Back-Reaction of Gravitational waves revisited

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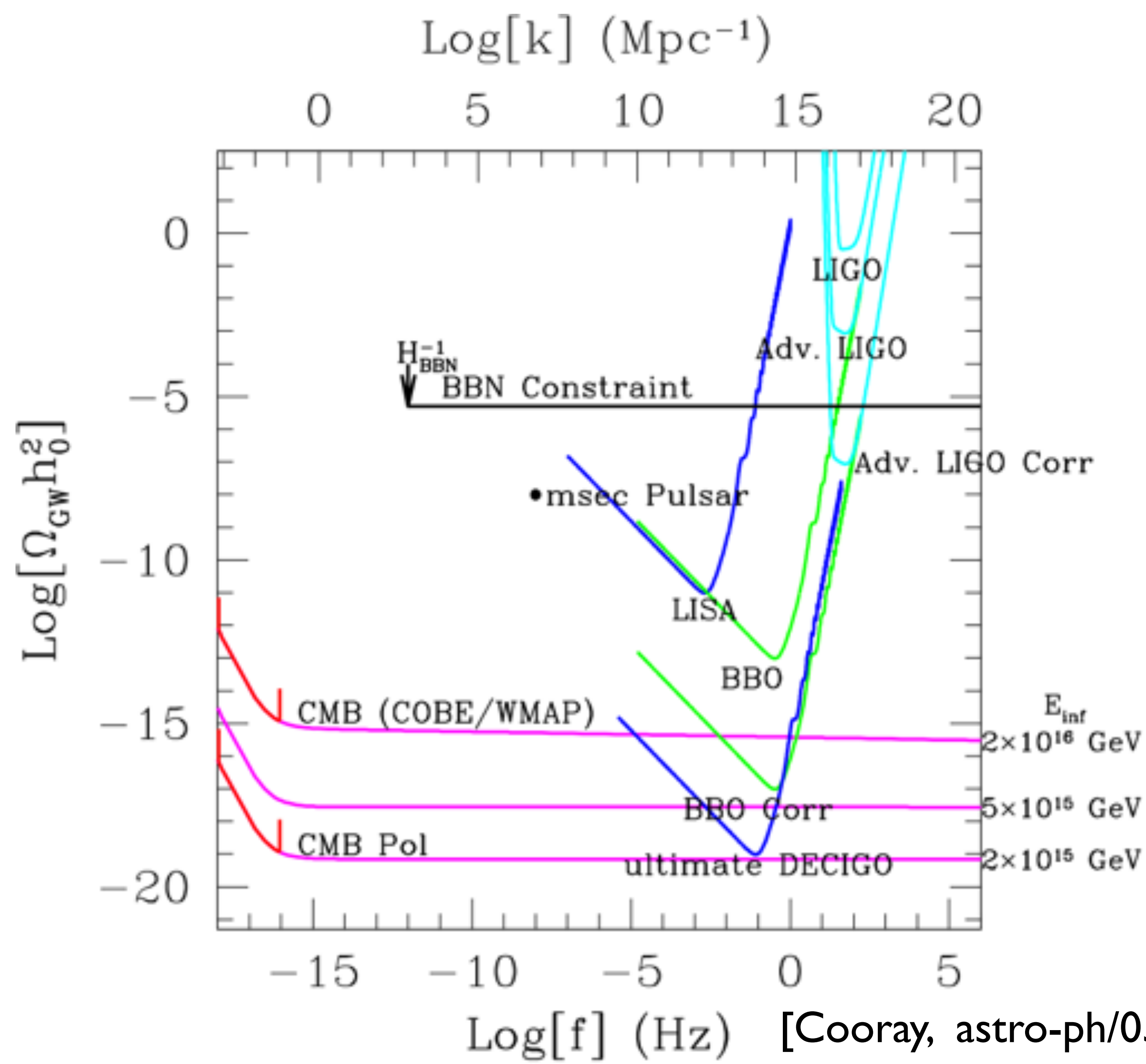
Gravitational waves as a probe of the early Universe

- Various sources of GWs in the early Universe (e.g., inflation) can be probed (constrained) from:

  - Interferometer
  - CMB
  - pulsar timing
  - BBN

  ⋮



[Cooray, astro-ph/0503118]

BBN constraint on GWs

- BBN can constrain the amount of extra radiation (or effective extra degrees of freedom N_{eff})

$$N_{\rm eff} = 2.85 \pm 0.28 \quad (1\sigma \text{ C.L.})$$

[Cyburt et al., 1505.01076]

- GWs behave as radiation (inside the horizon) and hence can be considered as extra radiation component.

- Fractional energy density can be constrained as:

$$\Omega_{\rm GW} h^2 \leq 5.6 \times 10^{-6} (N_{\rm eff}^{(\rm upper\ bound)} - 3)$$

- GWs (as extra radiation) can be considered as back-reaction of 2nd order tensor perturbations to the background.

Back-reaction of gravitational waves

- Tensor perturbations:

$$ds^2 = -a^2 d\tau^2 + a^2 \left[\gamma_{ij} + h_{ij}\right] dx^i dx^j$$

- We can calculate the "spatially averaged" Einstein tensor as

$$\langle G^{\mu}{}_{\nu} \rangle = \left\langle G^{(0)\mu}{}_{\nu} \right\rangle + \left\langle \delta G^{(1)\mu}{}_{\nu} \right\rangle + \left\langle \delta G^{(2)\mu}{}_{\nu} \right\rangle$$

back-reaction

(Spatially average is given by: $\langle A \rangle \equiv \frac{1}{V} \lim_{V \to \infty} \int A dV$

over constant time hypersurfaces.)

- $\left\langle G^{(2)\mu}{}_{\nu} \right\rangle$ can be regarded as "effective energy momentum tensor" in the background

Einstein tensor at 2nd order (tensor perturbations)

- 2nd Einstein tensor (after spatial average)

$$\left\langle \delta^{(2)} G^0{}_0 \right\rangle = \frac{1}{a^2}\left[\mathcal{H}\left\langle h^{km} h'_{km} \right\rangle + \frac{1}{8}\left\langle h'^{km} h'_{km} \right\rangle + \frac{1}{8}\left\langle h^{km,j} h_{km,j} \right\rangle \right],$$

$$\left\langle \delta^{(2)} G^0{}_i \right\rangle = \frac{1}{a^2}\left[\left\langle h^{mk} h'_{k[i,m]} \right\rangle - \frac{1}{4}\left\langle h'^{mk} h_{mk,i} \right\rangle \right],$$

$$\left\langle \delta^{(2)} G^i{}_0 \right\rangle = \frac{1}{a^2}\left[\frac{1}{4}\left\langle h'_{jk} h^{jk,i} \right\rangle - \left\langle h_{jk} h'^{k[i,j]} \right\rangle \right],$$

$$\left\langle \delta^{(2)} G^i{}_j \right\rangle = \frac{1}{a^2}\delta^i{}_j \left[\frac{3}{8}\left\langle h'_{km} h'^{km} \right\rangle - \frac{3}{8}\left\langle h^{km,n} h_{km,n} \right\rangle \right] + \frac{1}{a^2}\left[-\frac{1}{2}\left\langle h'^{ik} h'_{kj} \right\rangle - \frac{1}{4}\left\langle h^{km,i} h_{km,j} \right\rangle + \frac{1}{2}\left\langle h^{ik,m} h_{kj,m} \right\rangle \right],$$

→ Effective energy density and pressure

Effective energy density and pressure

- Effective energy density

$$\rho_{\rm GW} = \frac{1}{8\pi G}\left\langle \delta G^{(2)0}{}_{0}\right\rangle = \frac{1}{8\pi G a^2}\left(\frac{1}{8}\left\langle (h'_{ij})^2\right\rangle + \frac{1}{8}\left\langle (\nabla h_{ij})^2\right\rangle + \mathcal{H}\left\langle h^{ij} h'{}_{ij}\right\rangle\right)$$

- Effective pressure

$$\begin{aligned} p_{\rm GW} &= \frac{1}{3}\frac{1}{8\pi G}\left\langle G^{(2)i}{}_{i}\right\rangle + \frac{1}{3\mathcal{H}}\left\langle {}^{(2)}\Gamma^{\alpha}{}_{\alpha 0}\right\rangle (\rho^{(0)} + p^{(0)}) \\ &= \frac{1}{3}\frac{1}{8\pi G a^2}\left(-\frac{5}{8}\left\langle (h'_{ij})^2\right\rangle + \frac{7}{8}\left\langle (\nabla h_{ij})^2\right\rangle\right) + \frac{1}{2}\frac{1}{8\pi G a^2}\mathcal{H}\left(1 + w^{(0)}\right)\left\langle h^{ij} h'_{ij}\right\rangle \end{aligned}$$

On subhorizon scales, the average of the temporal and spatial gradient terms give the same contribution.

$$\rho_{\rm GW} = \frac{1}{4\cdot 8\pi G a^2}\left\langle (\nabla h_{ij})^2\right\rangle$$

$$p_{\rm GW} = \frac{1}{3\cdot 4\cdot 8\pi G a^2}\left\langle (\nabla h_{ij})^2\right\rangle$$

$$\Rightarrow \quad p_{\rm GW} = \frac{1}{3}\rho_{\rm GW}$$

[Abramo et al., gr-qc/9704037]

Effective energy density and pressure (super-horizon)

- Super-horizon evolutions can be considered separately for:

  - de Sitter era
  - Radiation-dominated (RD) era
  - Matter-dominated (MD) era

$$h_{ij}(\tau, \boldsymbol{x}) = \sum_{\lambda=+,\times} \int \frac{dk^3}{(2\pi)^3} \epsilon_{ij}^{\lambda}(\boldsymbol{k}) h_{\boldsymbol{k}}^{\lambda}(\tau) e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

- By solving the equation of motion $h_{\boldsymbol{k}}'' + 2\mathcal{H}h_{\boldsymbol{k}}' + k^2 h_{\boldsymbol{k}} = 0$, we can obtain super-horizon solutions for each era:

  - $h_{\boldsymbol{k}} = A_k \left(1 + \frac{1}{2}(k\tau)^2 + \cdots\right)$: de Sitter, $\mathcal{H} = -\frac{1}{\tau}$
  - $h_{\boldsymbol{k}} = A_k \left(1 - \frac{1}{6}(k\tau)^2 + \cdots\right)$: RD, $\mathcal{H} = \frac{1}{\tau}$
  - $h_{\boldsymbol{k}} = A_k \left(1 - \frac{1}{10}(k\tau)^2 + \cdots\right)$: MD, $\mathcal{H} = \frac{2}{\tau}$

Effective energy density and pressure (super-horizon)

- Effective energy density for each k mode is given by

$$\tilde{\rho}_{\rm GW}(k) = -\frac{7}{8}k^2\left\langle |A_k^2| \right\rangle \quad (\text{de Sitter}),$$
$$\tilde{\rho}_{\rm GW}(k) = -\frac{5}{24}k^2\left\langle |A_k^2| \right\rangle \quad (\text{RD}),$$
$$\tilde{\rho}_{\rm GW}(k) = -\frac{11}{40}k^2\left\langle |A_k^2| \right\rangle \quad (\text{MD}).$$

- In all these cases, the pressure is given by

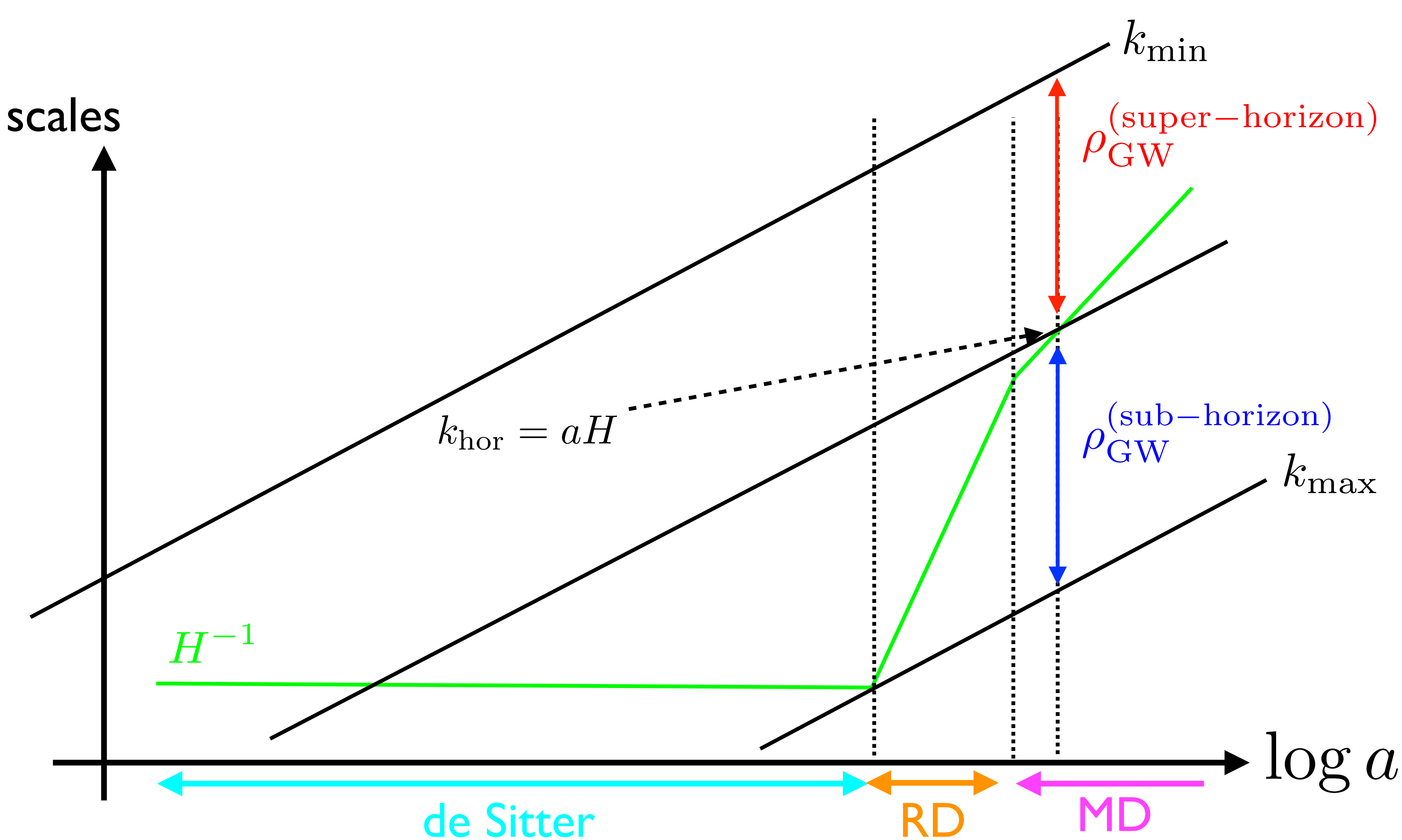
$$\tilde{p}_{\rm GW}(k) = -\frac{1}{3}\tilde{\rho}_{\rm GW}(k) \qquad \left(\tilde{\rho}_{\rm GW}(k) \propto a^{-2}\right)$$

→ Back-reaction of GWs on superhorizon scales behaves like the curvature. [Abramo et al., gr-qc/9704037]

Total (effective) energy density

Total energy density:

$$\begin{cases} \rho_{\rm GW}^{(\rm super-horizon)} = \dfrac{1}{8\pi G a^2} \displaystyle\int_{k_{\rm min}}^{k_{\rm hor}} \dfrac{dk}{k} \dfrac{\tilde{\rho}_{\rm GW}(k)}{|A_k|^2} \mathcal{P}_{\rm prim}(k) \\ \rho_{\rm GW}^{(\rm sub-horizon)} = \dfrac{1}{8\pi G a^2} \displaystyle\int_{k_{\rm hor}}^{k_{\rm max}} \dfrac{dk}{k} \dfrac{\tilde{\rho}_{\rm GW}(k)}{|A_k|^2} \mathcal{P}_{\rm prim}(k) \end{cases}$$



Sub-horizon mode during RD

- During RD, sub-horizon contribution of back-reaction is given by

$$\rho_{\rm GW}^{\rm (sub-horizon)}(t) = \frac{1}{3}\rho_c(t)\frac{A_T(k_*)}{8}\log\left(\frac{k_{\rm max}}{k_{\rm hor}}\right) \quad ({\rm for}\ \ n_T = 0)\,,$$

$$= \frac{1}{3}\rho_c(t)\ \frac{A_T(k_*)}{8n_T}\left[\left(\frac{k_{\rm max}}{k_*}\right)^{n_T} - \left(\frac{k_{\rm hor}}{k_*}\right)^{n_T}\right] \quad ({\rm for}\ \ n_T \neq 0)$$

(Primordial GW spectrum)

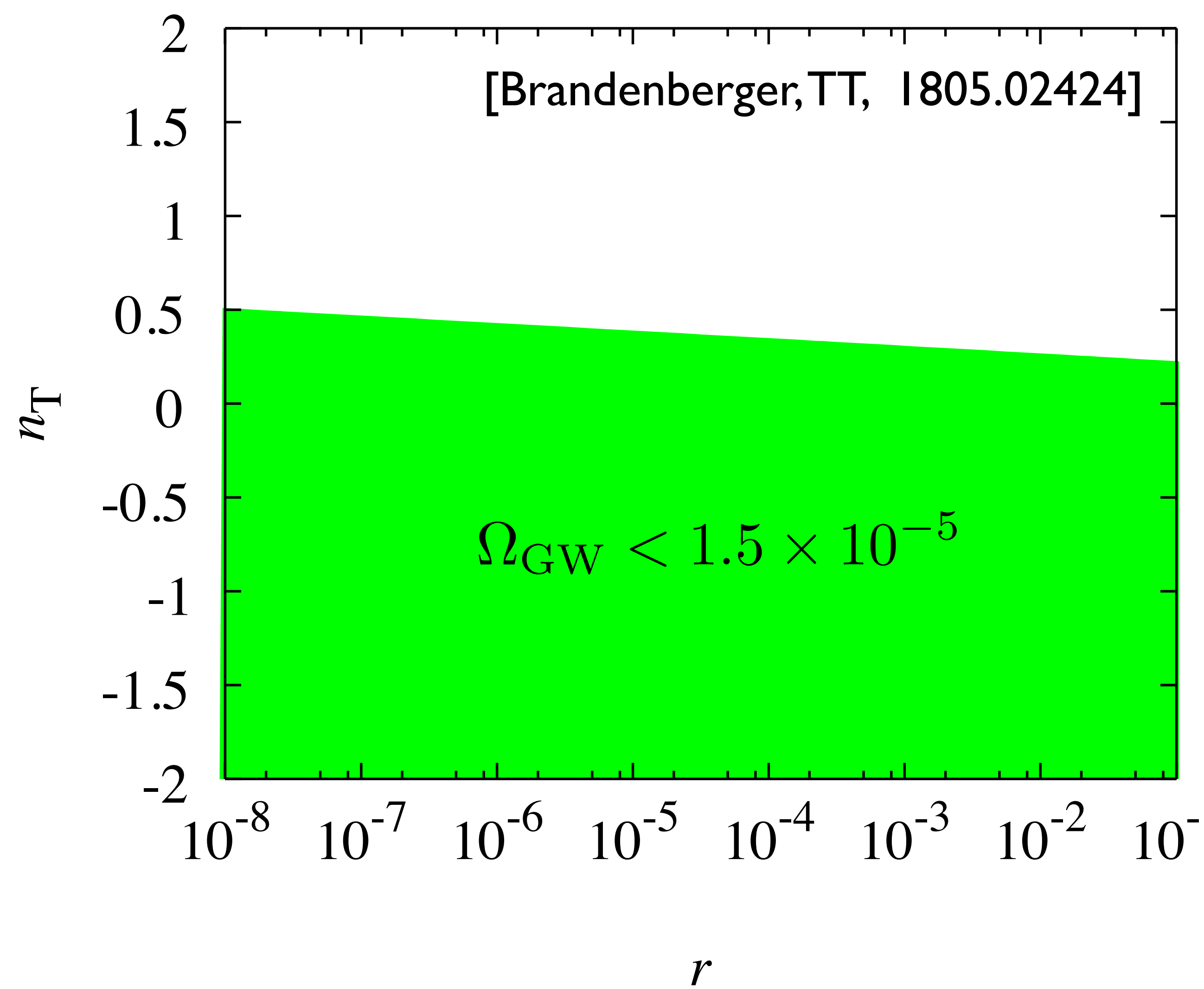
$$\mathcal{P}_{\rm prim}(k) = \sum_\lambda \frac{k^3}{\pi^2}|A_k^\lambda|^2 = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T}$$

By adopting BBN constraint:

$$\Omega_{\rm GW}^{\rm (sub-horizon)}h^2 \leq 5.6\times 10^{-6}(N_{\rm eff}-3)$$
$$\simeq 1.5\times 10^{-5}$$

→ we can obtain the bound on nT.

(See also [Stewart, Brandenberger 0711.4602; Kuroyanagi, TT, Yokoyama 1407.4785])



(We can also derive the constraint from MD case by requiring $|\rho_{\rm GW}| < \rho_c(t)$. However, less stringent than BBN.)

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- In fact, k_{max} ($=k_R$) and k_{hor} can also depend on n_T:

(if we assume a slow-roll inflation)

k_{max} :

$$\frac{k_{\rm max}}{k_*} = \frac{k_R}{k_*} = \frac{a_{\rm R}H_{\rm R}}{a_*H_*} = e^{N_*}\frac{H_{\rm R}}{H_*}$$

Assuming a slow-roll relation: $n_T = -2\epsilon = 2\frac{\dot{H}}{H^2} = 2\frac{1}{H}\frac{dH}{dN}$

$$\rightarrow\ H_R = H_*\exp\left[\frac{n_T}{2}N_*\right]$$

→ n_T can also depend through this relation.

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- In fact, k_{max} ($=k_R$) and k_{hor} can also depend on n_T:

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Values at the time of horizon crossing

k_R :

$$\frac{k_{\rm hor}}{k_*} = \frac{a_{\rm hor}H_{\rm hor}}{a_* H_*} \sim \frac{a_*}{a_{\rm hor}} \sim \frac{T_{\rm hor}}{T_R}$$

where $T_{\rm R} = \left(\frac{90}{\pi^2 g_*(t_{\rm end})}\right)^{1/4}\sqrt{H_R M_{\rm pl}}$

From the slow-roll equation: $H_R = H_* \exp\left[\frac{n_T}{2}N_*\right]$

→ n_T can also depend through this relation.

Sub-horizon mode during RD

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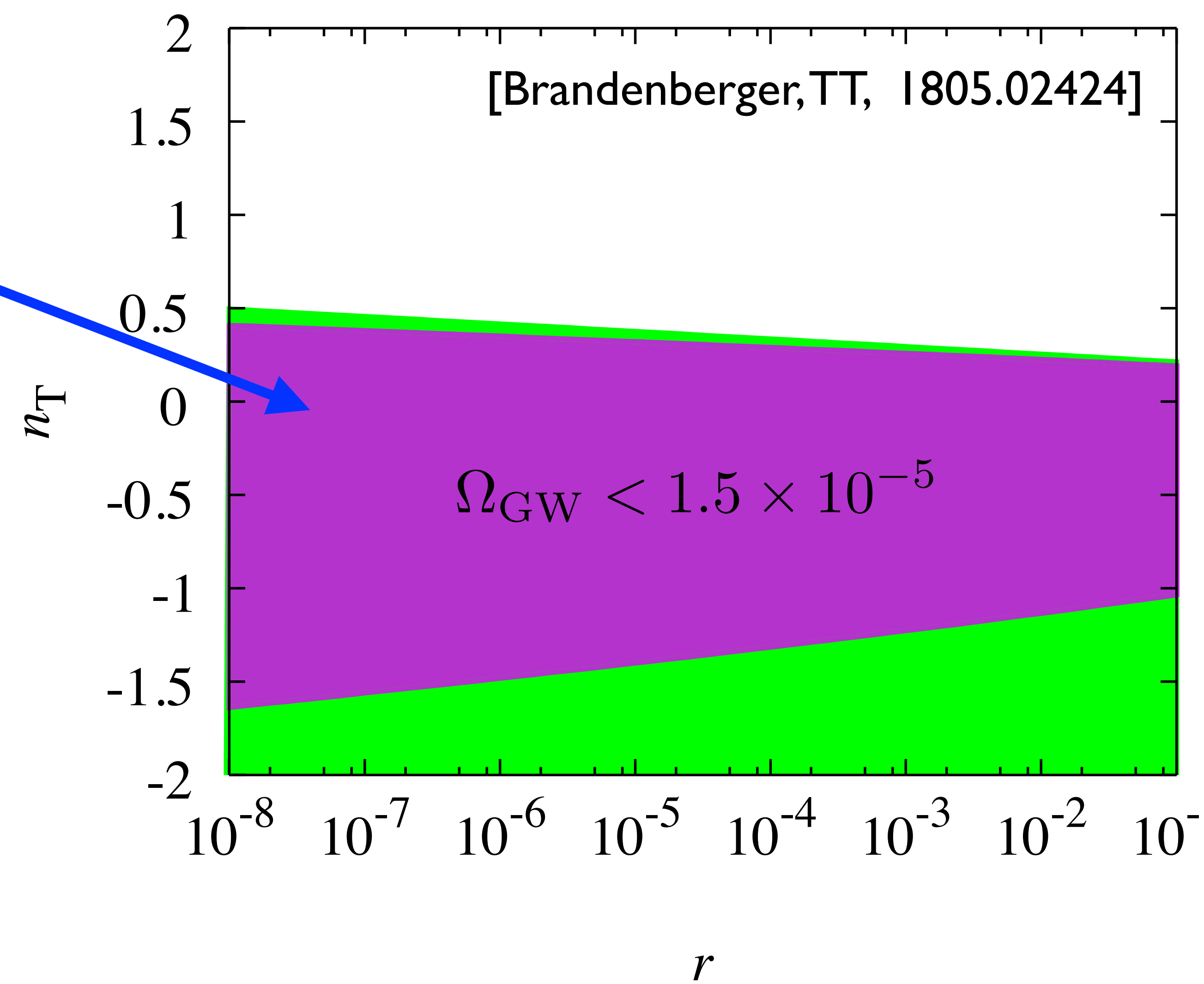
$$= \frac{1}{3}\rho_c(t)\ \frac{A_T(k_*)}{8n_T}\left[\left(\frac{k_{\rm max}}{k_*}\right)^{n_T} - \left(\frac{k_{\rm hor}}{k_*}\right)^{n_T}\right] \quad ({\rm for}\ \ n_T\neq 0)$$

Allowed region where the slow-roll relation $n_T = -2\varepsilon$ is adopted.



- Constrain gets severer.

- We can also constraint a negative n_T.

Super-Hubble mode in de Sitter phase

- During de Sitter phase, by requiring that $\left|\rho_{\rm GW}^{\rm (super-horizon)}\right| < \rho_{\rm crit}(t)$,
  we can put constraint on the total number of e-folds during inflation.

$$\rho_{\rm GW}^{\rm (super-horizon)}(t) \sim A_T(k_*)\rho_{\rm crit}(t)\left(\frac{k_{\rm min}}{k_*}\right)^{n_T}$$

$$\sim \left(\frac{a_*}{a_{\rm min}}\right)^{-|n_T|} \quad \text{(for negative } n_T\text{)}$$

$$= \exp\left[|n_T|(N_{\rm tot} - N_*)\right]$$

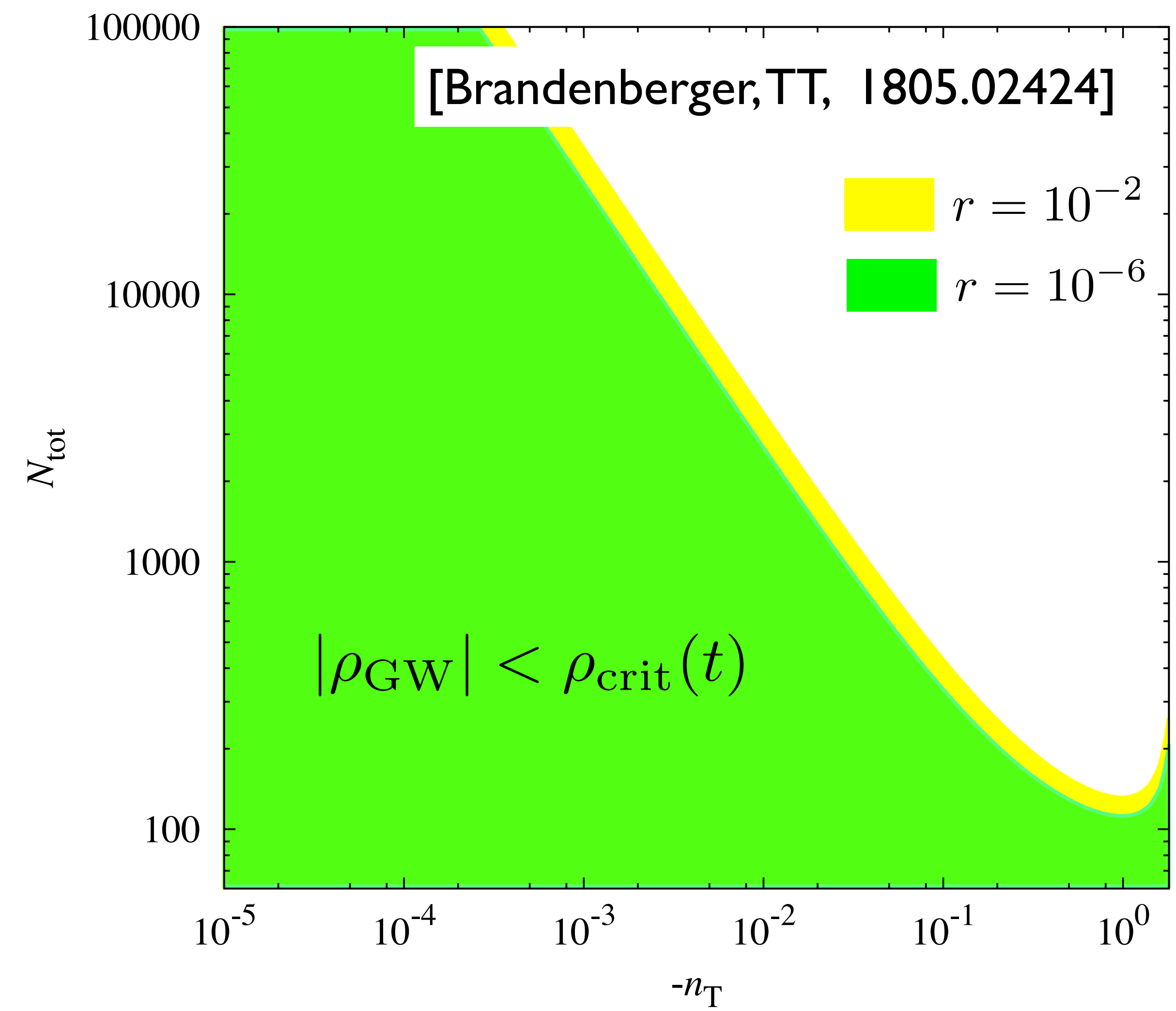
(Precise expression)

$$\rho_{\rm GW}^{\rm (super-horizon)}(t) = -\frac{\rho_c(t)}{3}\frac{7}{8}\frac{A_T}{2+n_T}\left(\frac{k_{\rm min}}{k_*}\right)^{n_T}\left[\left(\frac{k_{\rm hor}}{k_{\rm min}}\right)^{n_T} - \left(\frac{k_{\rm min}}{k_{\rm hor}}\right)^2\right]$$

→ When n_T is negative, the contribution becomes larger as we go back to the earlier time. ($k_{\rm hor}$ gets smaller and smaller.)

→ Constraint on the total number of e-folds during inflation.

(e.g., for $n_T \sim O(-0.01)$, $N_{\rm tot} < O(1000)$)



Summary

- Back-reaction of GWs behaves as radiation on sub-horizon scale, on the other hand, as curvature on super-horizon scales.

- We can derive the constraint on n_T by requiring that the back-reaction should be subdominant.

- By requiring that the back-reaction should be subdominant, we could derive the constraint on the total number of e-folds during inflation.
  (from super-Hubble modes in de Sitter phase.)

- We may should also calculate the back-reaction of GWs to (1st order) perturbations. (work in progress)