Large N limit and supersymmetric QCD

Di Xu

CTPU

Outline

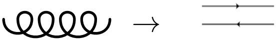
- Large N QCD
- Supersymmetric QCD (SQCD)

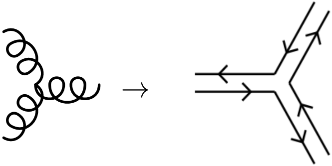
't Hooft suggested a strategy to introduce a small parameter into QCD. He pointed out that one can use the number of colors N to act as a free parameter. With an appropriate defination of "large N limit", $N \to \infty$ and $g^2 N = \lambda$ is fixed , the perturbation theory simplifies (only so-called planar diagrams survive, which is a subset diagrams of the full set) and presents nice features.

Start with the counting of powers of N in Feynman diagrams. First, write down the lagrangian of a non-abelian gauge theory with fermions.

$$L = -\frac{1}{2g^2} Tr[F_{\mu\nu}^2] + \bar{\psi}(i\partial\!\!\!/ + A\!\!\!/^a T^a - m)\psi, \tag{1}$$

where $F_{\mu\nu} = F_{\mu\nu}^a T^a$. One can read off the feynman rules and determine how the gauge coupling appears in propagators. The vertices and propagators have non-trivial g^2 dependence.


goes like $g^2 \approx N^{-1}$,


goes like $g^{-2} \approx N$.

Besides, Feynman diagrams will have factors of N resulting from internal color indices. The fermion propagator reads

$$\langle\psi^i(x)\bar{\psi}^j\rangle = \delta^{ij} S(x-y) \tag{2}$$

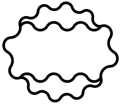
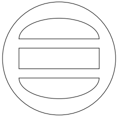
For gluon propagator

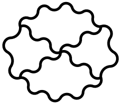
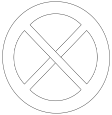
$$\langle A_\mu A_\nu\rangle = (T^a)^i_j (T^b)^l_k D_{\mu\nu}(x-y) \tag{3}$$

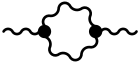
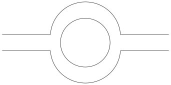
$$= \left(\frac{1}{2}\delta^i_j\delta^l_k - \frac{1}{2N}\delta^i_k\delta^l_j\right) D_{\mu\nu}(x-y)\delta^{ab} \tag{4}$$

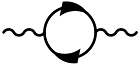
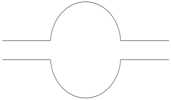
We can neglect second term in the large N limit. One can see for fermion propagator, one index flows in one direction for a particle, the opposite for an antiparticle, while for the gluon propagator one has two flowing in opposite directions. These can be represented by ’t Hooft’s double line notation as showed in the figure.

Here are some examples.

 $\rightarrow$  $\approx (g^2)^6 (\frac{1}{g^2})^4 N^4 \approx N^2$

 $\rightarrow$  $\approx (g^2)^6 (\frac{1}{g^2})^4 N^2 \approx N^0$

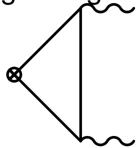
 $\rightarrow$  $\approx N g^4 \frac{1}{g^4} \approx N$

 $\rightarrow$  $\approx N^0$

These examples illustrate two general rules in large N expansion:

- non-planar diagrams (in double line notation, these are diagrams that could not be drawn without line crossings) are suppressed by factor of $\frac{1}{N^2}$
- diagrams with internal quark loop are suppressed by factor of $\frac{1}{N}$

In the 1970's Witten suggested that in the large N limit, the chiral anomaly could be treated as a perturbation, and the η' is a Goldstone boson in this limit. One could understand this by counting power of N in traingle diagram



$\approx N^0$

which is one power of N lower than $\frac{1}{g^2} F_{\mu\nu} F^{\mu\nu}$.

In particular, a Green's function with n insertions of $F\tilde{F}$ behaves as

$$\langle \left(\int F\tilde{F} \right)^n \rangle \sim N^{-n+2} \ . \qquad (5)$$

which now could be easily understood by counting powers of N.

Now to understand the mass of η', start from

$$\partial_\mu j_5^\mu = m_{\eta'}^2 f_\pi \tag{6}$$

The divergence of the current the can be written in terms of $F\tilde{F}$ using the anomaly equation. So the mass of the η' is proportional to the two point correlation function at zero momentum of $F\tilde{F}$ and $1/f_\pi^2$.

$$m_{\eta'}^2 \sim \frac{1}{f_\pi^2} \langle F\tilde{F}F\tilde{F} \rangle \tag{7}$$

Now since f_π^2 is order of N, one can see that the $m_{\eta'}^2 \sim \frac{1}{N}$.

The vacuum energy is of order N^2 and could be generally written as

$$E(\theta) = N^2 h(\theta/N) \tag{8}$$

The requirement that physics is periodic in θ and the formula above means that $E(\theta)$ is not smooth. This suggest that the pure gauge theory might exhibit a “branched” behavior.

Instantons in large N

A typical instanton contribution behaves like $e^{-\frac{8\pi^2}{g^2}}$, which in the large N limit, is exponentially suppressed.
But instanton contributions to the path integral are also generally infrared divergent. Consider QCD without flavors. The one-instanton contribution to $V(\theta)$ has the structure:

$$V(\theta) = \int d\rho \rho^{-5+\frac{11N}{3}} M^{\frac{11N}{3}} N e^{-\frac{8\pi^2}{g(M)^2}} \cos(\theta) \tag{9}$$

Suppose that the integral is cut off at $\rho \approx \Lambda^{-1}$ or $c\,\Lambda^{-1}$, then the result can be exponentially suppressed or enhanced by c^N. It is suggested that the most likely smooth limit for instanton effects in large N is zero.

Supersymmetric QCD (SQCD)

Supersymmetric QCD includes $SU(N_c)$ gauge group with N_f flavors of quark superfields in the fundmental representation of the gauge group and same number of superfields in the anti-fundmental representation of the gauge group. The lagrangian reads

$$\mathcal{L} = \int d^4\theta (Q_i^\dagger e^V Q_i + \bar{Q}_i e^V \bar{Q}_i^\dagger) - \frac{i}{16\pi} \int d^2\theta \tau W^{\alpha a} W_\alpha^a + h.c \qquad (10)$$

with $i = 1, ..., N_f$ and we take the superpotential to vanish. τ is a combination of the gauge coupling and the θ parameter which can be thought of as some spurion with non-zero vacuum expectation value

$$\langle \tau \rangle = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \qquad (11)$$

The theory has two anomalous global symmetries, the R symemtry and $U(1)$ flavor symmetry. One can combine them to form an anomaly-free symmetry $U(1)_{AF}$. Then the full global symmetry of the theory is

$$SU(N_f) \times SU(N_f) \times U_B(1) \times U_{AF}(1) \tag{12}$$

where the first $U(1)_B$ corresponds to baryon number conservation. The charges of fields under different $U(1)$ are givin in the table,

	$U(1)_B$	$U(1)_A$	$U(1)_R$	$U(1)_{AF}$
Q_i	+1	+1	0	$\frac{N_f - N_c}{N_f}$
ψ_{Q_i}	+1	+1	-1	$-\frac{N_c}{N_f}$
$\bar{Q}_i$	-1	+1	0	$\frac{N_f - N_c}{N_f}$
$\bar{\psi}_{Q_i}$	-1	+1	-1	$-\frac{N_c}{N_f}$
λ	0	0	+1	+1

Table: Charges of fields under different $U(1)$

In SQCD, the strongly interacting gaugino could undergoes condensation like quarks in QCD. Notice that $\langle\lambda^{\alpha a}\lambda^a_\alpha\rangle$ is also the scalar component of the superfield $W^{\alpha a}W^a_\alpha$. So one can calculate $\langle\lambda^{\alpha a}\lambda^a_\alpha\rangle$ by taking the derivative of the path integral with respect to the F term in τ.

$$\langle\lambda^{\alpha a}\lambda^a_\alpha\rangle = 16\pi\frac{\partial}{\partial F_\tau} logZ = 16\pi i\frac{\partial}{\partial\tau}W_{eff} \tag{13}$$

where W_{eff} is the effective superpotential after integrating out the gauge field. The exact result in pure gauge theory is

$$\langle\lambda\lambda\rangle = 32\pi^2\Lambda^3_{hol}e^{\frac{2\pi ik}{N}} \ . \tag{14}$$

θ and the η' Potential in SQCD

The vacuum energy is

$$V(\theta, k) \simeq m_\lambda |\Lambda_{hol}|^3 \cos\left(\frac{\theta + 2\pi k}{N}\right). \tag{15}$$

In terms of physical quantities,

$$m_\lambda \Lambda_{hol}^3 = N^2 m_{phys} \Lambda^3, \tag{16}$$

where $m_{phys} = g^2 m_\lambda$. Therefore, for very large N with θ and k fixed

$$V(\theta, k) \simeq N^2 m_{phys} |\Lambda|^3 \left(\frac{\theta + 2\pi k}{N}\right)^2. \tag{17}$$

This is compatible with the N-scaling and θ dependence of conventional large N story.

$N_f \ll N$ in supersymmetric QCD

Supersymmetric QCD with $N_f < N$ flavors possesses an $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ symmetry. Dynamically, a non-perturbative superpotential is generated,

$$W_{np} = (N - N_f)\frac{\Lambda_{hol}^{\frac{3N-N_f}{N-N_f}}}{(\det \bar{Q}Q)^{\frac{1}{N-N_f}}}. \tag{18}$$

Including supersymmetric mass terms for the quarks, the system has N supersymmetric vacua.

Turning on general soft breakings gives a set of theories which, in certain limits, should reduce to $SU(N)$ QCD with N_f flavors of fermionic quarks. Consider first adding only soft squark and gaugino masses

$$\delta V = \tilde{m}^2 \sum_f \left(|Q_f|^2 + |\bar{Q}_f|^2 \right) + m_\lambda \lambda\lambda. \tag{19}$$

With universal soft scalar mass terms, the first terms respect the full $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ symmetry of the supersymmetric theory. The gaugino mass term breaks the $U(1)_R$.

The η' potential at large N takes the form

$$V(\theta,\eta') = |m_\lambda|\Lambda_{hol}^3 \cos\left(\frac{\theta + 2\pi k + \frac{\eta'}{v}}{N}\right) + |m_q|\Lambda_{hol}^3 \cos\left(\frac{\eta'}{v} + \beta\right), \quad (20)$$

Vary the parameter

$$x = \frac{m_\lambda}{m_q}, \quad (21)$$

the number of local minima of the potential changes from N at small x to N_f at large x.

To see this explicitly, take the simplified case $|m_q|^2, |m_\lambda|^2 \ll \tilde{m}^2$, and $\tilde{m}^2$, m_q proportional to the unit matrix in flavor space. Then work out v. The potential for the η' then has the form:

$$V(\eta') = m_q \Lambda_{hol}^{\frac{3N-N_f}{N-N_f}} v_0^{-\frac{2N_f}{N-N_f}} \cos\left(\frac{N}{N-N_f}\eta'\right) + N m_\lambda \frac{\Lambda_{hol}^{\frac{3N-N_f}{N-N_f}}}{v_0^{\frac{2N_f}{N-N_f}}} \cos\left(\eta' \frac{N_f}{N-N_f}\right)$$

or, for $N \gg N_F$,

$$V(\eta') = m_q \Lambda_{hol}^3 v_0^{-\frac{2N_f}{N}} \cos(\eta') + N m_\lambda \Lambda_{hol}^3 \cos\left(\eta' \frac{N_f}{N}\right). \tag{23}$$

This potential is similar in structure to that for the ordinary η' proposed in large N.
It exhibits N vacua in the limit of small x, and N_f in the limit of large x.
Analogously, in ordinary QCD, the large-N η' potential has N_f vacua in the limit $m_q \ll \Lambda/N$, and N vacua in the opposite limit.

Instantons in SQCD

In SQCD with $N_f = N - 1$, the role of instantons in large N can be assessed sharply, exploiting the existence of a pseudomoduli space. The effective superpotential can be computed systematically, and infrared divergences are cut off by $Q\bar{Q}_{f\bar{f}} \equiv v^2 \delta_{f\bar{f}}$. The ρ integrals take the form

$$W \sim \int d\rho \, (\Lambda\rho)^{2N+1} (v^*)^{2N-2} \rho^{4N-5} e^{-c^2\rho^2|v|^2} \sim \frac{\Lambda^{2N+1}}{v^{2N-2}}. \qquad (24)$$

A careful analysis yields

$$W = \frac{\Lambda_{hol}^{2N+1}}{\det \bar{Q}Q} , \qquad (25)$$

which is naïvely of order e^{-N}.

However, v^2 also depends on Λ. For simplicity, taking all of the quarks to have equal mass,

$$v^N = \Lambda_{hol}^N \left(\frac{\Lambda_{hol}}{m_q}\right)^{\frac{1}{2}} . \tag{26}$$

At the stationary point,

$$\langle W \rangle = a\Lambda_{hol}^2 m_q \left[\frac{\Lambda_{hol}}{m_q}\right]^{1/N} . \tag{27}$$

This structure is dictated by symmetries and holomorphy.

Eq. (27) is notable. First, there is no exponential suppression with N:

$$\Lambda_{hol}^{2+\frac{1}{N}} = M^{2+\frac{1}{N}} e^{-\frac{8\pi^2}{g^2 N}+i\frac{\theta}{N}}. \tag{28}$$

Not only do the $e^{-\frac{8\pi^2}{g^2}}$ factors appear with a suitable power to avoid e^{-N} suppressions, but there are no factors like π^N or 2^N which might have obstructed a suitable large N limit.
We also note that in presence of a gaugino mass, we again find the usual formula for the vacuum energy,

$$E(\theta) = m_\lambda \langle W \rangle = m_\lambda \Lambda_{hol}^3 \cos\left(\frac{\theta + 2\pi k}{N}\right). \tag{29}$$

So in this case, we have agreement with expectations based on N counting of perturbative Feynman diagrams, yet the result arises entirely from an instanton!

Conclusion

- The presence of branches and the behavior in theories with matter (both with $N_f \ll N$ and $N_f \sim N$), are reflected in SBQCD
- In supersymmetric theories, instanton effects are sometimes calculable and do not fall off exponentially with N
- Branched structure in SBQCD is always associated with approximate discrete symmetries, which are badly broken in the nonsupersymmetric limit.
- In light of these differences, and to advance our understanding of nonperturbative phenomena in QCD, it would be of great interest to have additional lattice probes of the branched structure of large N QCD