The Effects of an Extended Neutrino Sphere on Supernova Neutrino Oscillations

Rasmus S. L. Hansen

MAX-PLANCK-INSTITUT
FÜR KERNPHYSIK
HEIDELBERG

NDM 2018, IBS, Korea
July 2, 2018
Janka et al. 2012

Diagrams illustrating the processes of shock stagnation, shock revival, explosion, and explosion and nucleosynthesis, focusing on the interactions of shock waves with proto-neutron stars.
Impact of neutrino oscillations

- Supernova explosion mechanism:
  \[ E(\nu_{\mu/\tau}) > E(\nu_e) \]

- Modify the neutrino signal:
  \[ \nu_e \rightarrow \nu_{\mu/\tau} \quad \nu_{\mu/\tau} \rightarrow \nu_e \]

- Nucleosynthesis depends on the neutron fraction:
  \[ p + \bar{\nu}_e \rightarrow n + e^+ \quad n + \nu_e \rightarrow p + e^- \]

(See also talk by Baha Balantekin on Friday)
Matter effect

Electron background shifts the energy eigenvalues.

\[ H = \frac{\Delta m^2}{2E} \begin{pmatrix} -c_{2\theta} & s_{2\theta} \\ s_{2\theta} & c_{2\theta} \end{pmatrix} + \sqrt{2} G_F n_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ m_{\text{eff.}}^2 \sim \nu_e \]

\[ m_1^2 \sim \nu_x \]

\[ m_2^2 \]

\[ n_e \]

vacuum

solar centre

Rasmus S. L. Hansen
Density matrix formalism

In the mean field approximation:

\[
\rho = \begin{pmatrix} \rho_{ee} & \rho_{ex} \\ \rho_{xe} & \rho_{xx} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{pmatrix} = \frac{1}{2} (P_0 + \vec{P} \cdot \vec{\sigma}).
\]

Similarly for the Hamiltonian:

\[
H = \frac{1}{2} \begin{pmatrix} V_z & V_x - iV_y \\ V_x + iV_y & -V_z \end{pmatrix} = \frac{1}{2} \vec{V} \cdot \vec{\sigma}.
\]

Equation of motion (in absence of collisions, see also poster by Hirokazu Sasaki):

\[
i \dot{\rho} = [H, \rho] \quad \Leftrightarrow \quad \dot{\vec{P}} = \vec{V} \times \vec{P}
\]
Collective oscillations

Normal oscillations:

\[ \text{prob}(\nu_e \rightarrow \nu_e) \propto \cos^2(\Delta m^2 L/4E). \]

Neutrino background:

\[ H_{\nu\nu} = \sqrt{2} G_F \int dp (\rho - \bar{\rho}). \]

Conversion independent of \( E \).

Non-linear problem \( \rightarrow \) hard to solve in a realistic setting.

Hannestad et al. 2006
Observation
Considering the individual neutrino, its oscillations in a SN is a linear problem.

Idea
How much of the neutrino-neutrino refraction can we describe using linear equations.

Ultimate goal
General conclusions about the behaviour of neutrino oscillations in presence of neutrino-neutrino refraction.

Methods
- Solve equations from first principles, analytic and numeric.
- Describe complicated systems using effective potentials.
General equations

Probe neutrino in arbitrary neutrino and matter background.

\[ H^{(p)} = \frac{1}{2} \left( -c_{2\theta} \omega_p + V_e + V_\nu \quad s_{2\theta} \omega_p + 2 \bar{V}_\nu e^{i\phi_B} \right) \left( s_{2\theta} \omega_p + 2 \bar{V}_\nu e^{-i\phi_B} \quad c_{2\theta} \omega_p - V_e - V_\nu \right), \tag{1} \]

\[ V_\nu = \int d\mathbf{k} V^0_{\nu}(\mathbf{k}) \left[ \rho_{ee}(\mathbf{k}) - \rho_{\tau \tau}(\mathbf{k}) \right], \]

\[ \bar{V}_\nu e^{i\phi_B} = \int d\mathbf{k} V^0_{\nu}(\mathbf{k}) \rho_{e\tau}(\mathbf{k}) \]

\[ V^0_{\nu}(\mathbf{k}) = \sqrt{2} G_F n(\mathbf{k}) (1 - \mathbf{v}_{bg} \cdot \mathbf{v}_p) \]
Rewrite the off-diagonal as $V'e^{i\phi'} = s_{2\theta}\omega_p + 2\bar{V}_\nu e^{i\phi_B}$. Apply the transformation $U = \text{diag} \left( e^{i\phi'/2}, e^{-i\phi'/2} \right)$.

$$H^{(p)} = \frac{1}{2} \begin{pmatrix} V' & V' \\ V' & -V' \end{pmatrix},$$

where

$$V' = \sqrt{4\bar{V}_\nu^2 + 4s_{2\theta}\omega_p \cos \phi_B \bar{V}_\nu + s_{2\theta}^2\omega_p^2},$$

$$V'^r = V_e + V_\nu + \dot{\phi}' - c_{2\theta}\omega_p.$$
Conditions for a large conversion

Our Ansatz is that every case where a large conversion happens can be described in terms of at least one of these frameworks:

1. **Resonance**
   Vanishing diagonal:
   \[ V_e + V_\nu + \dot{\phi}' - c_2 \theta \omega_p = 0. \]

2. **Adiabatic conversion**
   Fast oscillations in \( V' \) and \( V' \) can be removed by going to a rotating frame. This can result in a Hamiltonian describing adiabatic evolution.

3. **Parametric enhancement**
   Present if the period of oscillation equals the period of change of mixing angle.
Energy spectrum

\[ P_{\text{er}} \]

- \( \omega_k = 0.5 \omega_p \)
- \( \omega_k = \omega_p \)
- \( \omega_k = 1.5 \omega_p \)
- Box spectrum
- Analytic

RSLH and Smirnov, 2018
Neutrino emission

$\nu_{\mu/\tau}$ decouple first, then $\bar{\nu}_e$ and last $\nu_e$.

$\nu_{\mu/\tau}$ have:

Number sphere:

$$e^+ e^- \to \nu_{\mu/\tau} \bar{\nu}_{\mu/\tau}$$

Energy sphere:

$$e \nu_{\mu/\tau} \to e \nu_{\mu/\tau}$$

Transport sphere:

$$N \nu_{\mu/\tau} \to N \nu_{\mu/\tau}$$
Neutrino emission

$\nu_{\mu/\tau}$ decouple first, then $\bar{\nu}_e$ and last $\nu_e$.

$\nu_{\mu/\tau}$ have:
Number sphere:
$e^+ e^- \rightarrow \nu_{\mu/\tau} \bar{\nu}_{\mu/\tau}$

Energy sphere:
$e \, \nu_{\mu/\tau} \rightarrow e \, \nu_{\mu/\tau}$

Transport sphere:
$N \, \nu_{\mu/\tau} \rightarrow N \, \nu_{\mu/\tau}$

$\bar{\nu}_e$ and $\nu_e$ are dominated by absorption and emission from nucleons. ($n_n > n_p$)

Mean free path:
\[
\frac{1}{\lambda} \approx \frac{G_F^2}{\pi} (g_V^2 + 3g_A^2) E^2 n_N.
\]

Emissivity:
\[
j = \frac{1}{\lambda} \exp \left(-\frac{E}{T}\right)
\]
Extended source

Rough estimates:

- Neutrino sphere: $\sim 10\text{km}$.
- Width of neutrino sphere: $\sim 1\text{km}$.
- Oscillation length: $\sim \frac{1}{G_F n_e} \sim 10^{-8} - 10^{-7}\text{km}$.  

Average over emission region suppresses oscillatory terms by $10^7 - 10^8$.  
Parametric resonance is not removed as such.
Extended source - non-linear

\[ \rho_{ee} = \frac{1}{2}(1 + P_z) \]

\[ \Delta z = 0 \]
\[ \Delta z = \frac{3\pi \sin \beta}{\lambda} \]
\[ \Delta z = \frac{7\pi \sin \beta}{\lambda} \]
\[ \Delta z = \frac{21\pi \sin \beta}{\lambda} \]

\[ \text{emission region} \]

\[ |\rho_{ex} - H_{ex}| \]
Extended source - non-linear
Changing background - only matter

Coordinate system with $\vec{V}$ along z-axis:

$$\rho_{12}(r) = \rho_{12,\text{initial}} \exp \left( i \int_{r_e}^{r} \omega_m(r') dr' \right).$$

Average over emission point:

$$\langle \rho_{12}(r) \rangle = \int_{0}^{r} p(r_e) \frac{1}{2} \sin 2\theta_m(r_e) \exp \left( i \int_{r_e}^{r} \omega_m(r') dr' \right) dr_e,$$

where

$$p(r_e) = \frac{1}{\lambda(r_e)} \exp(-E/T) \exp \left( -\int_{r_e}^{\infty} \frac{1}{\lambda(r')} dr' \right).$$
Changing background - only matter

Include effect of damping, \( D \):
\[
\dot{P} = \vec{V} \times \vec{P} - D \vec{P}_T.
\]

For \( V_Z \) and \( D \) large and \( P_Z \approx 1 \):
\[
P_x \approx \frac{V_x V_Z}{V_Z^2 + D^2}, \quad P_y \approx \frac{-V_x D}{V_Z^2 + D^2}.
\]

Bell et al. 1998, Hannestad et al. 2012
Effect of non-adiabaticity

Solve $\dot{\vec{P}} = \vec{V} \times \vec{P} - D\vec{P}_T$ numerically:

Non-adiabatic effects for $D=0$:

$$\vec{P} = \begin{pmatrix} \frac{V_x}{V_z} + \frac{V_x \partial_r V_z}{V_z^3} \sin \left( \int V_z (r') dr' \right) \\ \frac{V_x \partial_r V_z}{V_z^3} \left( 1 - \cos \left( \int V_z (r') dr' \right) \right) \\ 1 + \frac{V_x^2 \partial_r V_z}{V_z^4} \sin \left( \int V_z (r') dr' \right) \end{pmatrix}.$$ 

$$|\rho_{12}| \approx \left| \frac{V_x \partial_r V_z}{2V_z^3} \right| \approx \left| \frac{V_x}{2V_z^2 r_0} \right|.$$
Linear stability analysis  Banerjee, Dighe, Raffelt 2011

Linear analysis demonstrate stability or instability.

\[
\rho = \frac{f_{\nu_e} + f_{\nu_x}}{2} + \frac{f_{\nu_e} - f_{\nu_x}}{2} \begin{pmatrix} s & S \\ S^* & -s \end{pmatrix}
\]

\[
|S| = |P_x + iP_y| \ll 1, \quad s^2 + S^2 = 1 \Rightarrow s = P_z \approx 1.
\]

Linearised equation:

\[
i \dot{S} = (\omega + \lambda + \mu)S - \mu \int d\Gamma'(1 - \nu \cdot \nu')S',
\]
Linear stability analysis

In Fourier space: \( S = e^{-i\Omega t} Q \)

\[
\Omega Q = (\omega + \lambda + \mu)Q - \mu \int d\Gamma' (1 - \nu \cdot \nu') Q'
\]

Unstable if \( \text{Im}(\Omega) \neq 0. \) (See also Capozzi et al. 2017)

Discrete modes: solve matrix equation.
(Continuos modes: Decompose in independent functions.)
Can also be formulated as a dispersion relation. (Izaguirre, Raffelt and Tamborra, 2016)
Simple model

Linearised equation: \((\omega = -1, \lambda = 30, \mu = 3)\)

\[
i \dot{\vec{S}} = \begin{pmatrix} -\omega + \lambda - \mu & \mu \\ -\mu & \omega + \lambda + \mu \end{pmatrix} \vec{S}
\]

Eigenvalues:

\[
\Omega = \lambda \pm \sqrt{\omega(2\mu + \omega)}
\]

Growth rate = \(\text{Im}(\Omega)\).

Emission point = \(\frac{2}{3}\Delta z\).

Start value = \(\frac{1}{\Delta z \lambda} \sin 2\theta_m\).

Does not work for large \(\Delta z\).
Multiple angles, in-homogeneous

Linear stability analysis of a more realistic model:

\[(\Omega + \mathbf{v} \cdot \mathbf{k})Q = (\omega + \lambda + \mu(\epsilon - \mathbf{v} \cdot \phi))Q - \mu \int d\Gamma'(1 - \mathbf{v} \cdot \mathbf{v}')g'Q'\]

- \(\mu\) and \(\lambda\) functions of \(r\).
- Multi angle matter effect.
- Homogeneous mode: \(k = 0\)

Chakraborty, RSLH, Izaguirre and Raffelt, 2015
Very fast flavour conversion  R. F. Sawyer

Chakraborty, RSLH, Izaguirre and Raffelt, 2016

Conversion on meter-scale.
Can also occur in a supernova.  Dasgupta, Mirizzi and Sen, 2016
Summary

- The non-negligible width of the neutrino sphere affects the neutrino state due to averaging over different emission points.

- The angle between the neutrino state and the Hamiltonian in polarization space is reduced by a factor $10^8$ at the neutrino sphere by the averaging.

- A small adiabaticity violation increases the angle significantly as the neutrino propagates out through the supernova.

- The onset of neutrino conversion can be analysed using linear stability analysis, and for a given model, it can be calculated if conversion has the potentially to occur.
Thanks for your attention!
Neutrino emission

Deleptonisation

Accretion

Cooling

Lang et al. 2016
Neutrino mixing

\[ \mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}} W^+_\mu \bar{\nu}_L \gamma^\mu l_L + h.c. \]

Interaction states and mass states are different:

\[ \nu^l = U \nu^i. \]

Mixing matrix:

\[
U = \begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{bmatrix}
\begin{bmatrix}
c_{13} & 0 & s_{13}e^{-i\delta_{\text{CP}}} \\
0 & 1 & 0 \\
-s_{13}e^{-i\delta_{\text{CP}}} & 0 & c_{13}
\end{bmatrix}
\begin{bmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
Collective oscillations

Can collective oscillations still occur? YES!
Very fast flavour conversion

Dasgupta, Mirizzi and Sen, 2016