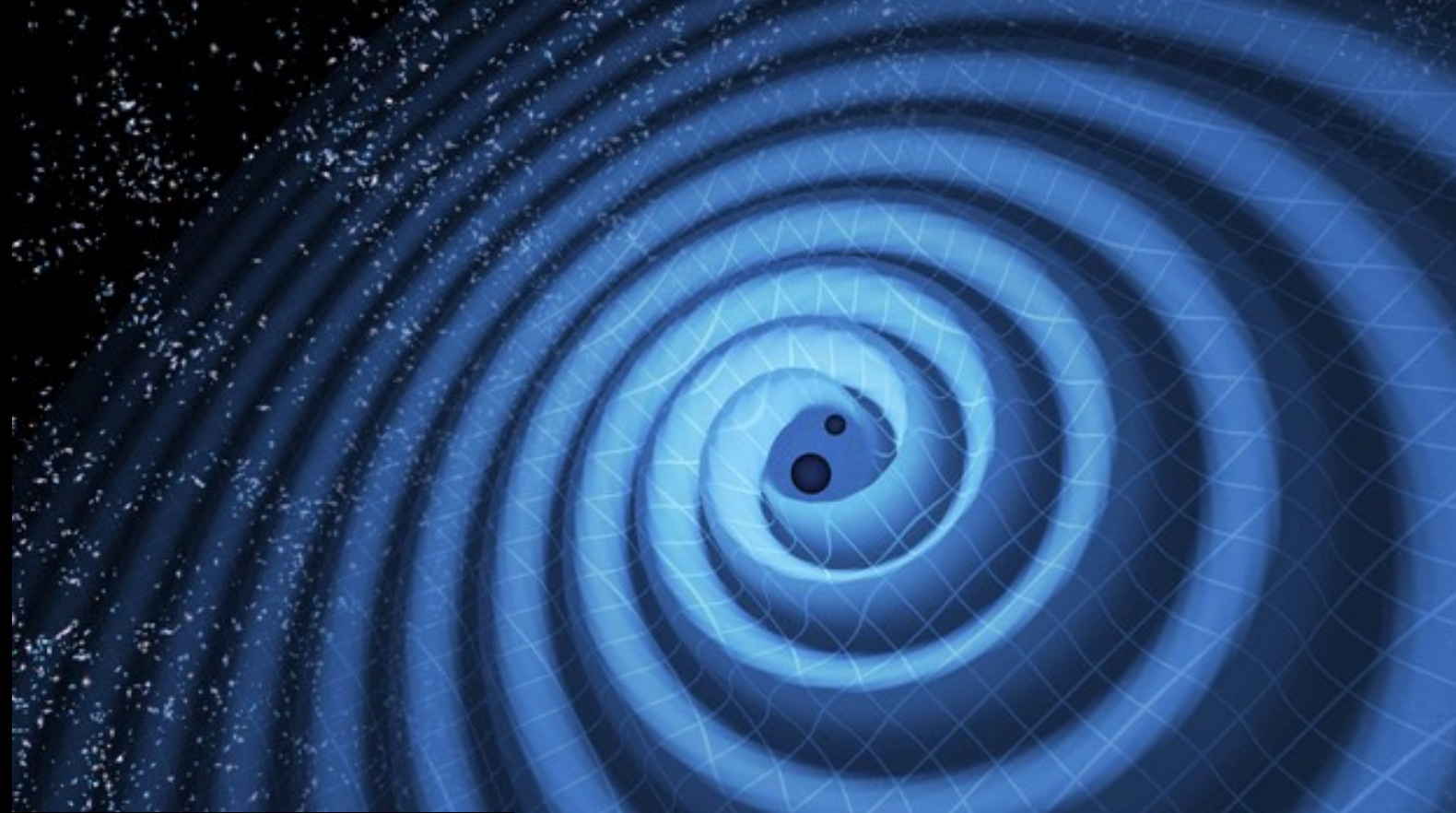


# Analytic Calculation of Gravitational Wave Spectrum induced from Primordial Curvature Perturbations

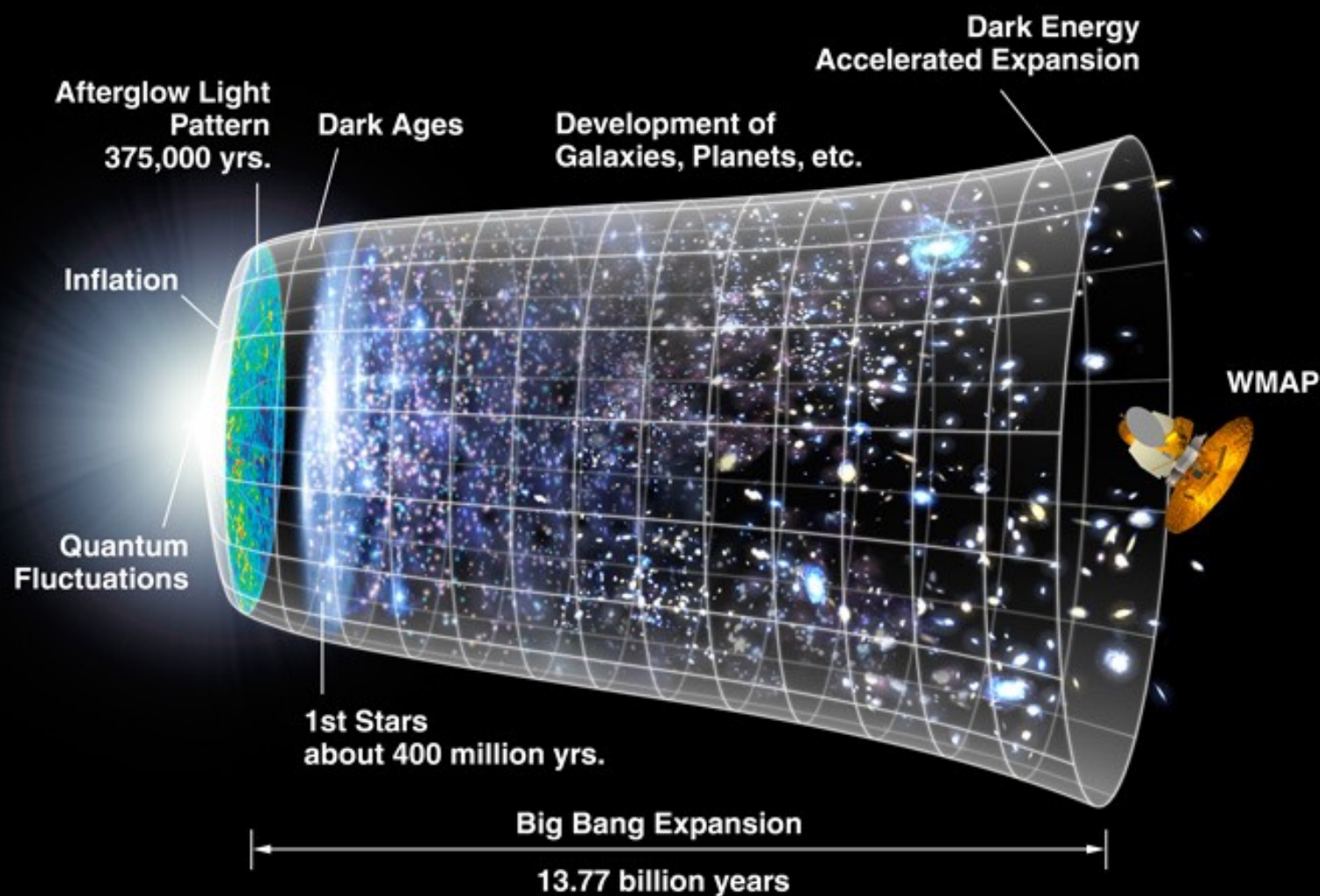
Takahiro Terada  
(KEK, JSPS fellow)

[Kohri, Terada, 1804.08577 [gr-qc], PRD 97 (2018) 123532]

# GW Astrophysics ↓ GW Cosmology



[Image credit: LIGO/T. Pyle]



We can probe  
the very early Universe  
or High Energy Physics.

[Image credit: NASA/WMAP Science Team]

NASA/WMAP Science Team

# What is the secondary GW?

[Ananda, Clarkson, Wands, gr-qc/0612013]

[Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

# What is the secondary GW?

[Ananda, Clarkson, Wands, gr-qc/0612013]

[Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

- Inflation produces  
scalar and tensor (1st-order) perturbations

$$ds^2 = -a^2(1 + 2\Phi)d\eta^2 + a^2 \left( (1 - 2\Phi)\delta_{ij} + \frac{1}{2}h_{ij} \right) dx^i dx^j$$

# What is the secondary GW?

[Ananda, Clarkson, Wands, gr-qc/0612013]

[Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

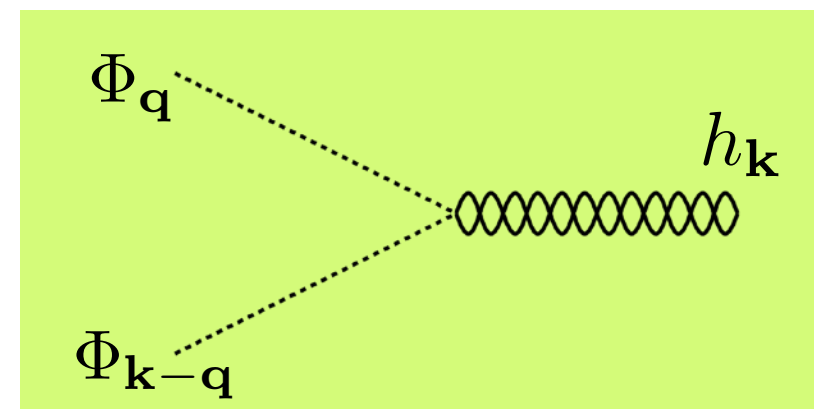
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

- Inflation produces  
scalar and tensor (1st-order) perturbations

$$ds^2 = -a^2(1 + 2\Phi)d\eta^2 + a^2 \left( (1 - 2\Phi)\delta_{ij} + \frac{1}{2}h_{ij} \right) dx^i dx^j$$

- (2nd-order tensor) = (1st-order scalar)<sup>2</sup>

$$h''_{\mathbf{k}}(\eta) + 2\mathcal{H}h'_{\mathbf{k}}(\eta) + k^2 h_{\mathbf{k}}(\eta) = 4S_{\mathbf{k}}(\eta)$$



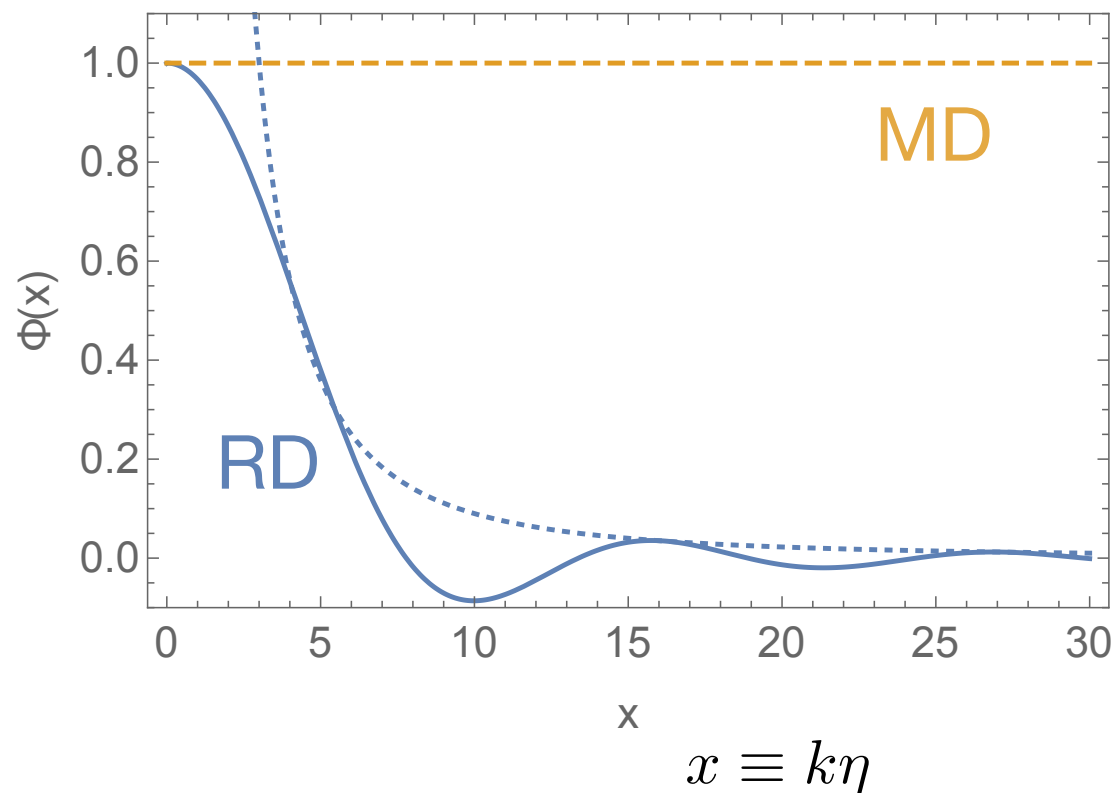
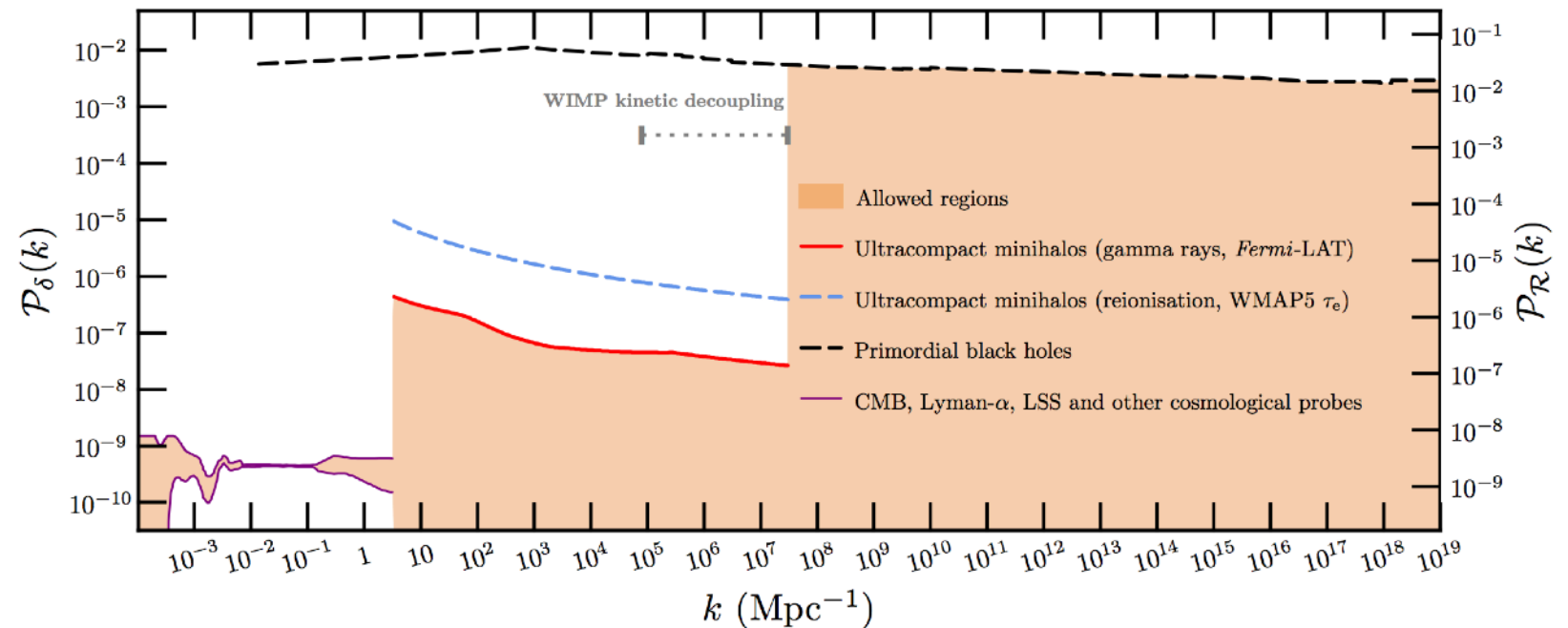
$$S_{\mathbf{k}} = \int \frac{d^3q}{(2\pi)^{3/2}} e_{ij}(\mathbf{k}) q_i q_j \left( 2\Phi_{\mathbf{q}}\Phi_{\mathbf{k}-\mathbf{q}} + \frac{4}{3(1+w)} (\mathcal{H}^{-1}\Phi'_{\mathbf{q}} + \Phi_{\mathbf{q}}) (\mathcal{H}^{-1}\Phi'_{\mathbf{k}-\mathbf{q}} + \Phi_{\mathbf{k}-\mathbf{q}}) \right)$$

# Enhanced scalar perturbations

[Bringmann, Scott, Akrami, 1110.2484]

## Initial condition

Enhanced primordial curvature perturbations are allowed at small scales.

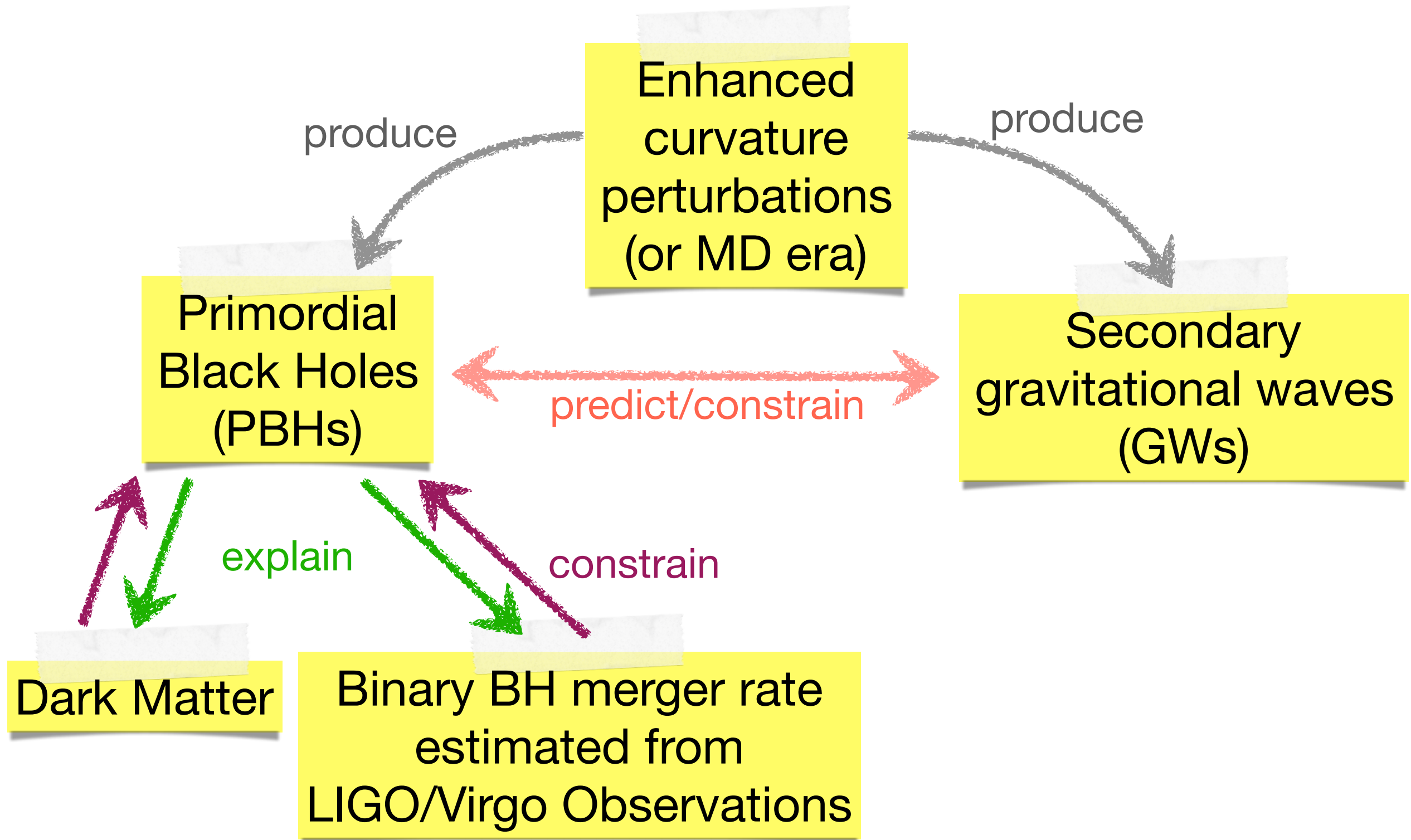


## Dynamics

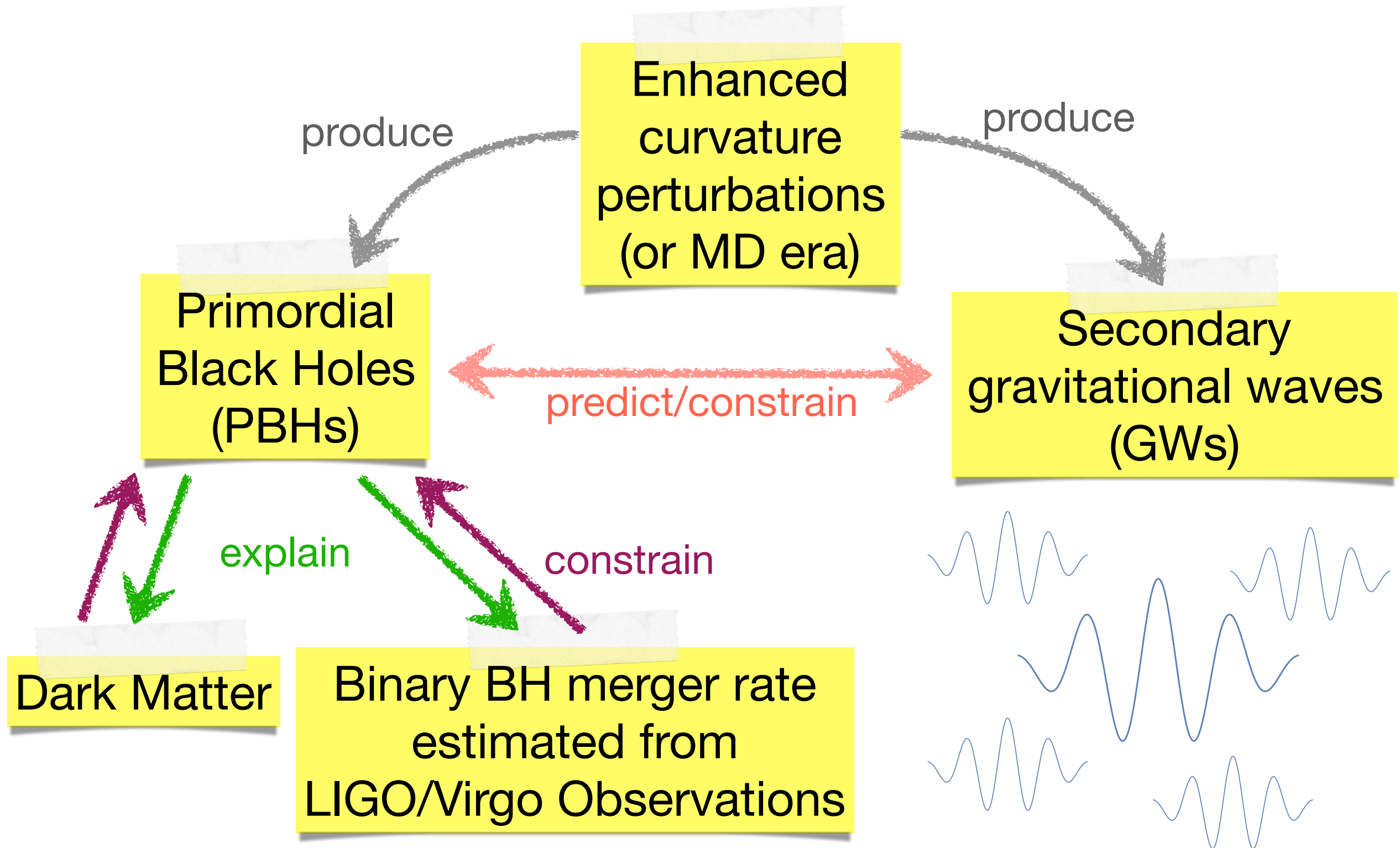
In a radiation-dominated (RD) era, the source decays.

In a matter-dominated (MD) era, the source does not decay.

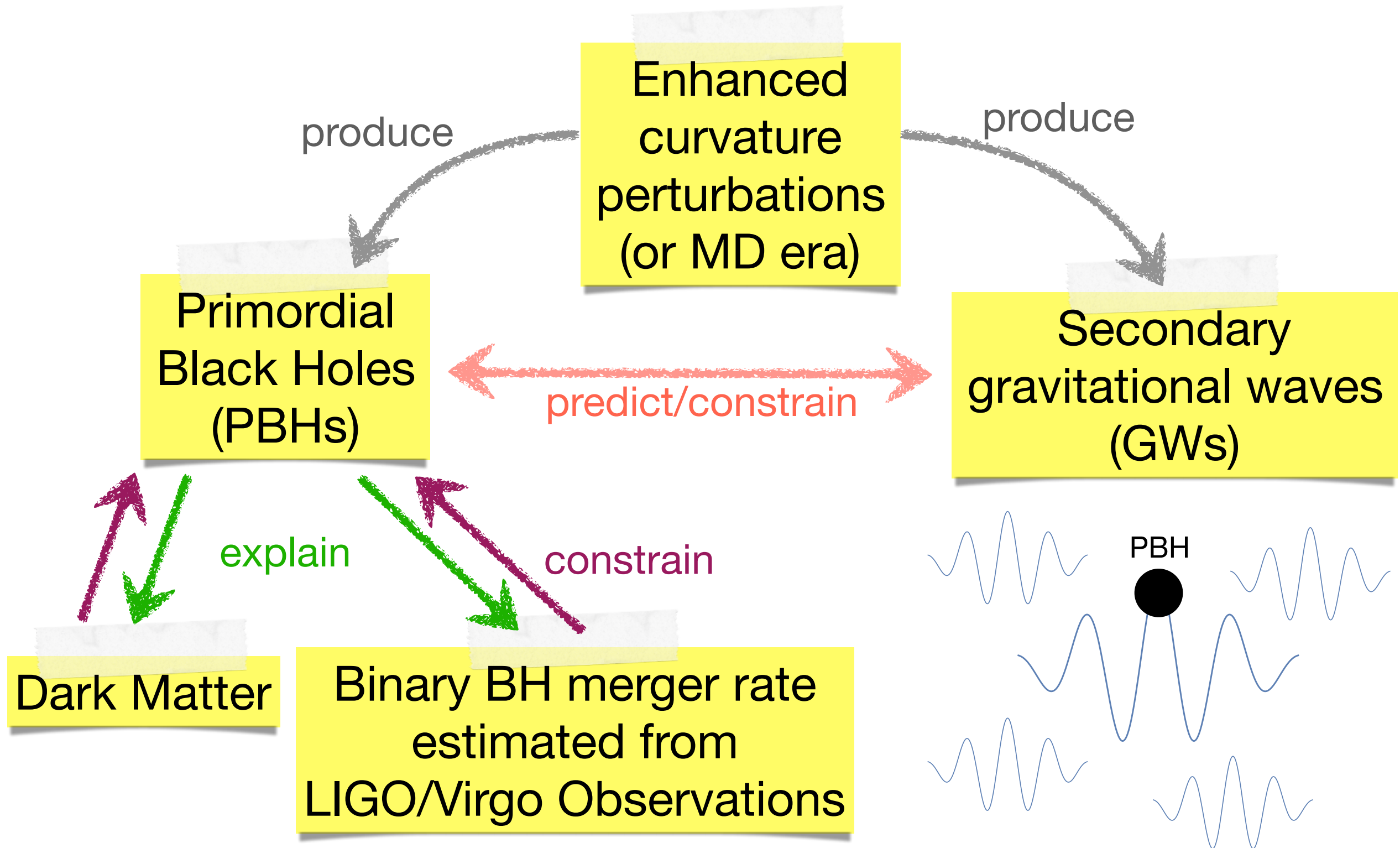
# Relation to PBH scenario



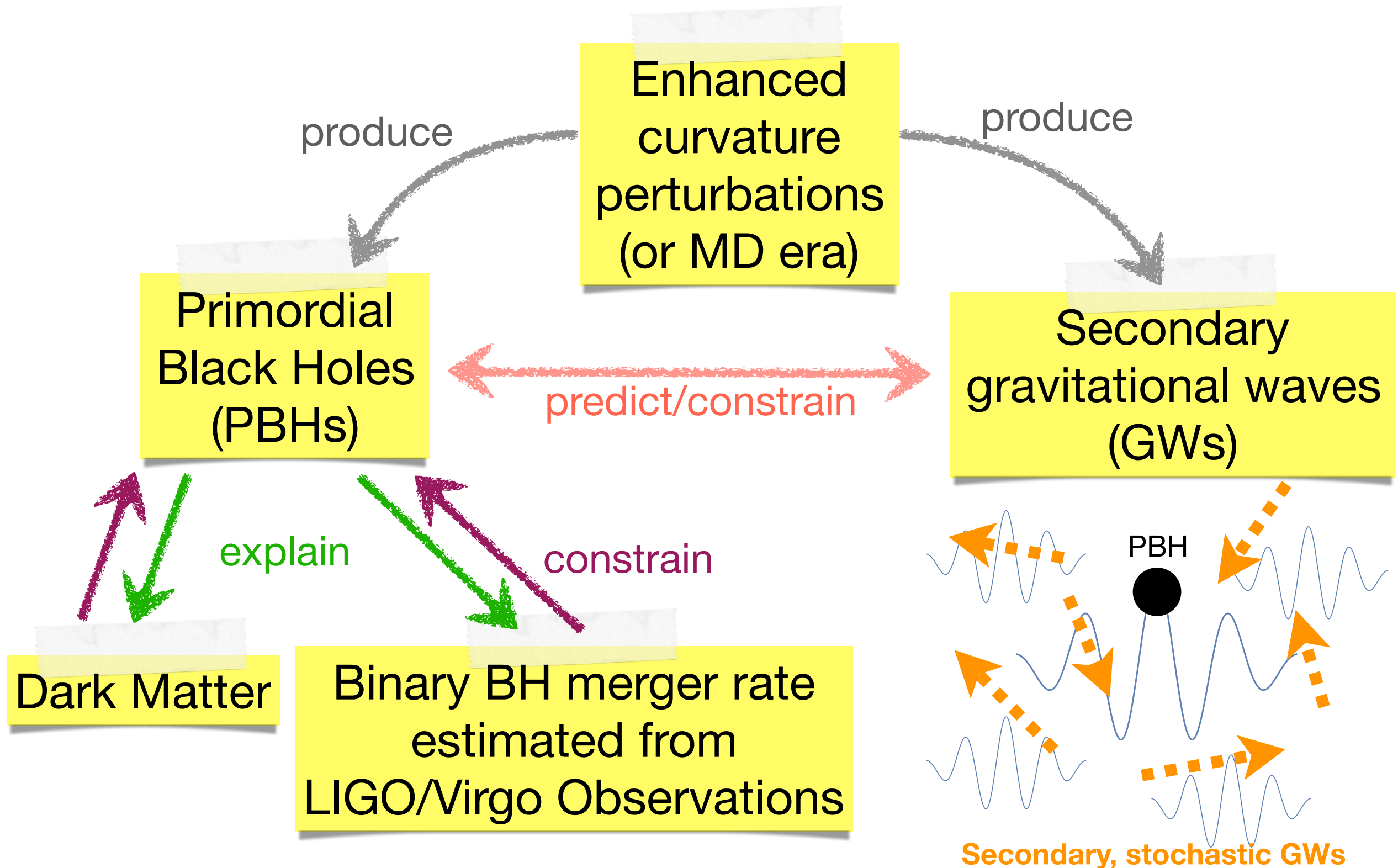
# Relation to PBH scenario



# Relation to PBH scenario



# Relation to PBH scenario



# Motivation (What we do)

The secondary GW power spectrum is obtained by **multiple integral** of an **oscillatory function**.

$$\mathcal{P}_h \sim \int dk \int dk' \left( \int dt f(k, k', t) \right)^2 \mathcal{P}_\zeta(k) \mathcal{P}_\zeta(k')$$

# Motivation (What we do)

The secondary GW power spectrum is obtained by **multiple integral** of an **oscillatory function**.

$$\mathcal{P}_h \sim \int dk \int dk' \left( \int dt f(k, k', t) \right)^2 \underbrace{\mathcal{P}_\zeta(k) \mathcal{P}_\zeta(k')}_{\text{primordial quantity}}$$

# Motivation (What we do)

The secondary GW power spectrum is obtained by **multiple integral** of an **oscillatory function**.

$$\mathcal{P}_h \sim \int dk \int dk' \underbrace{\left( \int dt f(k, k', t) \right)^2}_{\text{describing time evolution}} \underbrace{\mathcal{P}_\zeta(k) \mathcal{P}_\zeta(k')}_{\text{primordial quantity}}$$

“universal part”

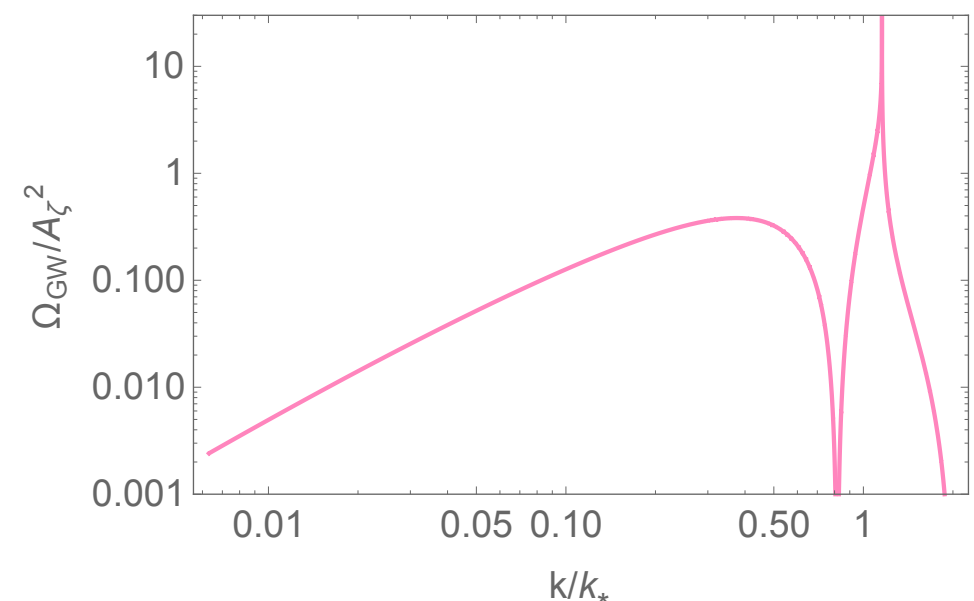
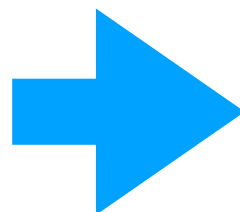
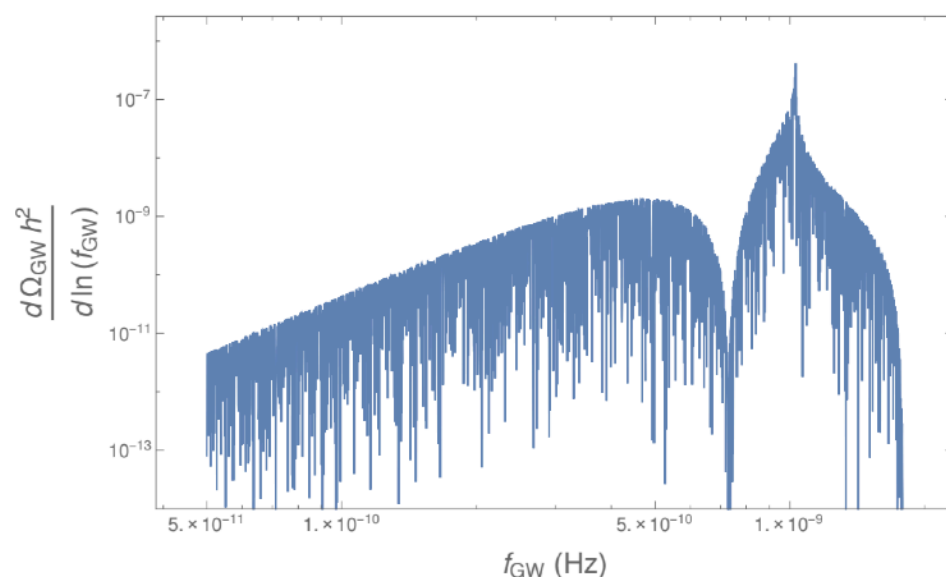
# Motivation (What we do)

The secondary GW power spectrum is obtained by **multiple integral** of an **oscillatory function**.

$$\mathcal{P}_h \sim \int dk \int dk' \underbrace{\left( \int dt f(k, k', t) \right)^2}_{\text{describing time evolution}} \underbrace{\mathcal{P}_\zeta(k) \mathcal{P}_\zeta(k')}_{\text{primordial quantity}}$$

“universal part”

[Orlofsky, Pierce, Wells, 1612.05279]



# Details (1/3)

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]  
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4vu} \right)^2 I^2(v, u, x) \mathcal{P}_\zeta(kv) \mathcal{P}_\zeta(ku)$$

where  $x \equiv k\eta$      $u = |\mathbf{k} - \tilde{\mathbf{k}}|/k$      $v = \tilde{k}/k$

# Details (1/3)

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]  
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4vu} \right)^2 I^2(v, u, x) \mathcal{P}_\zeta(kv) \mathcal{P}_\zeta(ku)$$

where  $x \equiv k\eta$      $u = |\mathbf{k} - \tilde{\mathbf{k}}|/k$      $v = \tilde{k}/k$

We want to analytically calculate this function.

$$\text{Definitions: } I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

# Details (1/3)

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]  
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4vu} \right)^2 I^2(v, u, x) \mathcal{P}_\zeta(kv) \mathcal{P}_\zeta(ku)$$

where  $x \equiv k\eta$      $u = |\mathbf{k} - \tilde{\mathbf{k}}|/k$      $v = \tilde{k}/k$

We want to analytically calculate this function.

$$\text{Definitions: } I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$f(v, u, \bar{x}) = \frac{6(w+1)}{3w+5} \Phi(v\bar{x}) \Phi(u\bar{x}) + \frac{6(1+3w)(w+1)}{(3w+5)^2} (\bar{x} \partial_{\bar{\eta}} \Phi(v\bar{x}) \Phi(u\bar{x}) + \bar{x} \partial_{\bar{\eta}} \Phi(u\bar{x}) \Phi(v\bar{x})) \\ + \frac{3(1+3w)^2(1+w)}{(3w+5)^2} \bar{x}^2 \partial_{\bar{\eta}} \Phi(v\bar{x}) \partial_{\bar{\eta}} \Phi(u\bar{x})$$

We have used  $\mathcal{H} = aH = 2/((1+3w)\eta)$      $\bar{x} \equiv k\bar{\eta}$      $w \equiv P/\rho$

# Details (2/3)

For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

# Details (2/3)

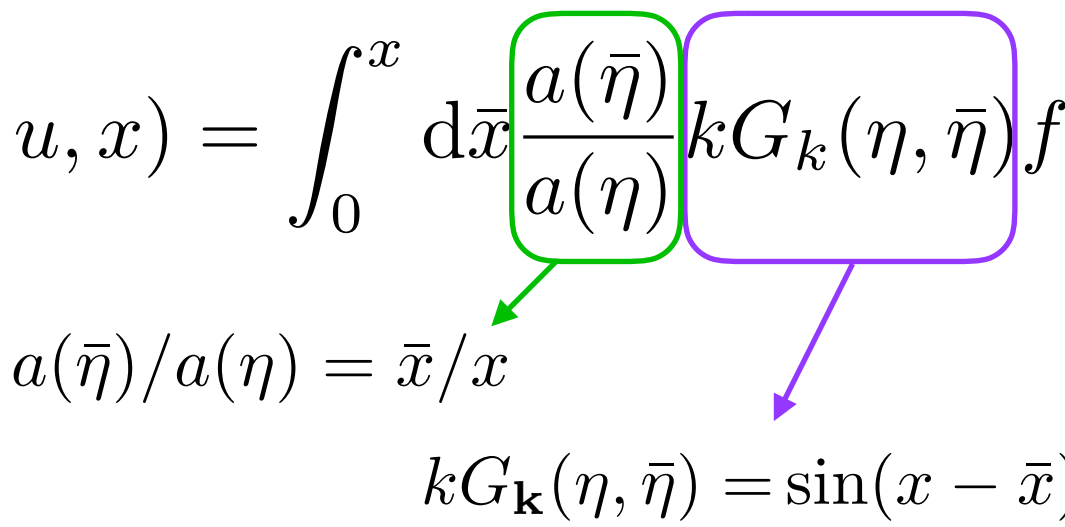
For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}/x$$

# Details (2/3)

For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$


$$a(\bar{\eta})/a(\eta) = \bar{x}/x$$

$$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \sin(x - \bar{x})$$

# Details (2/3)

For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}/x$$

$$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \sin(x - \bar{x})$$

$$\Phi(x) = \frac{9}{x^2} \left( \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3}) \right)$$

$$f_{\text{RD}}(v, u, x) = \frac{12}{u^3 v^3 x^6} \left( 18uvx^2 \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} + (54 - 6(u^2 + v^2)x^2 + u^2 v^2 x^4) \sin \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \right. \\ \left. + 2\sqrt{3}ux(v^2 x^2 - 9) \cos \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} + 2\sqrt{3}vx(u^2 x^2 - 9) \sin \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right)$$

# Details (3/3)

For definiteness, consider  $w = 1/3$  (RD era).

$$\begin{aligned}
 I_{\text{RD}}(v, u, x) = & \frac{3}{4u^3v^3x} \left( -\frac{4}{x^3} \left( uv(u^2 + v^2 - 3)x^3 \sin x - 6uvx^2 \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right. \right. \\
 & + 6\sqrt{3}ux \cos \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} + 6\sqrt{3}vx \sin \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} - 3(6 + (u^2 + v^2 - 3)x^2) \sin \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \Big) \\
 & + (u^2 + v^2 - 3)^2 \left( \sin x \left( \text{Ci} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) + \text{Ci} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \right. \\
 & - \text{Ci} \left( \left| 1 - \frac{v+u}{\sqrt{3}} \right| x \right) - \text{Ci} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) + \log \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \Big) \\
 & + \cos x \left( -\text{Si} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) - \text{Si} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \\
 & \left. \left. + \text{Si} \left( \left( 1 - \frac{v+u}{\sqrt{3}} \right) x \right) + \text{Si} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Si}(x) &= \int_0^x d\bar{x} \frac{\sin \bar{x}}{\bar{x}} \\
 \text{Ci}(x) &= - \int_x^\infty d\bar{x} \frac{\cos \bar{x}}{\bar{x}}
 \end{aligned}$$

# Details (3/3)

For definiteness, consider  $w = 1/3$  (RD era).

$$\begin{aligned}
 I_{\text{RD}}(v, u, x) = & \frac{3}{4u^3v^3x} \left( -\frac{4}{x^3} \left( uv(u^2 + v^2 - 3)x^3 \sin x - 6uvx^2 \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right. \right. \\
 & + 6\sqrt{3}ux \cos \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} + 6\sqrt{3}vx \sin \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} - 3(6 + (u^2 + v^2 - 3)x^2) \sin \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \Big) \\
 & + (u^2 + v^2 - 3)^2 \left( \sin x \left( \text{Ci} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) + \text{Ci} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \right. \\
 & - \text{Ci} \left( \left| 1 - \frac{v+u}{\sqrt{3}} \right| x \right) - \text{Ci} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) + \log \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \Big) \\
 & + \cos x \left( -\text{Si} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) - \text{Si} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \\
 & \left. \left. + \text{Si} \left( \left( 1 - \frac{v+u}{\sqrt{3}} \right) x \right) + \text{Si} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Si}(x) &= \int_0^x d\bar{x} \frac{\sin \bar{x}}{\bar{x}} \\
 \text{Ci}(x) &= - \int_x^\infty d\bar{x} \frac{\cos \bar{x}}{\bar{x}}
 \end{aligned}$$

**Oscillation average in the late-time limit**

$$\begin{aligned}
 \overline{I_{\text{RD}}^2(v, u, x \rightarrow \infty)} = & \frac{1}{2} \left( \frac{3(u^2 + v^2 - 3)}{4u^3v^3x} \right)^2 \left( \left( -4uv + (u^2 + v^2 - 3) \log \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \right)^2 \right. \\
 & \left. + \pi^2(u^2 + v^2 - 3)^2 \Theta(v + u - \sqrt{3}) \right)
 \end{aligned}$$

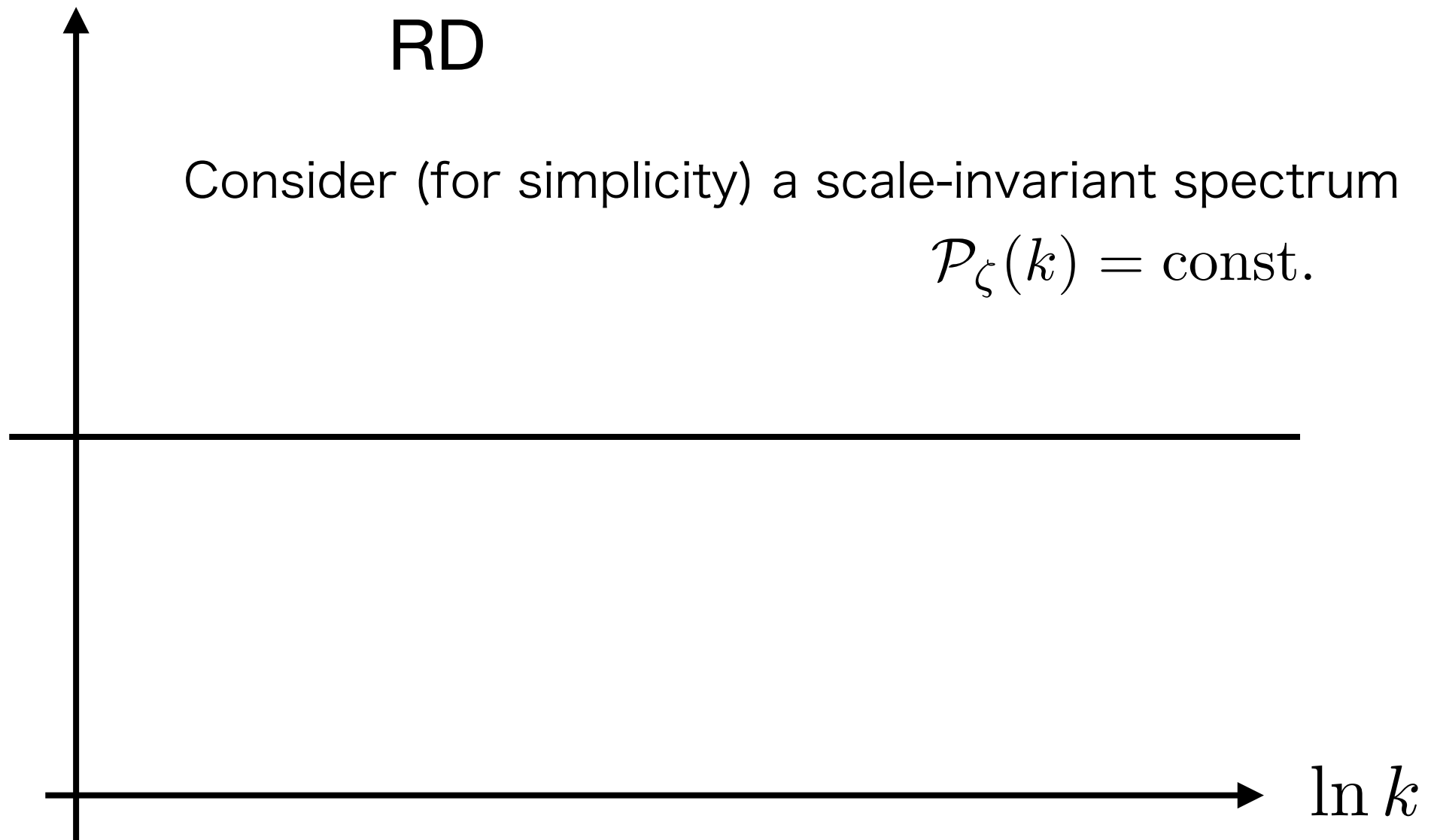
# General cases

$\ln \Omega_{\text{GW}}(k)$

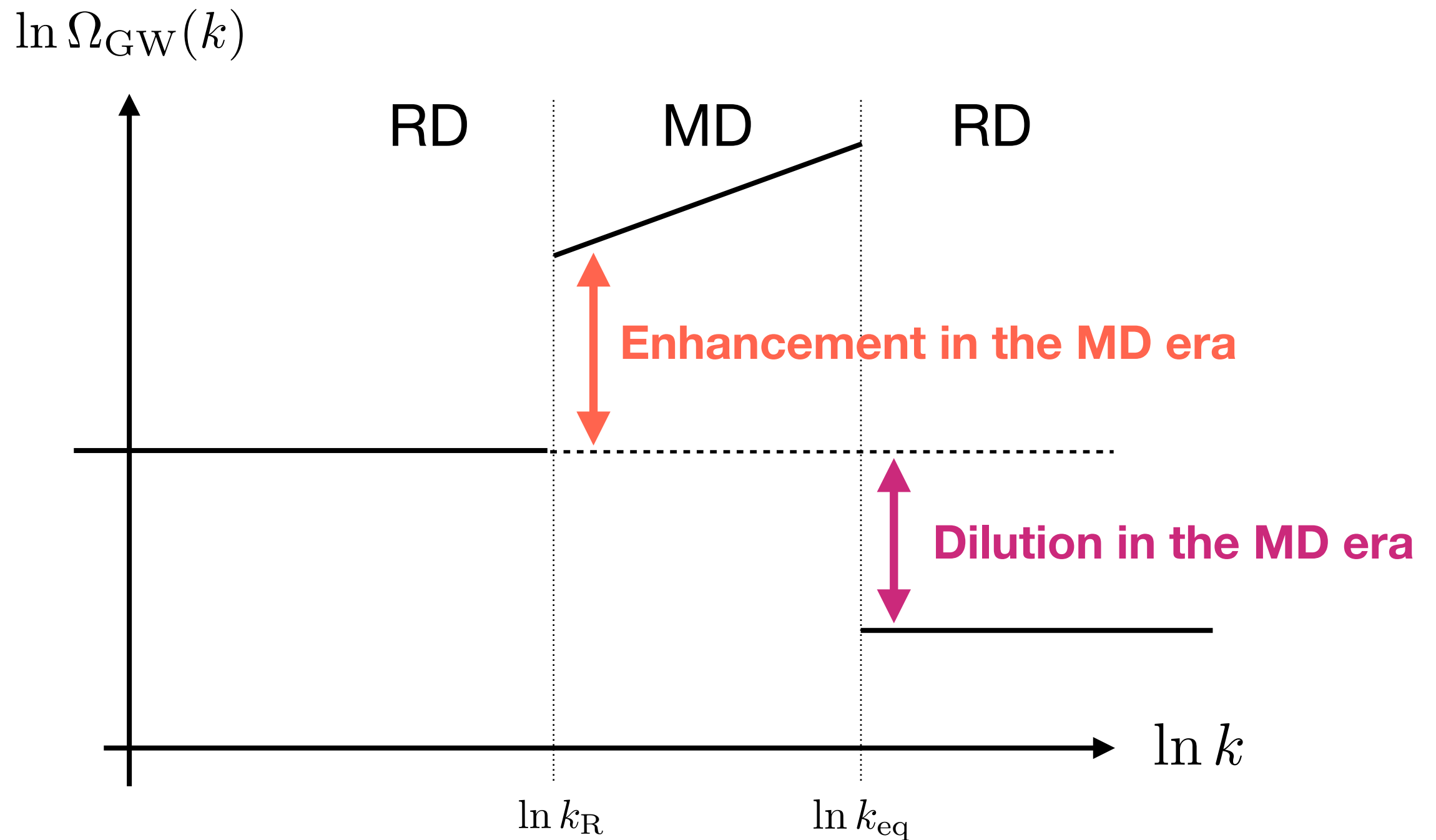
RD

Consider (for simplicity) a scale-invariant spectrum

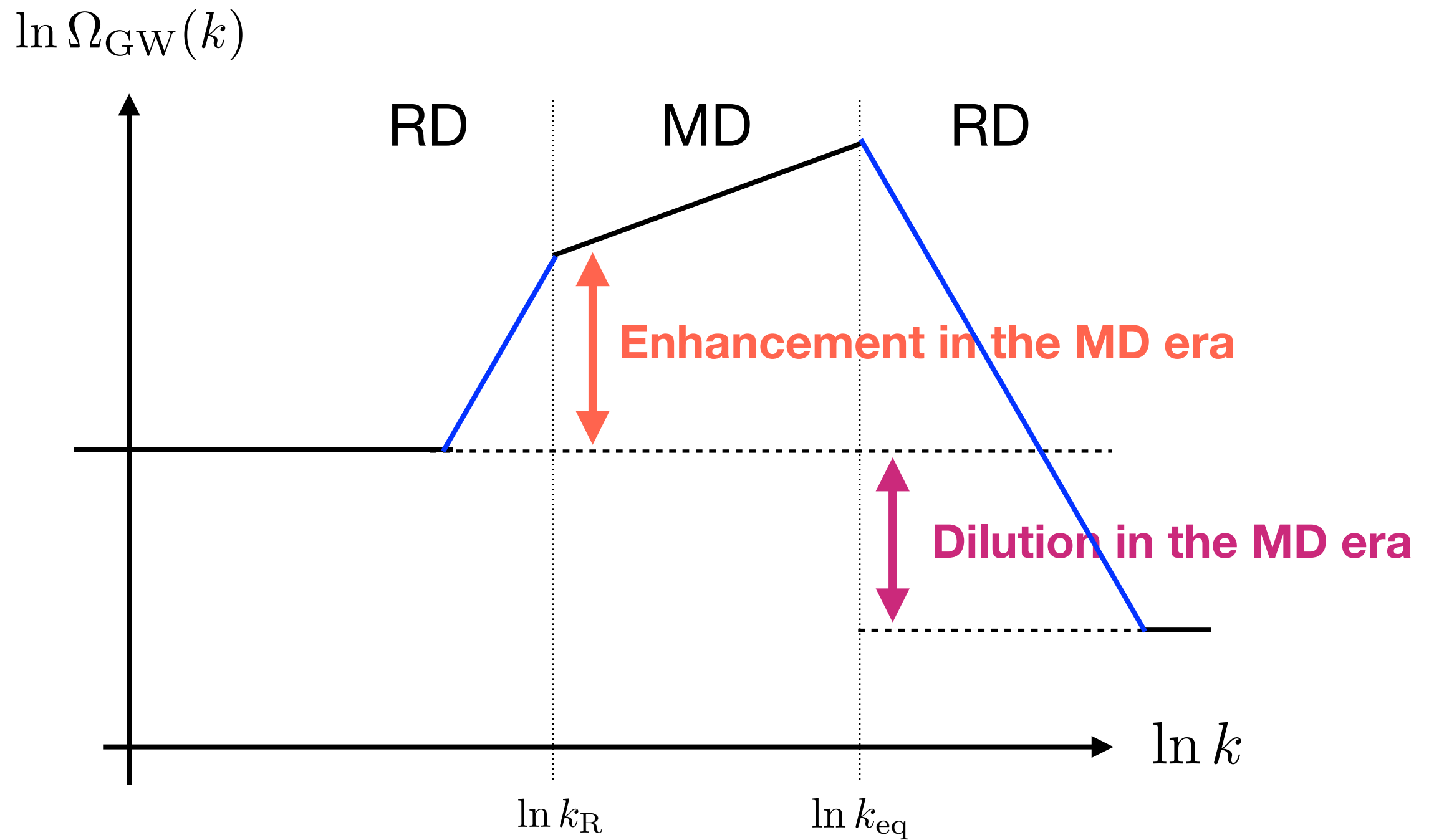
$$\mathcal{P}_\zeta(k) = \text{const.}$$



# General cases

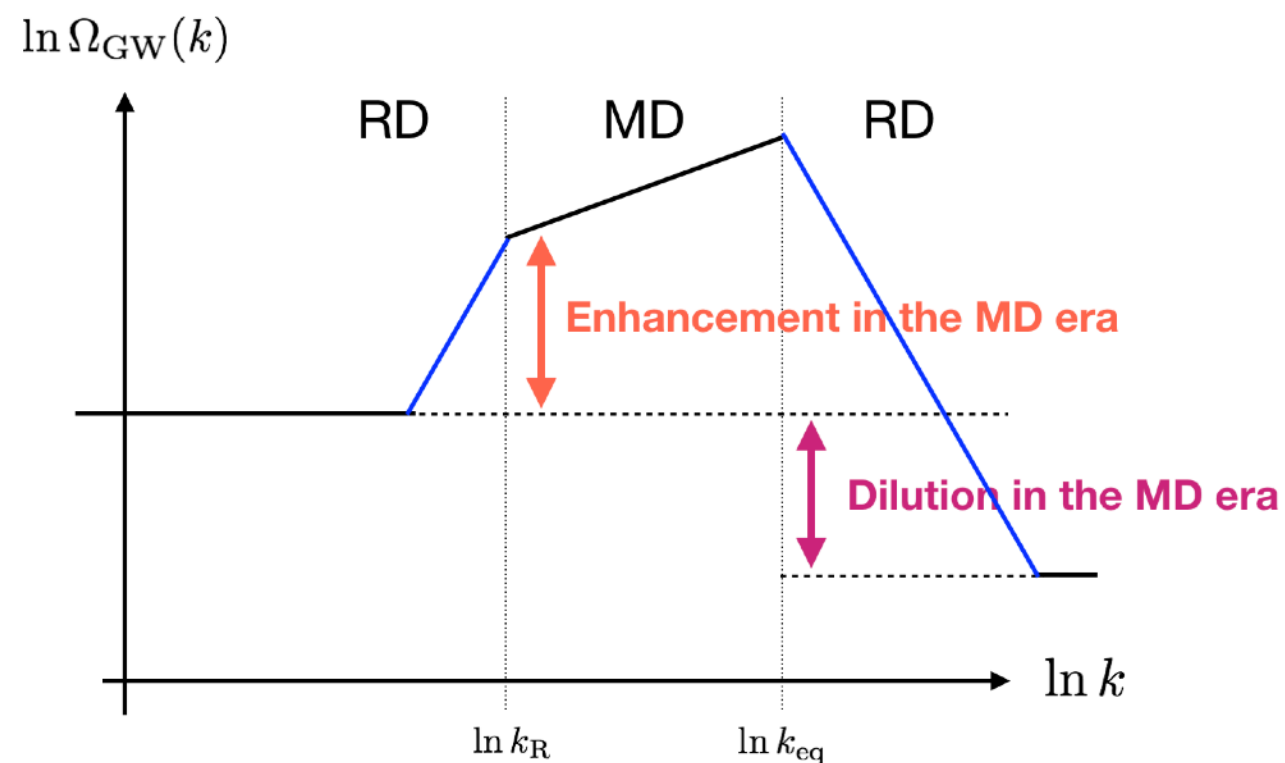


# General cases

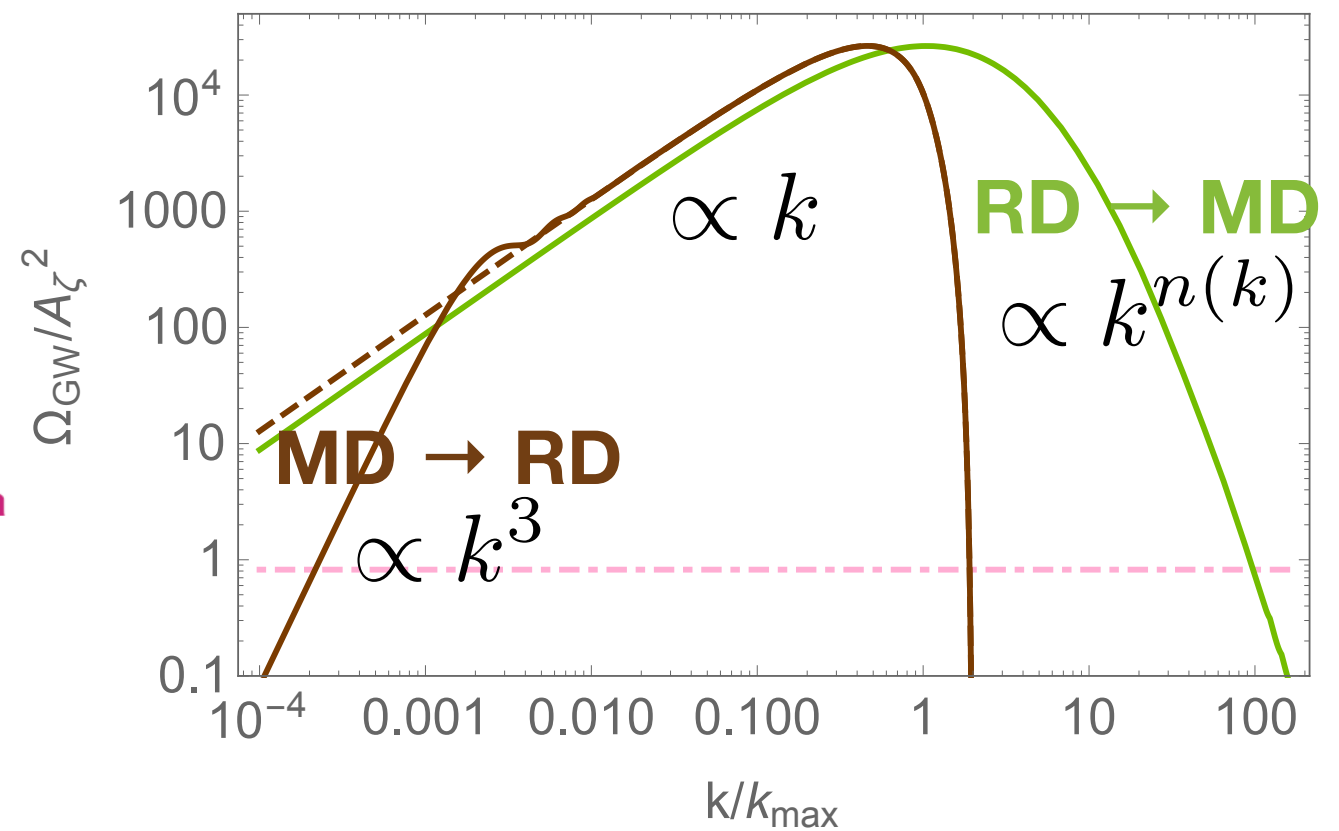


# General cases

**Schematic**



**Precise**

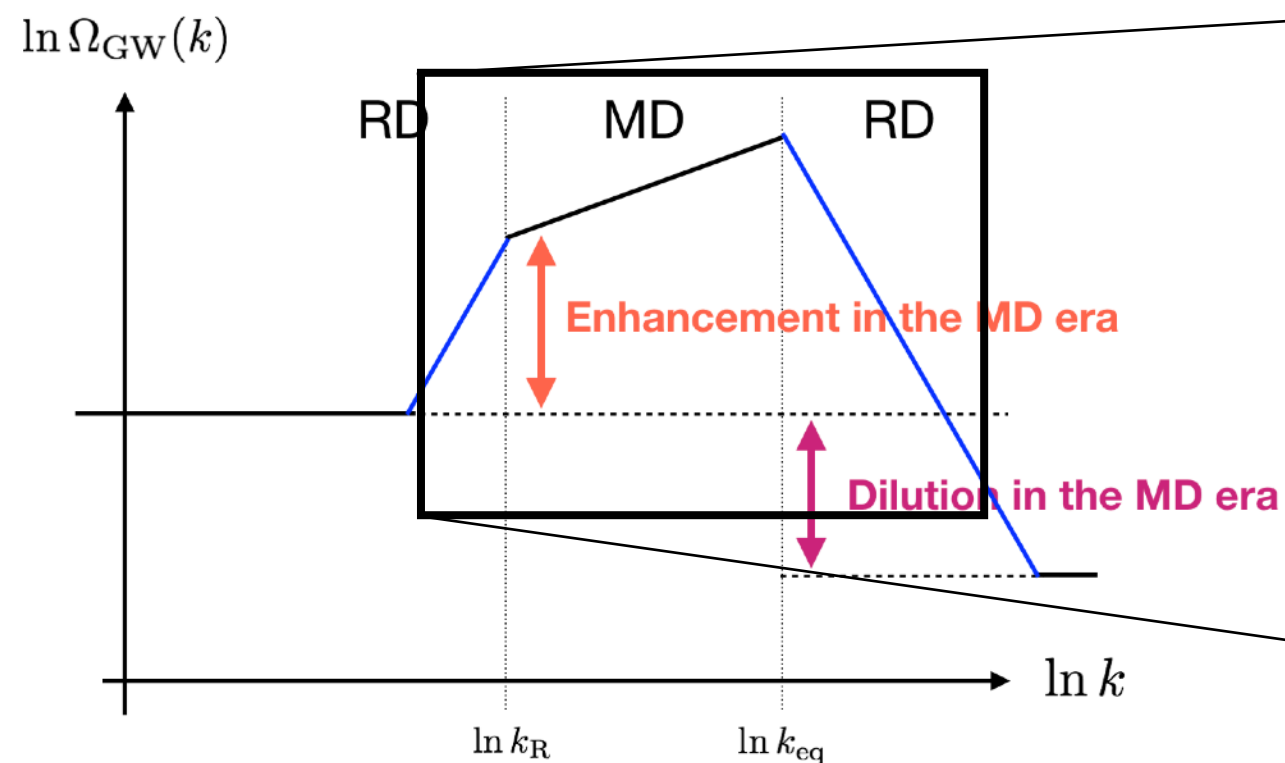


$$-6 \lesssim n(k) \lesssim -4$$

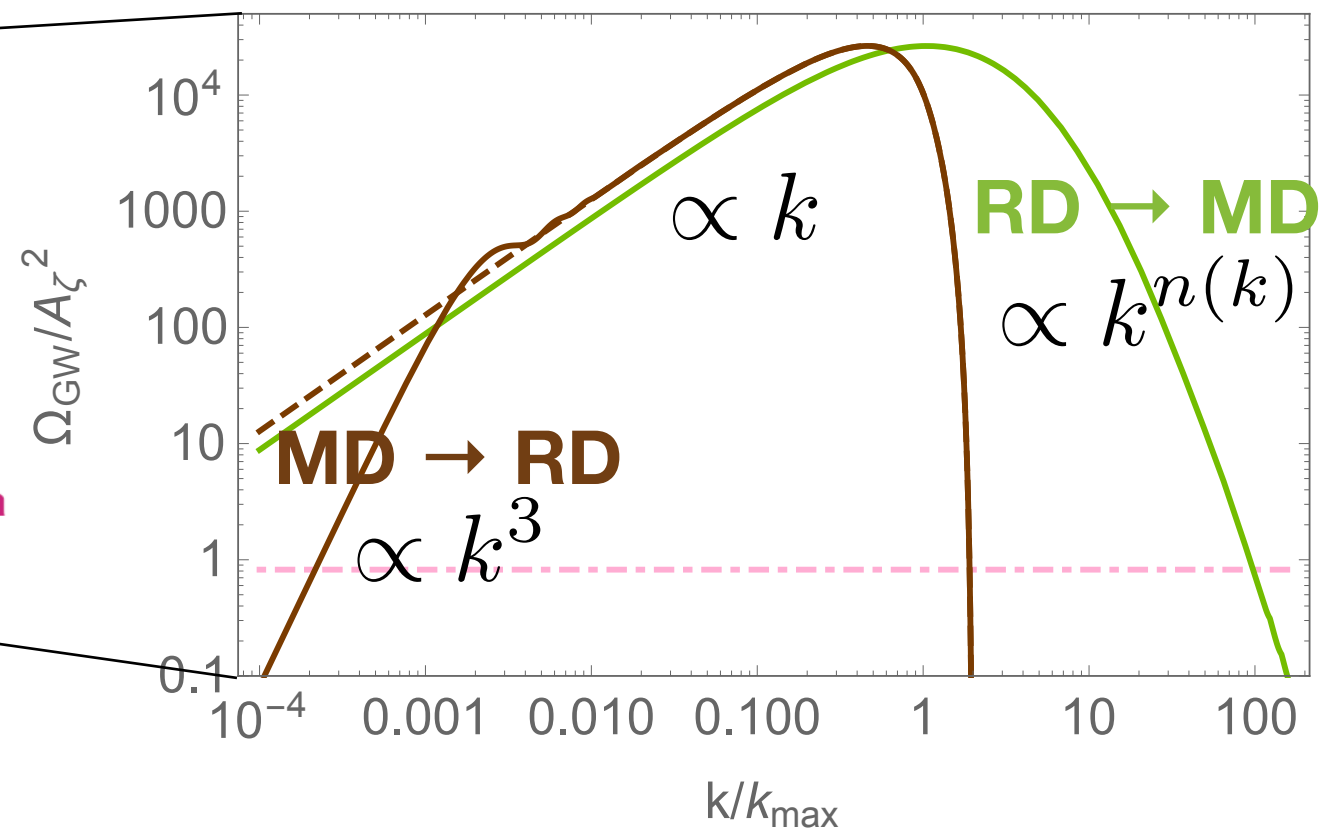
Scale-invariant primordial spectrum is assumed as a simple example.

# General cases

**Schematic**



**Precise**



$$-6 \lesssim n(k) \lesssim -4$$

Scale-invariant primordial spectrum is assumed as a simple example.

# Summary

# Summary

- **Secondary GW** is induced by primordial curvature perturbations. (important e.g. in PBH scenarios)

# Summary

- **Secondary GW** is induced by primordial curvature perturbations. (important e.g. in PBH scenarios)
- We **analytically calculated** the **universal part** of the power spectrum of the secondary GW.

# Summary

- **Secondary GW** is induced by primordial curvature perturbations. (important e.g. in PBH scenarios)
- We **analytically calculated** the **universal part** of the power spectrum of the secondary GW.
- Applicable to GWs induced in **RD**, **MD**, and **more general cases**.

# Summary

- **Secondary GW** is induced by primordial curvature perturbations. (important e.g. in PBH scenarios)
- We **analytically calculated** the **universal part** of the power spectrum of the secondary GW.
- Applicable to GWs induced in **RD**, **MD**, and **more general cases**.
- **Please use our results to write your papers!**

# Appendix:

## More quantitative explanations

- Basic things
- Radiation-dominated (RD) era
- Matter-dominated (MD) era
- More general cases

# Observable & Basic definitions

# Observable & Basic definitions

## Energy fraction

$$\Omega_{\text{GW}}(\eta, k) = \frac{1}{\rho_{\text{tot}}(\eta)} \frac{d\rho_{\text{GW}}(\eta)}{d \ln k} = \frac{1}{24} \left( \frac{k}{a(\eta)H(\eta)} \right)^2 \overline{\mathcal{P}_h(\eta, k)}$$

# Observable & Basic definitions

## Energy fraction

$$\Omega_{\text{GW}}(\eta, k) = \frac{1}{\rho_{\text{tot}}(\eta)} \frac{d\rho_{\text{GW}}(\eta)}{d \ln k} = \frac{1}{24} \left( \frac{k}{a(\eta)H(\eta)} \right)^2 \overline{\mathcal{P}_h(\eta, k)}$$

## Power spectrum

$$\langle h_{\mathbf{k}}^{\lambda}(\eta) h_{\mathbf{k}'}^{\lambda'}(\eta) \rangle = \delta_{\lambda\lambda'} \delta^3(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_h(\eta, k)$$

# Observable & Basic definitions

## Energy fraction

$$\Omega_{\text{GW}}(\eta, k) = \frac{1}{\rho_{\text{tot}}(\eta)} \frac{d\rho_{\text{GW}}(\eta)}{d \ln k} = \frac{1}{24} \left( \frac{k}{a(\eta)H(\eta)} \right)^2 \overline{\mathcal{P}_h(\eta, k)}$$

## Power spectrum

$$\langle h_{\mathbf{k}}^\lambda(\eta) h_{\mathbf{k}'}^{\lambda'}(\eta) \rangle = \delta_{\lambda\lambda'} \delta^3(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_h(\eta, k)$$

## Gravitational field: Green's function method

$$a(\eta)h_{\mathbf{k}}(\eta) = 4 \int^\eta d\bar{\eta} G_{\mathbf{k}}(\eta, \bar{\eta}) a(\bar{\eta}) S_{\mathbf{k}}(\bar{\eta})$$

$$G_{\mathbf{k}}''(\eta, \bar{\eta}) + \left( k^2 - \frac{a''(\eta)}{a(\eta)} \right) G_{\mathbf{k}}(\eta, \bar{\eta}) = \delta(\eta - \bar{\eta})$$

# Power spectrum & “universal part”

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]  
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4vu} \right)^2 I^2(v, u, x) \mathcal{P}_\zeta(kv) \mathcal{P}_\zeta(ku)$$

where  $x \equiv k\eta$      $u = |\mathbf{k} - \tilde{\mathbf{k}}|/k$      $v = \tilde{k}/k$

# Power spectrum & “universal part”

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]  
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4vu} \right)^2 I^2(v, u, x) \mathcal{P}_\zeta(kv) \mathcal{P}_\zeta(ku)$$

where  $x \equiv k\eta$      $u = |\mathbf{k} - \tilde{\mathbf{k}}|/k$      $v = \tilde{k}/k$

We want to analytically calculate this function.

$$\text{Definitions: } I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

# Power spectrum & “universal part”

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]  
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4 \int_0^\infty dv \int_{|1-v|}^{1+v} du \left( \frac{4v^2 - (1 + v^2 - u^2)^2}{4vu} \right)^2 I^2(v, u, x) \mathcal{P}_\zeta(kv) \mathcal{P}_\zeta(ku)$$

where  $x \equiv k\eta$      $u = |\mathbf{k} - \tilde{\mathbf{k}}|/k$      $v = \tilde{k}/k$

We want to analytically calculate this function.

$$\text{Definitions: } I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$f(v, u, \bar{x}) = \frac{6(w+1)}{3w+5} \Phi(v\bar{x}) \Phi(u\bar{x}) + \frac{6(1+3w)(w+1)}{(3w+5)^2} (\bar{x} \partial_{\bar{\eta}} \Phi(v\bar{x}) \Phi(u\bar{x}) + \bar{x} \partial_{\bar{\eta}} \Phi(u\bar{x}) \Phi(v\bar{x})) \\ + \frac{3(1+3w)^2(1+w)}{(3w+5)^2} \bar{x}^2 \partial_{\bar{\eta}} \Phi(v\bar{x}) \partial_{\bar{\eta}} \Phi(u\bar{x})$$

We have used  $\mathcal{H} = aH = 2/((1+3w)\eta)$      $\bar{x} \equiv k\bar{\eta}$      $w \equiv P/\rho$

# Analytic Calculation

For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

# Analytic Calculation

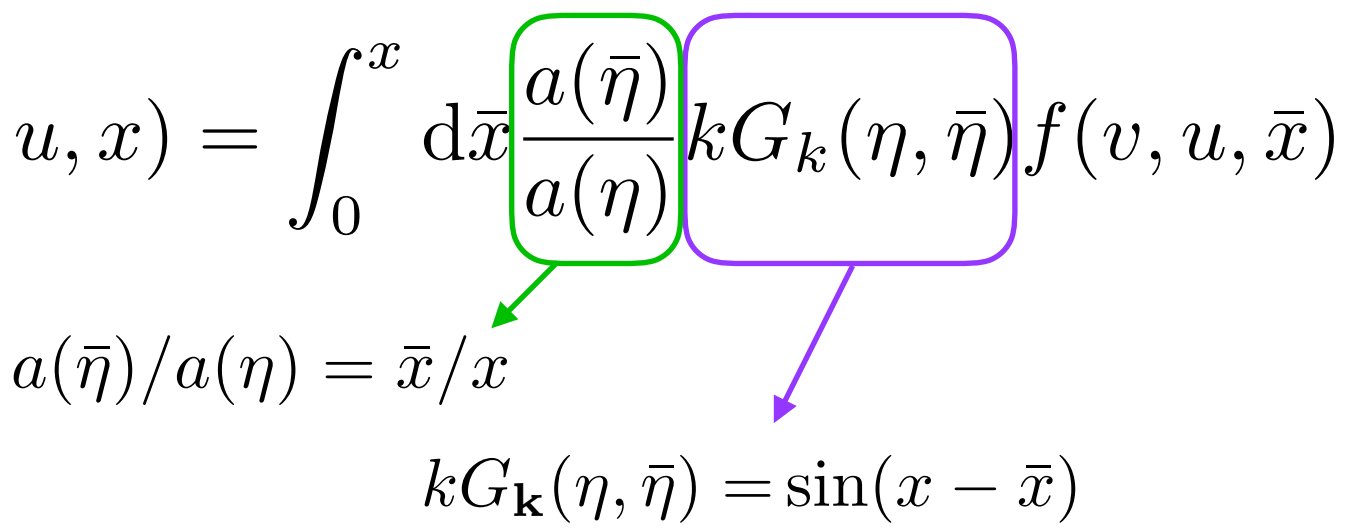
For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}/x$$

# Analytic Calculation

For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$


$$a(\bar{\eta})/a(\eta) = \bar{x}/x$$

$$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \sin(x - \bar{x})$$

# Analytic Calculation

For definiteness, consider  $w = 1/3$  (RD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}/x$$

$$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \sin(x - \bar{x})$$

$$\Phi(x) = \frac{9}{x^2} \left( \frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3}) \right)$$

$$f_{\text{RD}}(v, u, x) = \frac{12}{u^3 v^3 x^6} \left( 18uvx^2 \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} + (54 - 6(u^2 + v^2)x^2 + u^2 v^2 x^4) \sin \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \right. \\ \left. + 2\sqrt{3}ux(v^2 x^2 - 9) \cos \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} + 2\sqrt{3}vx(u^2 x^2 - 9) \sin \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right)$$

# Analytic Calculation

For definiteness, consider  $w = 1/3$  (RD era).

$$\begin{aligned}
 I_{\text{RD}}(v, u, x) = & \frac{3}{4u^3v^3x} \left( -\frac{4}{x^3} \left( uv(u^2 + v^2 - 3)x^3 \sin x - 6uvx^2 \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right. \right. \\
 & + 6\sqrt{3}ux \cos \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} + 6\sqrt{3}vx \sin \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} - 3(6 + (u^2 + v^2 - 3)x^2) \sin \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \Big) \\
 & + (u^2 + v^2 - 3)^2 \left( \sin x \left( \text{Ci} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) + \text{Ci} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \right. \\
 & - \text{Ci} \left( \left| 1 - \frac{v+u}{\sqrt{3}} \right| x \right) - \text{Ci} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) + \log \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \Big) \\
 & + \cos x \left( -\text{Si} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) - \text{Si} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \\
 & \left. \left. + \text{Si} \left( \left( 1 - \frac{v+u}{\sqrt{3}} \right) x \right) + \text{Si} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Si}(x) &= \int_0^x d\bar{x} \frac{\sin \bar{x}}{\bar{x}} \\
 \text{Ci}(x) &= - \int_x^\infty d\bar{x} \frac{\cos \bar{x}}{\bar{x}}
 \end{aligned}$$

# Analytic Calculation

For definiteness, consider  $w = 1/3$  (RD era).

$$\begin{aligned}
 I_{\text{RD}}(v, u, x) = & \frac{3}{4u^3v^3x} \left( -\frac{4}{x^3} \left( uv(u^2 + v^2 - 3)x^3 \sin x - 6uvx^2 \cos \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} \right. \right. \\
 & + 6\sqrt{3}ux \cos \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} + 6\sqrt{3}vx \sin \frac{ux}{\sqrt{3}} \cos \frac{vx}{\sqrt{3}} - 3(6 + (u^2 + v^2 - 3)x^2) \sin \frac{ux}{\sqrt{3}} \sin \frac{vx}{\sqrt{3}} \Big) \\
 & + (u^2 + v^2 - 3)^2 \left( \sin x \left( \text{Ci} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) + \text{Ci} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \right. \\
 & - \text{Ci} \left( \left| 1 - \frac{v+u}{\sqrt{3}} \right| x \right) - \text{Ci} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) + \log \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \Big) \\
 & + \cos x \left( -\text{Si} \left( \left( 1 - \frac{v-u}{\sqrt{3}} \right) x \right) - \text{Si} \left( \left( 1 + \frac{v-u}{\sqrt{3}} \right) x \right) \right. \\
 & \left. \left. + \text{Si} \left( \left( 1 - \frac{v+u}{\sqrt{3}} \right) x \right) + \text{Si} \left( \left( 1 + \frac{v+u}{\sqrt{3}} \right) x \right) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Si}(x) &= \int_0^x d\bar{x} \frac{\sin \bar{x}}{\bar{x}} \\
 \text{Ci}(x) &= - \int_x^\infty d\bar{x} \frac{\cos \bar{x}}{\bar{x}}
 \end{aligned}$$

**Oscillation average in the late-time limit**

$$\begin{aligned}
 \overline{I_{\text{RD}}^2(v, u, x \rightarrow \infty)} = & \frac{1}{2} \left( \frac{3(u^2 + v^2 - 3)}{4u^3v^3x} \right)^2 \left( \left( -4uv + (u^2 + v^2 - 3) \log \left| \frac{3 - (u+v)^2}{3 - (u-v)^2} \right| \right)^2 \right. \\
 & \left. + \pi^2(u^2 + v^2 - 3)^2 \Theta(v + u - \sqrt{3}) \right)
 \end{aligned}$$

Analytically

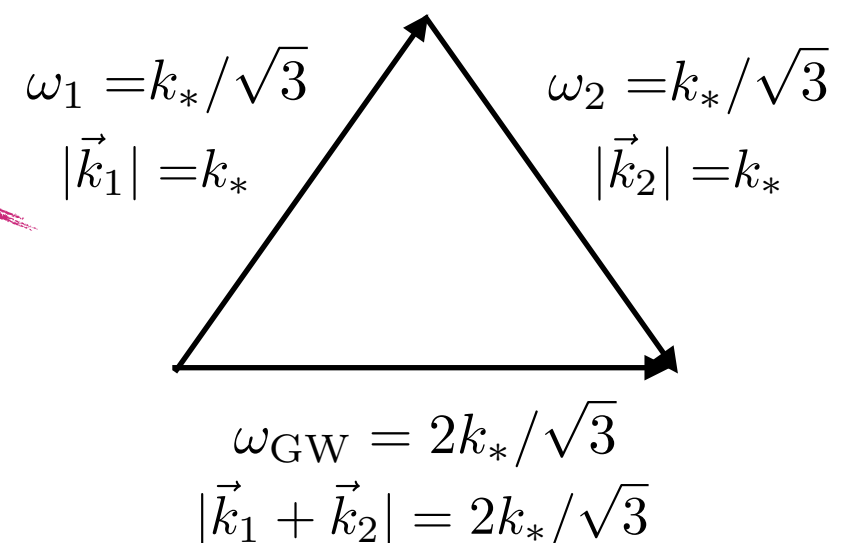
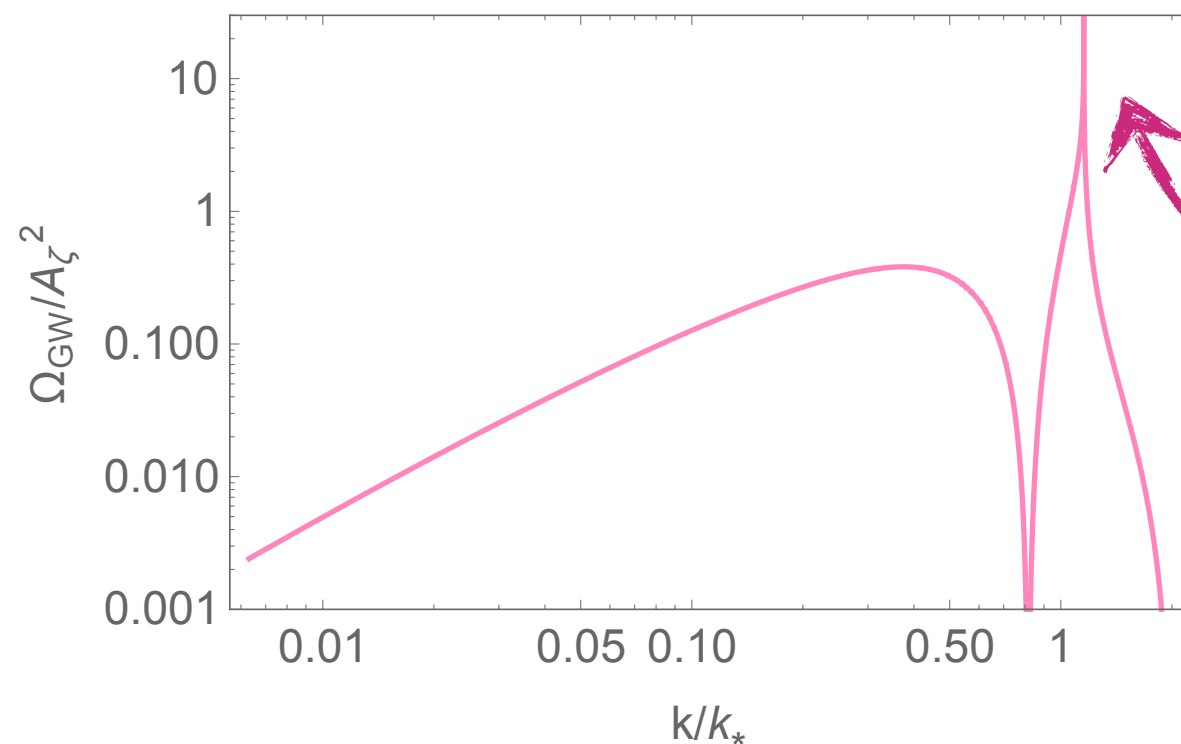
# Calculable GW spectrum

Example 1 in RD: Monochromatic case  $\mathcal{P}_\zeta(k) = A_\zeta \delta(\log k/k_*)$

$$\Omega_{\text{GW}}(\eta, k) = \frac{3A_\zeta^2}{64} \left( \frac{4 - \tilde{k}^2}{4} \right)^2 \tilde{k}^2 (3\tilde{k}^2 - 2)^2$$

$$\times \left( \pi^2 (3\tilde{k}^2 - 2)^2 \Theta(2\sqrt{3} - 3\tilde{k}) + \left( 4 + (3\tilde{k}^2 - 2) \log \left| 1 - \frac{4}{3\tilde{k}^2} \right| \right)^2 \right) \Theta(2 - \tilde{k})$$

$$\tilde{k} \equiv k/k_*$$



Analytically

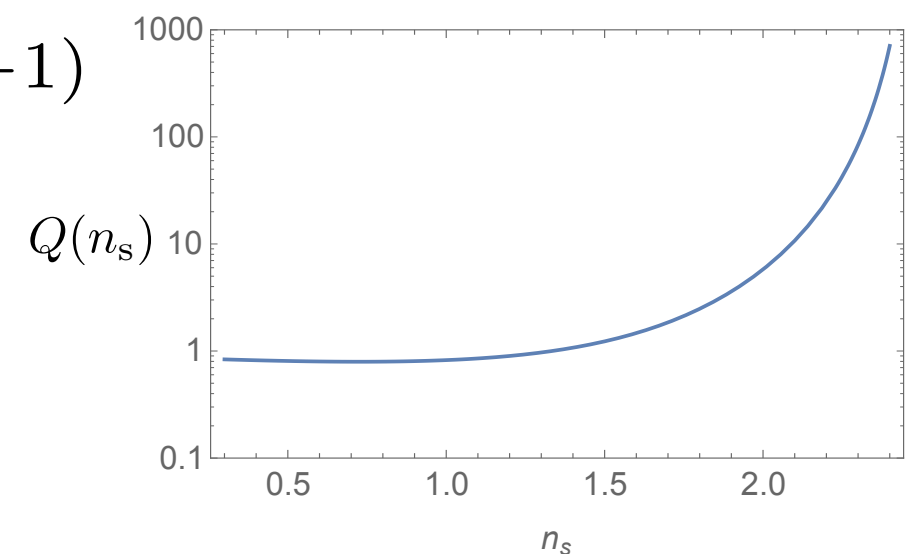
# Calculable GW spectrum

Example 2 in RD: Scale-invariant case  $\mathcal{P}_\zeta(k) = A_\zeta$

$$\Omega_{\text{GW}}(\eta, k) \simeq 0.8222 A_\zeta^2$$

Example 3 in RD: Power-law case  $\mathcal{P}_\zeta = A_\zeta \left( \frac{k}{k_*} \right)^{n_s - 1}$

$$\Omega_{\text{GW}}(\eta, k) = Q(n_s) A_\zeta^2 \left( \frac{k}{k_*} \right)^{2(n_s - 1)}$$



# Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider  $w = 0$  (MD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

# Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider  $w = 0$  (MD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}^2/x^2$$


# Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider  $w = 0$  (MD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$a(\bar{\eta})/a(\eta) = \bar{x}^2/x^2$

$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \frac{1}{x\bar{x}} ((1 + x\bar{x}) \sin(x - \bar{x}) - (x - \bar{x}) \cos(x - \bar{x}))$

# Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider  $w = 0$  (MD era).

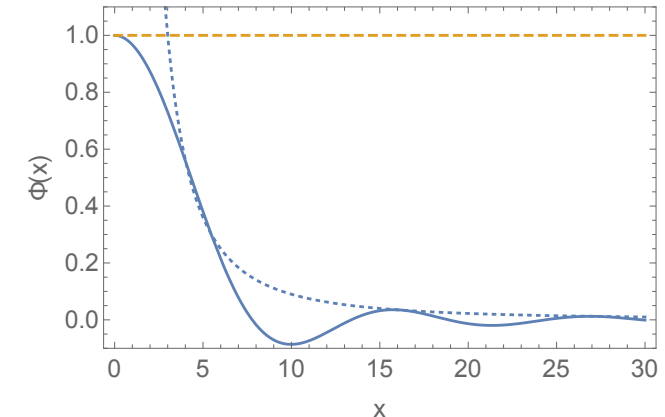
$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}^2/x^2$$

$$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \frac{1}{x\bar{x}} ((1 + x\bar{x}) \sin(x - \bar{x}) - (x - \bar{x}) \cos(x - \bar{x}))$$

$$\Phi(x) = 1$$

$$f_{\text{MD}}(v, u, x) = \frac{6}{5}$$



# Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider  $w = 0$  (MD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

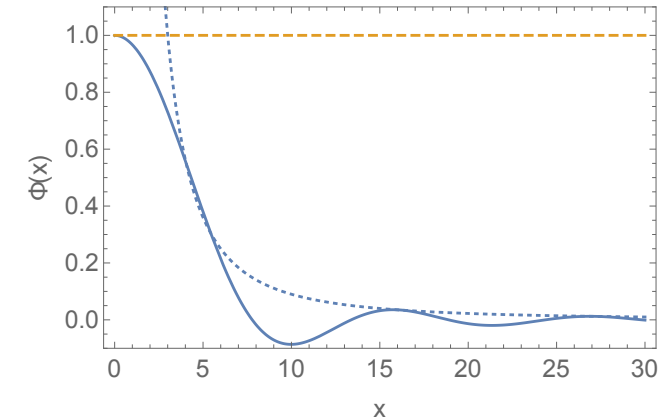
$$a(\bar{\eta})/a(\eta) = \bar{x}^2/x^2$$

$$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \frac{1}{x\bar{x}} ((1 + x\bar{x}) \sin(x - \bar{x}) - (x - \bar{x}) \cos(x - \bar{x}))$$

$$\Phi(x) = 1$$

$$f_{\text{MD}}(v, u, x) = \frac{6}{5}$$

**Non-decaying source!**



# Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider  $w = 0$  (MD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}^2/x^2$$

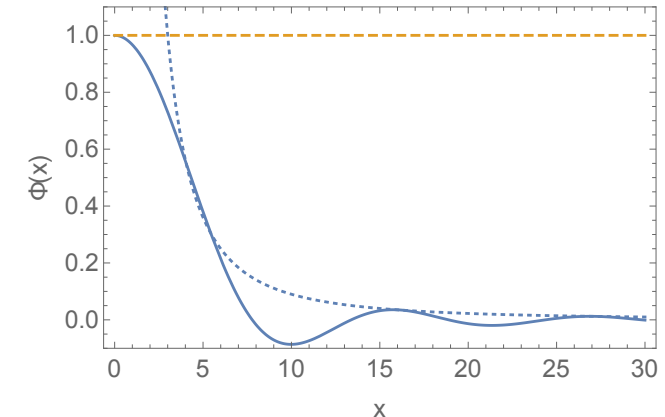
$$kG_{\mathbf{k}}(\eta, \bar{\eta}) = \frac{1}{x\bar{x}} ((1 + x\bar{x}) \sin(x - \bar{x}) - (x - \bar{x}) \cos(x - \bar{x}))$$

$$\Phi(x) = 1$$

$$f_{\text{MD}}(v, u, x) = \frac{6}{5}$$

**Non-decaying source!**

$$\overline{I_{\text{MD}}^2(v, u, x \rightarrow \infty)} = \frac{18}{25} \quad \leftarrow \text{Not diluted!}$$



# Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider  $w = 0$  (MD era).

$$I(v, u, x) = \int_0^x d\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$$

$$a(\bar{\eta})/a(\eta) = \bar{x}^2/x^2$$

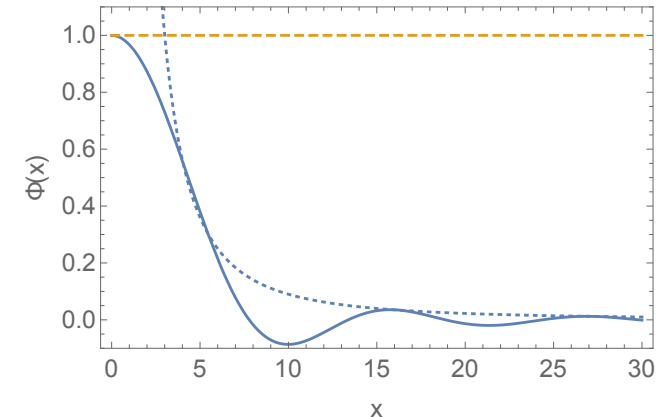
$$k G_{\mathbf{k}}(\eta, \bar{\eta}) = \frac{1}{x\bar{x}} ((1 + x\bar{x}) \sin(x - \bar{x}) - (x - \bar{x}) \cos(x - \bar{x}))$$

$$\Phi(x) = 1$$

$$f_{\text{MD}}(v, u, x) = \frac{6}{5}$$

**Non-decaying source!**

$$\overline{I_{\text{MD}}^2(v, u, x \rightarrow \infty)} = \frac{18}{25} \quad \leftarrow \text{Not diluted!}$$



**Constant metric distortion**

After MD

**Propagating GW**

# Calculable GW spectrum

Example 1 in MD: Monochromatic case     $\mathcal{P}_\zeta(k) = A_\zeta \delta(\log k/k_*)$

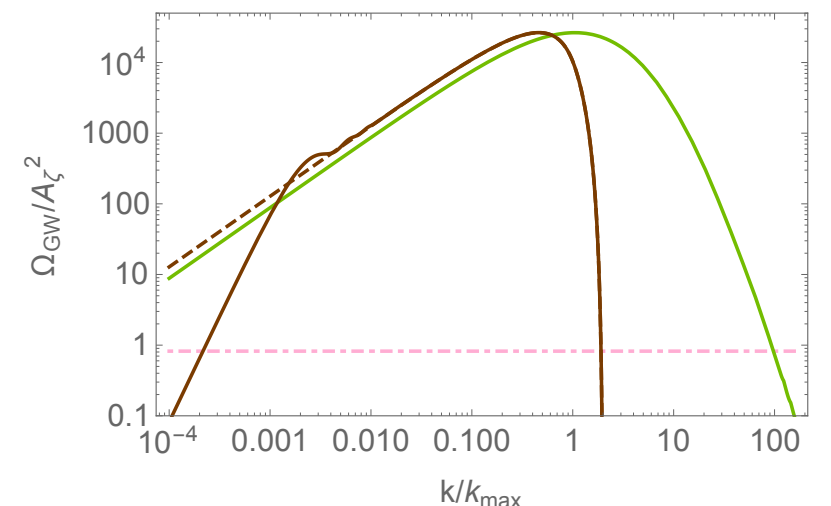
$$\Omega_{\text{GW}} = \frac{3}{25} \left( \frac{k_*}{aH} \right)^2 \left( 1 - \left( \frac{k}{2k_*} \right)^2 \right)^2 A_\zeta^2 \Theta(2k_* - k)$$

Example 2 in MD: Scale-invariant case with a cutoff

$$\mathcal{P}_\zeta(k) = A_\zeta \Theta(k_{\text{max}} - k)$$

$$\Omega_{\text{GW}} = \frac{A_\zeta^2}{14000} \left( \frac{k}{aH} \right)^2 \times \begin{cases} \left( 1792\tilde{k}^{-1} - 2520 + 768\tilde{k} + 105\tilde{k}^2 \right) & (0 < k \leq k_{\text{max}}) \\ \left( 1 - 2\tilde{k}^{-1} \right)^4 \left( 105\tilde{k}^2 + 72\tilde{k} + 16 - 32\tilde{k}^{-1} - 16\tilde{k}^{-2} \right) & (k_{\text{max}} < k \leq 2k_{\text{max}}) \end{cases}$$

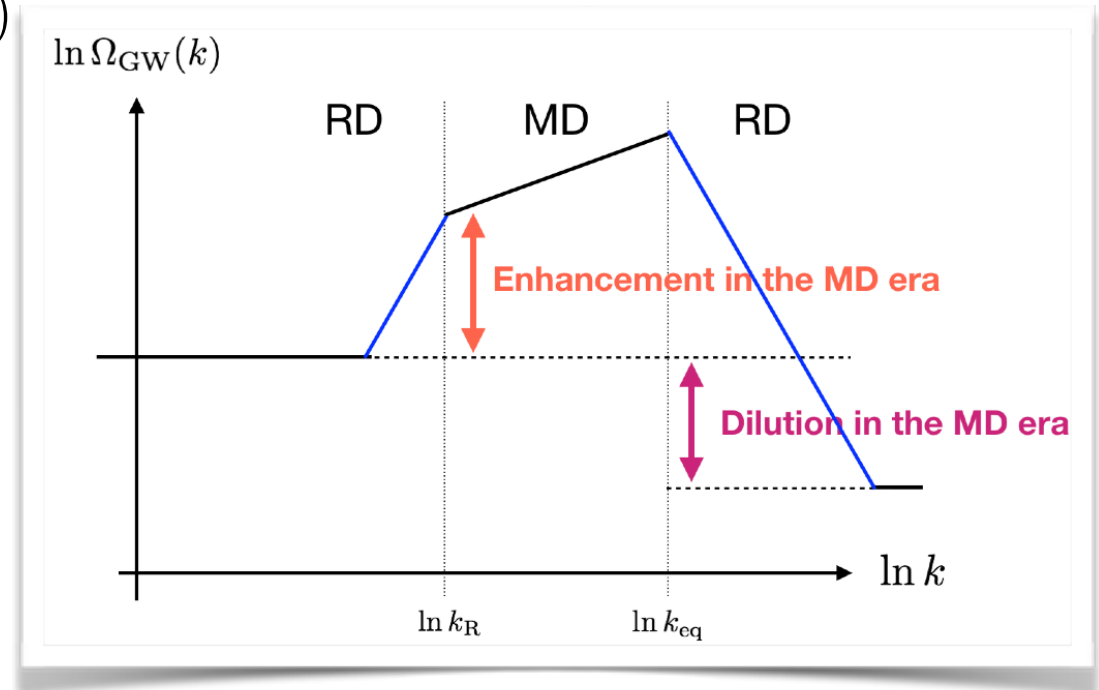
$$\tilde{k} \equiv k/k_*$$



# General cases

## The MD $\rightarrow$ RD transition

$$I(v, u, x) = \int_0^{x_R} d\bar{x} \left( \frac{x_R}{x} \right) \left( \frac{\bar{x}}{x_R} \right)^2 k G_k^{\text{MD} \rightarrow \text{RD}}(\eta, \bar{\eta}) f_{\text{MD}}(v, u, \bar{x}) \\ + \int_{x_R}^x d\bar{x} \left( \frac{\bar{x}}{x} \right) k G_k^{\text{RD}}(\eta, \bar{\eta}) f_{\text{MD} \rightarrow \text{RD}}(v, u, \bar{x})$$



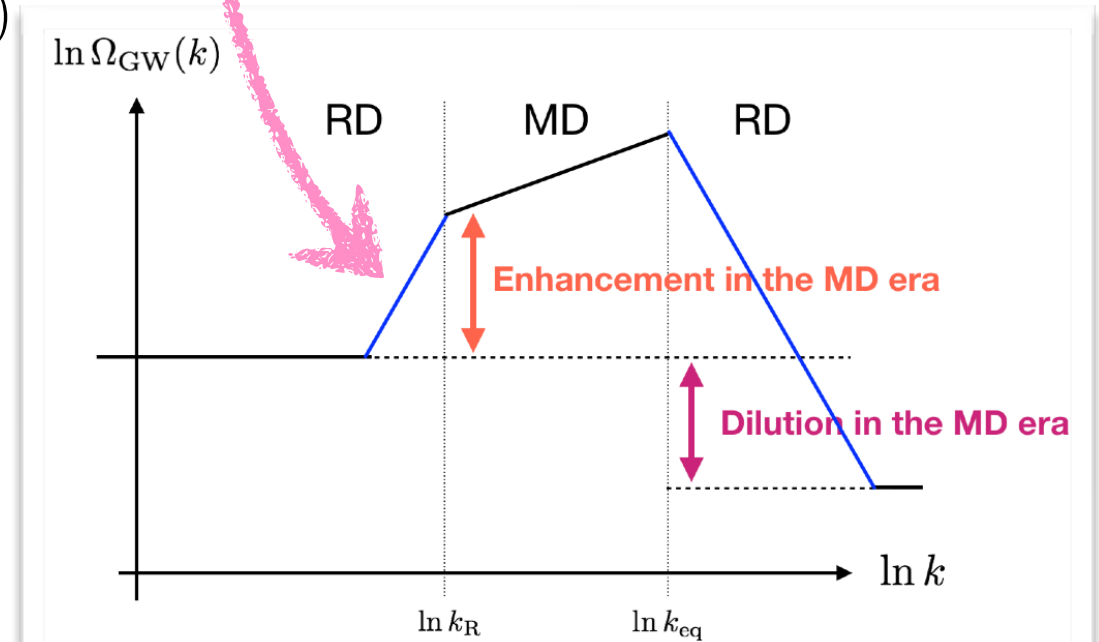
## The RD $\rightarrow$ MD transition

$$I(v, u, x) = \int_0^{x_{\text{eq}}} d\bar{x} \left( \frac{x_{\text{eq}}}{x} \right)^2 \left( \frac{\bar{x}}{x_{\text{eq}}} \right) k G_k^{\text{RD} \rightarrow \text{MD}}(\eta, \bar{\eta}) f_{\text{RD}}(v, u, \bar{x}) \\ + \int_{x_{\text{eq}}}^x d\bar{x} \left( \frac{\bar{x}}{x} \right)^2 k G_k^{\text{MD}}(\eta, \bar{\eta}) f_{\text{RD} \rightarrow \text{MD}}(v, u, \bar{x})$$

# General cases

## The MD $\rightarrow$ RD transition

$$I(v, u, x) = \int_0^{x_R} d\bar{x} \left( \frac{x_R}{x} \right) \left( \frac{\bar{x}}{x_R} \right)^2 k G_k^{\text{MD} \rightarrow \text{RD}}(\eta, \bar{\eta}) f_{\text{MD}}(v, u, \bar{x}) \\ + \int_{x_R}^x d\bar{x} \left( \frac{\bar{x}}{x} \right) k G_k^{\text{RD}}(\eta, \bar{\eta}) f_{\text{MD} \rightarrow \text{RD}}(v, u, \bar{x})$$



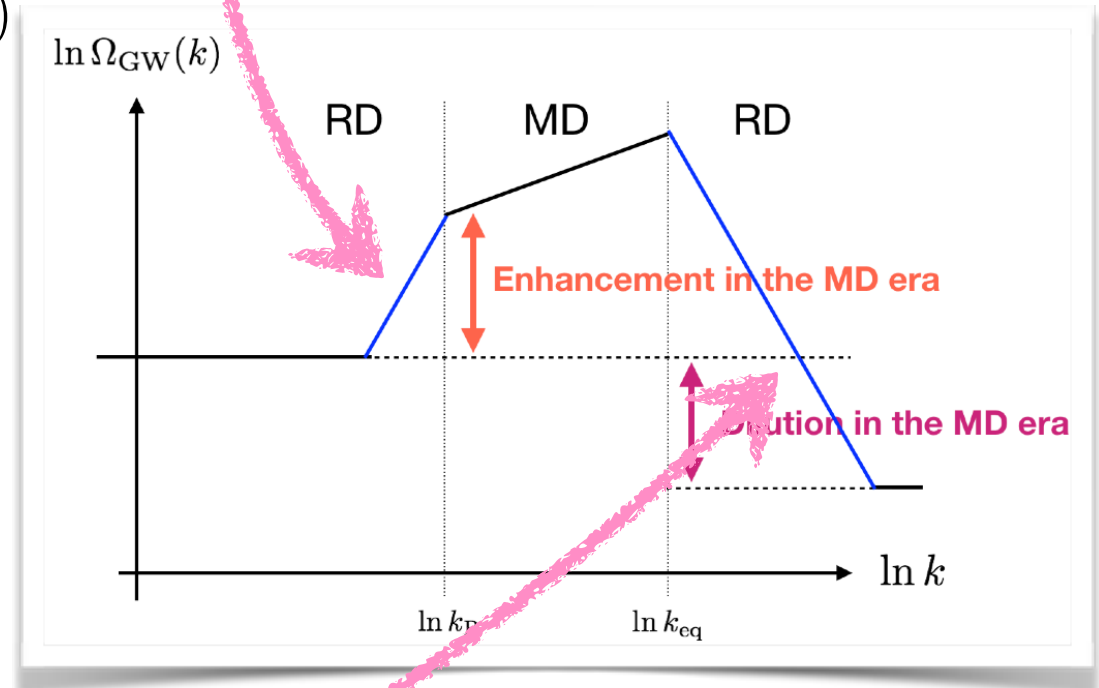
## The RD $\rightarrow$ MD transition

$$I(v, u, x) = \int_0^{x_{\text{eq}}} d\bar{x} \left( \frac{x_{\text{eq}}}{x} \right)^2 \left( \frac{\bar{x}}{x_{\text{eq}}} \right) k G_k^{\text{RD} \rightarrow \text{MD}}(\eta, \bar{\eta}) f_{\text{RD}}(v, u, \bar{x}) \\ + \int_{x_{\text{eq}}}^x d\bar{x} \left( \frac{\bar{x}}{x} \right)^2 k G_k^{\text{MD}}(\eta, \bar{\eta}) f_{\text{RD} \rightarrow \text{MD}}(v, u, \bar{x})$$

# General cases

## The MD → RD transition

$$I(v, u, x) = \int_0^{x_R} d\bar{x} \left( \frac{x_R}{x} \right) \left( \frac{\bar{x}}{x_R} \right)^2 k G_k^{\text{MD} \rightarrow \text{RD}}(\eta, \bar{\eta}) f_{\text{MD}}(v, u, \bar{x}) \\ + \int_{x_R}^x d\bar{x} \left( \frac{\bar{x}}{x} \right) k G_k^{\text{RD}}(\eta, \bar{\eta}) f_{\text{MD} \rightarrow \text{RD}}(v, u, \bar{x})$$



## The RD → MD transition

$$I(v, u, x) = \int_0^{x_{\text{eq}}} d\bar{x} \left( \frac{x_{\text{eq}}}{x} \right)^2 \left( \frac{\bar{x}}{x_{\text{eq}}} \right) k G_k^{\text{RD} \rightarrow \text{MD}}(\eta, \bar{\eta}) f_{\text{RD}}(v, u, \bar{x}) \\ + \int_{x_{\text{eq}}}^x d\bar{x} \left( \frac{\bar{x}}{x} \right)^2 k G_k^{\text{MD}}(\eta, \bar{\eta}) f_{\text{RD} \rightarrow \text{MD}}(v, u, \bar{x})$$

# General cases

## The MD → RD transition

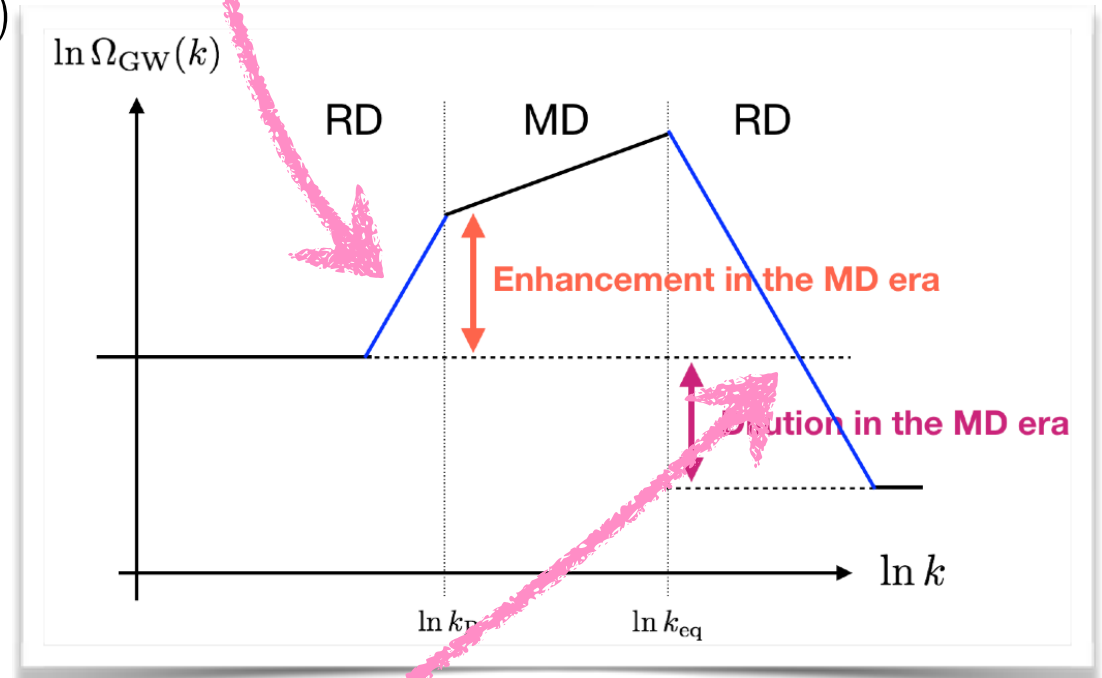
$$I(v, u, x) = \int_0^{x_R} d\bar{x} \left( \frac{x_R}{x} \right)^2 \left( \frac{\bar{x}}{x_R} \right) k G_k^{\text{RD}}(\eta, \bar{\eta}) f_{\text{MD} \rightarrow \text{RD}}(v, u, \bar{x})$$

$$\frac{3}{5x x_R^3} (3(2x_R^2 - 1) \cos x - 6x_R \sin x + 2x_R^4 \cos(x - x_R) + 4x_R^3 \sin(x - x_R) + 3 \cos(x - 2x_R))$$

## The RD → MD transition

$$I(v, u, x) = \int_0^{x_{\text{eq}}} d\bar{x} \left( \frac{x_{\text{eq}}}{x} \right)^2 \left( \frac{\bar{x}}{x_{\text{eq}}} \right) k G_k^{\text{RD} \rightarrow \text{MD}}(\eta, \bar{\eta}) f_{\text{RD}}(v, u, \bar{x})$$

$$+ \int_{x_{\text{eq}}}^x d\bar{x} \left( \frac{\bar{x}}{x} \right)^2 k G_k^{\text{MD}}(\eta, \bar{\eta}) f_{\text{RD} \rightarrow \text{MD}}(v, u, \bar{x})$$

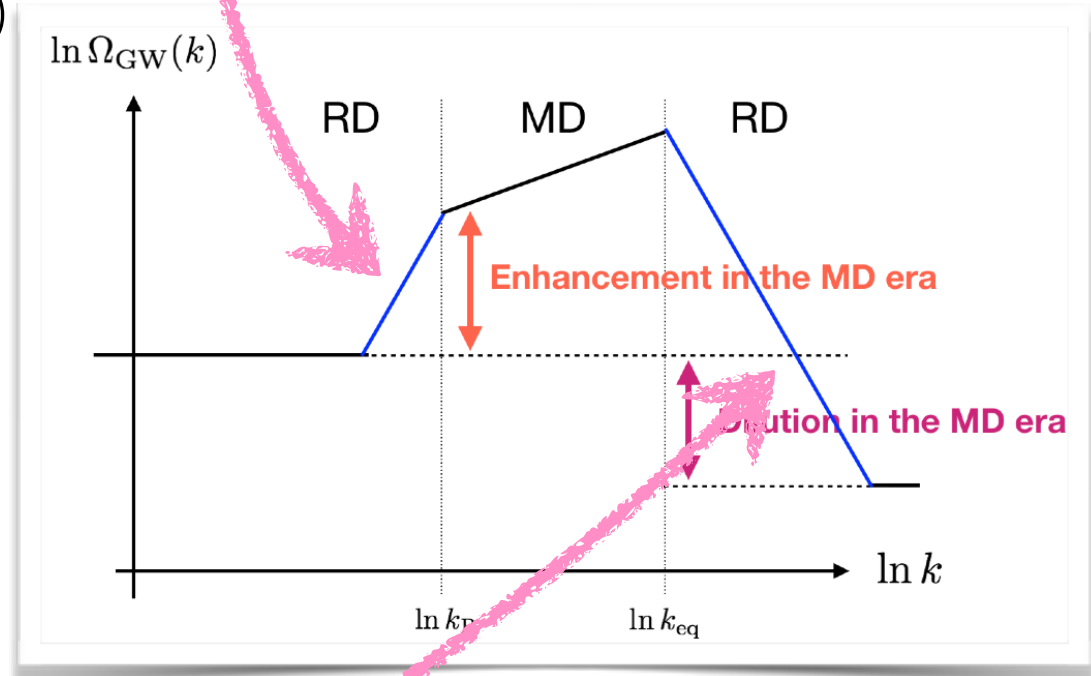


# General cases

## The MD $\rightarrow$ RD transition

$$I(v, u, x) = \int_{x_R}^{x_R} d\bar{x} \left( \frac{x_R}{x} \right) \kappa_{\text{GW}}^{\text{MD}}(\eta, \bar{\eta}) f_{\text{MD} \rightarrow \text{RD}}(v, u, \bar{x})$$

$$\frac{3}{5x x_R^3} (3(2x_R^2 - 1) \cos x - 6x_R \sin x + 2x_R^4 \cos(x - x_R) + 4x_R^3 \sin(x - x_R) + 3 \cos(x - 2x_R))$$



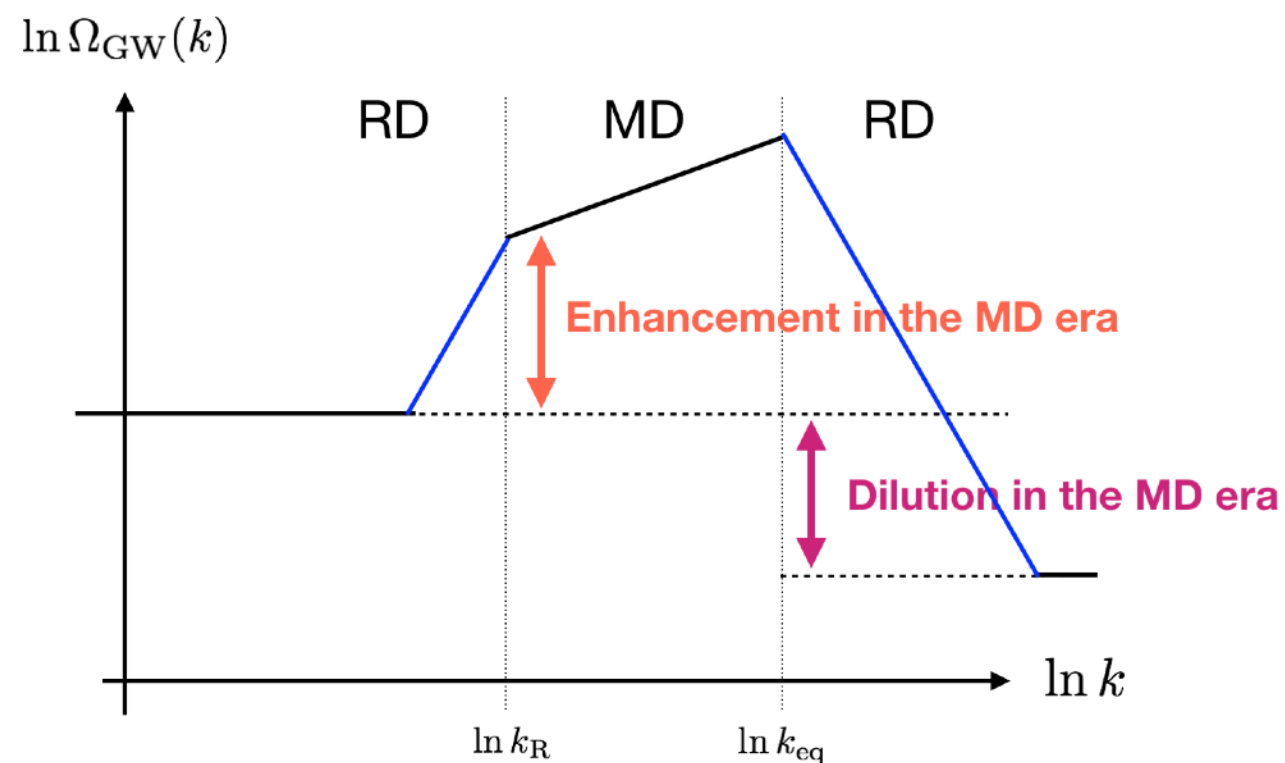
## The RD $\rightarrow$ MD transition

$$\frac{(x^3 - 3(x - x_{\text{eq}}) - x x_{\text{eq}}^2) \cos(x - x_{\text{eq}}) - (3 + 3x x_{\text{eq}} - x_{\text{eq}}^2) \sin(x - x_{\text{eq}})}{x^3} \times \frac{6 \ln(c_1 u x_{\text{eq}})}{5 (c_2 u x_{\text{eq}})^2} \frac{\ln(c_1 v x_{\text{eq}})}{(c_2 v x_{\text{eq}})^2}$$

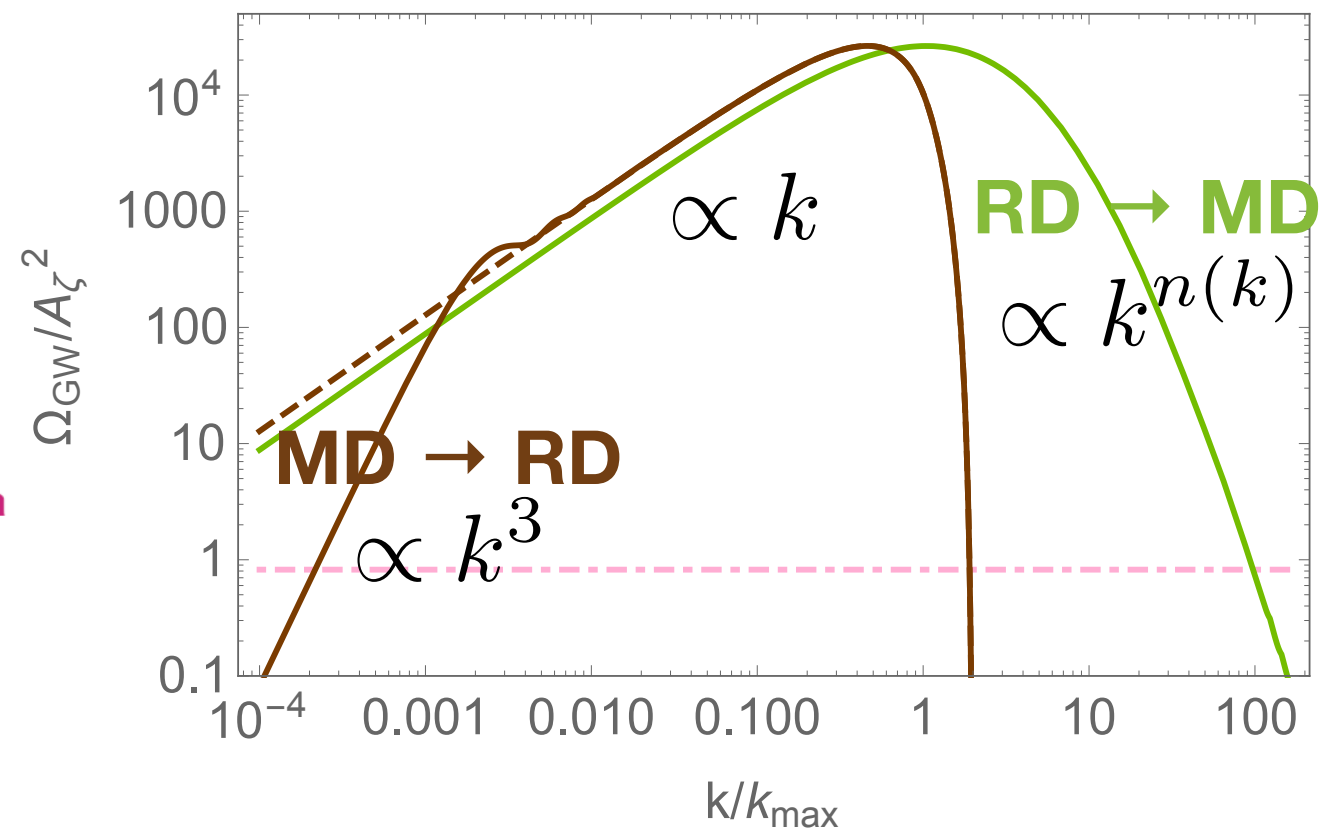
$$+ \int_{x_{\text{eq}}}^{x_{\text{eq}}} d\bar{x} \left( \frac{x_{\text{eq}}}{x} \right) \kappa_{\text{GW}}^{\text{RD}}(\eta, \bar{\eta}) f_{\text{RD} \rightarrow \text{MD}}(v, u, \bar{x})$$

# General cases

**Schematic**



**Precise**

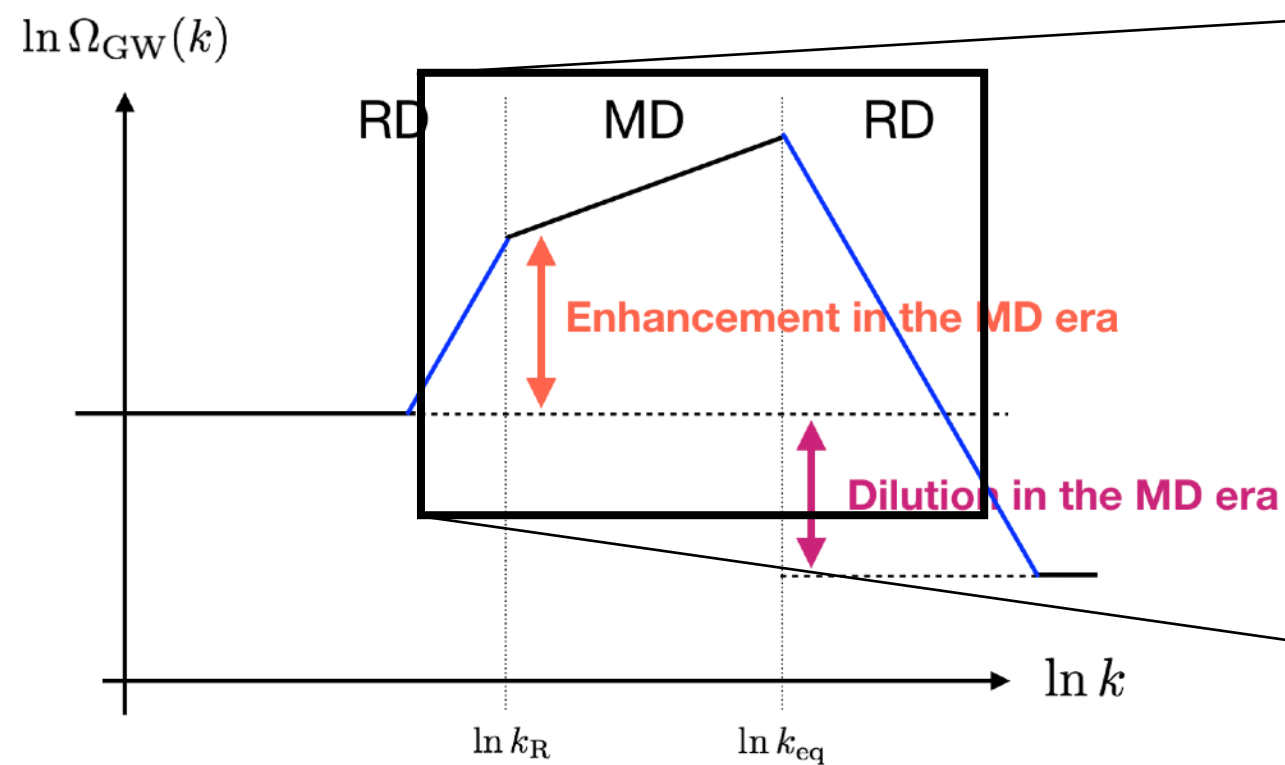


$$-6 \lesssim n(k) \lesssim -4$$

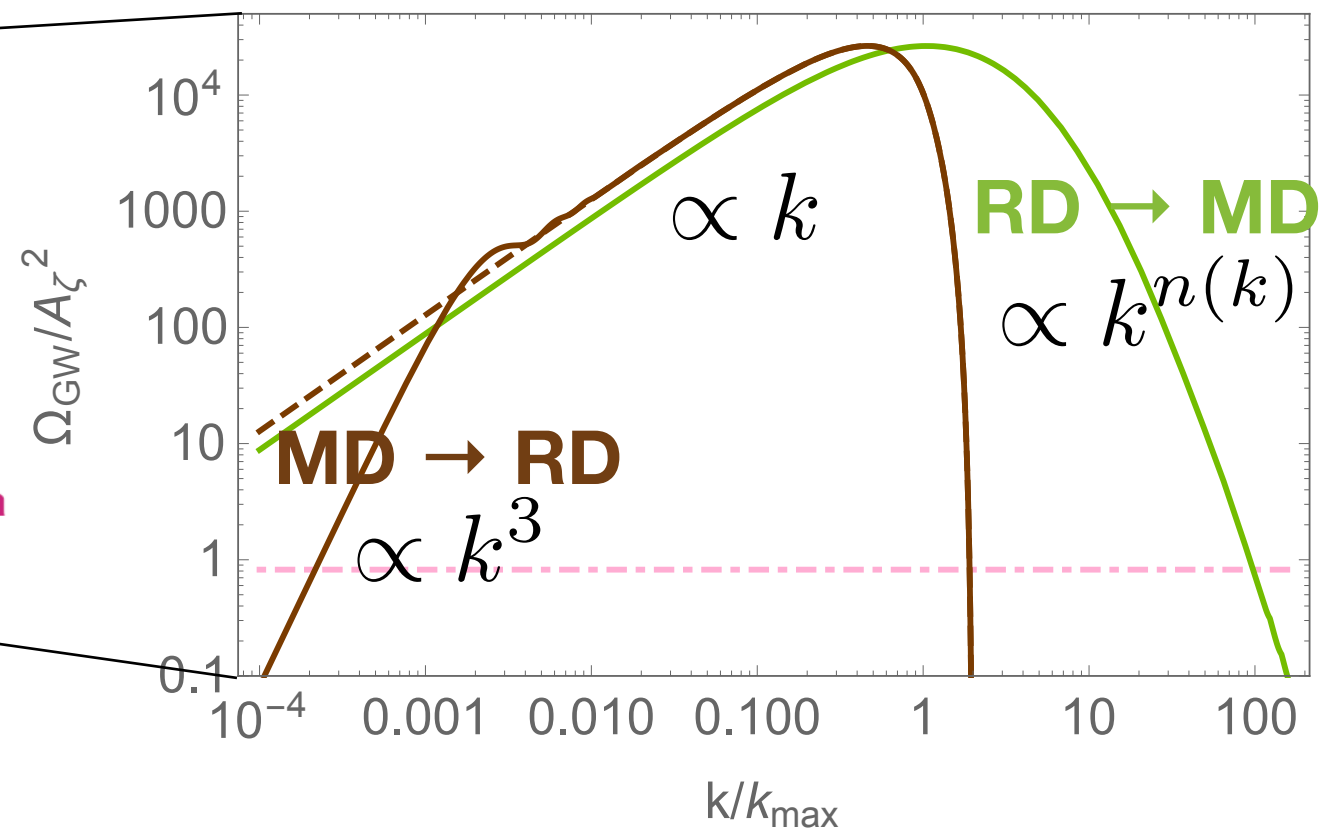
Scale-invariant primordial spectrum is assumed as a simple example.

# General cases

**Schematic**



**Precise**



$$-6 \lesssim n(k) \lesssim -4$$

Scale-invariant primordial spectrum is assumed as a simple example.

# APPLICATION: PRIMORDIAL BLACK HOLE SCENARIOS

[Kohri, Terada, 1802.06785]

# Current PBH abundance

## PBH fraction in CDM

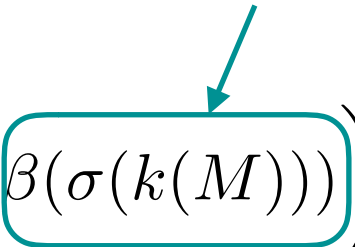
$$\begin{aligned} f_{\text{PBH}}(M) &= \frac{1}{\rho_{\text{CDM}}} \frac{d\rho_{\text{PBH}}}{d \ln M} \\ &= \left( \frac{g_*(T)}{g_*(T_{\text{eq}})} \frac{g_{*,s}(T_{\text{eq}})}{g_{*,s}(T)} \frac{T}{T_{\text{eq}}} \gamma \beta(\sigma(k(M))) \right) \bigg|_{T=\text{Min}[T_M, T_R]} \frac{\Omega_{\text{m}}}{\Omega_{\text{CDM}}} \end{aligned}$$

# Current PBH abundance

## PBH fraction in CDM

$$f_{\text{PBH}}(M) = \frac{1}{\rho_{\text{CDM}}} \frac{d\rho_{\text{PBH}}}{d \ln M}$$
$$= \left( \frac{g_*(T)}{g_*(T_{\text{eq}})} \frac{g_{*,s}(T_{\text{eq}})}{g_{*,s}(T)} \frac{T}{T_{\text{eq}}} \gamma \beta(\sigma(k(M))) \right) \bigg|_{T=\text{Min}[T_M, T_R]} \frac{\Omega_m}{\Omega_{\text{CDM}}}$$

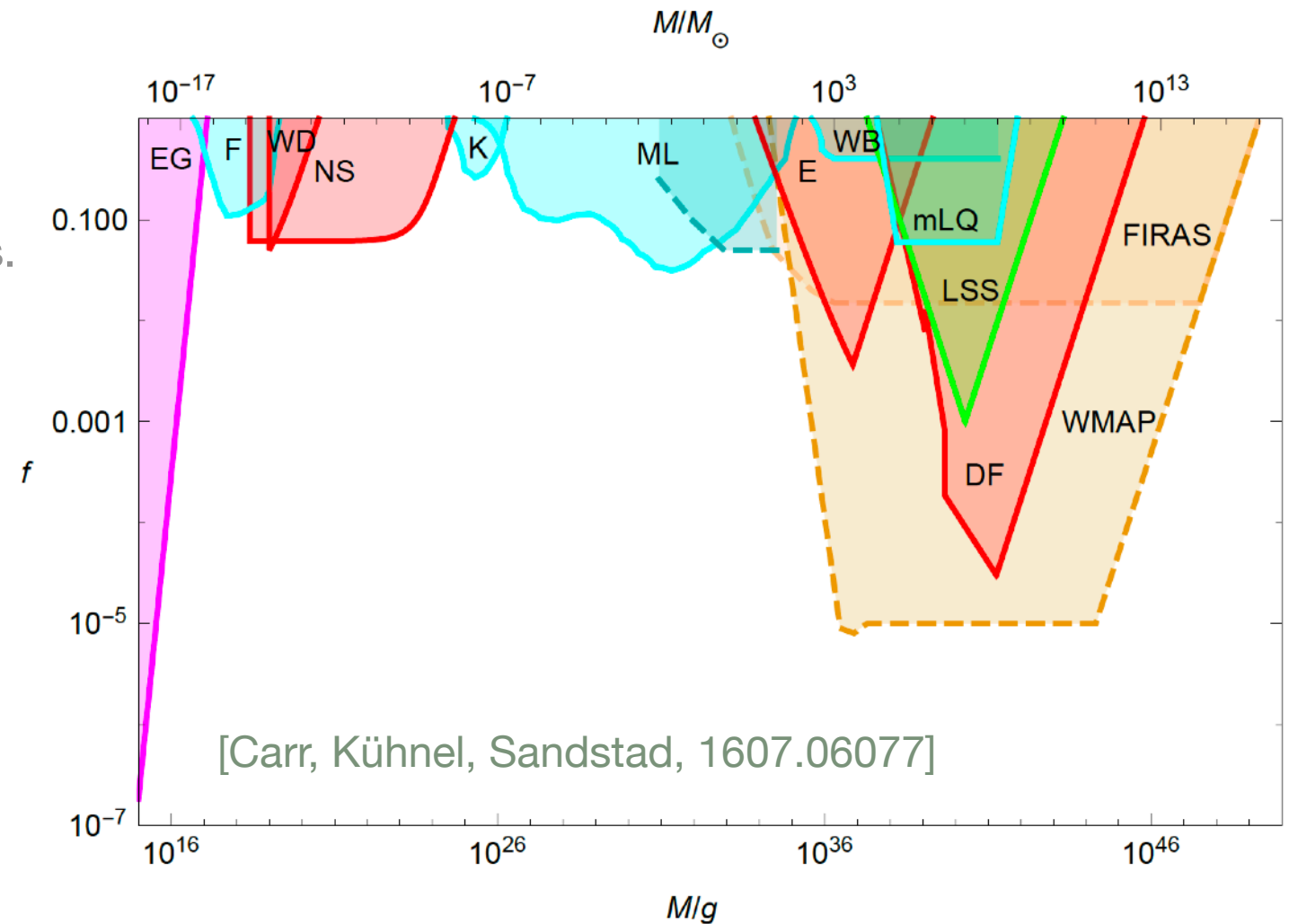
PBH formation probability



# Current PBH abundance

## Observational constraints

Note: These are not the latest constraints.



## PBH fraction in CDM

$$f_{\text{PBH}}(M) = \frac{1}{\rho_{\text{CDM}}} \frac{d\rho_{\text{PBH}}}{d \ln M}$$

$$= \left( \frac{g_*(T)}{g_*(T_{\text{eq}})} \frac{g_{*,s}(T_{\text{eq}})}{g_{*,s}(T)} \frac{T}{T_{\text{eq}}} \gamma \beta(\sigma(k(M))) \right) \bigg|_{T=\text{Min}[T_M, T_R]} \frac{\Omega_m}{\Omega_{\text{CDM}}}$$

PBH formation probability

# PBH formation probability

## Coarse-grained perturbations

[Young, Byrnes, Sasaki, 1405.7023]  
see also [Ando, Inomata, Kawasaki, 1802.06393]

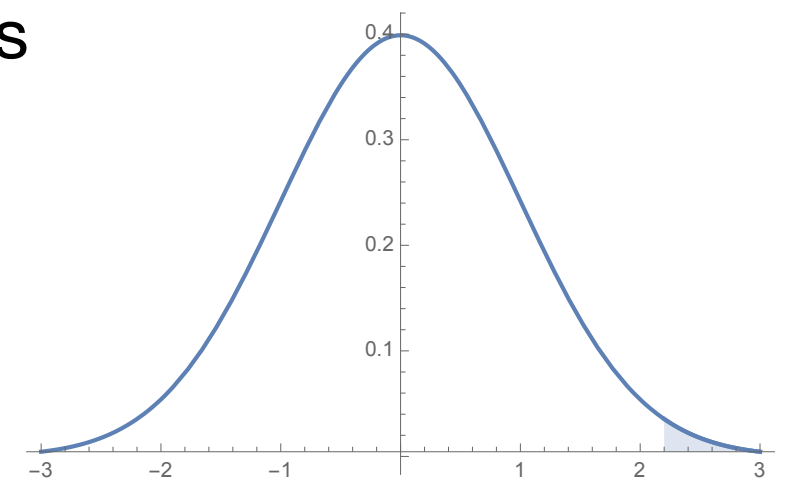
$$\sigma^2(k) = \int_{-\infty}^{\infty} d \ln q \, w^2 \left( \frac{q}{k} \right) \frac{4(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} \left( \frac{q}{k} \right)^4 T^2(q/k) P_{\zeta}(q)$$
$$\sim \frac{2(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} P_{\zeta}(k)$$

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

# PBH formation probability

## Coarse-grained perturbations

[Young, Byrnes, Sasaki, 1405.7023]  
see also [Ando, Inomata, Kawasaki, 1802.06393]

### Window function

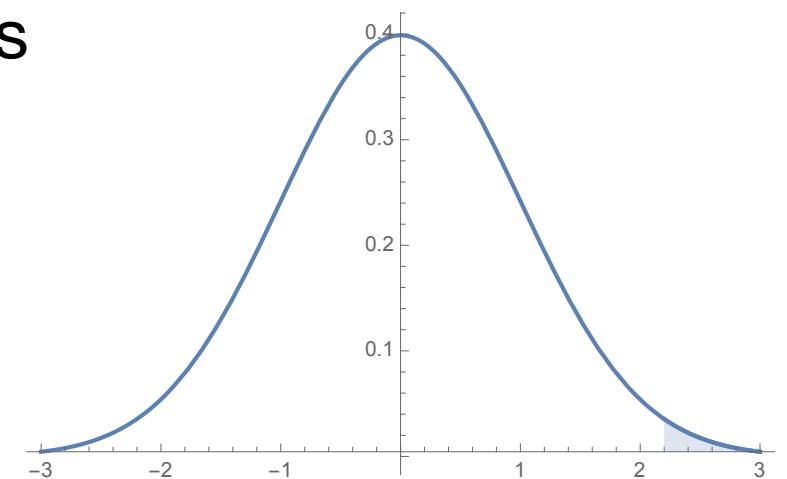
$$\sigma^2(k) = \int_{-\infty}^{\infty} d \ln q \, w^2 \left( \frac{q}{k} \right) \frac{4(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} \left( \frac{q}{k} \right)^4 T^2(q/k) P_{\zeta}(q)$$
$$\sim \frac{2(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} P_{\zeta}(k)$$

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{\delta^2}{2\sigma^2} \right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

# PBH formation probability

## Coarse-grained perturbations

[Young, Byrnes, Sasaki, 1405.7023]  
see also [Ando, Inomata, Kawasaki, 1802.06393]

Window function

$$\sigma^2(k) = \int_{-\infty}^{\infty} d \ln q \, w^2 \left( \frac{q}{k} \right) \frac{4(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} \left( \frac{q}{k} \right)^4 T^2(q/k) P_{\zeta}(q)$$

Transfer function

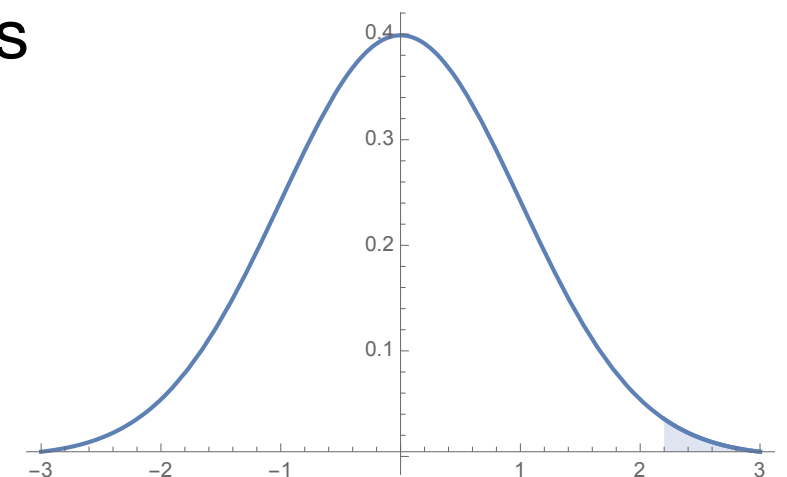
$$\sim \frac{2(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} P_{\zeta}(k)$$

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{\delta^2}{2\sigma^2} \right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

# PBH formation probability

## Coarse-grained perturbations

[Young, Byrnes, Sasaki, 1405.7023]  
see also [Ando, Inomata, Kawasaki, 1802.06393]

Window function

Primordial curvature perturbations

$$\sigma^2(k) = \int_{-\infty}^{\infty} d \ln q \, w^2 \left( \frac{q}{k} \right) \frac{4(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} \left( \frac{q}{k} \right)^4 T^2(q/k) P_{\zeta}(q)$$

Transfer function

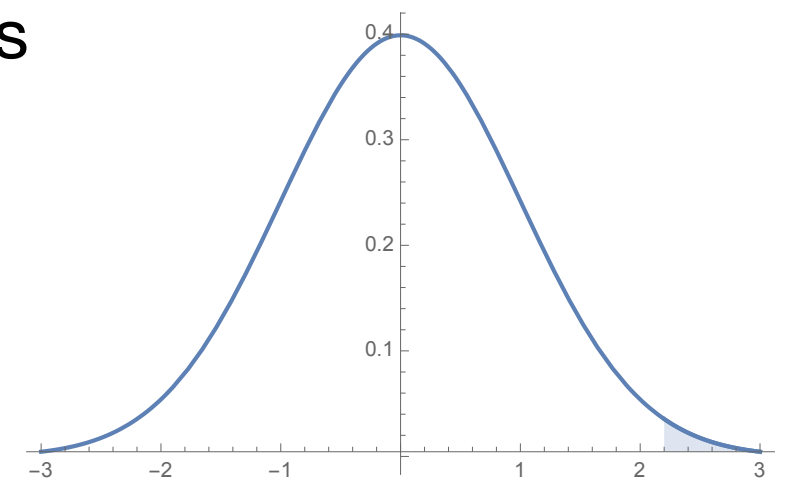
$$\sim \frac{2(1 + w_{\text{eos}})^2}{(5 + 3w_{\text{eos}})^2} P_{\zeta}(k)$$

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{\delta^2}{2\sigma^2} \right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

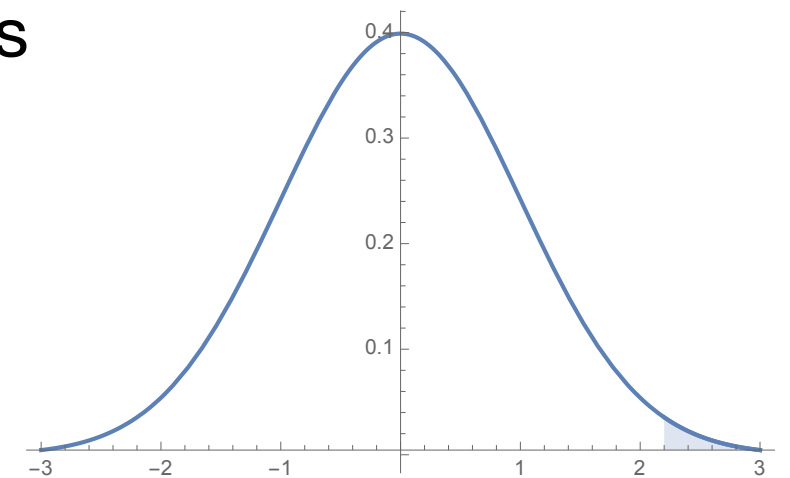
# PBH formation probability

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

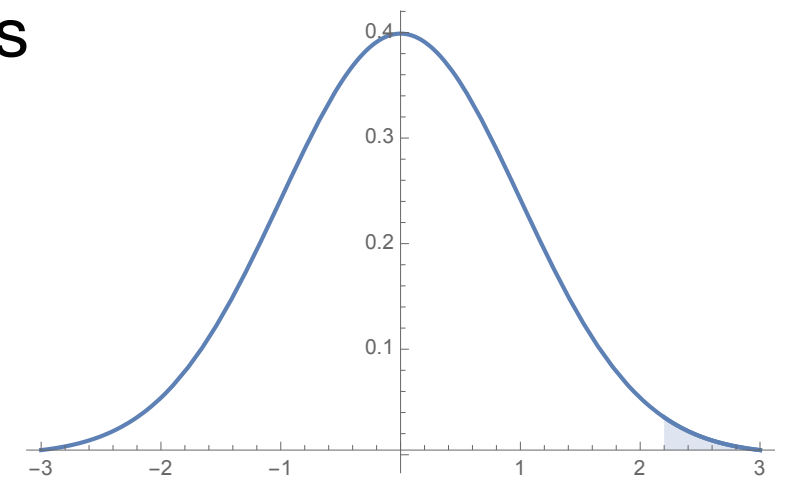
# PBH formation probability

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

## Formation probability in MD

[Khlopov, Polnarev, 1980, 1985]

$$\frac{\delta\rho}{\rho} \propto a(t)$$

Perturbation grows in the MD era → low threshold

Anisotropy & angular momentum affects the formation rate.

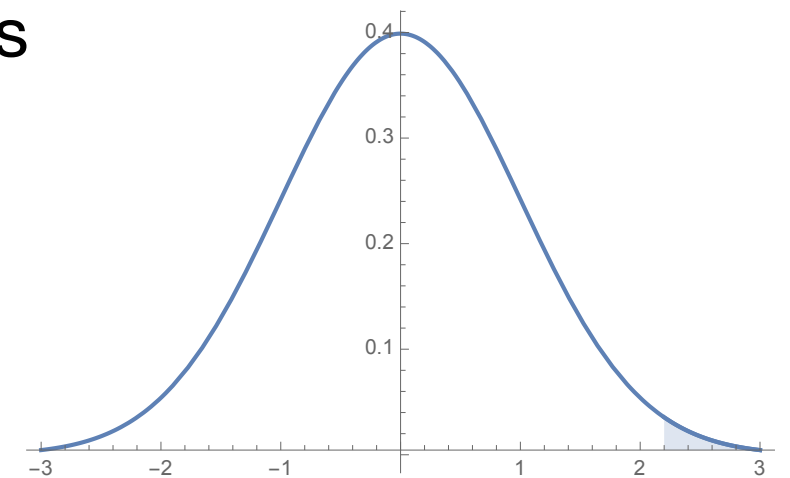
# PBH formation probability

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

## Formation probability in MD

[Khlopov, Polnarev, 1980, 1985]

$$\frac{\delta\rho}{\rho} \propto a(t)$$

Perturbation grows in the MD era → low threshold

Anisotropy & angular momentum affects the formation rate.

$$\beta(\sigma) = \begin{cases} 1.894 \times 10^{-6} \times f_q \mathcal{I}^6 \sigma^2 \exp\left(-0.1474 \frac{\mathcal{I}^{4/3}}{\sigma^{2/3}}\right) & (\sigma < 0.005) \\ 0.05556 \sigma^5 & (\sigma \geq 0.005) \end{cases}$$

[Harada, Yoo, Kohri, Nakao, Jhingan, 1609.01588]

[Harada, Yoo, Kohri, Nakao, 1707.03595]

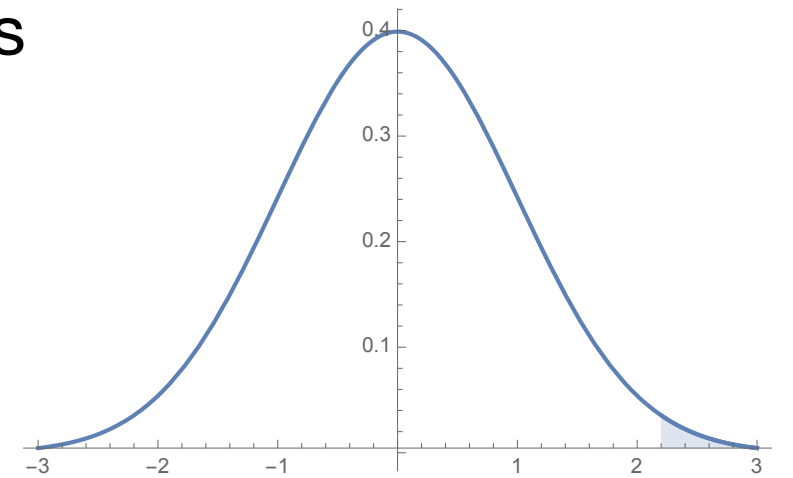
# PBH formation probability

## Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

## Formation probability in MD

[Khlopov, Polnarev, 1980, 1985]

$$\frac{\delta\rho}{\rho} \propto a(t)$$

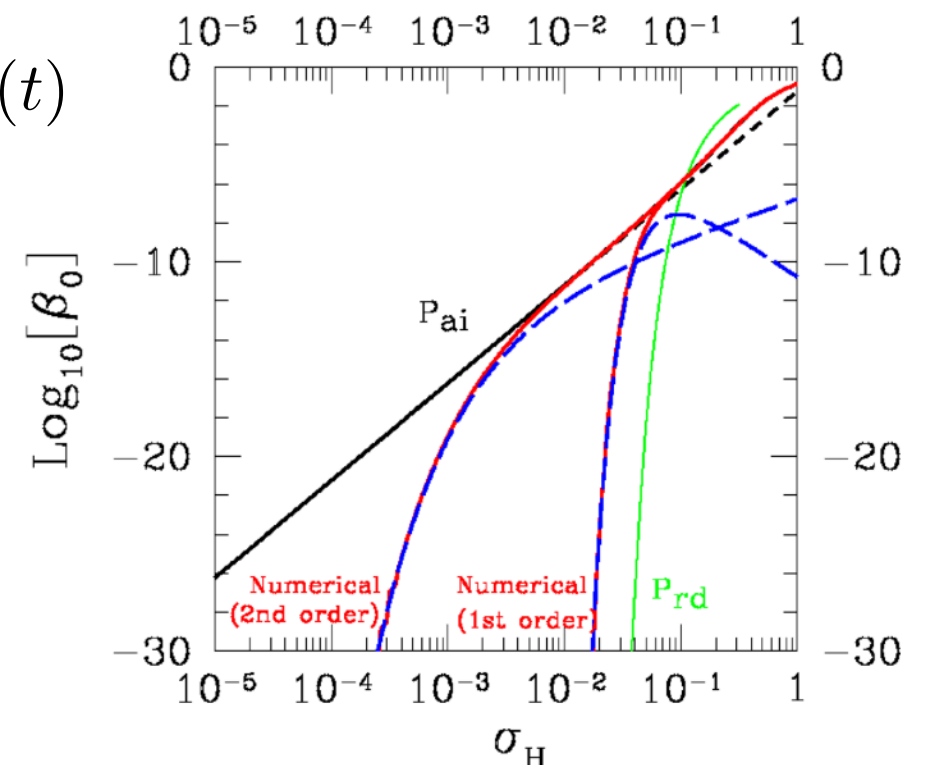
Perturbation grows in the MD era → low threshold

Anisotropy & angular momentum affects the formation rate.

$$\beta(\sigma) = \begin{cases} 1.894 \times 10^{-6} \times f_q \mathcal{I}^6 \sigma^2 \exp\left(-0.1474 \frac{\mathcal{I}^{4/3}}{\sigma^{2/3}}\right) & (\sigma < 0.005) \\ 0.05556 \sigma^5 & (\sigma \geq 0.005) \end{cases}$$

[Harada, Yoo, Kohri, Nakao, Jhingan, 1609.01588]

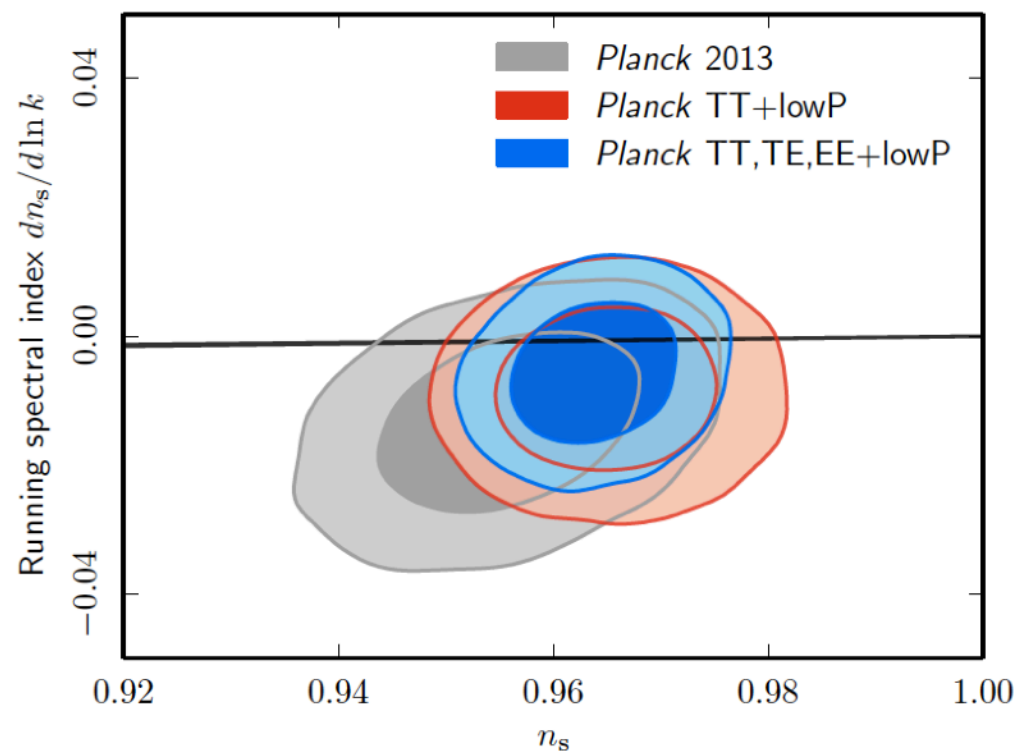
[Harada, Yoo, Kohri, Nakao, 1707.03595]



[Harada, Yoo, Kohri, Nakao, 1707.03595]

# Inflation with running spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left( \frac{k}{k_*} \right)^{n_s - 1 + \frac{\alpha_s}{2} \ln \frac{k}{k_*} + \frac{\beta_s}{6} \left( \ln \frac{k}{k_*} \right)^2 + \dots}$$



**Fig. 4.** Marginalized joint 68 % and 95 % CL for  $(n_s, dn_s/d \ln k)$  using *Planck* TT+lowP and *Planck* TT,TE,EE+lowP. For comparison, the thin black stripe shows the prediction for single-field monomial chaotic inflationary models with  $50 < N_* < 60$ .

Planck TT+lowP (TT,TE,EE+lowP)

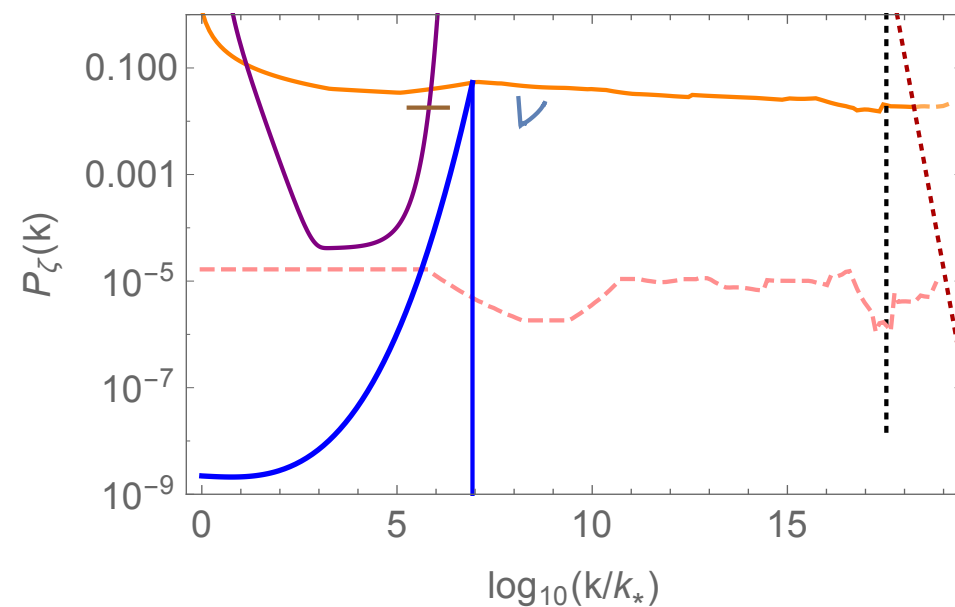
$$\begin{aligned} n_s &= 0.9569 \pm 0.0077 \quad (0.9586 \pm 0.0056), \\ dn_s/d \ln k &= 0.011^{+0.014}_{-0.013} \quad (0.009 \pm 0.010), \\ d^2 n_s/d \ln k^2 &= 0.029^{+0.015}_{-0.016} \quad (0.025 \pm 0.013), \end{aligned}$$

at 68% CL at the pivot scale  $k^*=0.05/\text{Mpc}$ .

# Interplay between PBH & GW

## PBH-for-LIGO scenario

### Curvature perturbations



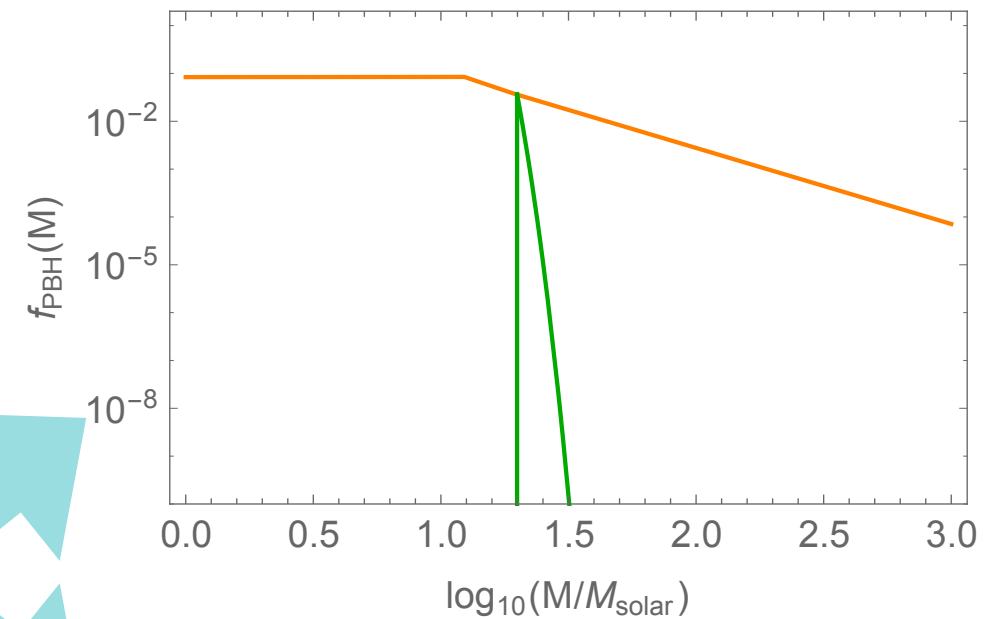
$$n_s = 0.96$$

$$\alpha_s = 0$$

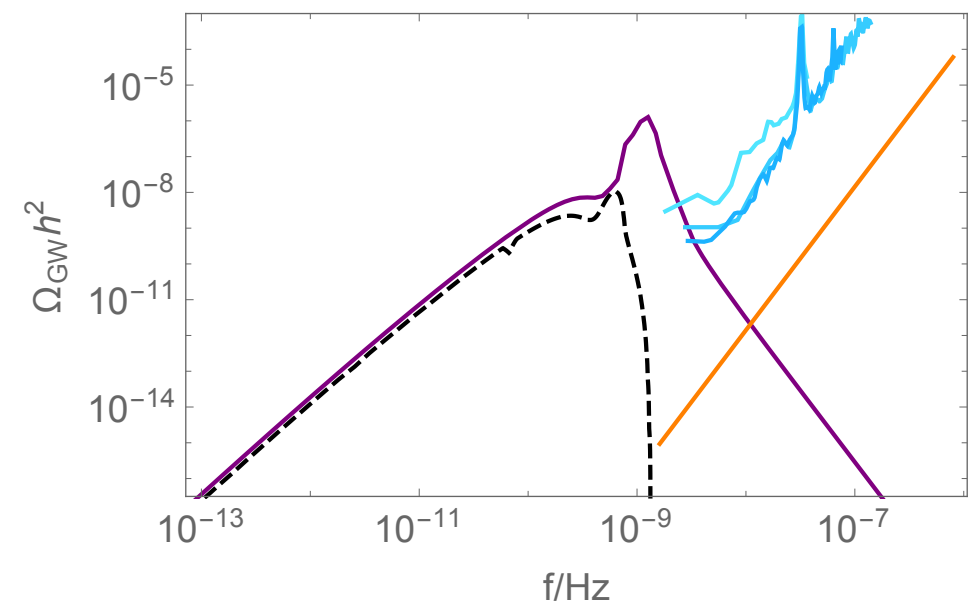
$$\beta_s = 0.026$$

$$T_R = 10^9 \text{ GeV}$$

### PBH mass spectrum



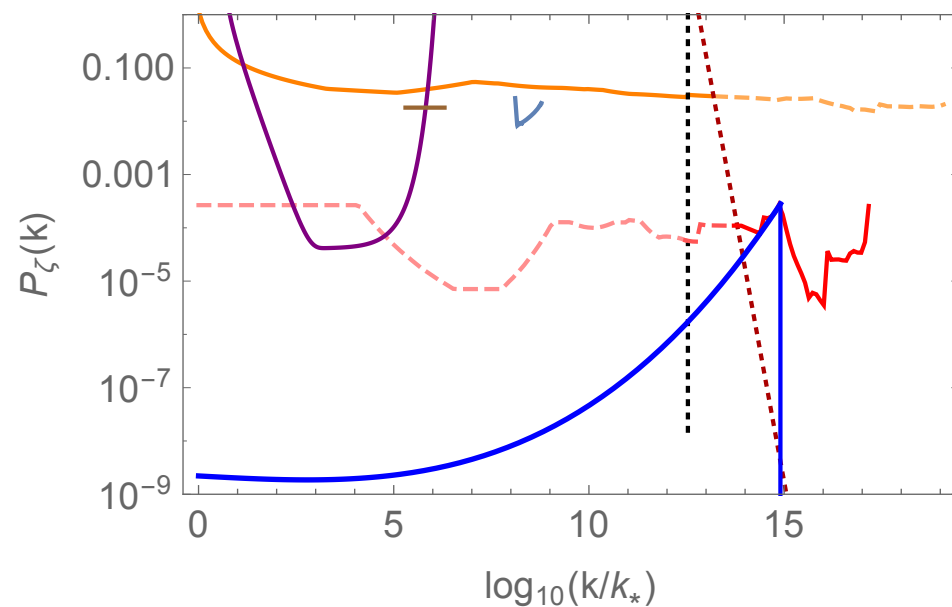
### Secondary GW spectrum



# Interplay between PBH & GW

## PBH-DM scenario

### Curvature perturbations



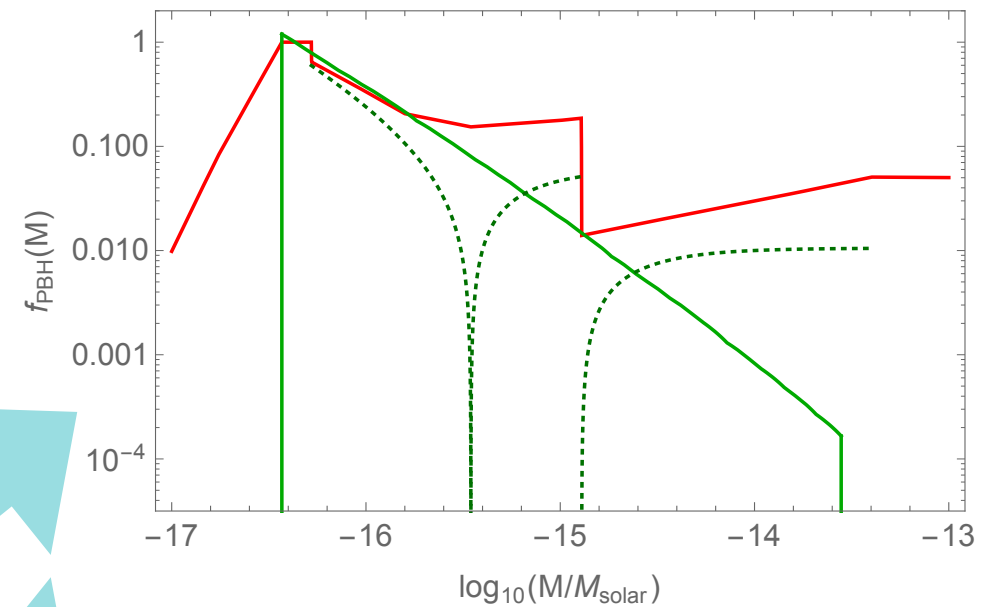
$$n_s = 0.96$$

$$\alpha_s = 0$$

$$\beta_s = 0.0019485$$

$$T_R = 10^4 \text{ GeV}$$

### PBH mass spectrum



### Secondary GW spectrum

