# Self-anisotropizing universe in Horndeski theory

Hiroaki W. H. Tahara

RESCEU, U Tokyo

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COSMO-18@IBS, Daejeon, Korea

HWHT, S.Nishi, T.Kobayashi, J.Yokoyama, JCAP07(2018)058

# Content

- In GR, inflation dilutes anisotropies.
  - *Wald's cosmological no-hair theorem*
  - We can say there is only an isotropic attractor.

- In some class of modified gravity,
  the theorem does not hold anymore.
  - Anisotropic attractors emerge.
  - See this with Horndeski theory.
    - General case
    - Specific case

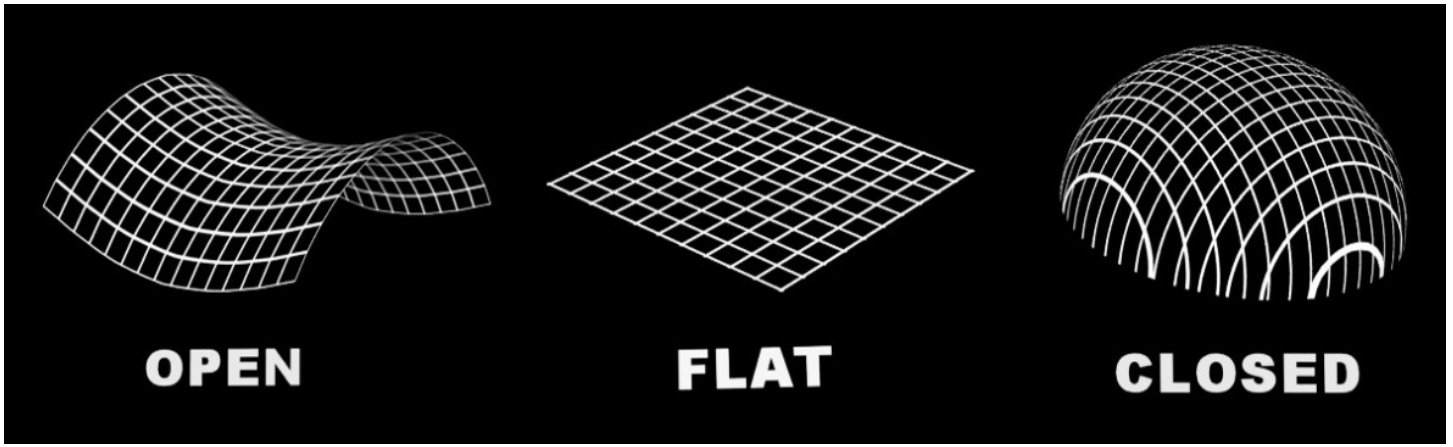
# Weak cosmological principle works well

- Spatially homogeneous and isotropic

⇒ FLRW metric

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$k = -1$ $k = 0$ $k = +1$

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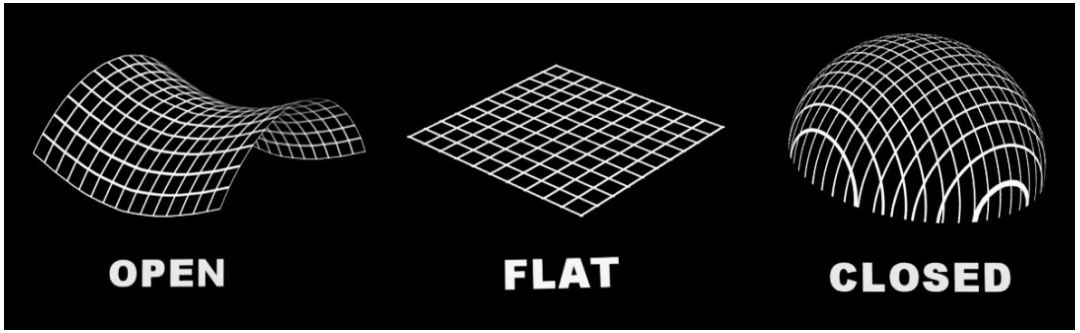
# To smooth out initial conditions

Very early universe may have:

Inhomogeneity,
Anisotropy… etc.

Inflation: efficient way to dilute them



# Anisotropies vanish with positive Λ

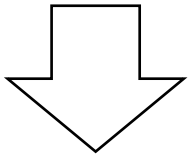
R.M. Wald, Phys. Rev. D 28 (1983) 2118

Bianchi models

all types of background anisotropies in 3-dim. space

$$
\begin{aligned}
&\text{metric}: && ds^2 = -dt^2 + \gamma_{ij}(t)\chi^i\chi^j \\
&\text{type I}: && \chi^1 = dx^1,\ \chi^2 = dx^2,\ \chi^3 = dx^3 \\
&\text{type II}: && \chi^1 = dx^2 - x^1 dx^3,\ \chi^2 = dx^3,\ \chi^3 = dx^1 \\
&\text{type III}: && \chi^1 = e^{-x^1}dx^2,\ \chi^2 = dx^3,\ \chi^3 = dx^1 \\
&\text{type IV}: && \chi^1 = e^{-x^1}dx^2 - x^1 e^{-x^1}dx^3,\ \chi^2 = e^{-x^1}dx^3,\ \chi^3 = dx^1 \\
&\text{type V}: && \chi^1 = e^{-x^1}dx^2,\ \chi^2 = e^{-x^1}dx^3,\ \chi^3 = dx^1 \\
&\qquad\vdots \\
&\text{type IX}: && \chi^1 = -\sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2,\ \chi^2 = \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2,\ \chi^3 = \cos x^1 dx^2 + dx^3
\end{aligned}
$$

$$T_{\mu\nu} = -\Lambda_0 g_{\mu\nu} + \tilde{T}_{\mu\nu}$$



$\Lambda_0$: positive cosmological constant

$\tilde{T}_{\mu\nu}$ : satisfies Strong and Dominant EC

$$\boxed{ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j}$$

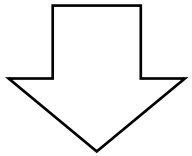
All of Bianchi types (except IX) approach FLRW metric

**Wald's cosmic no-hair theorem (in GR)**

# Introduce the simplest anisotropies

FLRW metric (k=0)

$$ds^2 = -dt^2 + a(t)^2 \left[dx^2 + dy^2 + dz^2\right]$$



Kasner metric (=diagonal Bianchi type I)

$$ds^2 = -dt^2 + a(t)^2 \left[e^{2(\beta_+ + \sqrt{3}\beta_-)}dx^2 + e^{2(\beta_+ - \sqrt{3}\beta_-)}dy^2 + e^{-4\beta_+}dz^2\right]$$

Homogeneous: $a = a(t),\ \beta_+ = \beta_+(t),\ \beta_- = \beta_-(t)$

Anisotropic: $\dot{\beta}_+ \neq 0\ ,\ \dot{\beta}_- \neq 0$

(different expansion rates in directions x, y and z )

# “Anisotropic” Friedmann eqns.

General relativity

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R+\mathcal{L}_{\rm m}) \quad \text{or} \quad G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

↓ Kasner metric

$$ds^2 = -dt^2 + a(t)^2\left[e^{2(\beta_+ + \sqrt{3}\beta_-)}dx^2 + e^{2(\beta_+ - \sqrt{3}\beta_-)}dy^2 + e^{-4\beta_+}dz^2\right]$$

From (0,0) component

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \underbrace{\dot\beta_+^2 + \dot\beta_-^2}_{\equiv \rho_{\rm aniso}} \qquad T_{00} = \rho$$

From (i, j) component

$$\frac{d}{dt}\left[a^3\dot\beta_\pm\right] = 0 \qquad \rho_{\rm aniso} \equiv \dot\beta_+^2 + \dot\beta_-^2 \propto a^{-6}$$

# Anisotropy vanishes faster

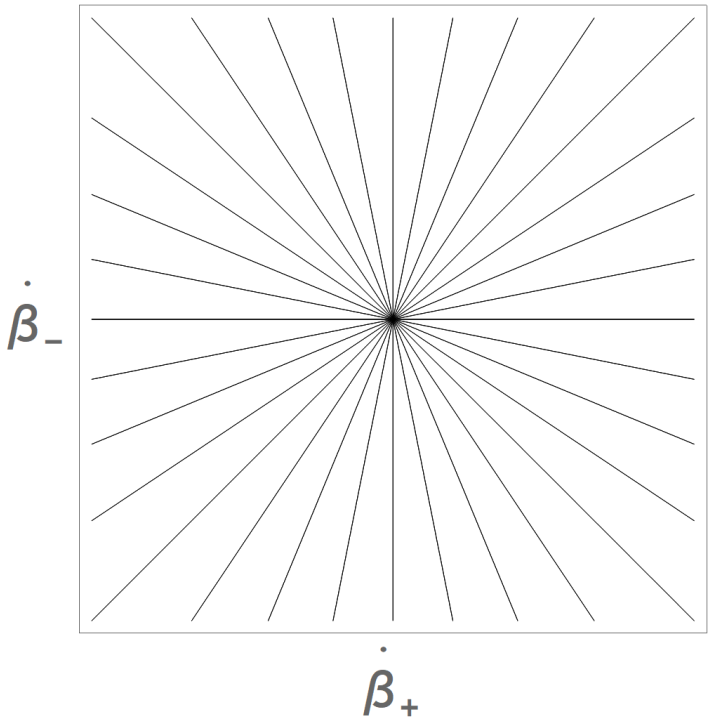
$$\rho_\Lambda \propto a^0$$

$$\rho_{\rm curvature} \propto a^{-2}$$

$$\rho_{\rm matter} \propto a^{-3}$$

$$\rho_{\rm radiation} \propto a^{-4}$$

$$\boxed{\rho_{\rm aniso} \equiv \dot{\beta}_+^2 + \dot{\beta}_-^2 \propto a^{-6}}$$




In GR, expansion decreases anisotropies efficiently.

# Horndeski theory

- Discovered by Horndeski (1974):
  - As the most general tensor theory with a scalar field which has the second-order EoMs.
- Rediscovered by Deffayet+ (2011):
  - In a different context (Galileon).
- Equivalence is indicated by Kobayashi+ (2011).

# Horndeski action

Horndeski 1974, Deffayet+ 2011, Kobayashi+ 2011

The most general single-field scalar-tensor theory with the second-order EoM.

$$S = \int \mathrm{d}^4x\sqrt{-g}\sum_{i=2}^{5}\tilde{\mathcal{L}}_i,$$

Feature:
- No Ostrogradsky ghost
- 4 functional DOF

$$\tilde{\mathcal{L}}_2 = G_2(\phi, X),$$
$$\tilde{\mathcal{L}}_3 = -G_3(\phi, X)\Box\phi,$$
$$\tilde{\mathcal{L}}_4 = G_4(\phi, X)\mathcal{R} + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2],$$
$$\tilde{\mathcal{L}}_5 = G_5(\phi, X)\mathcal{G}_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3]$$

$$G_{5X} \equiv \frac{\partial G_5}{\partial X}$$

$X \equiv -(\partial\phi)^2/2$ : a canonical kinetic term

$G_2(\phi, X), G_3(\phi, X), G_4(\phi, X), G_5(\phi, X)$ : arbitrary functions

# Horndeski theory includes:

- General relativity

$$G_4 = \frac{1}{16\pi G}\ ,\ G_3 = G_5 = 0$$

- Brans-Dicke gravity

$$G_2 = \frac{1}{8\pi}\left(\frac{\omega}{\phi}X - U(\phi)\right),\ G_4 = \frac{\phi}{16\pi},\ G_3 = G_5 = 0$$

- Kaluza-Klein compactification of higher-dimensional Lovelock gravity

$$G_5 \propto X$$

# Behavior of anisotropies

We start with Kasner metric:

$$ds^2 = -dt^2 + a^2(t)\left[e^{2(\beta_+(t)+\sqrt{3}\beta_-(t))}dx^2 + e^{2(\beta_+(t)-\sqrt{3}\beta_-(t))}dy^2 + e^{-4\beta_+(t)}dz^2\right]$$

Substituting this in the Horndeski action, we get:

$$S^{(0)} = \int dtd^3x\ a^3\left[G_2 + (\ddot{\phi} + 3H\dot{\phi})G_3 - 6(G_4 - \dot{\phi}^2 G_{4X})(H^2 - \dot{\beta}_+^2 - \dot{\beta}_-^2) - 6H\dot{\phi}G_{4\phi}\right.$$
$$\left. + \dot{\phi}^3 G_{5X}(H^3 - 3H(\dot{\beta}_+^2 + \dot{\beta}_-^2) + 2(3\dot{\beta}_+\dot{\beta}_-^2 - \dot{\beta}_+^3)) - 3\dot{\phi}^2 G_{5\phi}(H^2 - (\dot{\beta}_+^2 + \dot{\beta}_-^2))\right],$$

As seen, it has $\beta_\pm$ only as their time derivatives:

$$P_{\beta_+} = 6a^3\left[\mathcal{G}_{\rm T}\dot{\beta}_+ - \dot{\phi}^3 G_{5X}(\dot{\beta}_+^2 - \dot{\beta}_-^2)\right],$$
$$P_{\beta_-} = 6a^3\left[\mathcal{G}_{\rm T}\dot{\beta}_- + 2\dot{\phi}^3 G_{5X}\dot{\beta}_+\dot{\beta}_-\right].$$

Where $\mathcal{G}_{\rm T} \equiv 2G_4 - 2\dot{\phi}^2 G_{4X} - H\dot{\phi}^3 G_{5X} + \dot{\phi}^2 G_{5\phi}$

# Conservation of momenta of $\beta_\pm$

$$P_{\beta_+} = 6a^3 \left[ \mathcal{G}_{\rm T}\dot{\beta}_+ - \dot{\phi}^3 G_{5X}(\dot{\beta}_+^2 - \dot{\beta}_-^2) \right],$$

$$P_{\beta_-} = 6a^3 \left[ \mathcal{G}_{\rm T}\dot{\beta}_- + 2\dot{\phi}^3 G_{5X}\dot{\beta}_+\dot{\beta}_- \right].$$

As a(t) increses, these approach zero.

→ [ … ]=0 gives the position of attractor.

(i) G5X=0 (or GR)

$$(\dot{\beta}_+, \dot{\beta}_-) = (0,0)$$

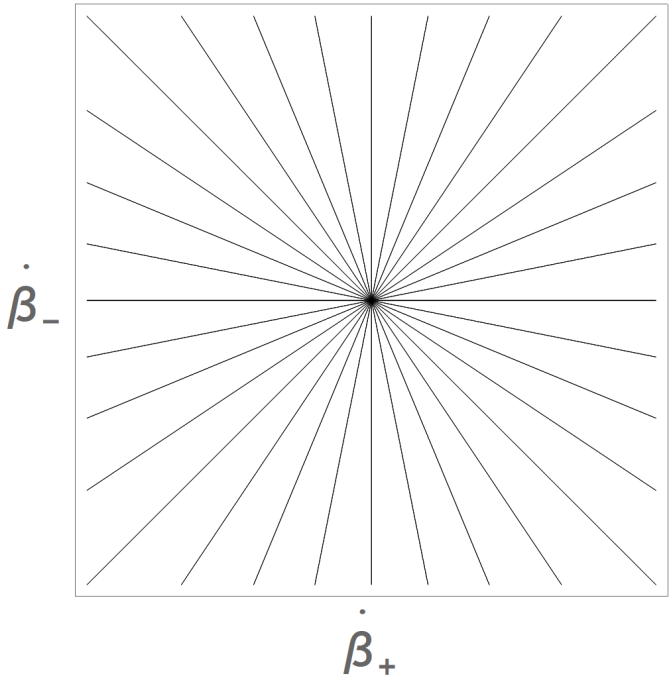
c.f. $\dfrac{d}{dt}\left[a^3\dot{\beta}_\pm\right] = 0$ in GR

(ii) G5X≠0

$$(\dot{\beta}_+, \dot{\beta}_-) = (0,0)\ ,\ b(-1,0)\ ,\ b(1/2, \pm\sqrt{3}/2) \quad \left(b = \frac{\mathcal{G}_{\rm T}}{-\dot{\phi}^3 G_{5X}}\right)$$

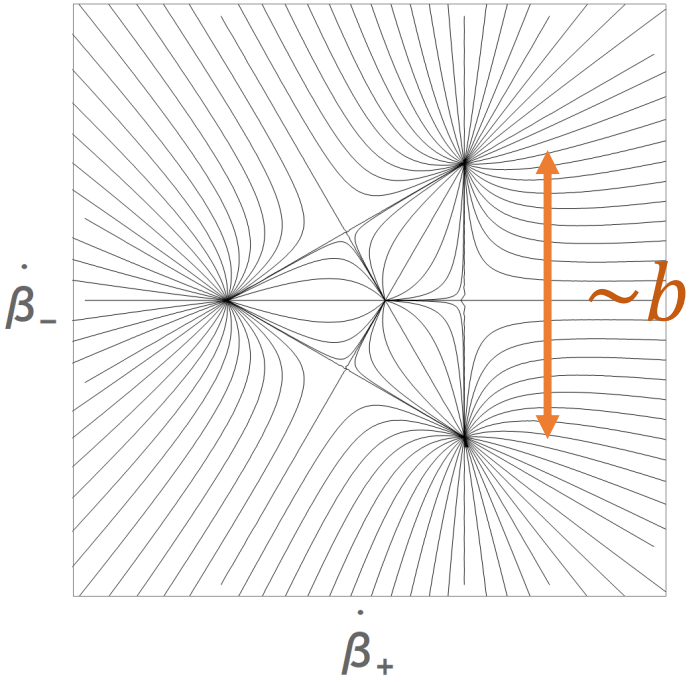
# Evolution of $\dot{\beta}_{\pm}$

**G5X=0 (or GR)**




**G5X≠0**

$$\left( b = \frac{\mathcal{G}_{\rm T}}{-\dot{\phi}^3 G_{5X}} \right)$$




We can draw trajectories with

$$\frac{P_{\beta_+}}{P_{\beta_-}} = \frac{\mathcal{G}_{\rm T}\dot{\beta}_+ - \dot{\phi}^3 G_{5X}(\dot{\beta}_+^2 - \dot{\beta}_-^2)}{\mathcal{G}_{\rm T}\dot{\beta}_- + 2\dot{\phi}^3 G_{5X}\dot{\beta}_+\dot{\beta}_-}$$

# To fix the triangle in a phase space

- We normalize $\dot{\beta}_{\pm}$ :

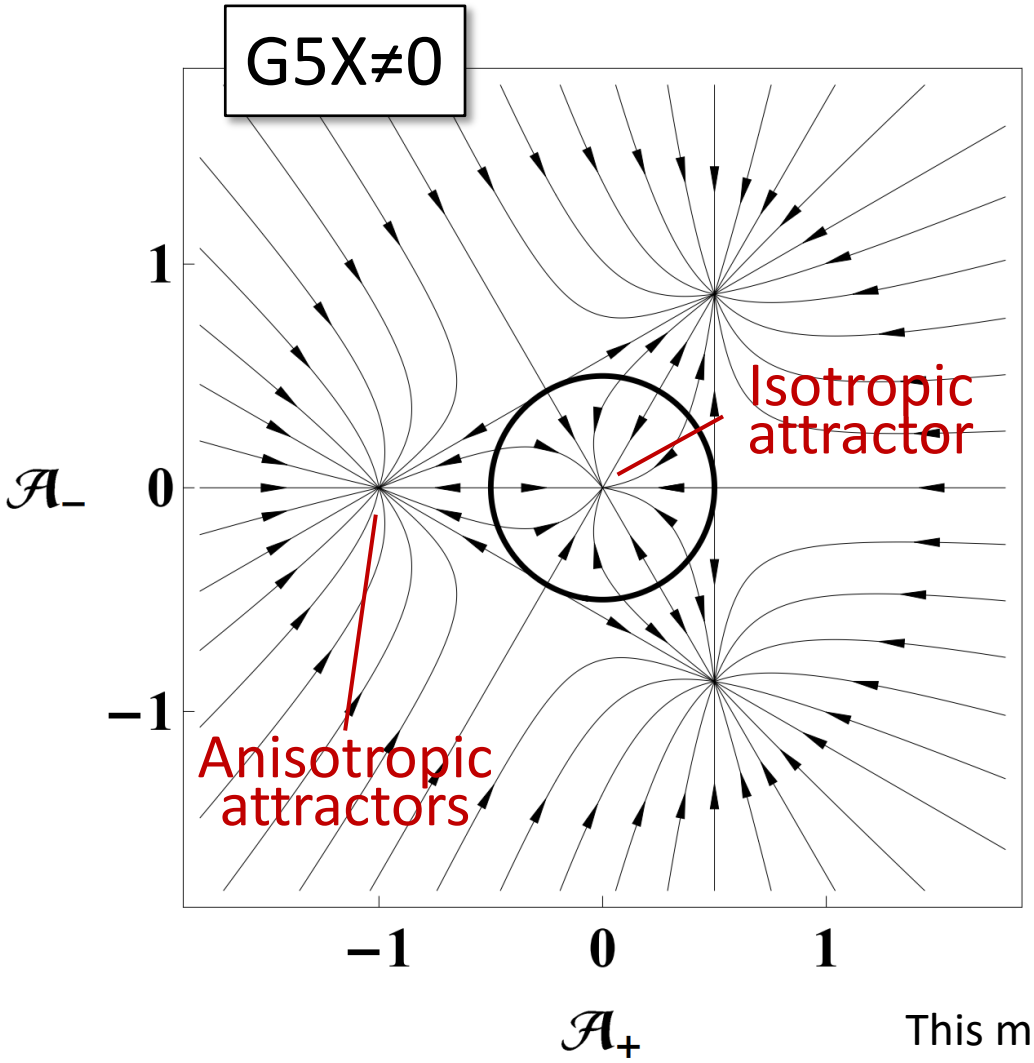
$$P_{\beta_+} = 6a^3 \left[ \mathcal{G}_{\rm T} \dot{\beta}_+ - \dot{\phi}^3 G_{5X} (\dot{\beta}_+^2 - \dot{\beta}_-^2) \right],$$
$$P_{\beta_-} = 6a^3 \left[ \mathcal{G}_{\rm T} \dot{\beta}_- + 2\dot{\phi}^3 G_{5X} \dot{\beta}_+ \dot{\beta}_- \right].$$

$$\mathcal{A}_{\pm} = \frac{-\dot{\phi}^3 G_{5X}}{\mathcal{G}_{\rm T}} \dot{\beta}_{\pm}$$

$$P_{\beta_+} = \tau \left[ \mathcal{A}_+ + \mathcal{A}_+^2 - \mathcal{A}_-^2 \right],$$
$$P_{\beta_-} = \tau \left[ \mathcal{A}_- - 2\mathcal{A}_+ \mathcal{A}_- \right].$$

where $\tau = 6a^3 \dfrac{\mathcal{G}_{\rm T}^2}{-\dot{\phi}^3 G_{5X}}$

# Evolution of $\mathcal{A}_{\pm}$




$$\tau = 6a^3 \frac{\mathcal{G}_{\rm T}^2}{-\dot{\phi}^3 G_{5X}}$$

As $|\tau|$ increases:

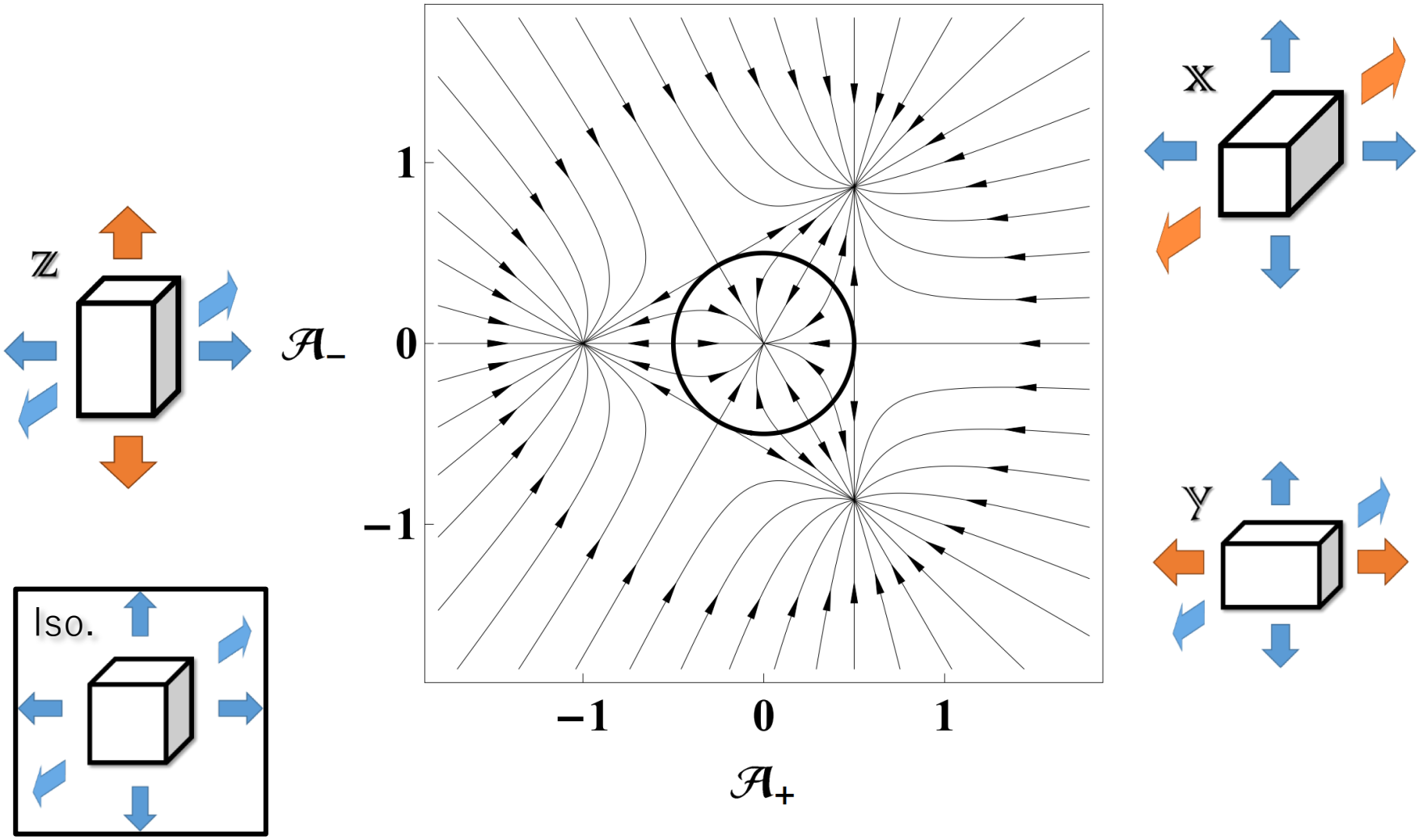
Inside of the circle
→ Isotropic attractor

Outside of the circle
→ Anisotropic attractor

We call such solutions "self-anisotropizing universe"

This might challenge the robustness of inflation?

# Anisotropic attractors

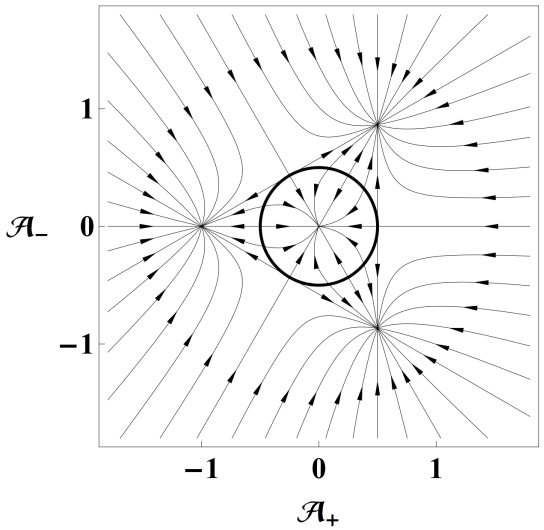


# An example

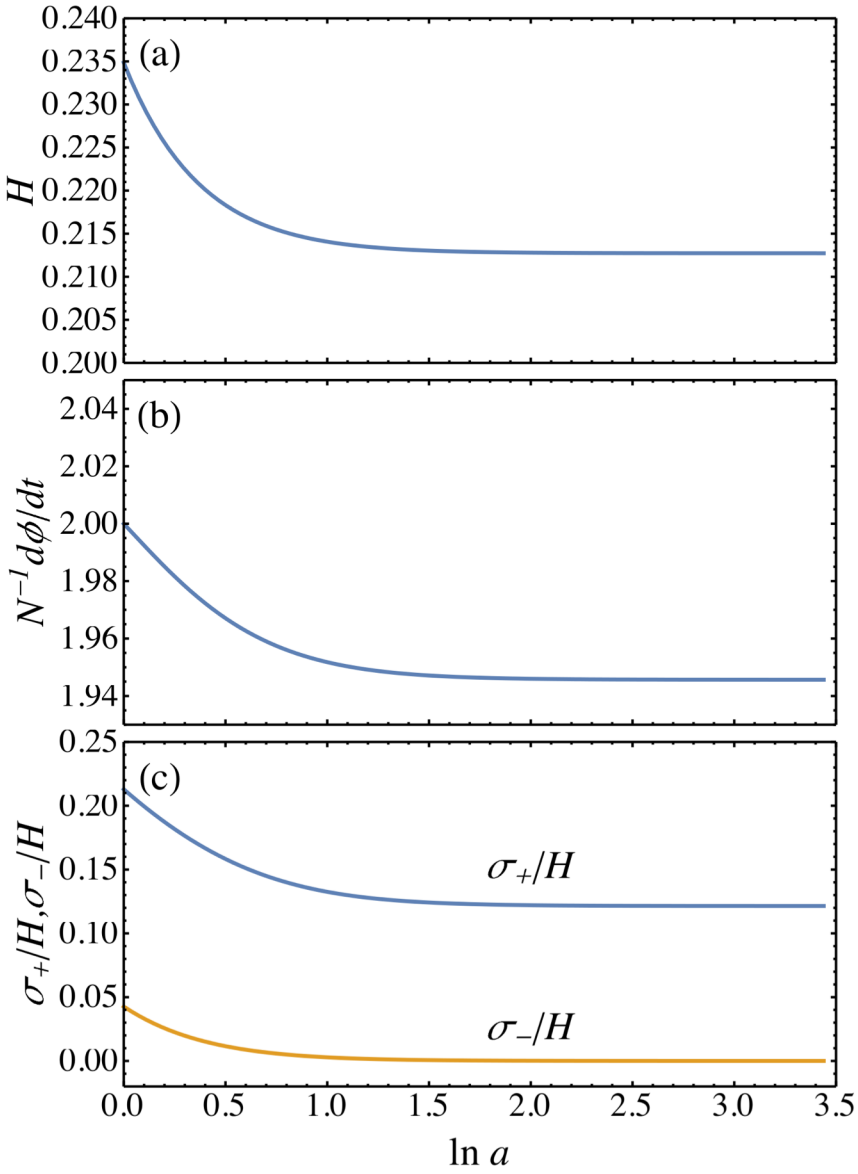
$$G_2 = -V_0, \quad G_3 = 0,$$

$$G_4 = \frac{M^2}{2} + g_4 X, \quad G_5 = g_5 X$$

$V_0 = 0.1,\ M = 1,\ g_4 = -0.2,$ and $g_5 = 1$




$$\tau = 6a^3 \frac{\mathcal{G}_{\rm T}^2}{-\dot{\phi}^3 G_{5X}}$$




$$H \equiv \frac{d \ln a}{dt} \quad , \quad \sigma_\pm \equiv \frac{d\beta_\pm}{dt}$$

# Conclusion

- A positive cosmological constant isotropizes homogeneous universes in GR.

- G5X changes the dynamics of spacetime and the anisotropic attractors emerge.
  - For example, slow-roll inflation can satisfy the condition to approach attractors.
    → Anisotropic inflation
  - Terminal state is depend on the initial condition.