Einstein Double Field Equations

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Core idea: string theory predicts its own gravity rather than GR

In General Relativity the metric $g_{\mu\nu}$ is the only geometric and gravitational field, whereas in string theory the closed-string massless sector comprises a two-form potential $B_{\mu\nu}$ and the string dilaton ϕ in addition to the metric $g_{\mu\nu}$. Furthermore, these three fields transform into each other under T-duality. This hints at a natural augmentation of GR: upon treating the whole closed string massless sector as stringy graviton fields, Double Field Theory [1, 2] may evolve into 'Stringy Gravity'. Equipped with an $\mathbf{O}(D,D)$ covariant differential geometry beyond Riemann [3], we spell out the definitions of the stringy Einstein curvature tensor and the stringy Energy-Momentum tensor. Equating them, all the equations of motion of the closed string massless sector are unified into a single expression [4],

$$G_{AB} = 8\pi G T_{AB} \tag{1}$$

which we dub the Einstein Double Field Equations.

Double Field Theory as Stringy Gravity

• **Built-in symmetries & Notation:**

– $\mathbf{O}(D,D)$ T-duality

– DFT diffeomorphisms (ordinary diffeomorphisms plus B-field gauge symmetry)

– Twofold local Lorentz symmetries, $\mathbf{Spin}(1,D{-}1)\times\mathbf{Spin}(D{-}1,1)$

⇒ Two locally inertial frames exist separately for the left and the right modes.

Index	Representation	Metric (raising/lowering indices)
$A,B,\cdots,M,N,\cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$p,q,\cdots$	$\mathbf{Spin}(1,D{-}1)$ vector	$\eta_{pq}=\mathrm{diag}(-++\cdots+)$
$\alpha,\beta,\cdots$	$\mathbf{Spin}(1,D{-}1)$ spinor	$C_{\alpha\beta}$, $(\gamma^p)^T=C\gamma^pC^{-1}$
$\bar{p},\bar{q},\cdots$	$\mathbf{Spin}(D{-}1,1)$ vector	$\bar{\eta}_{\bar{p}\bar{q}}=\mathrm{diag}(+--\cdots-)$
$\bar{\alpha},\bar{\beta},\cdots$	$\mathbf{Spin}(D{-}1,1)$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}}$, $(\bar{\gamma}^{\bar{p}})^T=\bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

The $\mathbf{O}(D,D)$ metric $\mathcal{J}_{AB}$ divides doubled coordinates into two: $x^A=(\tilde{x}_\mu,x^\nu)$, $\partial_A=(\tilde{\partial}^\mu,\partial_\nu)$.

• **Doubled-yet-gauged spacetime:**
The doubled coordinates are 'gauged' through a certain equivalence relation, $x^A\sim x^A+\Delta^A$, such that each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$ [5]. This implies a section condition, $\partial_A\partial^A=0$, which can be conveniently solved by setting $\tilde{\partial}^\mu\equiv 0$.

• **Stringy graviton fields (closed-string massless sector), $\{d, V_{Mp}, \bar{V}_{N\bar{q}}\}$:**
Defining properties of the DFT-metric,

$$\mathcal{H}_{MN}=\mathcal{H}_{NM}\,,\qquad \mathcal{H}_K{}^L\mathcal{H}_M{}^N\mathcal{J}_{LN}=\mathcal{J}_{KM}\,, \tag{2}$$

set a pair of symmetric and orthogonal projectors,

$$P_{MN}=P_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}+\mathcal{H}_{MN})\,,\qquad P_L{}^MP_M{}^N=P_L{}^N\,,$$
$$\bar{P}_{MN}=\bar{P}_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}-\mathcal{H}_{MN})\,,\qquad \bar{P}_L{}^M\bar{P}_M{}^N=\bar{P}_L{}^N\,,\qquad P_L{}^M\bar{P}_M{}^N=0\,.$$

Further, taking the "square roots" of the projectors, we acquire a pair of DFT vielbeins,

$$P_{MN}=V_M{}^pV_N{}^q\eta_{pq}\,,\qquad \bar{P}_{MN}=\bar{V}_M{}^{\bar{p}}\bar{V}_N{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

satisfying their own defining properties,

$$V_{Mp}V^M{}_q=\eta_{pq}\,,\quad \bar{V}_{M\bar{p}}\bar{V}^M{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\quad V_{Mp}\bar{V}^M{}_{\bar{q}}=0\,,\quad V_M{}^pV_{Np}+\bar{V}_M{}^{\bar{p}}\bar{V}_{N\bar{p}}=\mathcal{J}_{MN}\,.$$

The most general solutions to (2) can be classified by two non-negative integers $(n,\bar{n})$ [6],

$$\mathcal{H}_{MN}=\begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^\mu X_\lambda^i-\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu}+X_\kappa^iY_i^\nu-\bar{X}_\kappa^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho-2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^\rho\end{pmatrix}$$

where $1\le i\le n$, $1\le\bar{\imath},i\le\bar{n}$ and

$$H^{\mu\nu}X^i_\nu=0\,,\quad H^{\mu\nu}\bar{X}^{\bar{\imath}}_\nu=0\,,\quad K_{\mu\nu}Y_i^\nu=0\,,\quad K_{\mu\nu}\bar{Y}_{\bar{\imath}}^\nu=0\,,\quad H^{\mu\rho}K_{\rho\nu}+Y_i^\mu X_\nu^i+\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\nu^{\bar{\imath}}=\delta^\mu{}_\nu\,.$$

Strings become chiral and anti-chiral over n and $\bar{n}$ directions: $X^i_\mu\partial_+x^\mu=0$, $\bar{X}^{\bar{\imath}}_\mu\partial_-x^\mu=0$. Examples include $(0,0)$ Riemannian geometry as $K_{\mu\nu}=g_{\mu\nu}$, $H^{\mu\nu}=g^{\mu\nu}$, $(1,1)$ Gomis-Ooguri non-relativistic background, $(1,0)$ Newton-Cartan gravity, and $(D-1,0)$ Carroll gravity.

• **Covariant derivative:**
The 'master' covariant derivative, $\mathcal{D}_A=\partial_A+\Gamma_A+\Phi_A+\bar{\Phi}_A$, is characterized by compatibility:

$$\mathcal{D}_Ad=\mathcal{D}_AV_{Bp}=\mathcal{D}_A\bar{V}_{B\bar{p}}=0\,,\quad \mathcal{D}_A\mathcal{J}_{BC}=\mathcal{D}_A\eta_{pq}=\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}}=\mathcal{D}_AC_{\alpha\beta}=\mathcal{D}_A\bar{C}_{\bar{\alpha}\bar{\beta}}=0\,.$$

The stringy Christoffel symbols are [3]

$$\begin{aligned}\Gamma_{CAB}=&\;2\left(P\partial_CP\bar{P}\right)_{[AB]}+2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E-P_{[A}{}^DP_{B]}{}^E\right)\partial_DP_{EC}\\ &-4\left(\tfrac{1}{P_M{}^M-1}P_{C[A}P_{B]}{}^D+\tfrac{1}{\bar{P}_M{}^M-1}\bar{P}_{C[A}\bar{P}_{B]}{}^D\right)\left(\partial_Dd+(P\partial^EP\bar{P})_{[ED]}\right)\,,\end{aligned}$$

and the spin connections are $\Phi_{Apq}=V^B{}_p(\partial_AV_{Bq}+\Gamma_{AB}{}^CV_{Cq})$, $\bar{\Phi}_{A\bar{p}\bar{q}}=\bar{V}^B{}_{\bar{p}}(\partial_A\bar{V}_{B\bar{q}}+\Gamma_{AB}{}^C\bar{V}_{C\bar{q}})$.
In Stringy Gravity, there are no normal coordinates where Γ_{CAB} would vanish point-wise: the Equivalence Principle holds for point particles but is generically broken for strings (i.e. extended objects).

• **Scalar and 'Ricci' curvatures:**
The semi-covariant Riemann curvature in Stringy Gravity is defined by

$$S_{ABCD}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^E{}_{AB}\Gamma_{ECD}\right)\,,$$

where $R_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{ACE}\Gamma_B{}^E{}_D-\Gamma_{BCE}\Gamma_A{}^E{}_D$ (the "field strength" of Γ_{CAB}).
The completely covariant 'Ricci' and scalar curvatures are, with $S_{AB}=S_{ACB}{}^C$,

$$S_{p\bar{q}}:=V^A{}_p\bar{V}^B{}_{\bar{q}}S_{AB}\,,\qquad S_{(0)}:=\left(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{CD}\right)S_{ABCD}\,.$$

While $e^{-2d}S_{(0)}$ corresponds to the original DFT Lagrangian density [1, 2], or the 'pure' Stringy Gravity, the master covariant derivative fixes its minimal coupling to extra matter fields, *e.g.* type II maximally supersymmetric DFT [7] or the Standard Model [8].

Derivation of Einstein Double Field Equations

Variation of the action for Stringy Gravity coupled to generic matter fields, Υ_a, gives

$$\begin{aligned}&\delta\int e^{-2d}\Big[\tfrac{1}{16\pi G}S_{(0)}+L_{\rm matter}\Big]\\ =&\int e^{-2d}\Big[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{8\pi G}\delta d(S_{(0)}-8\pi GT_{(0)})+\delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\Big]\\ =&\int e^{-2d}\Big[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\{G_{AB}-8\pi GT_{AB}\}+(\hat{\mathcal{L}}_\xi\Upsilon_a)\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\Big]\,,\end{aligned}$$

where the second line is for generic variations and the third line is specifically for diffeomorphic transformations. We are naturally led to define

$$K_{p\bar{q}}:=\frac{1}{2}\left(V_{Ap}\frac{\delta L_{\rm matter}}{\delta\bar{V}_A{}^{\bar{q}}}-\bar{V}_{A\bar{q}}\frac{\delta L_{\rm matter}}{\delta V_A{}^p}\right)\,,\qquad T_{(0)}:=e^{2d}\times\frac{\delta\left(e^{-2d}L_{\rm matter}\right)}{\delta d}\,,$$

and subsequently the stringy *Einstein curvature*, G_{AB}, and *Energy Momentum tensor*, T_{AB},

$$G_{AB}=4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,,\qquad \mathcal{D}_AG^{AB}=0\qquad(\text{off-shell})\,,$$
$$T_{AB}:=4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,,\qquad \mathcal{D}_AT^{AB}=0\qquad(\text{on-shell})\,.$$

The equations of motion of the stringy graviton fields are thus unified into a single expression, the *Einstein Double Field Equations* (1). Note that $G_A{}^A=-DS_{(0)}$, $T_A{}^A=-DT_{(0)}$.
Restricting to the $(0,0)$ Riemannian background, the Einstein Double Field Equations reduce to

$$\begin{aligned}R_{\mu\nu}+2\nabla_\mu(\partial_\nu\phi)-\tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}&=8\pi GK_{(\mu\nu)}\,,\\ \nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right)&=16\pi Ge^{-2\phi}K_{[\mu\nu]}\,,\\ R+4\Box\phi-4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}&=8\pi GT_{(0)}\,,\end{aligned}$$

which imply the conservation law, $\mathcal{D}_AT^{AB}=0$, given explicitly by

$$\nabla^\mu K_{(\mu\nu)}-2\partial^\mu\phi\,K_{(\mu\nu)}+\tfrac{1}{2}H_\nu{}^{\lambda\mu}K_{[\lambda\mu]}-\tfrac{1}{2}\partial_\nu T_{(0)}=0\,,\qquad \nabla^\mu\left(e^{-2\phi}K_{[\mu\nu]}\right)=0\,.$$

The Einstein Double Field Equations also govern the dynamics of other non-Riemannian cases, $(n,\bar{n})\neq(0,0)$, where the Riemannian metric, $g_{\mu\nu}$, cannot be defined.

Examples

– *Pure Stringy Gravity with cosmological constant:*

$$\tfrac{1}{16\pi G}e^{-2d}\left(S_{(0)}-2\Lambda_{\rm DFT}\right)\,,\qquad K_{p\bar{q}}=0\,,\qquad T_{(0)}=\tfrac{1}{4\pi G}\Lambda_{\rm DFT}\,.$$

– *RR sector, given by a* $\mathbf{Spin}(1,9)\times\mathbf{Spin}(9,1)$ *bi-spinorial potential,* $\mathcal{C}^\alpha{}_{\bar{\alpha}}$ *:*

$$L_{\rm RR}=\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})\,,\qquad K_{p\bar{q}}=-\tfrac{1}{4}{\rm Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})\,,\qquad T_{(0)}=0\,,$$

where $\mathcal{F}=\mathcal{D}_+\mathcal{C}=\gamma^p\mathcal{D}_p\mathcal{C}+\gamma^{(11)}\mathcal{D}_{\bar{p}}\mathcal{C}\bar{\gamma}^{\bar{p}}$ is the RR flux set by an $\mathbf{O}(D,D)$ covariant "H-twisted" cohomology, $(\mathcal{D}_+)^2=0$, and $\bar{\mathcal{F}}=\bar{C}^{-1}\mathcal{F}^TC$ is its charge conjugate [7].

– *Spinor field:* $L_\psi=\bar{\psi}\gamma^p\mathcal{D}_p\psi+m_\psi\bar{\psi}\psi$, $K_{p\bar{q}}=-\tfrac{1}{4}(\bar{\psi}\gamma_p\mathcal{D}_{\bar{q}}\psi-\mathcal{D}_{\bar{q}}\bar{\psi}\gamma_p\psi)$, $T_{(0)}=0$.

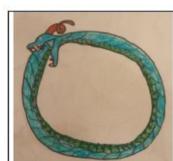
– *Green-Schwarz superstring (κ-symmetric):*

$$e^{-2d}L_{\rm string}=\tfrac{1}{4\pi\alpha'}\int{\rm d}^2\sigma\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_i^M\Pi_j^N\mathcal{H}_{MN}-\epsilon^{ij}D_iy^M(\mathcal{A}_{jM}-i\Sigma_{jM})\right]\delta^D\big(x-y(\sigma)\big)\,,$$
$$K_{p\bar{q}}(x)=\tfrac{1}{4\pi\alpha'}\int{\rm d}^2\sigma\sqrt{-h}h^{ij}(\Pi_i^MV_{Mp})(\Pi_j^N\bar{V}_{N\bar{q}})\,e^{2d}\delta^D\big(x-y(\sigma)\big)\,,\qquad T_{(0)}=0\,,$$

where $\Sigma_i^M=\bar{\theta}\gamma^M\partial_i\theta+\bar{\theta}'\bar{\gamma}^M\partial_i\theta'$ and $\Pi_i^M=\partial_iy^M-\mathcal{A}_i^M-i\Sigma_i^M$ (doubled-yet-gauged) [9].

Gravitational effect

The regular spherical solution to the $D=4$ Einstein Double Field Equations shows that Stringy Gravity modifies GR (Schwarzschild geometry), in particular at "short" dimensionless scales, R/MG, *i.e.* distance normalized by mass times Newton constant. This might shed new light upon the dark matter/energy problems, as they arise essentially from "short distance" observations. Furthermore, it would be intriguing to view the B-field and DFT dilaton d as 'dark gravitons', since they decouple from the geodesic motion of point particles, which should be defined in string frame [10].

	Electron ($R\simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1{\rm AU}/M_\odot G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M\propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim10^{5}$	0^+

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