G}}{e}\right)^{2/5}\rho^{7/5}.$

shows noticeable difference even in the non-relativistic, Newtonian regime: $k_B T < \mu_0 g L \ll \mu_0 c^2.$

## Then, when can we observe the gravity effect?

**A: Only when the macroscopic effects must be unavoidable.**

Macroscopic: size > kinetic energy/gravitational force

1. **System is being kept in thermal equilibrium compulsory.**

2. **The (self) gravity (or curvature) increases equally or faster than the inverse of system size. (e.g., Palatini *f(R)* gravity near the star surface. This is impossible in GR.)**

3. **The size of the system is forced to be macroscopic.**
   **Ex) The de Broglie wavelength of the particles is very large (light, slowly moving particles e.g., the scalar dark matter).**
   **→ Require quantum mechanical treatment.**

4. **Near an event horizon where the gravity diverges.**
   **→ Require general relativistic treatment.**

# Self-gravitating sphere in thermal equilibrium

1. Self-gravitating ideal gas in a box of radius $r = r_\star$.

2. Assume that the <u>whole system is at the same temperature.</u>
   (isothermal system in the presence of gravity)

## Strategy

1. Do statistics as if the gravitational potential is given.
   → relation between the density and pressure = EoS

2. Solve gravitational EoM:
   → get the gravitational potential.

3. Insert the obtained potential back to the statistics:
   → get physical quantities.

# Self-gravitating sphere in thermal equilibrium

## Statistics with a given the potential $V(r)$

$$\log Z_1 = -\beta V_0 + \frac{3}{2}\log\left(\frac{\mu_0}{2\pi\beta\hbar^2}\right) + \log\frac{4\pi r_\star^3}{3} + \log\mathcal{Z},$$

$$\mathcal{Z} = \frac{3}{4\pi r_\star^3}\int e^{-\beta(V(r)-V_0)}d^3r.$$

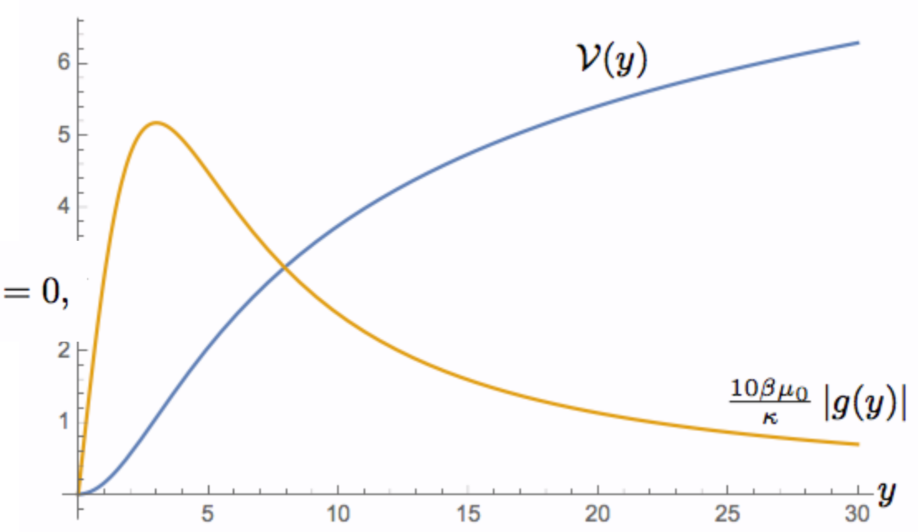
**Density:** $$\rho(r) \equiv \mu_0 \times n(r) = \bar{\rho}\times\frac{e^{-\beta(V(r)-V_0)}}{\mathcal{Z}},$$

**Pressure:** $$P(r) = \mu_0^{-1}k_B T\rho(r).$$

## Determine the potential:

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G r^2\rho(r) \quad \Rightarrow \mathcal{V}''(y) + \frac{2}{y}\mathcal{V}'(y) - e^{-\mathcal{V}} = 0,$$

$$\kappa \equiv \beta^{1/2}\times\left(\frac{4\pi G\mu_0\bar{\rho}}{\mathcal{Z}}\right)^{1/2}, \qquad V(r) \equiv V_0 + \beta^{-1}\mathcal{V}(\kappa r),$$



# Re-inserting the potential to the partition function,

**Gravity part of the partition fn:** $\mathcal{Z} \equiv \mathcal{Z}(X) = \frac{3}{X^3}\int_0^X dy\, y^2 e^{-\mathcal{V}(y)}, \qquad X = \kappa r_\star.$

**Asymptotic forms:**

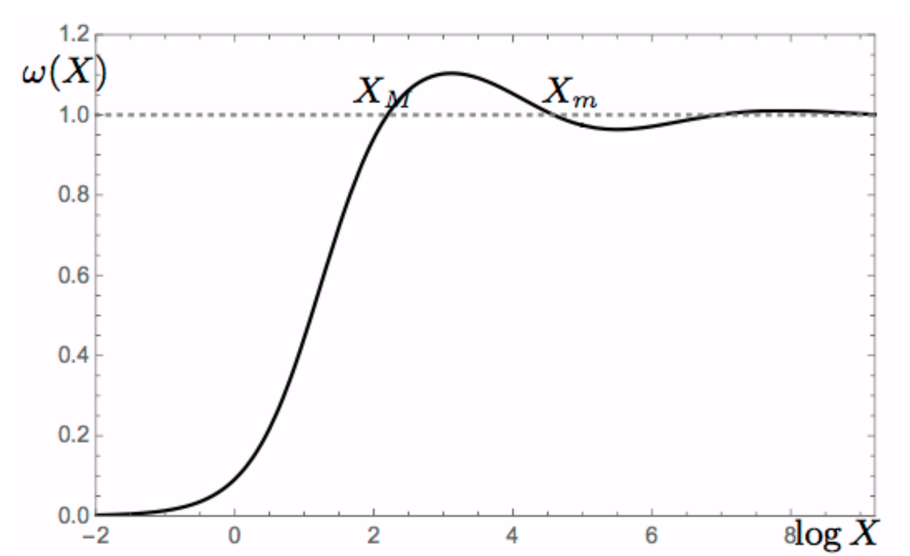
$$\mathcal{Z}(X) \approx \begin{cases} 1 - \frac{X^2}{10} + \cdots, & X < 1 \\ \frac{6}{X^2}\left[1 - c\sqrt{\frac{2}{X}}\cos\left(\frac{\sqrt{7}}{2}\log X - \phi'\right) + \cdots\right], & X \gg 1 \end{cases},$$

$c$ and $\phi$ are determined approximately to be 0.5892 and 5.221

**Total energy:** $U_N = -N\left(\frac{\partial \log Z_1}{\partial \beta}\right)_V = \frac{3Nk_BT}{2} + \Omega,$

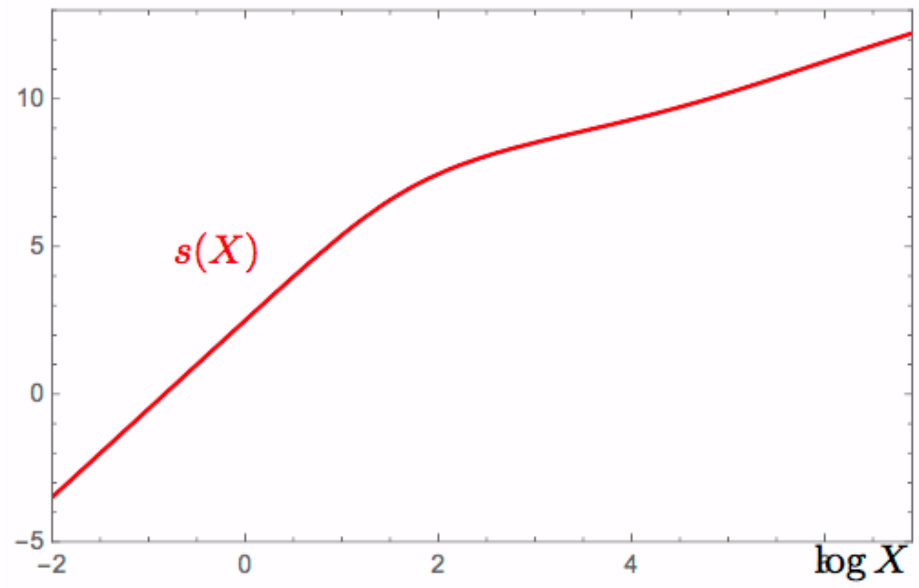
**Gravitational potential energy:**

$$\Omega \equiv NV_0 + Nk_BT\,\omega(X); \qquad \omega(X) = -\frac{X\mathcal{Z}'}{2\mathcal{Z}} = \frac{3}{2}\left(1 - \frac{e^{-\mathcal{V}(X)}}{\mathcal{Z}(X)}\right).$$



# Entropy:

$$\frac{S_N}{Nk_B} = \log\frac{4\pi}{3N(\kappa\hbar/\mu_0 c)^3} + \frac{3}{2}\log\frac{k_B T}{2\pi\mu_0 c^2} + s(X); \qquad s(X) \equiv \frac{5}{2} + 3\log X + \omega(X) + \log\mathcal{Z}.$$

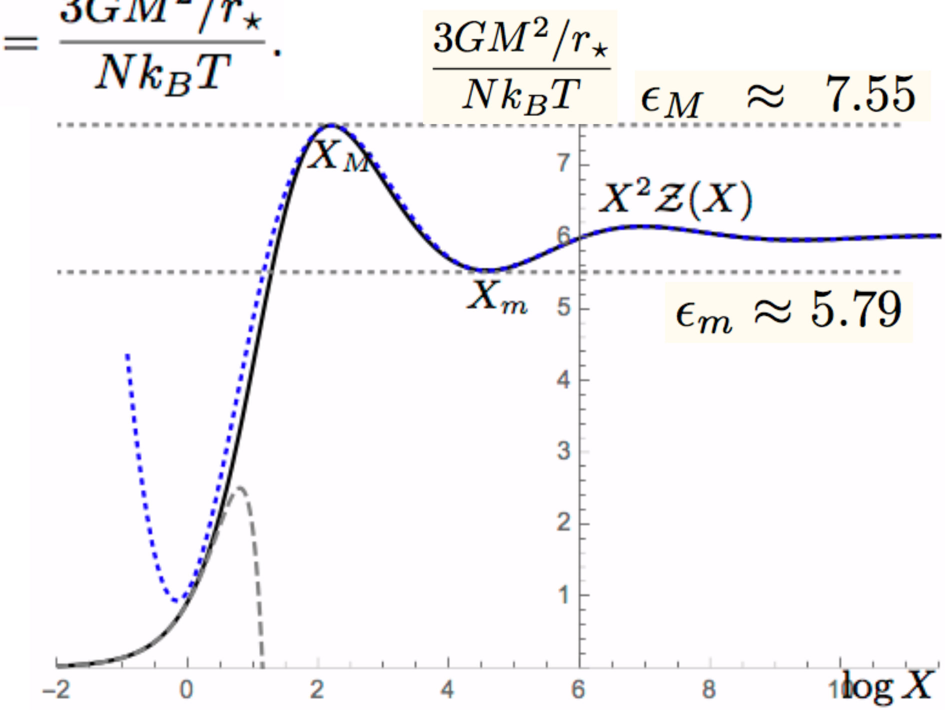



**The entropy increases monotonically. (No negative entropy problem)**

# Constraint from self-gravitating condition

$X$ is not an independent parameter but dependent on the temperature and size.

$$\epsilon(X) \equiv X^2 \mathcal{Z}(X) = \frac{3GM^2/r_\star}{Nk_BT}.$$




$$\epsilon'(X) = 2X\mathcal{Z}(X)(1-\omega(X)) = 0.$$

$X$ is uniquely determined only when

$$3GM^2/r_\star < \epsilon_m N k_B T,$$

No static spherically symmetric (stable or not) configuration exist when the system is too dense:

$$3GM^2/(r_\star\ N k_B T) > \epsilon_M$$

Therefore, <u>low temperature, extremely dense stars do not exist</u> in this theory.

## Heat capacity

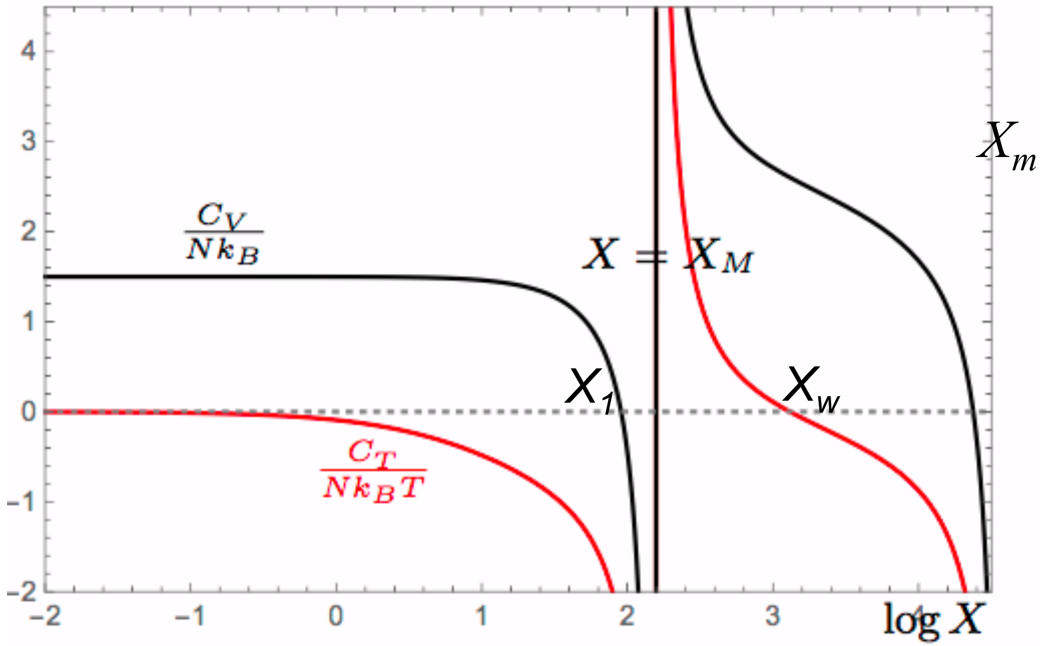
**Heat capacity:** $$\frac{C_V}{Nk_B} \equiv \frac{1}{Nk_B}\left(\frac{\partial U_N}{\partial T}\right)_{r_\star} = \frac{3}{2} + \omega - \frac{X\omega'}{2(1-\omega)}.$$

$$\frac{C_T}{Nk_BT} = \frac{1}{Nk_BT}\left(\frac{\partial \Omega}{\partial \log r_\star}\right)_T = -\frac{X\omega'}{2(1-\omega)}.$$

**The system is unstable for $X_1 < X < X_M$.**

**The heat capacity for fixed $T$ is negative for $X < X_M$.**

**Both decreases for $0 < X < X_M$.**




**The heat capacity for fixed $T$ is positive for $X_M < X < X_w$. (implication?)**

# Equation of State

$$\bar{P} = \bar{K}(X)\rho^{5/3}; \qquad \bar{K}(X) \equiv K \times \left(\mathcal{Z}e^{\omega}\right)^{-2/3},$$

$$\bar{K}(X) \approx K \times \begin{cases} 1 + \frac{X^2}{10} + \cdots, & X < 1 \\ \frac{X^{4/3}}{(6\sqrt{e})^{2/3}} \left[1 - \frac{\sqrt{7}c}{3\sqrt{X}} \sin\left(\frac{\sqrt{7}}{2}\log X - \phi' - \arctan\sqrt{7}\right) + \cdots\right], & X \gg 1 \end{cases}.$$

**For small $X$:**

$$\bar{P} = K\rho^{5/3}\left(1 + \frac{NG(36\pi M^2)^{1/3}}{10}\frac{\rho^{4/3}}{\bar{P}} + \cdots\right).$$

**The correction term is order $X^2 \sim \frac{3GM^2/r_\star}{Nk_BT}$**

**For large $X$:**

$$\bar{P} \approx K\rho^{5/3}\frac{(X(Tr_\star))^{4/3}}{(6\sqrt{e})^{2/3}},$$

**where $X$ should be determined from the relation,** $\frac{\cos\left(\frac{\sqrt{7}}{2}\log X - \phi'\right)}{\sqrt{X}} = \frac{1}{\sqrt{2}c}\left(1 - \frac{GM^2/r_\star}{2k_BT}\right).$

$$2N\,k_B\,T \approx GM^2/r_*$$

# Conclusion

| | Locally | Macroscopically |
|---|---|---|
| $PV = Nk_BT,$ | Kept | kept |
| $P = K\rho^{\gamma}$ | kept | modified |

**However, there are some cases when the macroscopic effects can be observable.**

**As an example, we deal a self-gravitating sphere in thermal equilibrium.**

Thanks, All Participants.