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Group Theoretic Approach to Theory of Fermion Production

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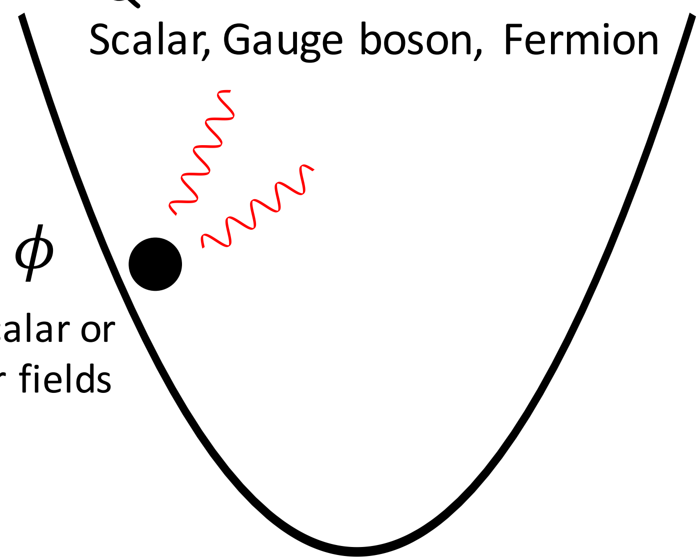
Based on
Min, SON, Suh 1808.00939

Particle Production



- Preheating via parametric resonance or excitation in post-inflationary era

  Kofman, Linde, Starobinsky 97'

- Axion-inflation via gauge boson ($\phi F\tilde{F}$) or fermion ($\partial_\mu \phi\, j^{\mu 5}$) production

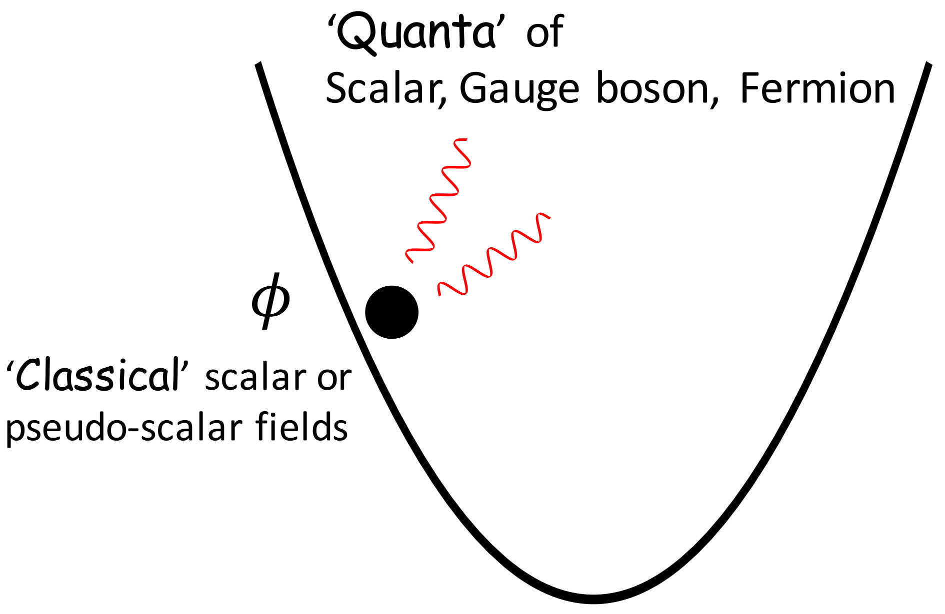
  Anbor, Sorbo 10' Adshead, Pearce, Peloso, Roberts, Sorbo 18'

- Gravitational waves from preheating

  Many literature (hard to list all here)

List goes on .....

Particle Production



We will focus on the 'Reformulation' of theory of fermion production

Traditional Approach
To
Theory of Fermion Production

called technique of ‘Bogoliubov’ coefficient

The model

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[\bar{\psi}\left(i\,e^{\mu}{}_{a}\,\gamma^{a}D_{\mu} - m + g(\phi)\right)\psi + \frac{1}{2}\left(\partial_{\mu}\phi\right)^{2} - V(\phi)\right]$$

On the metric:

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2 = a(t)^2(d\tau^2 - d\mathbf{x}^2)$$

Under rescaling $\psi \to a^{-3/2}\psi$

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - ma + \underline{g(\phi)}\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^4V(\phi)$$

Common Interaction type in literature

$$g(\phi) = \begin{cases} h\phi & : \text{Yukawa-type coupling} \\ \dfrac{1}{f}\gamma^{\mu}\gamma^{5}\partial_{\mu}\phi & : \text{derivative coupling} \end{cases}$$

More stuff to talk about !

We will assume spatially homogenous scalar field : $\partial_{\mu}\phi = \dot{\phi}$

We will not distinguish t and τ unless it is necessary

Fermion Production is formulated in Hamiltonian formalism

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - ma - \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

A subtlety with derivative coupling

$$\Pi_{\psi} = \frac{\delta\mathcal{L}}{\delta\dot{\psi}} = i\psi^{+} \qquad \Pi_{\phi} = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = a^{2}\dot{\phi} - \frac{1}{f}\bar{\psi}\gamma^{0}\gamma^{5}\psi$$

$$\begin{aligned}\mathcal{H} &= \Pi_{\psi}\dot{\psi} + \Pi_{\phi}\dot{\phi} - \mathcal{L} \\ &= \bar{\psi}\left(-i\gamma^{i}\partial_{i} + ma + \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi - \frac{1}{2a^{2}}\frac{(\bar{\psi}\gamma^{0}\gamma^{5}\psi)^{2}}{f^{2}} + \frac{1}{2a^{2}}\Pi_{\phi}^{2} + a^{5}V(\phi)\end{aligned}$$

Adshead, Sfakianakis 15'

Definition of particle number is ambiguous

Massless limit is not manifest

A way out: field redefinition

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - ma - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

Adshead, Sfakianakis 15'

$$\psi \to e^{-i\gamma^5\phi/f}\psi$$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - \underbrace{ma\cos\frac{2\phi}{f}}_{=m_R} + i\,\underbrace{ma\sin\frac{2\phi}{f}}_{=m_I}\gamma^5\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

Adshead, Pearce, Peloso, Roberts, Sorbo 18'

Hamiltonian formalism

$$\Pi_\psi = \frac{\delta\mathcal{L}}{\delta\dot{\psi}} = i\psi^+ \qquad \Pi_\phi = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = a^2\dot{\phi}$$

$$\mathcal{H} = \bar{\psi}\left(-i\gamma^i\partial_i + m_R - i\,m_I\gamma^5\right)\psi + \frac{1}{2a^2}\Pi_\phi^2 + a^4V(\phi)$$

No ψ - dependence in conjugate momentum Π_ϕ

Entire fermion sector is quadratic in ψ

: particle number is unambiguously defined

Massless limit is manifest

Fermion production

$$\mathcal{H} = \bar{\psi}\left(-i\,\gamma^i \partial_i + m_R - i\,m_I \gamma^5\right)\psi + \frac{1}{2a^2}\Pi_\phi^2 + a^4 V(\phi)$$

Garbrecht, Prokopec, Schmidt 02'

To estimate Fermion Production, we quantize ψ
while keeping pseudo-scalar as a classical field

Quantum field ψ

We follow notation and convention in
Adshead, Pearce, Peloso, Roberts, Sorbo 18'

$$\psi = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{r=\pm} \left[U_r(\mathbf{k},t)\,a_r(\mathbf{k}) + V_r(-\mathbf{k},t)\,b_r^+(-\mathbf{k})\right]$$

$$U_r = \begin{pmatrix} u_r(\mathbf{k},t)\ \chi_r(\mathbf{k}) \\ r\ v_r(\mathbf{k},t)\,\chi_r(\mathbf{k}) \end{pmatrix}, \quad V_r = C\,\bar{U}_r^T \qquad \text{with } C = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$$

$$\chi_r(\mathbf{k}) = \frac{k + r\,\vec{\sigma}\cdot\mathbf{k}}{\sqrt{2k(k+k_3)}}\bar{\chi}_r \qquad \text{where } \bar{\chi}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\chi}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

** helicity basis for an arbitrary $\mathbf{k}$

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

$$B_r = \frac{r\, e^{ir\varphi_k}}{2}[2\, m_R u_r v_r - k(u_r^2 - v_r^2) - i r m_I (u_r^2 + v_r^2)]$$

Fermion number density for a particle with helicity r

$$n_{r,k} = \langle 0 | a_r^+(\mathbf{k}; t) a_r(\mathbf{k}; t) | 0 \rangle$$

w/ $a_r(\mathbf{k}; t)$, $a_r^+(\mathbf{k}; t)$ are diagonalized $a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$ at $t \neq 0$

At $t = 0$

$a_r(\mathbf{k})|0\rangle = 0$

$a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$
$\leftrightarrow$ one-particle state
due to $B_r = 0$

At $t \neq 0$

$a_r(\mathbf{k}; t)|0\rangle \neq 0$

$a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$
$\not\leftrightarrow$ one-particle state
anymore due to $B_r \neq 0$

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

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Fermion number density for a particle with helicity r

$$n_{r,k} = \langle 0| a_r^+(\mathbf{k};t) a_r(\mathbf{k};t) |0\rangle \boxed{= |\beta_r|^2}$$

w/ $a_r(\mathbf{k};t)$, $a_r^+(\mathbf{k};t)$ are diagonalized $a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$ at $t \neq 0$

$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

Bogoliubov coeff.

$$a_r(\mathbf{k};t) = \alpha_r\, a_r(\mathbf{k}) - \beta_r^*\, b_r^+(\mathbf{k})$$

$$b_r^+(\mathbf{k};t) = \beta_r\, a_r(\mathbf{k}) + \alpha_r^*\, b_r^+(\mathbf{k})$$

Diag. ops at $t \neq 0$

In terms of diag. ops at $t = 0$

looks too technical … Any simplication?

$$n_{r,k} = \langle 0|a_r^{+}(\mathbf{k};t)a_r(\mathbf{k};t)|0\rangle$$
$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega}Re(u_r^* v_r) - \frac{r\, m_I}{2\omega}Im(u_r^* v_r)$$

Solving EOM of u_r, v_r with correct initial condition is another source of confusion

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Solving EOM of u_r, v_r with correct initial condition is another source of confusion

Recall a Fourier mode in 'helicity' basis

$$\psi \sim U_r(\mathbf{k},t)a_r(\mathbf{k}) + V_r(-\mathbf{k},t)b_r^+(-\mathbf{k})$$

$$U_r = \begin{pmatrix} u_r(\mathbf{k},t)\ \chi_r(\mathbf{k}) \\ r\ v_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} u_r \\ r v_r \end{pmatrix} \otimes \chi_r \equiv \xi_r \otimes \chi_r$$

looks too technical … Any simplication?

$$n_{r,k} = \langle 0| a_r^+(\mathbf{k};t) a_r(\mathbf{k};t)|0\rangle$$
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Then we realize that

$$\zeta_{r\,1} = \frac{1}{2} r(u_r^* v_r + u_r v_r^*) = r\; Re(u_r^* v_r)$$
$$\zeta_{r\,2} = -\frac{i}{2} r(u_r^* v_r - u_r v_r^*) = r\; Im(u_r^* v_r)$$
$$\zeta_{r\,3} = \frac{1}{2}(|u_r|^2 - |v_r|^2)$$

$$\longrightarrow \quad \vec{\zeta}_r = \xi_r^+ \vec{\sigma}\, \xi_r$$

collapses into one vector

$$n_{r,k} = \langle 0 | a_r^{+}(\mathbf{k};t) a_r(\mathbf{k};t) | 0 \rangle$$

$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r\, m_I}{2\omega} Im(u_r^* v_r)$$

$$\mathbf{q} = rk\ \hat{x}_1 + m_I\ \hat{x}_2 + m_R\ \hat{x}_3$$

* We will see the origin of this vector later

$$\vec{\zeta}_r = \xi_r^{+} \vec{\sigma}\ \xi_r \qquad \text{w/ } \xi_r \equiv \begin{pmatrix} u_r \\ r v_r \end{pmatrix}$$

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$$\zeta_{r\,3} = \frac{1}{2}(|u_r|^2 - |v_r|^2)$$

$$n_{r,k}(t) \ = \ \frac{1}{2}\left(1 - \frac{\mathbf{q}\cdot\vec{\zeta}_r}{|\mathbf{q}|}\right) = \frac{1}{2}(1 - \cos\theta)$$

$\vec{\zeta}_r$, $\mathbf{q}$ behave like vector reps of SO(3) !

What is this mysterious SO(3)?

Group Theoretic Approach

Lorentz Group

Weyl Representation

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = -\sigma_3 \otimes I_2$$

Spinor rep. satisfying Lorentz algebra

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$J_i \equiv \frac{1}{2}\epsilon_{ijk} S^{jk} = \frac{1}{2} I_2 \otimes \sigma_i \text{ (space rotation) ,} \qquad K_i \equiv S^{i0} = \frac{i}{2}\sigma_3 \otimes \sigma_i \text{ (boost)}$$

$$\psi \sim \xi_r \otimes \chi_r \;\rightarrow\; e^{-i\vec{\theta}\cdot\vec{J}}\psi = \xi \otimes \boxed{e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\chi_r}$$

Universally acts on ψ_L and ψ_R

On the other hand

$$(J_{L,R})_i = \frac{J_i \mp i\,K_i}{\sqrt{2}} = \frac{1}{2}(I_2 \pm \sigma_3) \otimes \frac{\sigma_i}{2} \quad : \quad SU(2)_L \times SU(2)_R$$

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

: Rep. of $SU(2)_L \times SU(2)_R$ is constructed as a 'tensor sum'

‘Reparametrization’ Group

While γ^μ is fixed and only ψ transforms in the Lorentz group,

$$\gamma^\mu \to \gamma^\mu, \quad \psi \to \Lambda_{1/2}\psi\,,$$

there is a freedom in choosing a representation of the gamma matrices.
This freedom is totally unphysical.

Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}1_4$$

$$\gamma^\mu \to U\gamma^\mu U^{-1} \quad : \quad \mathrm{GL(4,C)}$$

Dirac Theory

We assign the transformation of ψ, $\psi \to U\psi$

$$\mathcal{L} = \psi^+\gamma^0\left(i\gamma^\mu\partial_\mu - m\right)\psi$$

$$\to \mathcal{L} = \psi^+\, U^+U\gamma^0U^{-1}\left(iU\gamma^\mu U^{-1}\partial_\mu - m\right)U\psi$$

$$U^+U = UU^+ = 1 \quad : \quad \mathrm{U(4)}$$

We consider the following subgroup of $U(4)$

$$SU(2)_1 \times SU(2)_2 \times U(1) \subset U(4)$$

The rep of subgroup is constructed as a 'tensor product' of two $SU(2)$'s and phase rotation, e.g.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{= U_1} \otimes U_2 = \begin{pmatrix} a_{11}U_2 & a_{12}U_2 \\ a_{21}U_2 & a_{22}U_2 \end{pmatrix}$$

Under $SU(2)_1 \otimes SU(2)_2$ transformation (we associate $U(1)$ with ξ_r)

$$\psi \sim \xi_r \otimes \chi_r \to (U_1 \otimes U_2)(\xi_r \otimes \chi_r) = (U_1\xi_r) \otimes (U_2\chi_r)$$

Universally acts on ψ_L and ψ_R

This is what we are looking for

Looks similar to space rotation of Lorentz group.

But it can not be identified with SU(2) space rotation

E.g. $\bar{\psi}\gamma^\mu\psi \to \psi^+U^+U\gamma^0U^{-1}U\gamma^\mu U^{-1}U\psi = \bar{\psi}\gamma^\mu\psi$

$$\bar{\psi}\gamma^\mu\psi \to \bar{\psi}\,\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}\psi = \Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\mu\psi$$

A well-known example of $SU(2)_1$

Weyl Representation $\psi_{\text{Weyl}} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = -\sigma_3 \otimes I_2$$

Dirac Representation $\psi_{\text{Dirac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \sigma_3 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2$$

Two representations are related via a similarity transformation

$$\gamma^\mu_{\text{Weyl}} \ \rightarrow \ U_1 \gamma^\mu_{\text{Weyl}} U_1^{-1} = \gamma^\mu_{\text{Dirac}}$$

$$\psi_{\text{Weyl}} \ \rightarrow U_1 \psi_{\text{Weyl}} = \psi_{\text{Dirac}}$$

w/ $U_1(\pi/2) = e^{i\frac{\pi}{2}\frac{\sigma_y}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Previously mysterious group that we were looking for is

$$SU(2)_1 \times U(1)$$

We will drop subscript from now on

This is what our group theoretic approach is based on

$SU(2)_2$ does not play any important role. We ignore it

Fermion production
in `Inertial Frame'

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - m_R + i\,m_I\gamma^5\right)\psi + \cdots$$

Group Theoretic Approach

Dirac equation in inertial frame

$$\left(i\,\gamma^{\mu}\partial_{\mu} - m_R + i\,m_I\gamma^5\right)\psi = 0$$

EOM in tensor form for a Fourier mode can be written as (using $(\vec{\sigma}\cdot\mathbf{k})\chi_r = rk\chi_r$)

$$[(i\,\sigma_3\partial_t - irk\sigma_2 - m_R I_2 + im_I\sigma_1)\otimes I_2](\xi_r\otimes\chi_r) = 0$$

Gives rise to EOM of fundamental rep.

$$\partial_t\xi_r \;=\; -i(\mathbf{q}\cdot\vec{\sigma})\xi_r$$

: it is called Weyl equation in condensed matter physics

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

SU(2) fundamental ($\partial_t\xi_r$)

SU(2) embedding of SO(3) vector $\mathbf{q}$ ($\mathbf{q}\cdot\vec{\sigma}$)

Group Theoretic Approach

✓ Fundamental rep. of SU(2)

$$\xi_r \equiv \begin{pmatrix} u_r \\ r v_r \end{pmatrix}$$

- EOM of fundamental rep.

$$\partial_t \xi_r \; = \; -i(\mathbf{q} \cdot \vec{\sigma})\xi_r$$

SU(2) embedding
of SO(3) vector

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

Group Theoretic Approach

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$$\xi_r \equiv \begin{pmatrix} u_r \\ r v_r \end{pmatrix}$$

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SU(2) embedding of SO(3) vector

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

✓ In terms of SO(3) ~ SU(2) reps

Bilinear of ξ_r : $\xi_r^+ A\, \xi_r$

w/ A = arbitrary 2×2 complex matrix

$\xi^+\,\xi\ (=1)$: scalar

$\vec{\zeta}_r = \xi^+\vec{\sigma}\,\xi$: vector

the only non-trivial rep.

- EOM of vector rep.

$$\partial_t \zeta_{r\,i} = \frac{1}{2}\xi_r^+[i\mathbf{q}\cdot\vec{\sigma}, \sigma_i]\xi_r = 2\epsilon_{ijk}q_j\zeta_{r\,k}$$

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r$$

Analog to classical precession motion

Quantum mechanical fermion production

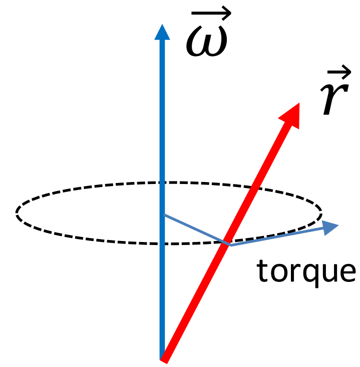
$$\frac{1}{2}\frac{d\vec{\zeta}_r}{dt} = \mathbf{q}\times\vec{\zeta}_r$$

$\mathbf{q}$ as angular velocity

$$? = \boldsymbol{q}\cdot\vec{\zeta}_r$$

Classical precession of a vector $\vec{r}$ with angular velocity $\vec{\omega}$

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$



E.g. when $\vec{r} = \mathbf{M}$ (magnetization),

$$\vec{\omega} = \vec{\omega}_{\mathbf{M}} = -\gamma\mathbf{B}$$

$$\frac{d\mathbf{M}}{dt} = \vec{\omega}_{\mathbf{M}}\times\mathbf{M}$$: called block eq.

$$E = \vec{\omega}_{\mathbf{M}}\cdot\mathbf{M}$$

Particle number density

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

$$B_r = \frac{r\, e^{ir\varphi_k}}{2}[2\, m_R u_r v_r - k(u_r^2 - v_r^2) - irm_I(u_r^2 + v_r^2)]$$

Now it is clear that each matrix element should be a function of $\mathbf{q}$ and $\vec{\zeta}_r$ in our group theoretic approach

Diagonal element

$$A_r = \mathbf{q} \cdot \vec{\zeta}_r$$
$$= \omega \cos\theta$$

Off-diagonal element

$$|B_r| = \left|\mathbf{q} \times \vec{\zeta}_r\right|$$
$$= \omega \sin\theta$$

One can easily see why eigenvalues are $\pm\omega = \pm|\mathbf{q}|$

$$|\mathbf{q}| = \omega = \sqrt{k^2 + m^2}$$

Particle number density

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \mathbf{q} \cdot \vec{\zeta}_r\,, \quad |B_r| = \left|\mathbf{q} \times \vec{\zeta}_r\right|$$

In our approach, a few group properties can uniquely determine fermion number density

$$n_{r,k} = \langle 0|a_r^+(\mathbf{k};t)a_r(\mathbf{k};t)|0\rangle = |\beta_r|^2 = f(\mathbf{q} \cdot \vec{\zeta}_r, |\mathbf{q}|)$$

1. It should be at most linear in $\vec{\zeta}_r$ (note $\left|\vec{\zeta}_r\right| = 1$)

$$n_{r,k} = A \pm B \frac{\mathbf{q} \cdot \vec{\zeta}_r}{|\mathbf{q}|}$$

which gives rise to inequality,

$$A - B \leq n_{r,k} \leq A + B$$

2. Pauli-blocking

$$0 \leq n_{r,k} \leq 1$$

$$\longrightarrow \quad n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q} \cdot \vec{\zeta}_r}{|\mathbf{q}|}\right)$$

$'-'$ sign chosen for the consistency with the form of A_r

(** agrees with our explicit computation)

Solution of EOM

Closed form of solution is available

3×3 rep. of SO(3)

$$\frac{1}{2}\partial_t \vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r = (\mathbf{q}\cdot\mathbf{L})\vec{\zeta}_r \qquad n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q}\cdot\vec{\zeta}_r}{|\mathbf{q}|}\right)$$

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

- Initial condition (↔ zero particle number) at $t = t_0$ is straightforward than other approach

$$\vec{\zeta}_r(t_0, t_0) = \frac{\mathbf{q}(t_0)}{|\mathbf{q}(t_0)|}$$

- Just like solving Schrödinger eq. for the unitary op., EOM can be iteratively solved

$$\vec{\zeta}_r(t, t_0) = T\exp\left(\int_{t_0}^{t} dt'\,(\mathbf{q}\cdot\mathbf{L})(t')\right)\frac{\mathbf{q}(t_0)}{|\mathbf{q}(t_0)|}$$

Expanding involves commutators of $\mathbf{q}\cdot\mathbf{L}$

WKB solution might be the case with vanishing commutators

Switching to `Rotating Frame'

Via $\psi \to e^{+i\gamma^5\phi/f}\psi$

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^\mu\partial_\mu - ma - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

Equivalent to, in terms of $\vec{\zeta}_r$,

$$\vec{\zeta}_r \to R(t)\vec{\zeta}_r\,, \qquad R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi/f & -\sin 2\phi/f \\ 0 & \sin 2\phi/f & \cos 2\phi/f \end{pmatrix}$$

Switching to `Rotating Frame'

Via $\psi \to e^{+i\gamma^5\phi/f}\psi$

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^\mu\partial_\mu - ma - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

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- ✓ This rotating frame is non-inertial frame
- ✓ Needs to supplement extra terms, e.g. Coriolis , centrifugal forces etc, to keep physics independent

EOM in `Rotating Frame'

Under $\vec{\zeta}_r \rightarrow R(t)\vec{\zeta}_r$,

Similarly to the classical mechanics, EOM transforms like

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r = (\mathbf{q}\cdot\mathbf{L})\vec{\zeta}_r \qquad \rightarrow \qquad \frac{1}{2}\partial_t(R\vec{\zeta}_r) = (\mathbf{q}\cdot\mathbf{L})(R\vec{\zeta}_r)$$

$$\frac{1}{2}\partial_t\vec{\zeta}_r = R^T(\mathbf{q}\cdot\mathbf{L})R\ \vec{\zeta}_r - \frac{1}{2}R^T\dot{R}\vec{\zeta}_r$$

w/ $\left(R^T\dot{R}\right)_{ij} \equiv \epsilon_{ijk}\omega_{\zeta_r k}$

EOM in `Rotating Frame'

Under $\vec{\zeta}_r \to R(t)\vec{\zeta}_r$,

Similarly to the classical mechanics, EOM transforms like

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r = (\mathbf{q}\cdot\mathbf{L})\vec{\zeta}_r \quad \to \quad \frac{1}{2}\partial_t(R\vec{\zeta}_r) = (\mathbf{q}\cdot\mathbf{L})(R\vec{\zeta}_r)$$

$$\frac{1}{2}\partial_t\vec{\zeta}_r = R^T(\mathbf{q}\cdot\mathbf{L})R\,\vec{\zeta}_r - \frac{1}{2}R^T\dot{R}\vec{\zeta}_r$$

w/ $\left(R^T\dot{R}\right)_{ij} \equiv \epsilon_{ijk}\omega_{\zeta_r k}$

EOM can be brought back to the universal form

$$\frac{1}{2}\partial_t\vec{\zeta}_r = R\mathbf{q}\times\vec{\zeta}_r + \frac{1}{2}\vec{\omega}_{\zeta_r}\times\vec{\zeta}_r = \left(R\mathbf{q} + \vec{\omega}_{\zeta_r}\right)\times\vec{\zeta}_r = \mathbf{q}'\times\vec{\zeta}_r$$

$$\mathbf{q}' = \left(rk + \frac{\dot{\phi}}{f}\right)\hat{x}_1 + ma\,\hat{x}_3$$

: different basis amounts to choose different angular velocity

Particle number density in `Rotating (non-inertial) Frame'

Particle number density in rotating frame

$$n_{r,k} = \langle 0|a_r^+(\mathbf{k};t)a_r(\mathbf{k};t)|0\rangle = f(\mathbf{q}'\cdot\vec{\zeta}_r, |\mathbf{q}'|)$$

It should be at most linear in $\vec{\zeta}_r$.
Higher order terms should vanish to match to the one in inertial frame in $\dot{\phi}\to 0$ limit

$$n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q}'\cdot\vec{\zeta}_r}{|\mathbf{q}'|}\right)$$

* does not take into account of quartic coupling etc..

: matches to the quadratic term

$$\mathcal{H}_\psi = \bar{\psi}\left(-i\gamma^i\partial_i + ma + \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi - \frac{1}{2a^2}\frac{(\bar{\psi}\gamma^0\gamma^5\psi)^2}{f^2}$$

See Adshead, Sfakianakis 15' for a related discussion

1. It looks like particle numbers are different in two different frames.
2. Establishing the 'final' particle number as a basis-independent quantity seems very non-trivial, e.g. Inertial frame vs. Non-inertial frame

Summary

We proposed a new group theoretic approach to theory of fermion production

1. Based on the ‘Reparametrization’ group of gamma matrcies

   a. Totally unphysical symmetry (that we never cared) provides us with totally different viewpoint of a very complicated process such as fermion production

2. Insightful visualization of quantum mechanical fermion production dynamics.

   a. Dynamics is analogous to the classical precession.
   b. Crystal clear initial condition unlike the traditional approach.
   c. Systematic comparison between Exact solution vs WKB solution.

3. This approach applies to any fermion system

   a. Possible extension is gravitino production, fermion production from gravitational background, fermion production in extra-dim. spacetime