Preliminaries: The DDM Ensemble

Characteristics, Parametrizations, Scaling Relations, and Equations of State

Recall basic DDM framework:

The dark matter of the universe consists of N states, with N>>1.

- No state individually needs to carry the full Ω_{CDM} so long as the sum of their abundances matches Ω_{CDM}.
- In particular, each state can have a very small abundance.
- This allows states to exhibit different lifetimes! As long as those with larger abundances have larger lifetimes (and vice versa), phenomenological constraints can be satisfied.

Usual dark-matter scenarios are nothing but a limiting N=1 case of this more general framework. However, taking N>>1 leaves room for our states to exhibit a whole spectrum of decay widths (lifetimes) without running afoul of phenomenological and cosmological constraints.

Can outline the salient features of this scenario more quantitatively...

In general, universe progresses through four distinct phases

- Inflation
- Reheating (matter-dominated, where matter = inflaton)
- Radiation-dominated
- Matter-dominated (current epoch)

In general, consider "stuff" with equation of state $p = w\,\rho$.
This "stuff" will have an abundance $\Omega = \rho / \rho_{crit}$ which scales with time as...

- w= 0 for matter
- w= -1 for vacuum energy
- w= +1/3 for radiation
- w= -1/3 for curvature

$$\Omega \sim \begin{cases} t^{(1-3w)/2} & \text{RD phase} \\ t^{-2w} & \text{MD and reheating phases} \\ \exp[-3H(1+w)t] & \text{inflationary phase .} \end{cases}$$

For concreteness, assume individual DM components in our scenario are described by scalars ϕ_i , $i=1,...,N$ with

- masses m_i
- decay widths Γ_i describing decays into SM states.

In FRW universe, these fields will evolve according to...

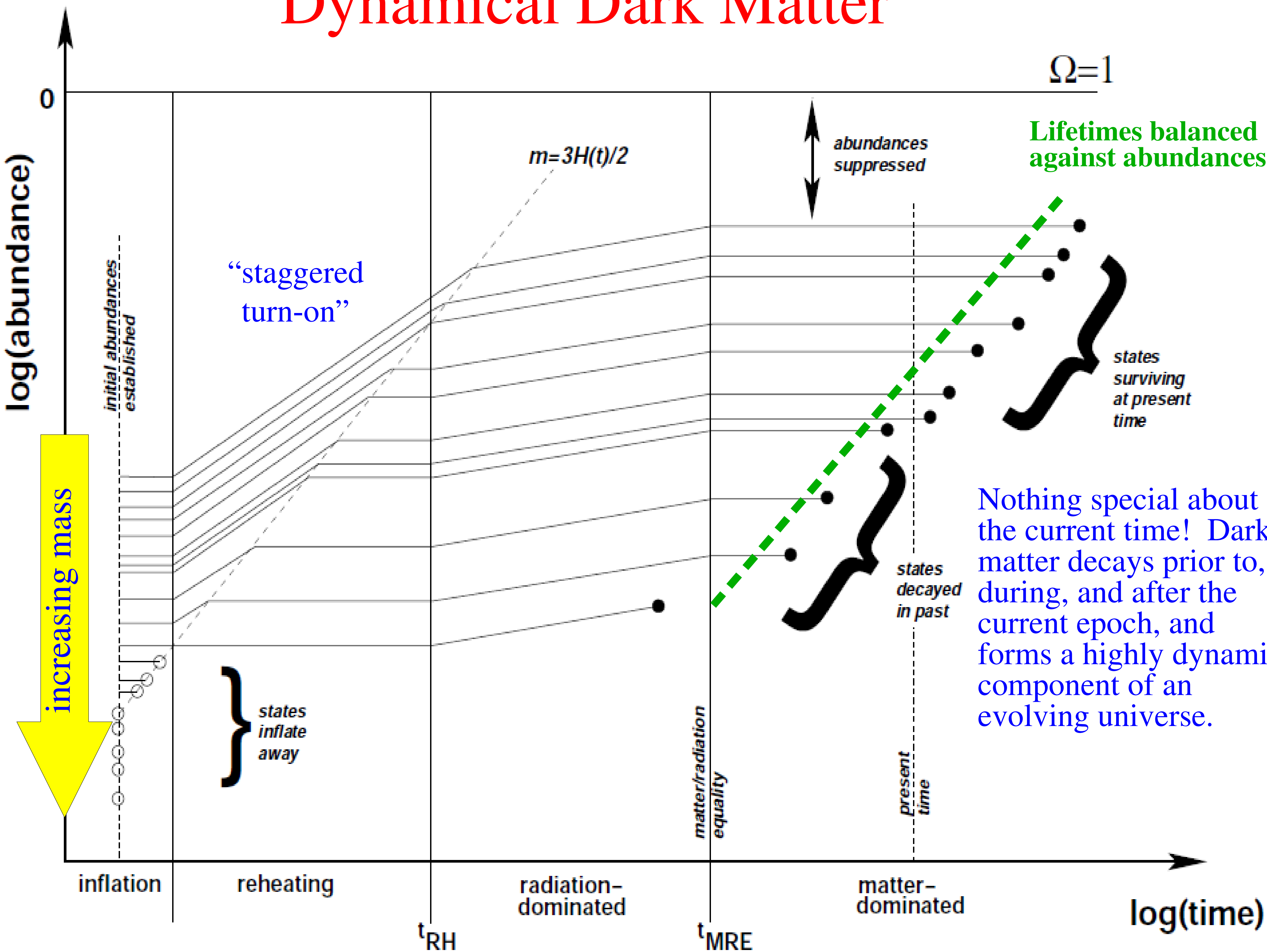
$$\ddot{\phi}_i + [3H(t) + \Gamma_i]\dot{\phi}_i + m_i^2\phi_i \;=\; 0$$

Hubble parameter:
H(t) ~ 1/t (FRW)

Transition from overdamped to underdamped oscillation...

- Transition from vacuum energy (w= -1) to matter (w=0).
- Occurs when $3H(t) = 2m_i$ ⇨ $t \sim 1/m_i$
- Heavier states "turn on" first, lighter states later.

Dynamical Dark Matter



Nothing special about the current time! Dark matter decays prior to, during, and after the current epoch, and forms a highly dynamic component of an evolving universe.

How to characterize a particular DDM configuration?

Introduce two "complementary" parameters:

- Total abundance at any moment: $\Omega_{\rm tot}(t) \equiv \sum_i \Omega_i(t)$

- *Distribution* of that total abundance: how much is $\Omega_{\rm tot}$ shared between a dominant component Ω_0 and all others?

  Define

$$\eta \equiv 1 - \frac{\Omega_0}{\Omega_{\rm tot}}$$

where $\Omega_0 \equiv \max_i\{\Omega_i\}$

Thus

$0 \le \eta \le 1$

- $\eta=0$ signifies one dominant component (standard picture)
- $\eta>0$ quantifies departure from standard picture

Each of these quantities will have a unique time-dependence in the DDM framework.

Start with η :

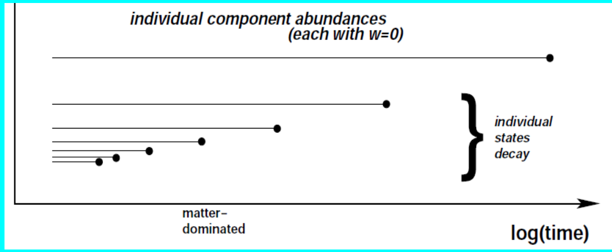
- Initial value of η is set when initial abundances established
- If during inflation, heavy modes inflate away ⇒ η **decreases!**
- During staggered turn-on, identity of "turned-on" state carrying largest abundance keeps changing (new state becomes dominant, former dominant becomes sub-dominant) ⇒ η **can increase or decrease!**
- Dark-matter decay widths are larger for heavier states which have smaller abundances ⇒ η **decreases! This effect is eventually the only one that survives at late times.**

Thus, there is always a time after which η decreases monotonically. It is nevertheless a fundamental property of the DDM framework that η is presumed significantly different from zero at the present time.

Now look at time-dependence of Ω_{tot} .

- Indeed, one important signature of the dynamical nature of dark matter in this framework is that Ω_{tot} is a time-evolving quantity ---- *even during the current matter-dominated epoch*!
- Within such a framework, it is therefore only to be regarded as an accident that Ω_{tot} happens to match the observed $\Omega_{CDM} = 0.26$ at the present time.
- Moreover, the time-dependence of Ω_{tot} in this framework will essentially give us an "effective" equation of state for our decaying DDM ensemble of dark-matter states.

To calculate time-dependence of Ω_{tot} ,
let's focus on the final, MD era:



$$\Omega_i(t) \;=\; \Omega_i\,\Theta(\tau_i - t)$$ where $$\tau_i \equiv \Gamma_i^{-1}$$

$$\frac{d\Omega_{\text{tot}}(t)}{dt} \;=\; \sum_i \Omega_i \frac{d}{dt}\Theta(\tau_i - t) \;=\; -\sum_i \Omega_i \delta(\tau_i - t)$$

Now let's replace sum over states by an integral

$$\sum_i \;\Longrightarrow\; \int d\tau\, n_\tau(\tau)$$

density of states per unit τ .

$$\frac{d\Omega_{\text{tot}}(t)}{dt} \;=\; -\int d\tau\, \Omega(\tau)\, n_\tau(\tau)\, \delta(\tau - t)$$
$$=\; -\Omega(t) n_\tau(t)\ .$$

To go further, let us parametrize the spectrum of ensemble components in terms of their scaling behavior as function of decay width ---

$$\Omega(\Gamma) \sim A\Gamma^{\alpha}$$

$$\alpha < 0$$

$$\eta_{\Gamma}(\Gamma) \sim B\Gamma^{\beta}$$

density of states *per unit* Γ

$$n_{\tau} = n_{\Gamma}\left|\frac{d\Gamma}{d\tau}\right| = \Gamma^2 n_{\Gamma}$$

We then have

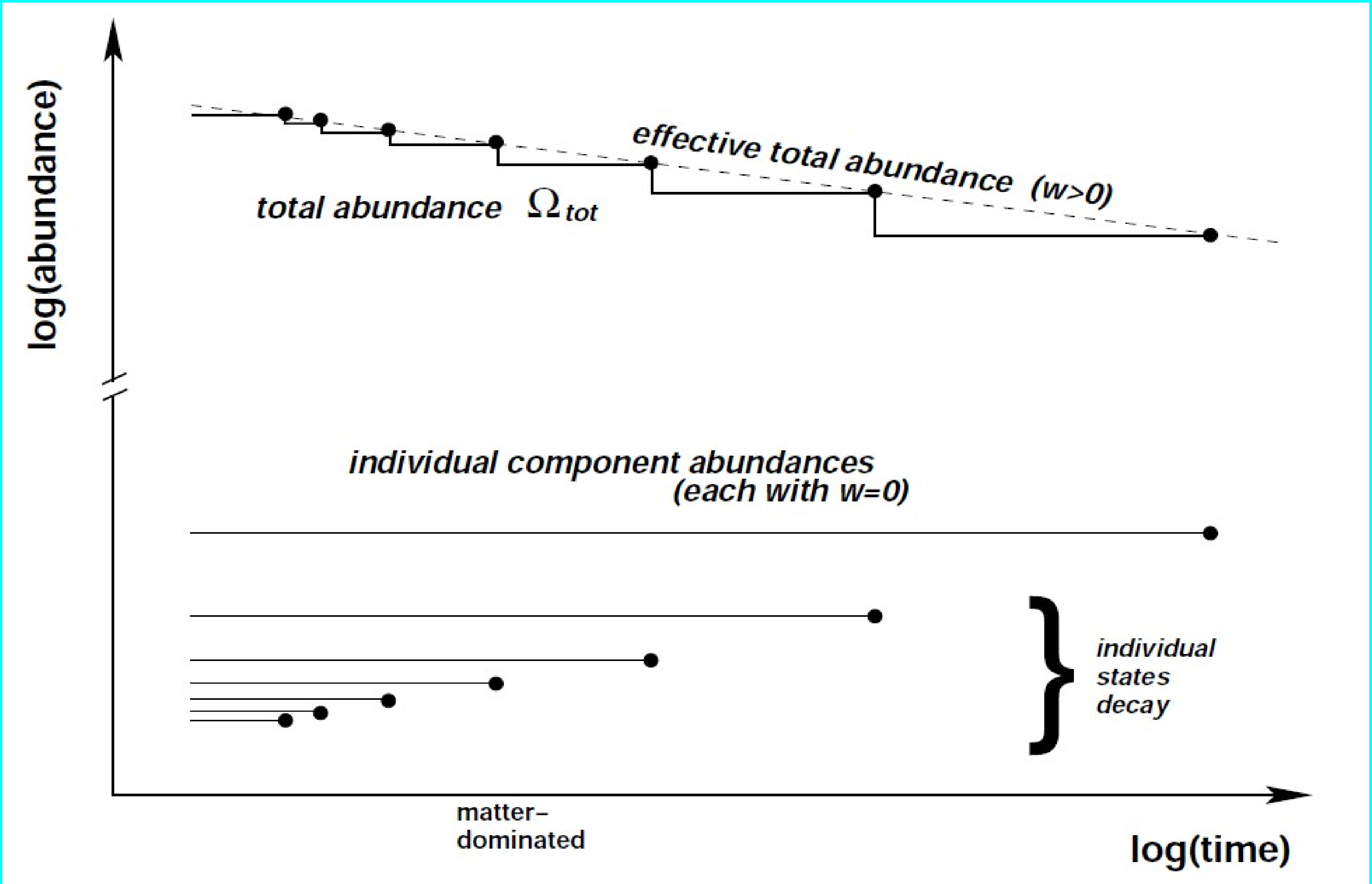
$$\Omega(\Gamma)n_{\tau}(\Gamma) \sim AB\Gamma^{\alpha+\beta+2}$$

or

$$\Omega(\tau)n_{\tau}(\tau) \sim AB\tau^{-\alpha-\beta-2}$$

$$\frac{d\Omega_{\rm tot}(t)}{dt} = -AB\,t^{-\alpha-\beta-2}$$

General result!



Sketch shown for $\alpha + \beta > -1$, with $\alpha < 0$ and $\beta > 0$

This means that the DDM ensemble as a whole has a non-zero "effective" equation-of-state parameter $w_{\rm eff}(t)$. In general, we can define...

$$w_{\rm eff}(t) \equiv -\left(\frac{1}{3H}\frac{d\log\rho_{\rm tot}}{dt}+1\right)$$

$$= \begin{cases} -\frac{1}{2}\left(\frac{d\log\Omega_{\rm tot}}{d\log t}\right) & \text{for RH/MD eras} \\ -\frac{2}{3}\left(\frac{d\log\Omega_{\rm tot}}{d\log t}\right)+\frac{1}{3} & \text{for RD era} \end{cases} .$$

We then find the results

- **For** $x \equiv \alpha+\beta \neq -1$ **:**

$$w_{\rm eff}(t) = \frac{(1+x)w_*}{2w_*+(1+x-2w_*)(t/t_{\rm now})^{1+x}}$$

where

$$w_* \equiv w_{\rm eff}(t_{\rm now}) = \frac{AB}{2\Omega_{\rm CDM}t_{\rm now}^{1+x}}$$

- **For** $x = -1$ **:**

$$w_{\rm eff}(t) = \frac{w_*}{1-2w_*\log(t/t_{\rm now})}$$

where

$$w_* \equiv w_{\rm eff}(t_{\rm now}) = \frac{AB}{2\Omega_{\rm CDM}}$$

These are "effective" equations of state for the entire DDM ensemble!

If the DDM model in question is to be in rough agreement with cosmological observations, we expect that w_* today should be fairly small (since traditional dark "matter" has $w = 0$).

We also expect that the function $w_{\text{eff}}(t)$ should not have experienced strong variations within the recent past.

Given the previous functional forms for $w_{\text{eff}}(t)$, this implies that the situations which are likely to be phenomenologically preferred are those with

$$x \equiv \alpha + \beta \lesssim -1$$

However, depending on the detailed properties of the particular DDM scenario under study, values of x slightly above -1 may also be acceptable.

At first glance, it might seem difficult (or at best fine-tuned) to arrange a collection of states which are not only suitable candidates for dark matter but in which the abundances and SM decay widths are precisely balanced in the required manner, with suitable values of x.

However, it turns out that there is one group of states for which such a balancing act can occur naturally...

An infinite tower of Kaluza-Klein (KK) states living in the bulk of large extra spacetime dimensions!

- SM restricted to brane → all bulk states can interact with the SM only gravitationally → natural candidates for dark matter!
- From 4D perspective, this “dark matter” appears as infinite tower of KK states.
- As we shall see, a suitable balancing of abundances and lifetimes can occur --- *even if the stability of the KK tower itself is entirely unprotected*!

Thus, a KK tower is a natural example of a DDM ensemble!

This feature ultimately emerges as the consequence of the non-trivial interplay between physics in the bulk and physics on the brane.

For example, let us consider a very simple "bare-bones" setup:

Universe has a single, flat extra dimension of length R, one massless bulk field Φ, and SM lives on a brane located at y=0...

SM fields

$$S = \int d^4x dy \left[\mathcal{L}_{\rm bulk}(\Phi) + \delta(y)\mathcal{L}_{\rm brane}(\psi_i, \Phi)\right]$$

where

$$\mathcal{L}_{\rm bulk} = \frac{1}{2}\partial_M \Phi^* \partial^M \Phi$$

no mass term in bulk

$$\mathcal{L}_{\rm int} \supset -\frac{1}{2}m^2|\Phi|^2$$

effective mass term induced by SM dynamics on brane

Now do KK reduction for Z_2 orbifold (line segment) of radius R:

$$\Phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{k=0}^{\infty} r_k\, \phi_k(x^\mu) \cos\left(\frac{ny}{R}\right)$$

$$r_k \equiv \begin{cases} 1 & \text{for } k=0 \\ \sqrt{2} & \text{for } k>0 \end{cases}$$

These are the individual constituents of our ensemble!

The masses of these states are the eigenvalues λ_k of the KK mass matrix.

Further assume…

- Standard misalignment production in the bulk for 5D bulk field Φ

  ⇒ sets initial abundance Ω_k for each KK mode ϕ_k
- Decay width Γ_k for decay $\phi_k \to$ SM as determined through standard leading-order brane/bulk interactions

e.g.,

$$\frac{1}{\hat{f}}\,(\partial_\mu \phi')\,\overline{\psi}\gamma^\mu \psi\ , \qquad \frac{1}{\hat{f}}\,\phi'\, F_{\mu\nu}F^{\mu\nu}$$

if Φ is CP-even
e.g., moduli…

$$\frac{1}{\hat{f}}\,(\partial_\mu \phi')\,\overline{\psi}\gamma^\mu \gamma^5 \psi\ , \qquad \frac{1}{\hat{f}}\,\phi'\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

if Φ is CP-odd
e.g., axions…

where ϕ' is projection of Φ onto brane:

$$\phi' \equiv \Phi(y)\big|_{y=0} = \sum_{k=0}^{\infty} r_k \phi_k$$

⇒

$$\Gamma_\lambda \sim \frac{\lambda^3}{\hat{f}^2}\,(\tilde{\lambda}^2 A_\lambda)^2 \ \sim\ \frac{\lambda^3}{\hat{f}^2}$$

Combining our results for Ω_λ and Γ_λ ,
we obtain the following *product relations*
across our KK towers:

Different possibilities depending on the conditions under which abundances were established

$$\begin{aligned} \text{instantaneous}: &\quad \Omega_\lambda \Gamma_\lambda^{2/3} \sim \text{constant} \\ \text{staggered (RD era)}: &\quad \Omega_\lambda \Gamma_\lambda^{7/6} \sim \text{constant} \\ \text{staggered (reheating/MD era)}: &\quad \Omega_\lambda \Gamma_\lambda^{4/3} \sim \text{constant} \end{aligned}$$

In all cases, decay widths are balanced against abundances, as promised!
This is a universal feature for such KK towers.

Indeed, for a generic KK tower,
we find the following values of $x = \alpha + \beta$:

	large $\tilde{\lambda}$	small $\tilde{\lambda}$
instantaneous	$-4/3$	$-4/5$
staggered (RD era)	$-11/6$	$-11/10$
staggered (RH/MD eras)	-2	$-6/5$

TABLE I: Values of the equation-of-state parameter $x \equiv \alpha + \beta$ for different portions of a general KK tower with different "turn-on" phenomenologies. We observe that KK towers naturally give rise to values $x \lesssim -1$, which is precisely the range favored phenomenologically.

... precisely in the phenomenologically preferred range!!

Thus, KK towers provide particularly compelling realizations of DDM ensembles!

Final note:

In this discussion, we have parametrized the cosmological abundances of the states across the DDM ensemble directly in terms of their corresponding decay widths:

$$\Omega(\Gamma) \sim A\Gamma^{\alpha} \qquad \alpha < 0$$

However, for many purposes it proves useful to adopt a general parametrization for the *masses* of the states in the DDM ensemble, then relate the abundances and lifetimes of these states to their masses.

Parametrize DDM masses:

$$m_n = m_0 + (\Delta m)\, n^{\delta} \qquad \Delta m, \delta > 0$$

e.g., $(m_0, \Delta m, \delta) = \begin{cases} (m, 1/R, 1) & \text{for } mR \ll 1 \\ (m, 1/(2mR^2), 2) & \text{for } mR \gg 1 \end{cases}$ KK tower w/ radius *R*, 4D mass *m*

$= (M_0, M_s, 2)$ for towers of string *oscillator* states and/or excited hadronic resonances

M_s: "string" scale (related to Regge slope)

M_0: related to Regge intercept

We then typically assume scaling relations of the forms...

Cosmological abundances

$$\Omega_n = \Omega_0 \left(\frac{m_n}{m_0}\right)^{\gamma}$$

Decay widths

$$\Gamma_n = \Gamma_0 \left(\frac{m_n}{m_0}\right)^{y}$$

e.g., $\mathcal{O}_n \sim c_n \chi_n \mathcal{O}_{\rm SM}/\Lambda^{d-4}$

→ $y = 2d - 7$

Our structural DDM constraints then take the forms...

1 $\gamma y < 0$

Inverse balancing of lifetimes against abundances across the ensemble (usually $\gamma<0$, $y>0$)

2 $-1 \lesssim \frac{1}{y}\left(\gamma + \frac{1}{\delta}\right) < 0$

Viable DDM effective equation of state

(also ensures finite $\Omega_{\rm tot}$ for infinite DDM ensembles)