Self-heating dark matter:

semi-annihilation + self-interaction

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Based on
AK, Hee Jung Kim, Hyungjin Kim, and Toyokazu Sekiguchi, PRL, 2018
AK and Hee Jung Kim (+ Hitoshi Murayama), in preparation

Oct. 8, 2019 @ IBS-MultiDark-IPPP workshop

Dark matter

Accumulated evidences from observations of the Universe

Known properties

- long-lived over the age of the Universe
- accounting for about $30\,\%$ of the present energy density of the Universe $\Omega_{\rm dm}h^2 \simeq 5\Omega_{\rm baryon}h^2 \simeq 0.12$
- feebly-interacting with photon and baryon
- not too hot to smear out primordial density contrast

Weakly-interacting massive particle (WIMP) paradigm

- satisfy above properties w/ new $\mathbb{Z}_2$symmetry
- originate from electroweak-scale new physics solving the hierarchy problem - WIMP miracle
- intensively investigated in direct/indirect detection experiments

No naturalness so far…

LHC discovery of $125\,\mathrm{GeV}$ Higgs and null-detection of top partners

- something wrong in the hierarchy problem and postulated solutions (including but not only electroweak-scale SUSY)

WIMP is no more as a miracle as we expected…

- new theoretical reasoning for WIMP? Arkani-Hamed and Dimopoulos, JHEP, 2005
  - mini-split SUSY Giudice and Romanino, NPB, 2004
    - sfermions $> 100\,\mathrm{TeV}$; gauginos $\sim \mathrm{TeV}$ Wells, PRD, 2005
      - $125\,\mathrm{GeV}$ Higgs
      - dark matter
      - precise grand unification

No convincing signals in direct/indirect detection experiments so far

- observational hints: small-scale issues in structure formation?

Bullock and Bolyan-Kolchin, Ann. Rev. Astron. Astrophys., 2018

What I will discuss

Self-interacting dark matter (SIDM) part 1

- observations of different size halos $M \sim 10^{9-14}\, M_\odot$
- velocity dependence of the cross section

Self-heating dark matter (SHDM): halos part 2

- evaporation through semi-annihilation
- no core collapse unlike pure SIDM

Self-heating dark matter: early Universe part 3

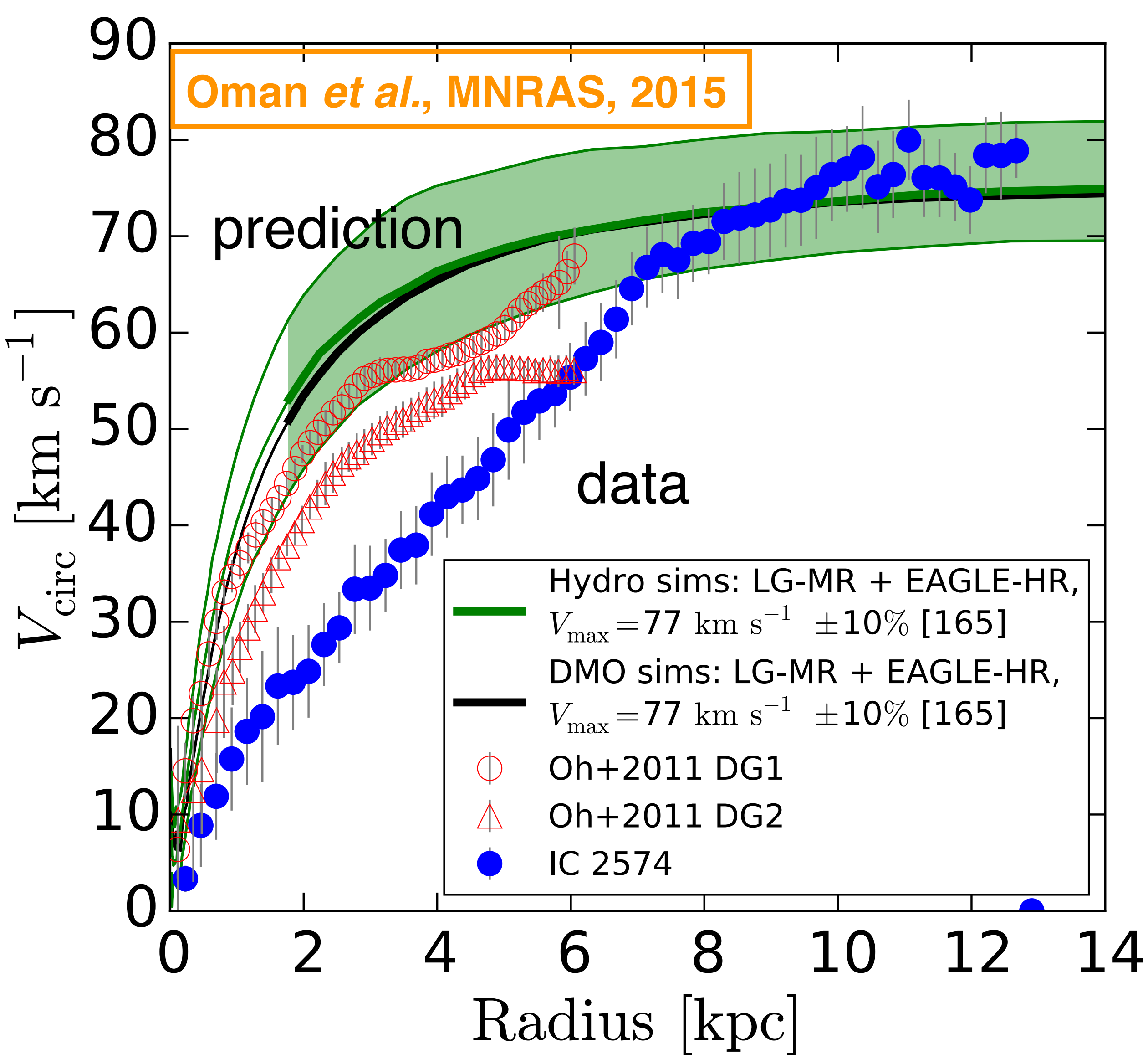
- thermal freeze-out
- suppression of the linear matter spectrum

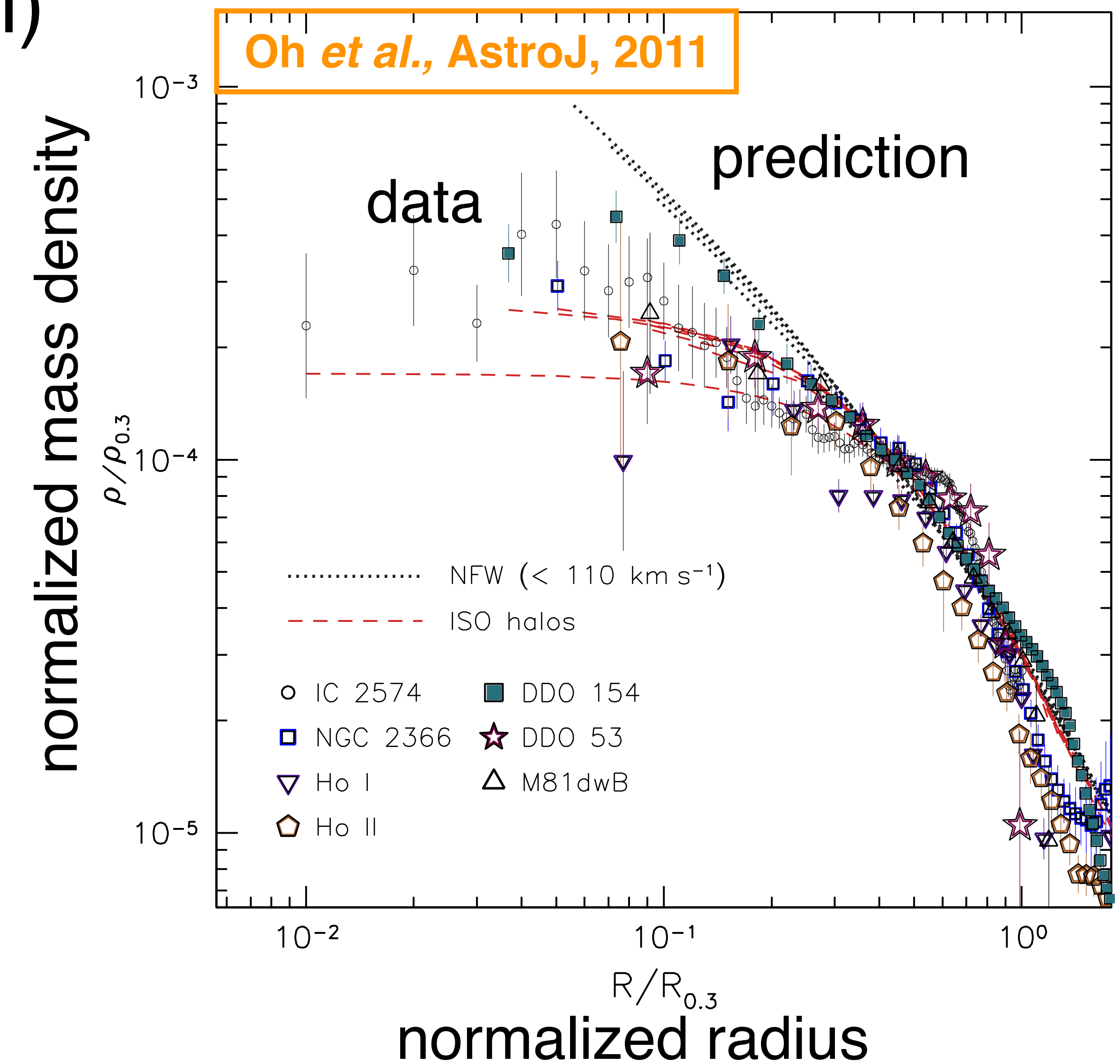
Self-interacting dark matter

Core-cusp problem

Collisionless cold dark matter (CDM)
→ cuspy profile

Observed dwarf/law-surface brightness (LSB) galaxies
→ cored profile $M \sim 10^{10\text{-}11} M_\odot$





a.k.a. inner mass deficit problem

- overpredict the circular velocity by a factor of ~ 2 (~ 4 in mass)

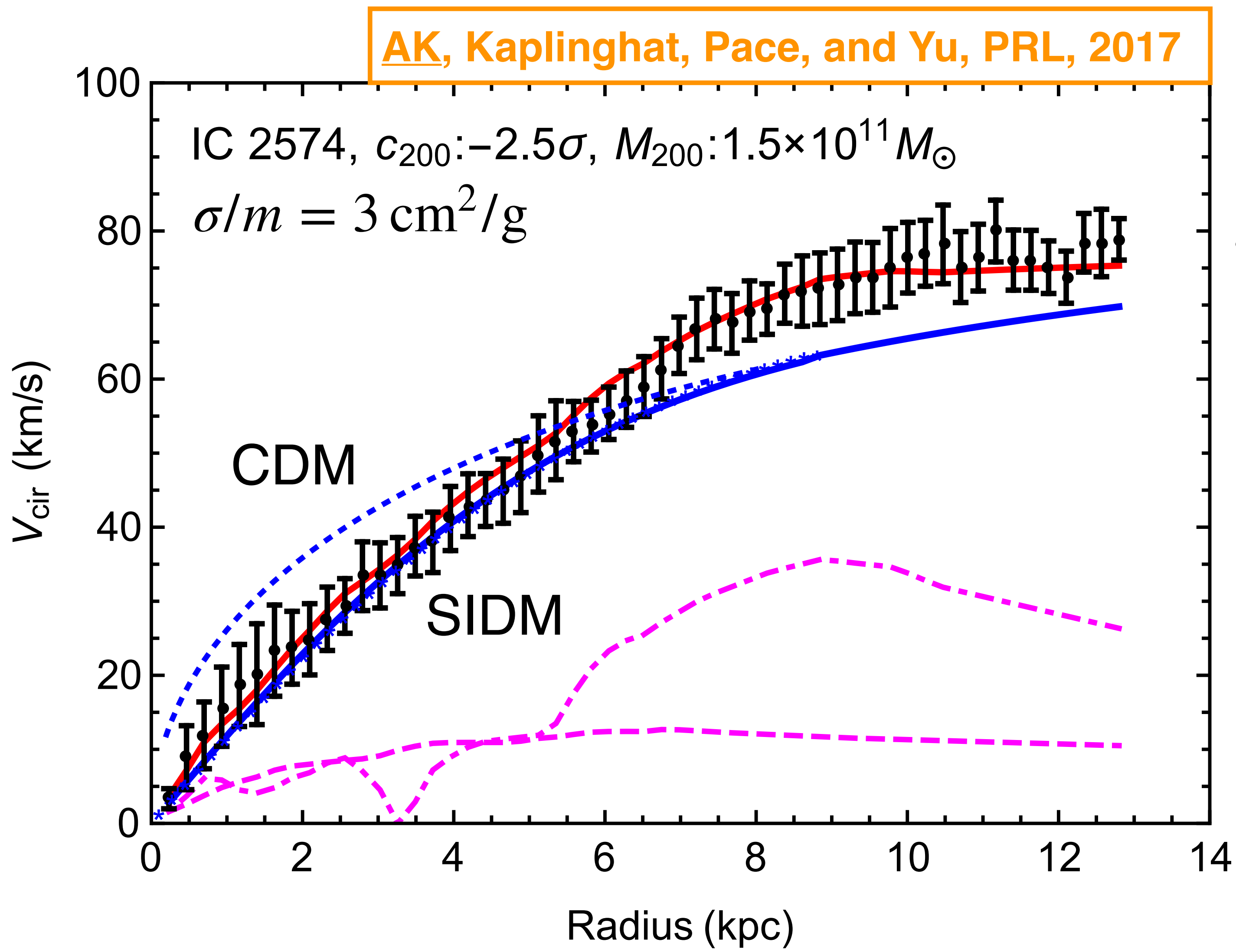
Self-interacting dark matter

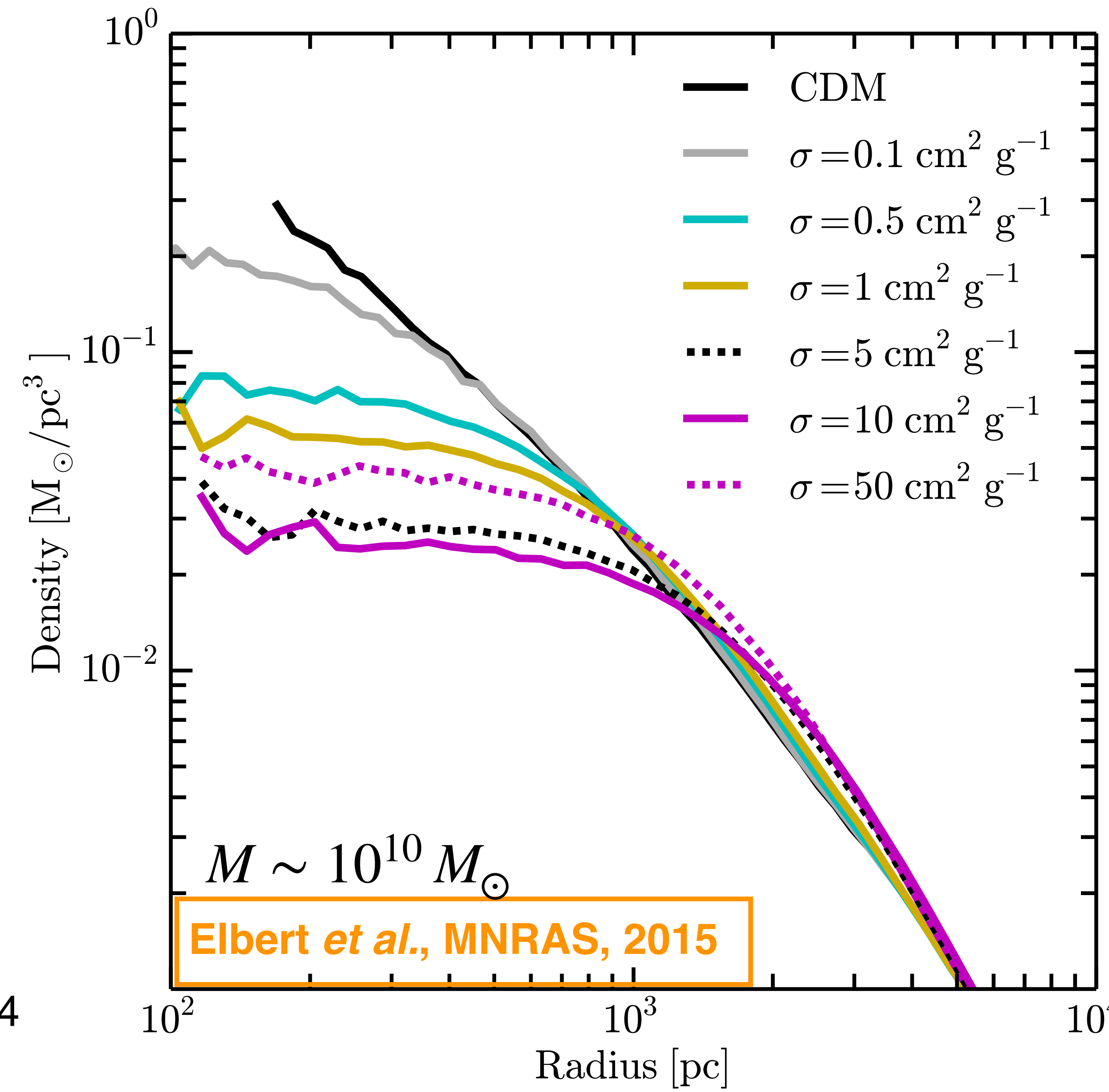
The issues may be attributed to incomplete understanding of complex astrophysical processes (subgrid physics)

The issues may indicate alternatives to CDM (WIMPs)

- self-interacting dark matter (SIDM) $\sigma/m \sim 1\,\mathrm{cm}^2/\mathrm{g}$

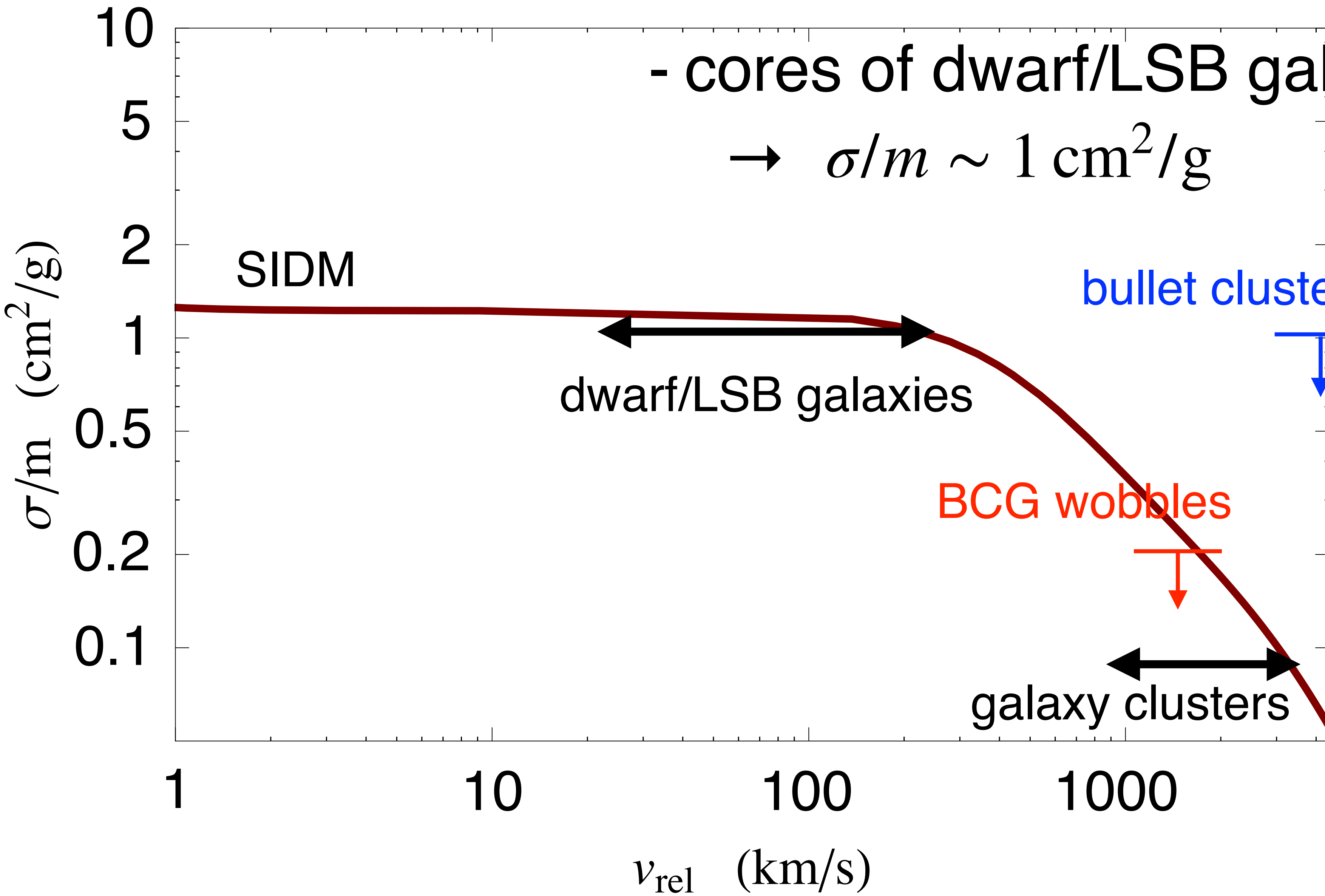
- reduce the central mass density





Dark matter halos as particle colliders

SIDM cross section inferred by a variety of halos diminishes toward higher velocity



- cores of dwarf/LSB galaxies $M \sim 10^{10\text{-}11}\, M_\odot$

$\rightarrow\ \sigma/m \sim 1\,\mathrm{cm^2/g}$

- offset of bullet cluster

$\rightarrow\ \sigma/m \lesssim 1\,\mathrm{cm^2/g}$

Randall *et al.*, ApJ, 2008

- offsets of brightest cluster galaxy (BCG)

$\rightarrow\ \sigma/m \lesssim 0.2\,\mathrm{cm^2/g}$

Harvey, Robertson, Massay, and McCarthy, MNRAS, 2019

- cores in galaxy clusters $M \sim 10^{14\text{-}15}\, M_\odot$

$\rightarrow\ \sigma/m \sim 0.1\,\mathrm{cm^2/g}$

Kaplinghat, Tulin, and Yu, PRL, 2016

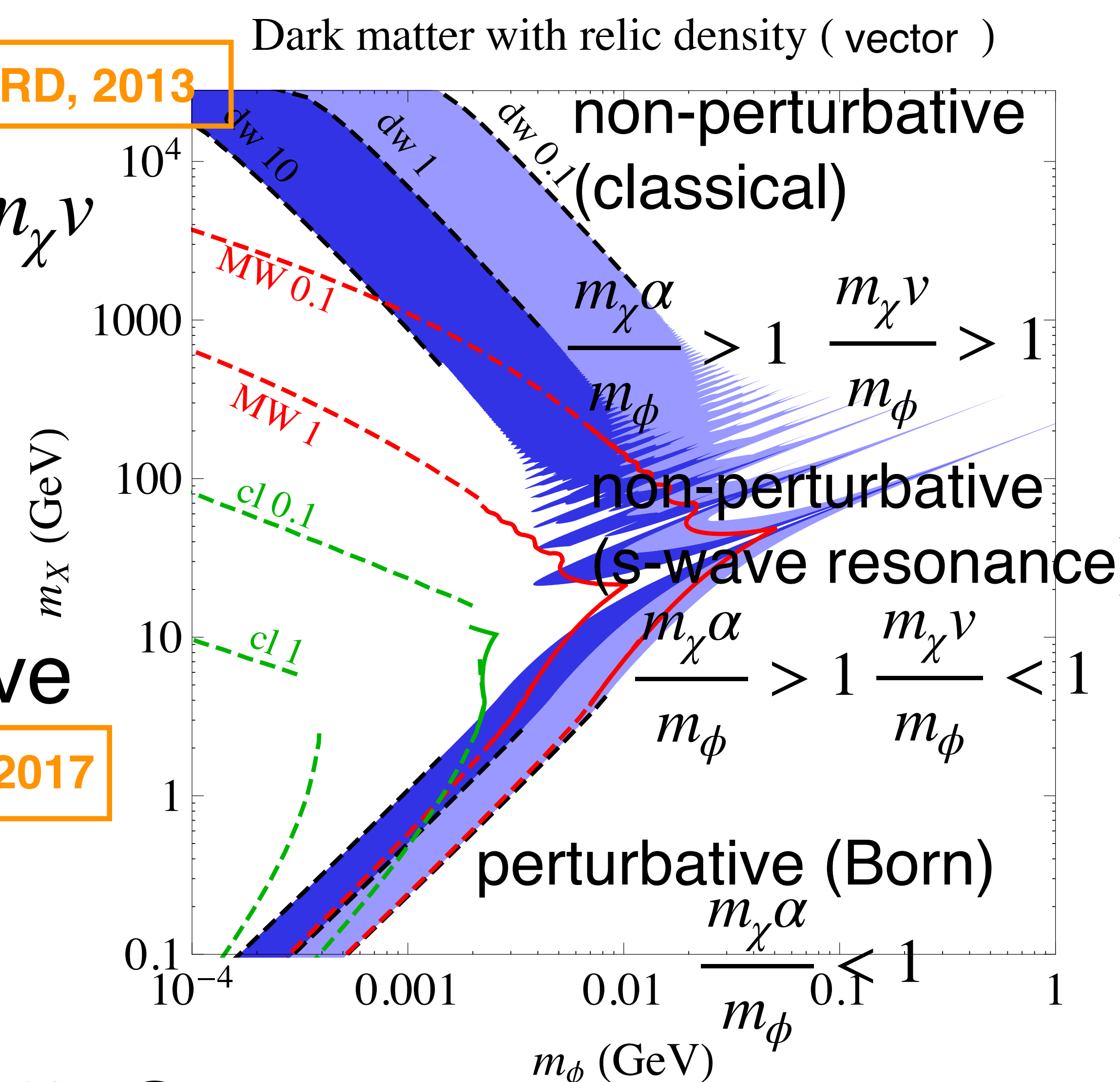
Origin of velocity dependence

Light mediator $m_\phi < m_\chi$ **Tulin, Yu, and Zurek, PRD, 2013**

- suppressed at high velocity $m_\phi < m_\chi v$
  like the Rutherford scattering

- visible decay of ϕ
  ❌ indirect detection experiments
  if χ annihilates into ϕ via s-wave

Bringmann, Kahlhoefer, Schmidt-Hoberg, and Walia, PRL, 2017



- hierarchical two scales

QCD $\sigma_{\rm SIDM}/m(v) \simeq \sigma_{n\text{-}p}/m/10(v/10)$

- previous figure

$$k \cot \delta \simeq -\frac{1}{a} + \frac{1}{2}k^2 r \quad -a \gg r$$

- deuteron

→ composite asymmetric DM

Steve's talk

$\Lambda_{\rm QCD'} \sim \Lambda_{\rm QCD}$

Ibe, AK, Kobayashi, Kuwahara, and Nakano, PRD, 2019

Others

- puffy dark matter

Chu, Garcia-Cely, and Murayama, arXiv:1901.00075

- resonant dark matter

Chu, Garcia-Cely, and Murayama, PRL, 2019

Exothermic dark matter

Dark fusion - $\boxed{\text{dark nuclei}}$ McDermott, PRL, 2018

- eXciting dark matter (XDM)

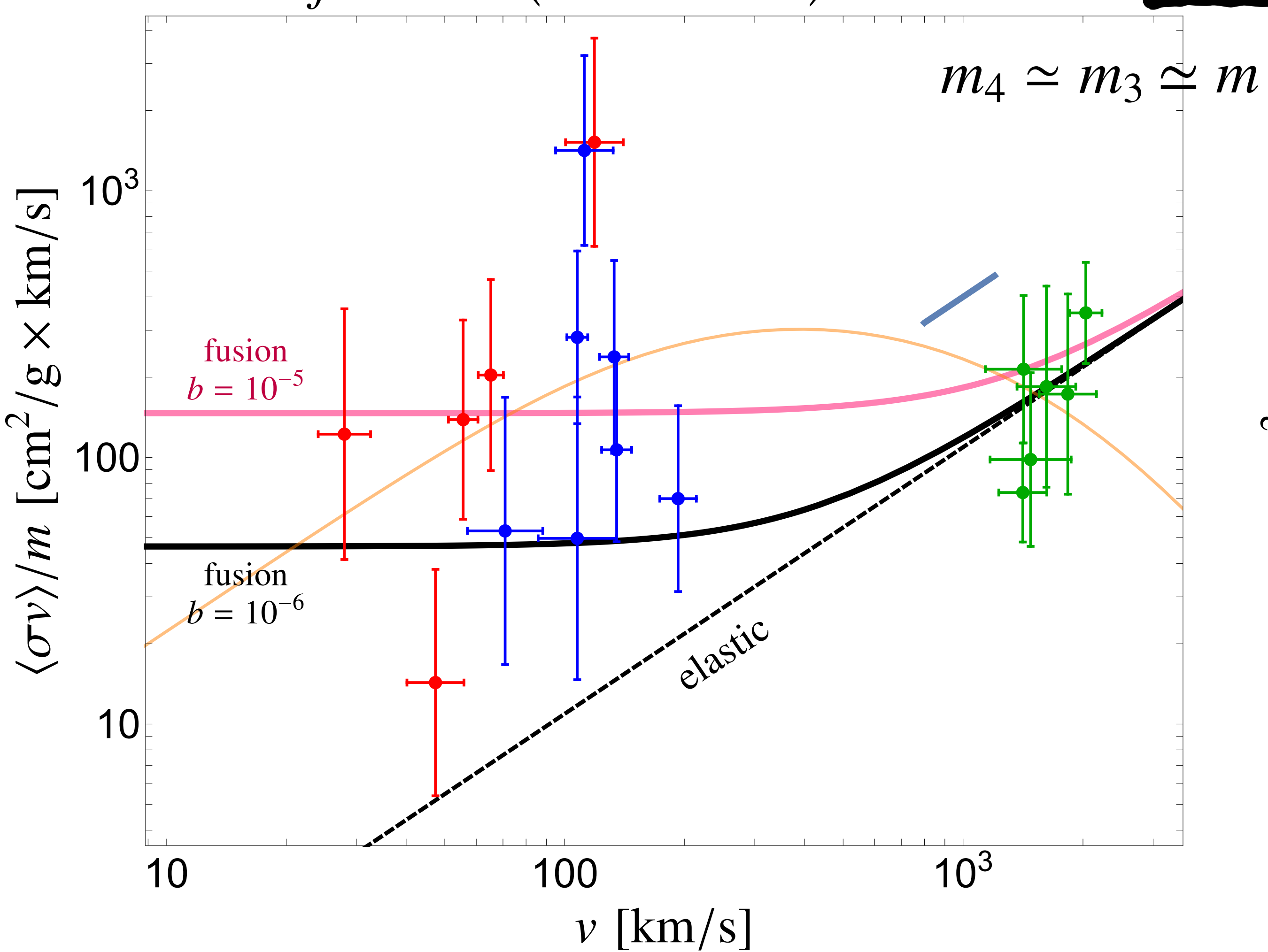
Loeb and Weiner, PRL, 2011

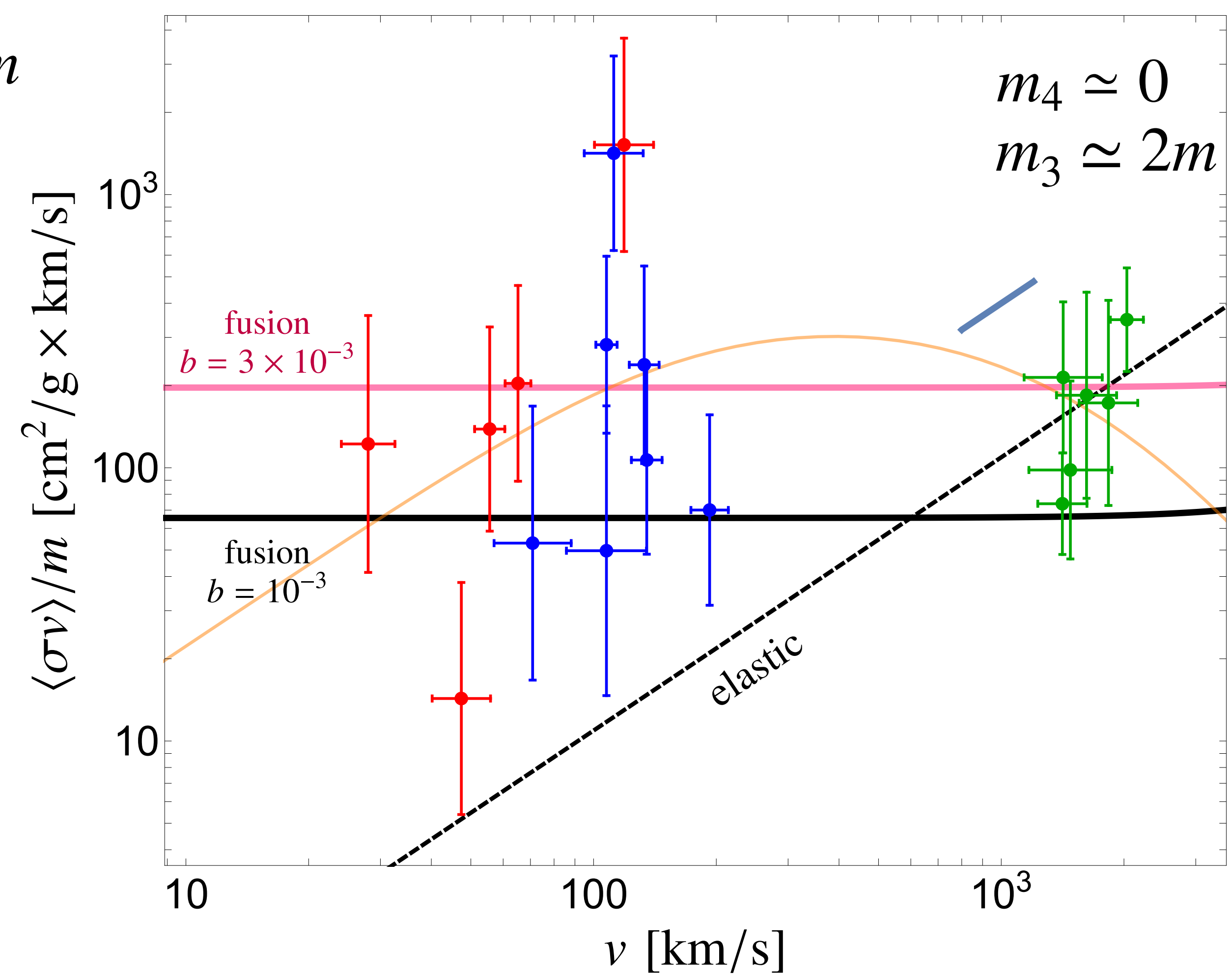
Schutz and Slatyer, JCAP, 2015

$$\boxed{1} + \boxed{2} \to \boxed{3} + \boxed{4} \qquad \Delta = 1 - \frac{m_3 + m_4}{m_1 + m_2} > 0$$

$$\frac{d\sigma}{d\Omega} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s}\frac{|p_f|}{|p_i|} \qquad p_i \propto v \qquad 0 < \Delta \ll 1 \qquad m_1 \simeq m_2 \simeq m$$

$$p_f \simeq m\left(2\Delta + v^2\right)^{1/2} \qquad \boxed{\Delta \text{ vs } v^2} \qquad p_f \simeq m\left(2\Delta + v^2\right)$$





Inelastic self-interacting dark matter

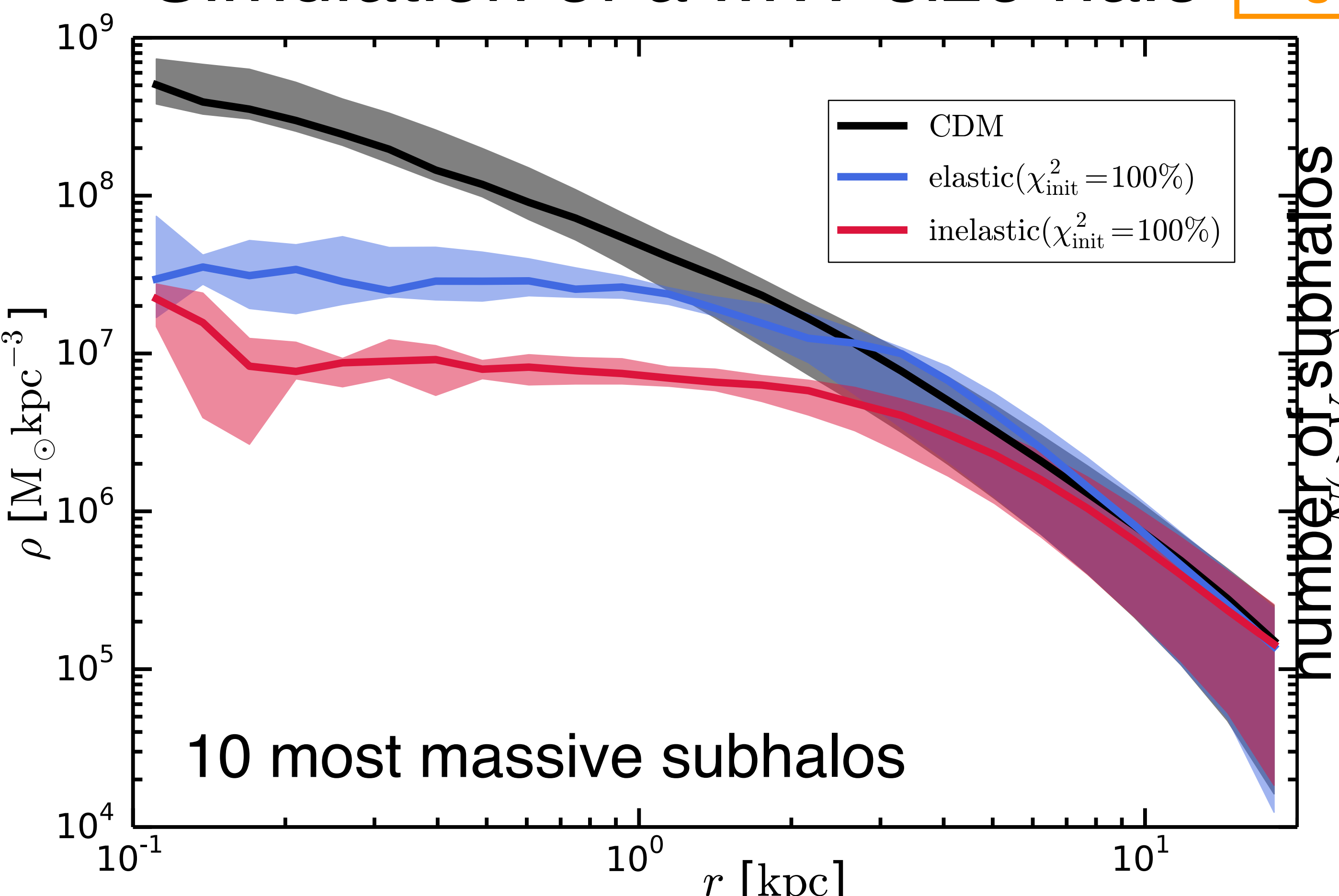
One more important effect: heating - make a difference from pure SIDM

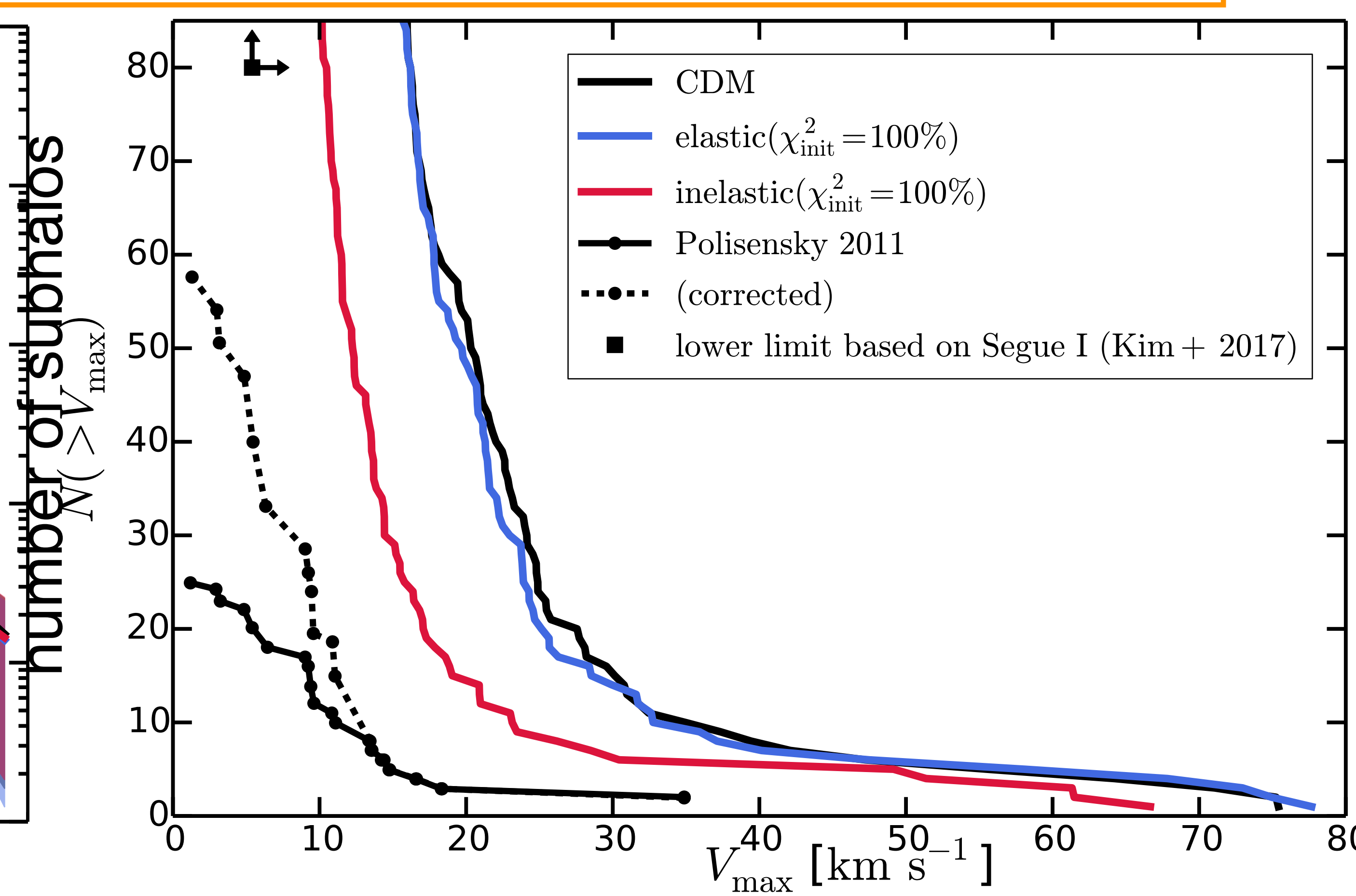
$1 + 2 \rightarrow \boxed{3} + 4$

- boosted w/ kinetic energy $\sim \Delta$

→ DM evaporation from (the inner region of) a halo

Simulation of a MW-size halo **Vogelsberger, Zavala, Schutz and Slatyer, MNRAS, 2019**





$\boxed{\Delta \text{ vs } v^2}$ - bigger impact on a smaller-size halo

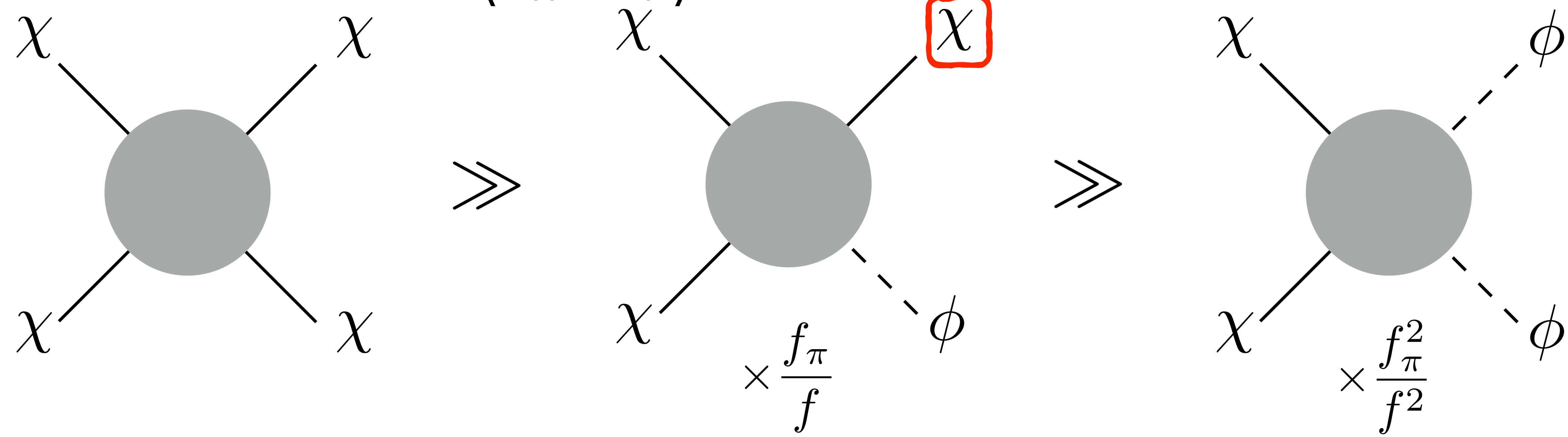
- difficult to cover a wide range w/ a simulation

Self-heating dark matter: halos

Self-heating dark matter

Self-heating dark matter (SHDM):
elastic self-scattering + semi-annihilation

$\sigma_{\rm self}/m \sim 1\,{\rm cm}^2/{\rm g}$ $\quad \langle \sigma_{\rm semi} v_{\rm rel} \rangle = 6 \times 10^{-26}\,{\rm cm}^3/{\rm s}$



e.g. variant of strongly-interacting massive particles (SIMPs)

- hidden pions in a hidden confinement sector

Hochberg, Kuflik, Murayama, Volansky, and Wacker, PRL, 2015

- unbroken flavor symmetry → the stability of pions

- η'-axion-like particle (ALP) mixing through the anomalous coupling → freeze-out through semi-annihilation

AK, Kim, and Sekiguchi, PRD, 2017

Gravothermal evolution

DM fluid: Lagrangian picture

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho$$ - equation of continuity

$$\frac{\partial \left(\rho \nu^2\right)}{\partial r} + \frac{GM\rho}{r^2} = 0$$ - hydrostatic equilibrium

$$\frac{3}{\nu}\left(\frac{\partial \nu}{\partial t}\right)_M - \frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_M = \frac{1}{\nu^2}\boxed{\frac{\delta u}{\delta t}}$$ - energy conservation

- nature of DM

c.f. Chu and Garcia-Cely, JCAP, 2018

Eulerian picture

only semi-annihilation

$\nu^2(r,t) = \text{const}\,.$

Initial condition: NFW profile

$$\rho_{\text{NFW}} = \frac{\rho_s}{r/r_s(1+r/r_s)^2}$$ w/ concentration-mass relation $\leftrightarrow$ ρ_s - r_s

Dutton and Macciò, MNRAS, 2014

Self-heating: energy deposit

Pure SIDM - heat conduction

Koda and Shapiro, MNRAS, 2011

$$\frac{\delta u_{\rm cond}}{\delta t} = -\frac{1}{4\pi r^2 \rho}\frac{\partial L}{\partial r} \qquad \frac{L}{4\pi r^2} = -\kappa m \frac{\partial}{\partial r}\nu^2 \qquad \kappa^{-1} = \kappa_{\rm SMFP}^{-1} + \kappa_{\rm LMFP}^{-1}$$

- conductivity

$$\kappa_{\rm SMFP} = \frac{75\sqrt{\pi}}{64}\frac{\rho\lambda^2}{a m t_{\rm self}} \qquad t_{\rm self} = \frac{m}{a\rho\sigma_{\rm self}\nu} \qquad a = \sqrt{\frac{16}{\pi}}$$

- collisional limit $\lambda \ll H$

$$\kappa_{\rm LMFP} = \frac{3}{2}C\frac{\rho H^2}{m t_{\rm self}} \qquad C = 0.75$$

- free-streaming limit $\lambda \ll H$

$$\lambda = \frac{m}{\sigma_{\rm self}\rho}$$

- free-streaming length

$$H = \sqrt{\frac{\nu^2}{4\pi G\rho}}$$

- Jeans (gravitational) length

Energy deposit through semi-annihilation

$$\frac{\delta u_{\rm semi}}{\delta t} = \frac{\rho\left\langle\sigma_{\rm semi}v_{\rm rel}\right\rangle}{m}\frac{\xi\,\delta E}{m} \qquad \delta E = \frac{m}{4}$$

$$\xi = b \times \frac{r}{\lambda}\bigg|_{r=r_s}$$

- deposit efficiency w/ a fudge factor b

$$\simeq 0.0002\, b\left(\frac{r_s}{3.43\ {\rm kpc}}\right)\left(\frac{\rho_s}{0.011\ M_\odot/{\rm pc}^3}\right)\left(\frac{\sigma_{\rm self}/m}{0.1\ {\rm cm}^2/{\rm g}}\right)$$

Sample halos

SHDM $\delta u = \delta u_{\rm semi} + \delta u_{\rm cond}$

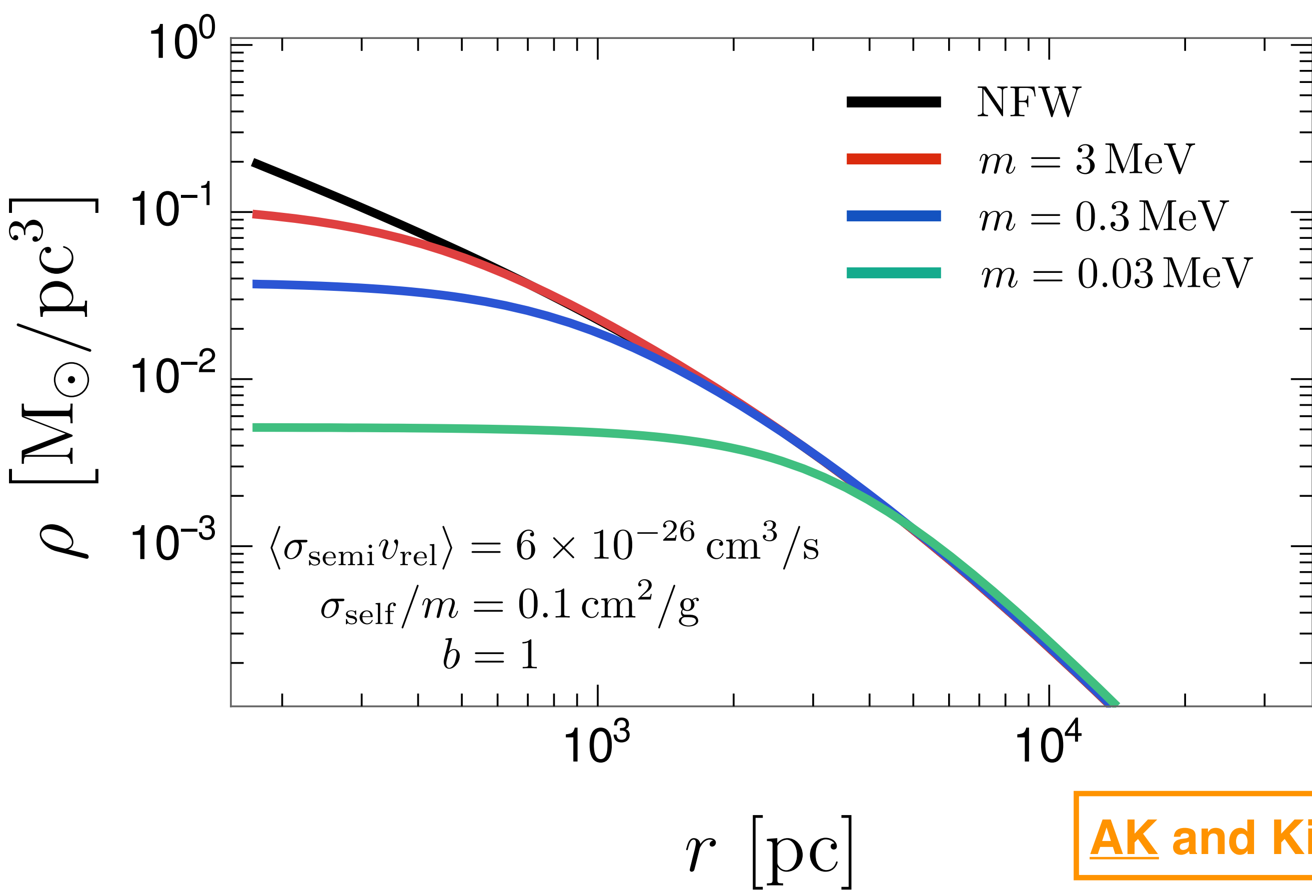
$$\frac{1}{\nu^2}\frac{\delta u_{\rm semi}}{\delta t} = \frac{\rho\left\langle \sigma_{\rm semi} v_{\rm rel}\right\rangle}{m}\frac{\xi \delta E}{m\nu^2} = \left(\frac{\nu_{\rm NFW}(r_s)}{\nu}\right)^2 \left(\frac{\rho}{\rho_{\rm NFW}(r_s)}\right)\frac{1}{t_{\mathscr{J}}}$$

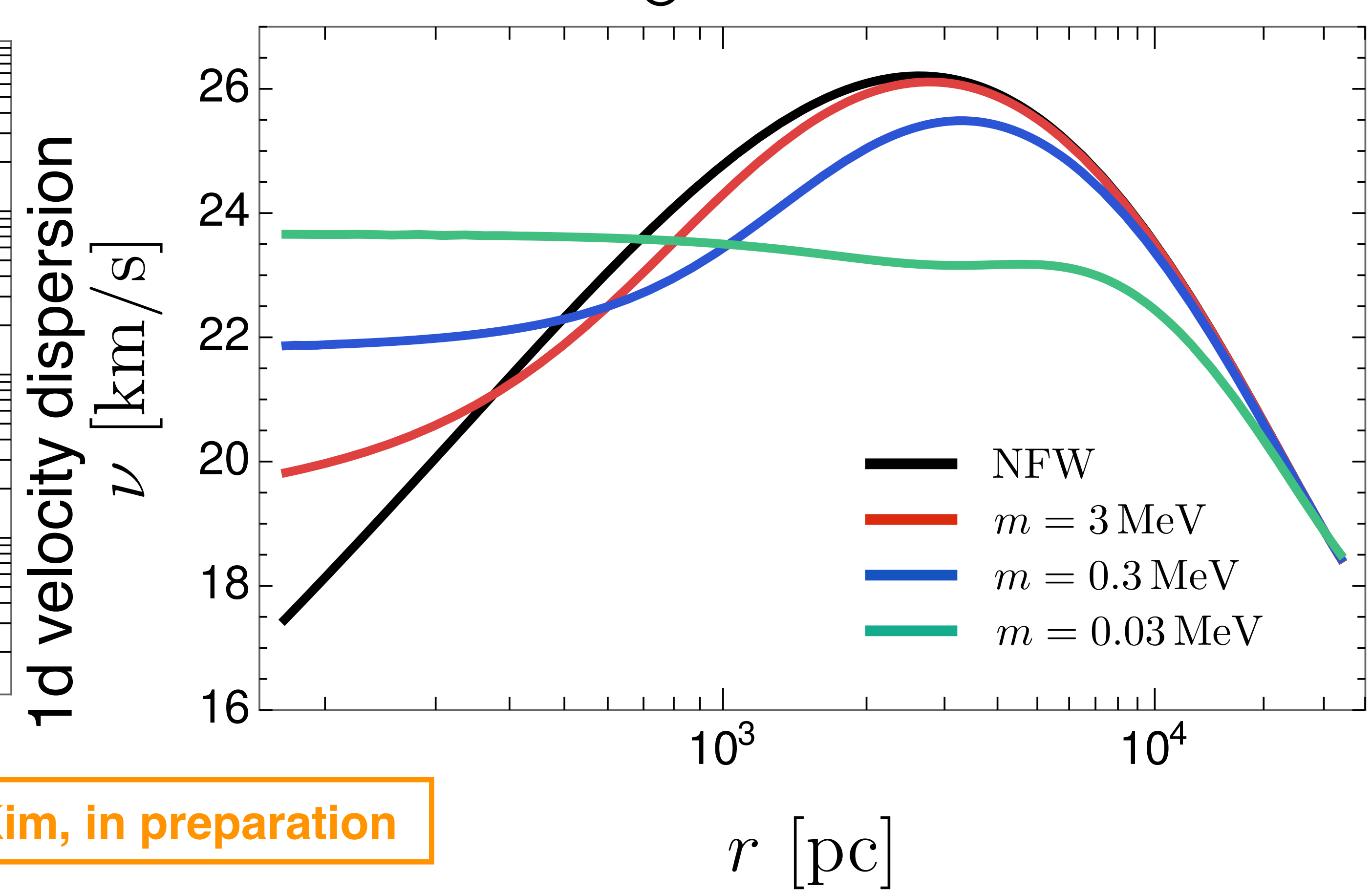
- heating time scale

$$t_{\mathscr{J}} \simeq \frac{17\,{\rm Gyr}}{b}\left(\frac{M_{200}}{10^{10}\,{\rm M}_\odot}\right)^{0.68}\left(\frac{m}{0.1\,{\rm MeV}}\right)\left(\frac{6\times 10^{-26}{\rm cm}^3/{\rm s}}{\left\langle \sigma_{\rm semi} v_{\rm rel}\right\rangle}\right)\left(\frac{0.1\,{\rm cm}^2/{\rm g}}{\sigma_{\rm self}/m}\right)$$

For given $\sigma_{\rm self}/m$ and $\left\langle \sigma_{\rm semi} v_{\rm rel}\right\rangle$, smaller m has a bigger impact

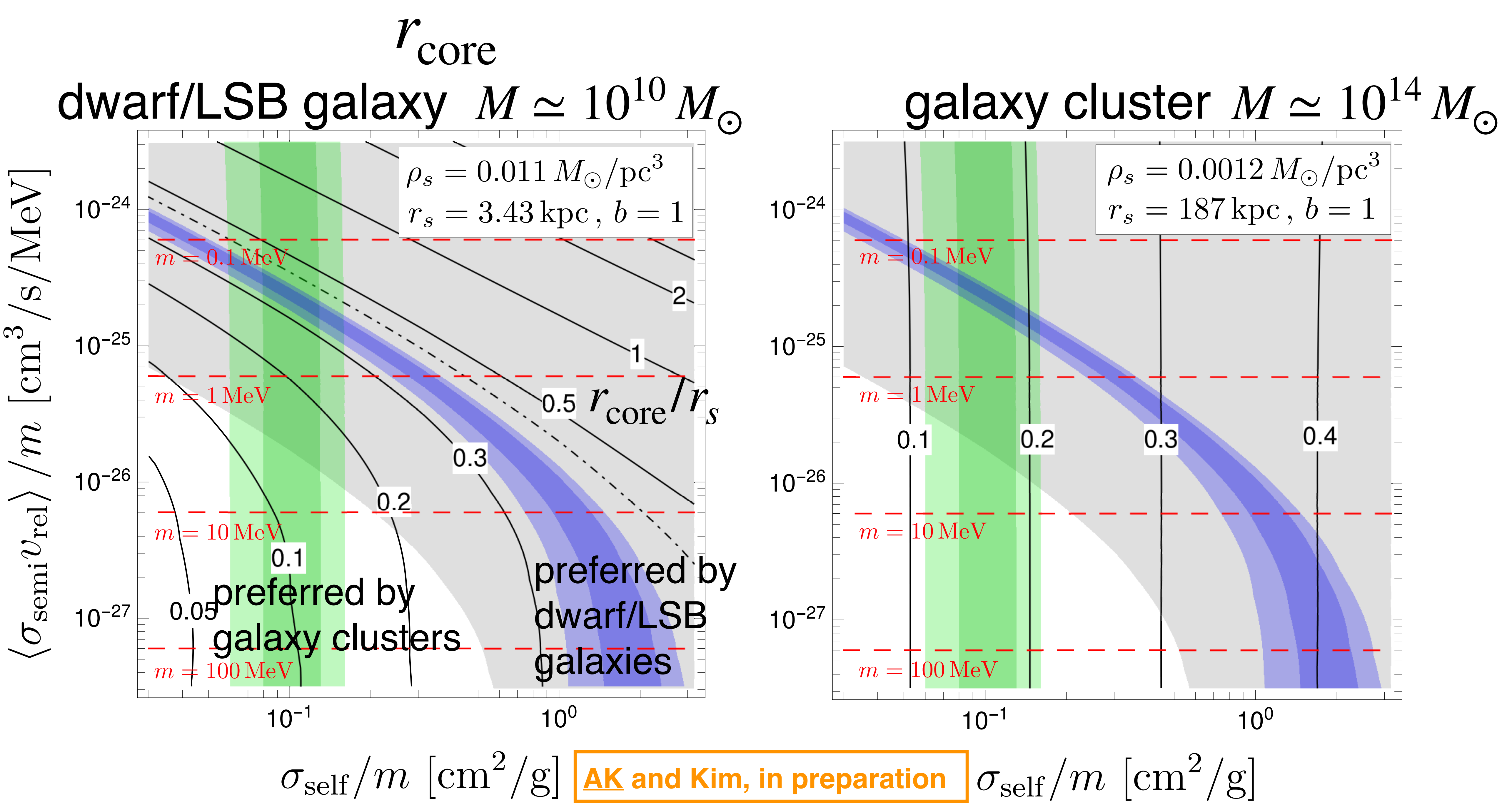
dwarf/LSB galaxy $M \simeq 10^{10}\,M_\odot$





AK and Kim, in preparation

Core size - cross sections



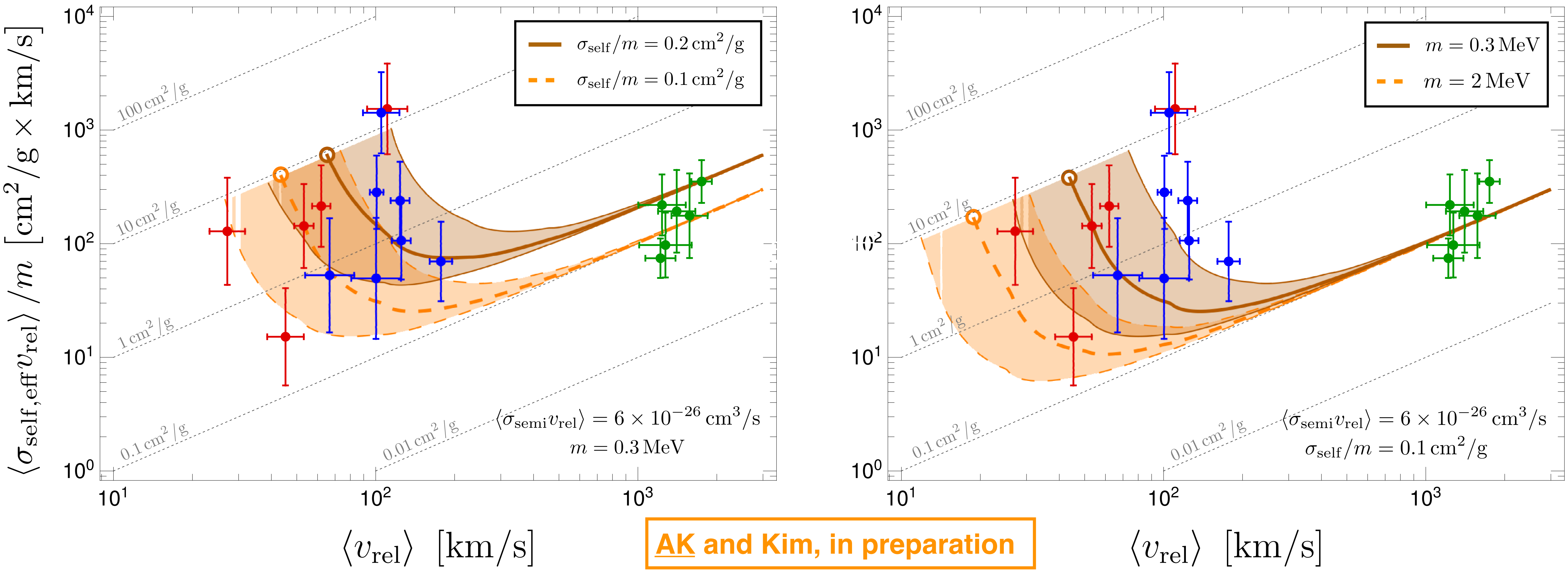
Given $r_{\rm core}$, smaller $\sigma_{\rm self}/m$ for larger $\langle \sigma_{\rm semi} v_{\rm rel} \rangle / m$, i.e., smaller m

Preferred parameter: $\sigma_{\rm self}/m \sim 0.1\,\mathrm{cm}^2/\mathrm{g}$ $\ m \sim O(1)\,\mathrm{MeV}$

Effective velocity dependence

Effective self-scattering cross section $\sigma_{\rm self,eff}/m$

defined so that $r_{\rm core}({\rm SHDM}) = r_{\rm core}(\text{pure SIDM})$



AK and Kim, in preparation

Sharp increase toward a smaller-size halo

SHDM ↔ pure SIDM? No!

No solution to $r_c({\rm SHDM}) = r_c(\text{pure SIDM})$ for $\sigma_{\rm self,eff}/m > 10\,{\rm cm^2/g}$

SHDM ≠ pure SIDM

Two regimes of pure SIDM

Ann and Shapiro, MNRAS, 2005

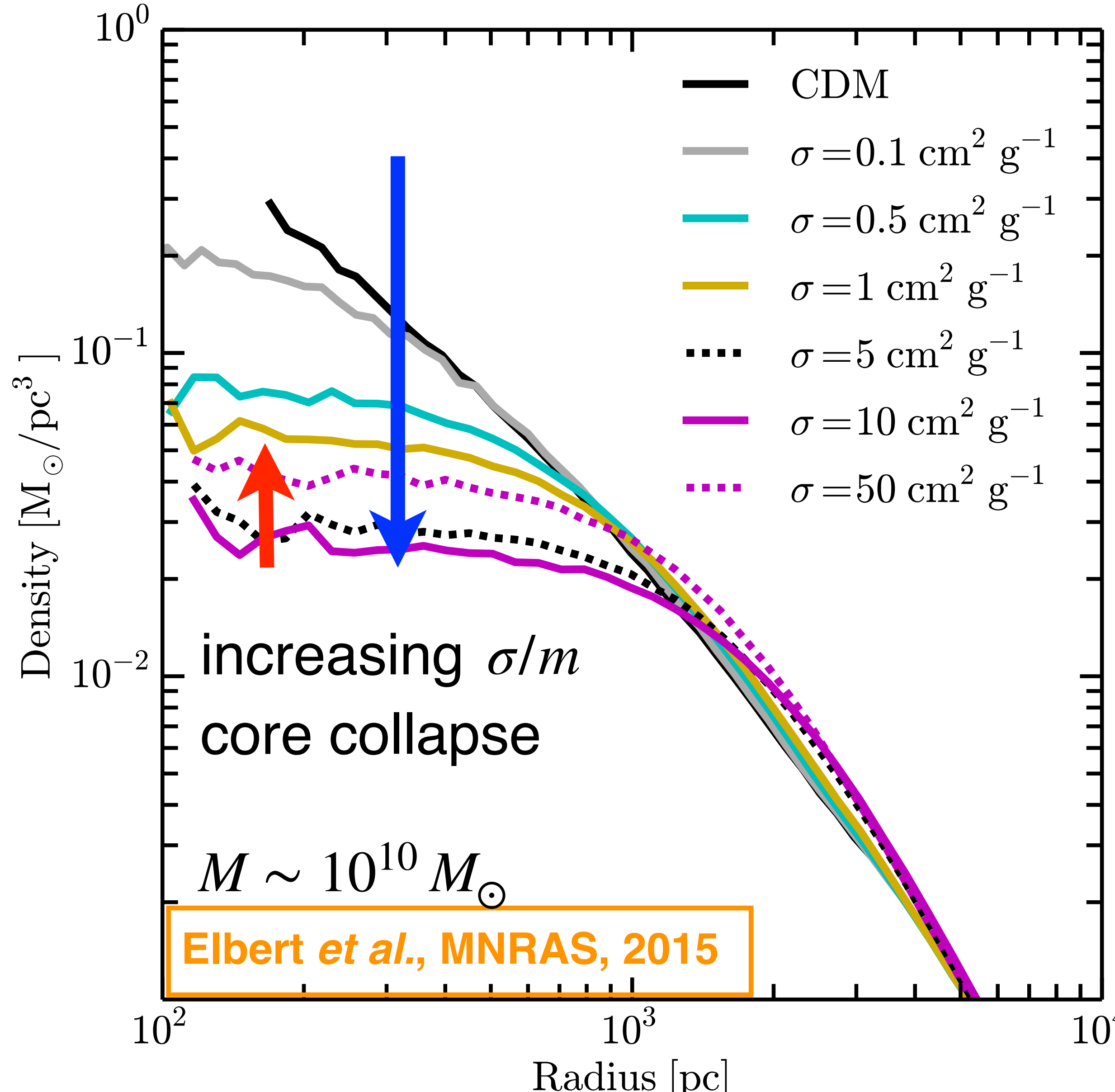
- free-streaming limit $\lambda \gg H$

  decreasing σ/m → CDM

- collisional limit $\lambda \ll H$

  increasing σ/m → CDM!

→ maximal core size $r_{\rm core} \lesssim 0.5 r_s$



Monotonically escalating impacts toward smaller-size halos in SHDM

- heating

$$\rho_{\rm core}(t) \sim \rho_s \cdot \left(t_{\mathcal{J}}/t\right) \quad r_{\rm core}(t) \simeq r_s \cdot \left(t/t_{\mathcal{J}}\right)^{0.57}$$

$$t_{\mathcal{J}} \simeq \frac{17\,\mathrm{Gyr}}{b}\left(\frac{\boxed{M_{200}}}{10^{10}\,\mathrm{M_\odot}}\right)^{0.68}\left(\frac{m}{0.1\,\mathrm{MeV}}\right)\left(\frac{6\times10^{-26}\mathrm{cm^3/s}}{\langle\sigma_{\rm semi}v_{\rm rel}\rangle}\right)\left(\frac{0.1\,\mathrm{cm^2/g}}{\sigma_{\rm self}/m}\right)$$

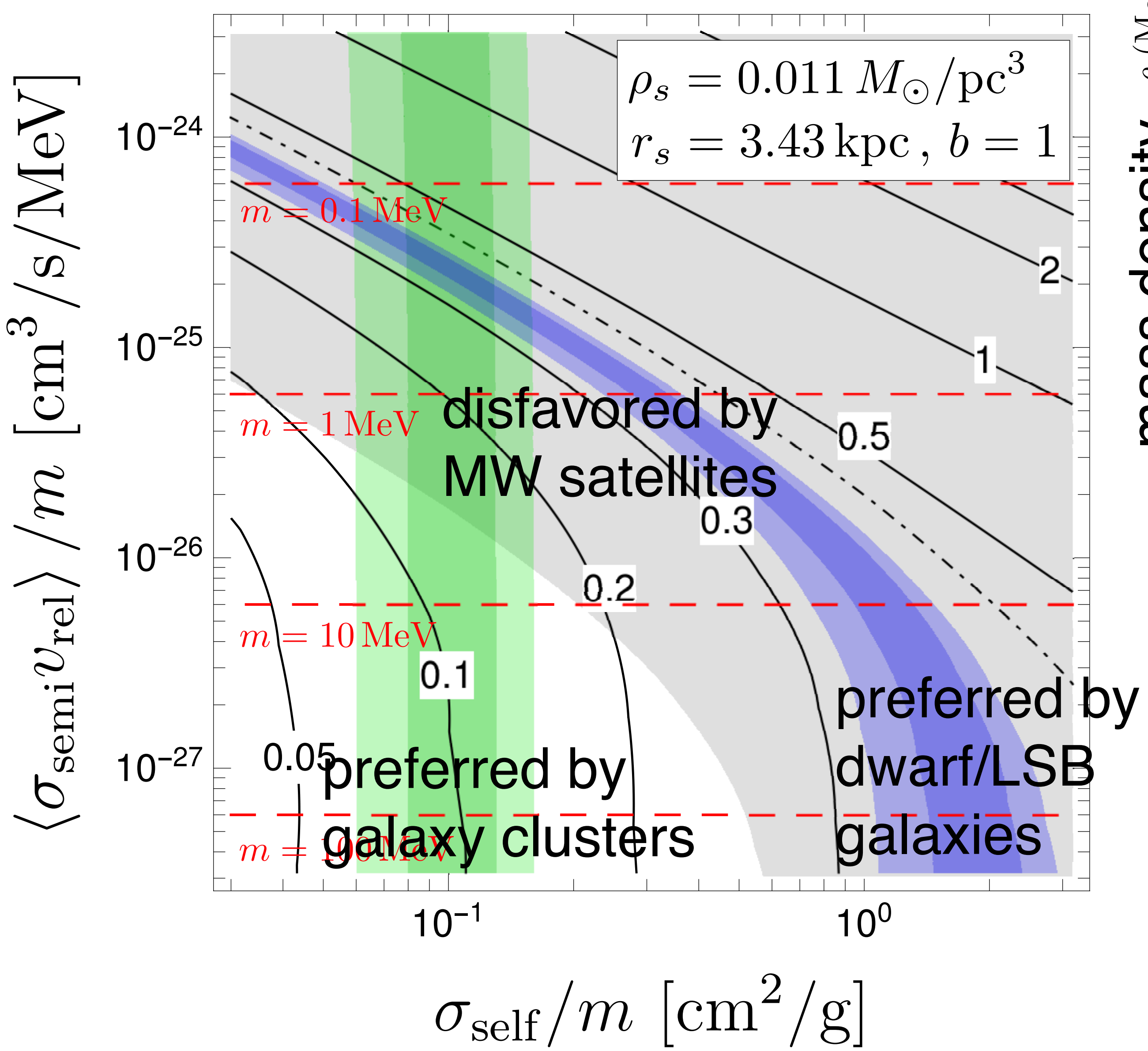
- differentiate SHDM and pure SIDM by smaller-size halos

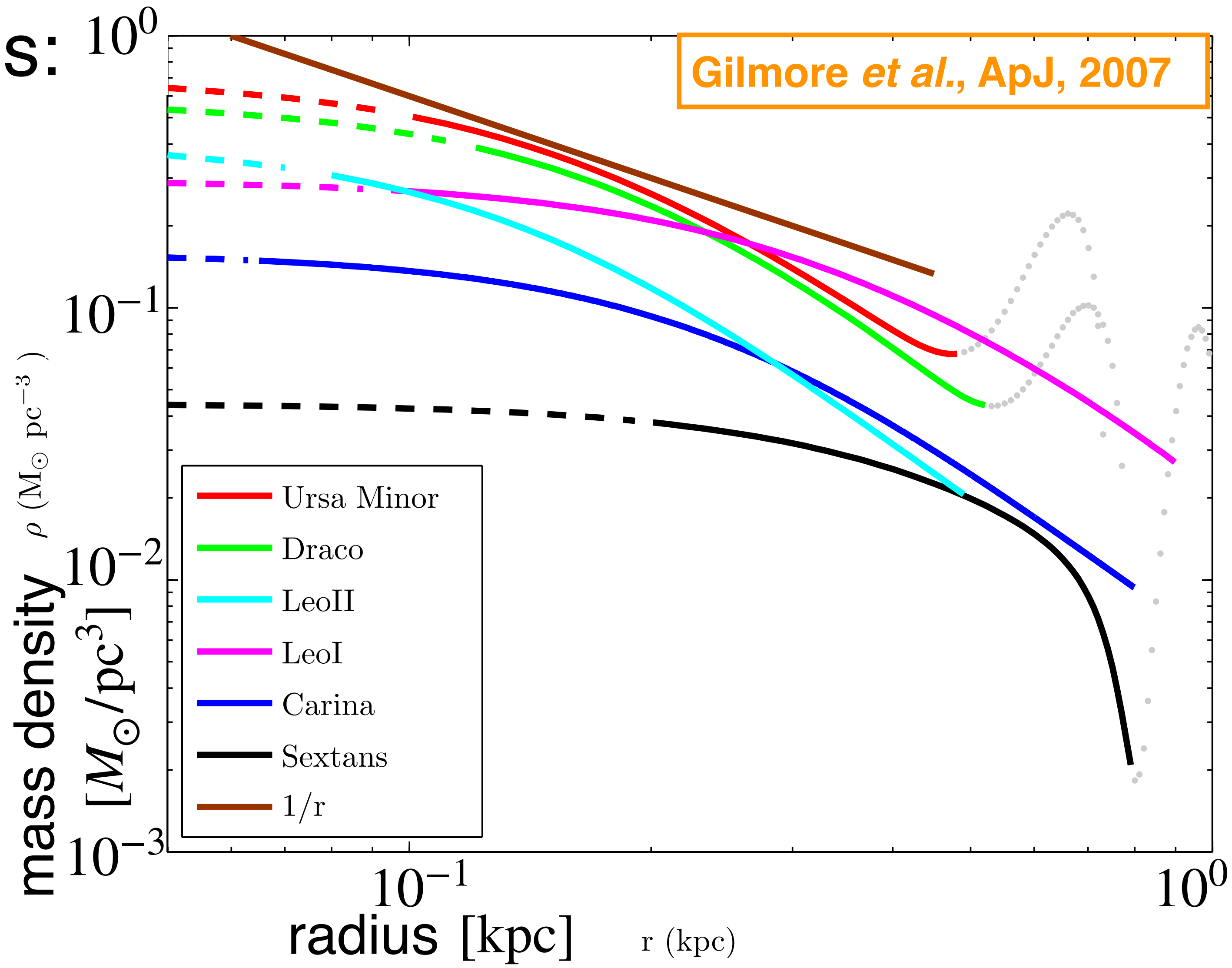
Milky-Way satellites

Infall (before tidal-stripping) mass:

$M \sim 10^9\, M_\odot$ Wolf *et al.*, MNRAS, 2010

Disfavored if $\rho_{\rm core} < \rho_{\rm core,obs}$

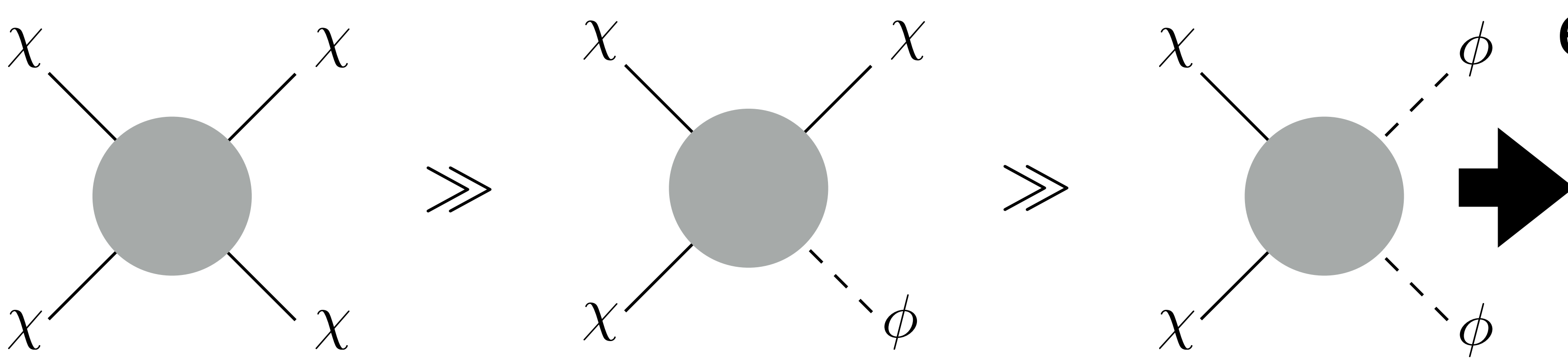




Self-heating (or generically exothermic) dark matter could not explain MW satellites, dwarf/LSB galaxies, and galaxy clusters at the same time

Self-heating dark matter: early Universe

Co-evolution equations



efficient self-scattering

$$f_\chi = \frac{n_\chi}{n_\chi^{\rm eq}(T_\chi)} \exp(-E_\chi/T_\chi)$$

Co-evolution equations: AK, Kim, Kim, and Sekiguchi, PRL, 2018

$$T_\chi = T_\phi$$

$$\dot{n}_\chi + 3Hn_\chi = -n_\chi \langle \sigma_{\rm semi} v \rangle_{T_\chi T_\chi} \left[n_\chi - \mathcal{J}(T_\chi, T_\phi) n_\chi^{\rm eq}(T_\chi) \right]$$

Boltzmann equation for WIMP freeze-out

$$\dot{T}_\chi + 3HT_\chi \left(\frac{T_\chi}{\sigma_E} \right)^2 = -\left(\frac{T_\chi}{\sigma_E} \right)^2 \frac{n_\phi^{\rm eq}(T_\chi)}{n_\chi^{\rm eq}(T_\chi)} \langle \Delta E \sigma_{\rm inv} v \rangle_{T_\chi, T_\phi = T_\chi} \times \left[n_\chi - n_\chi^{\rm eq}(T_\chi) \mathcal{K}(T_\chi, T_\phi) \right]$$

- adiabatic cooling

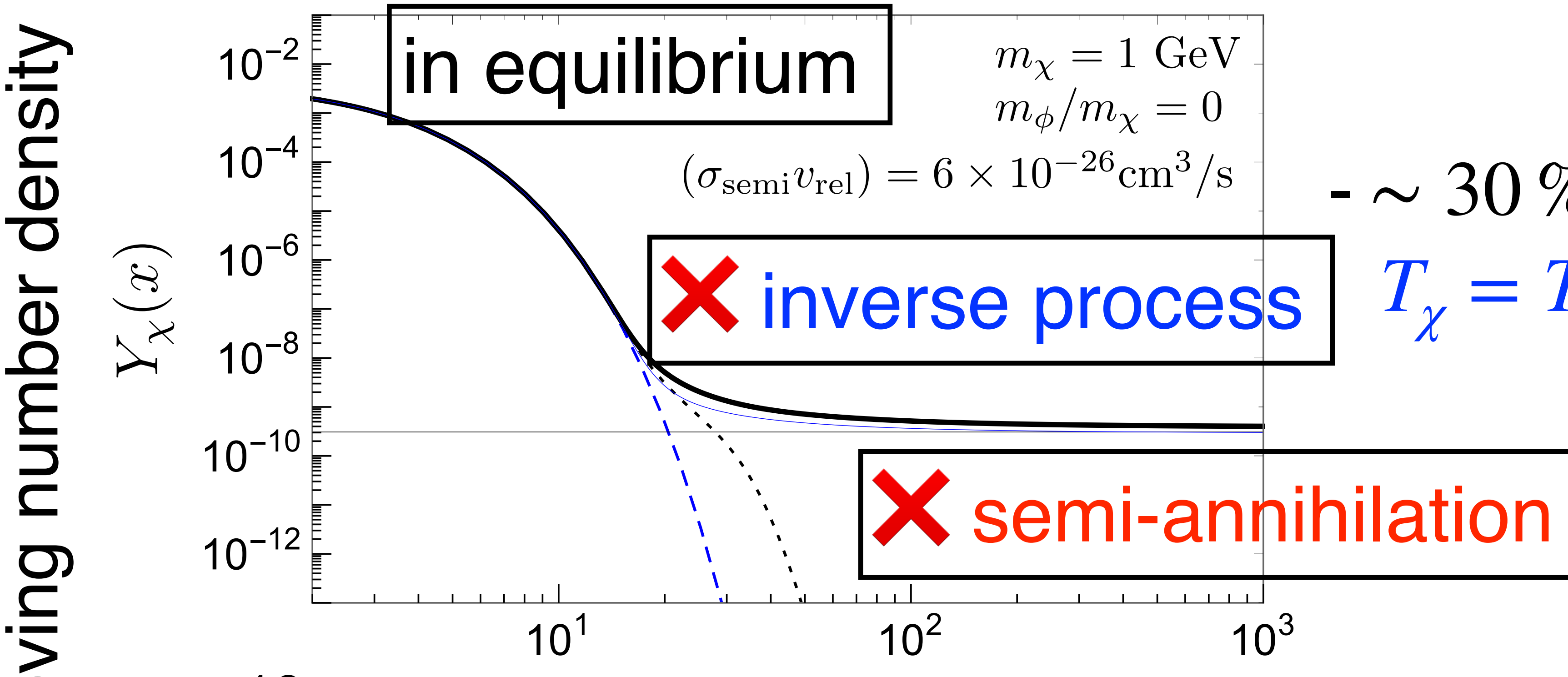
- heating through semi-annihilation

relativistic: $\sigma_E^2 = 3T_\chi \rightarrow T_\chi \propto 1/a$

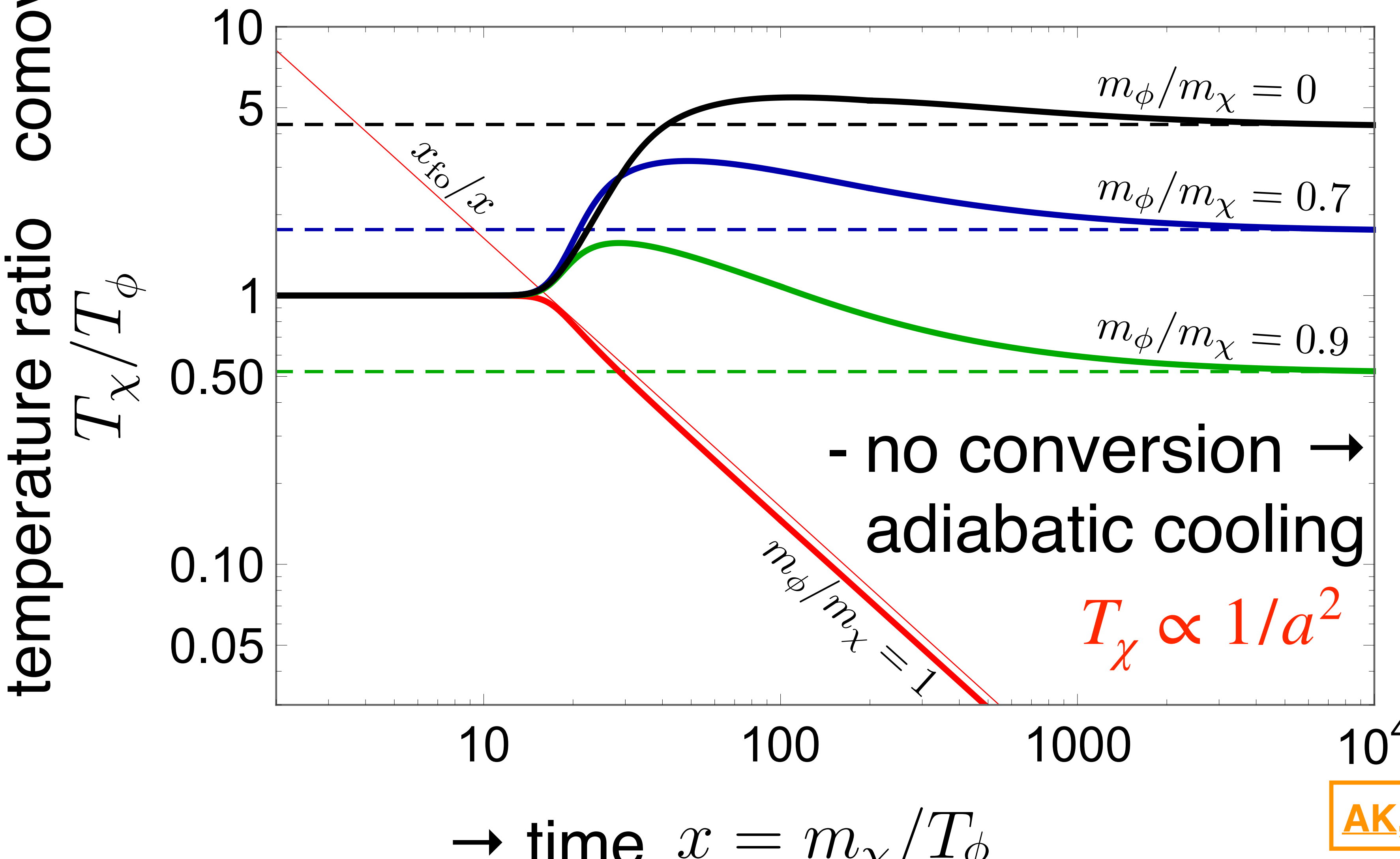
non-relativistic: $\sigma_E^2 = 3/2T_\chi \rightarrow T_\chi \propto 1/a^2$

$$\sigma_E^2 = \langle E_\chi^2 \rangle - \langle E_\chi \rangle^2$$

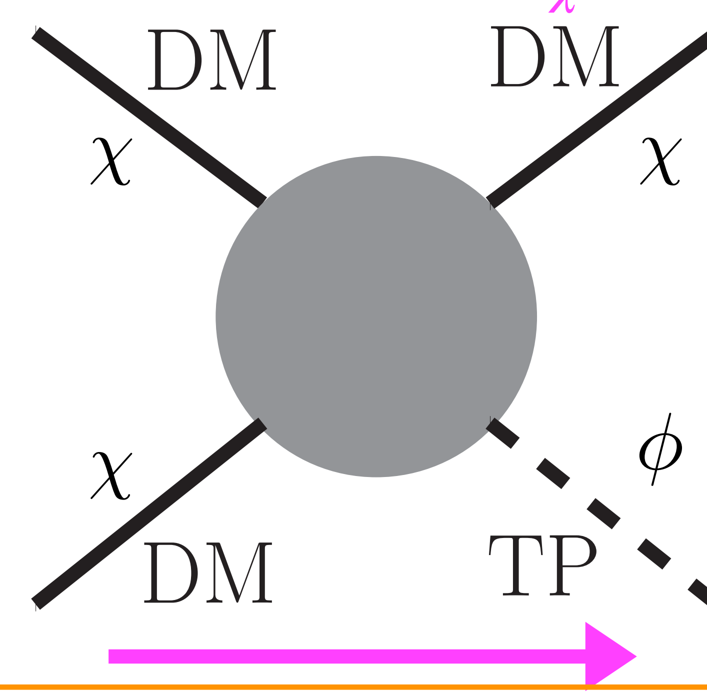
Freeze-out



- $\sim 30\,\%$ difference from $T_\chi = T_\phi \propto 1/a$

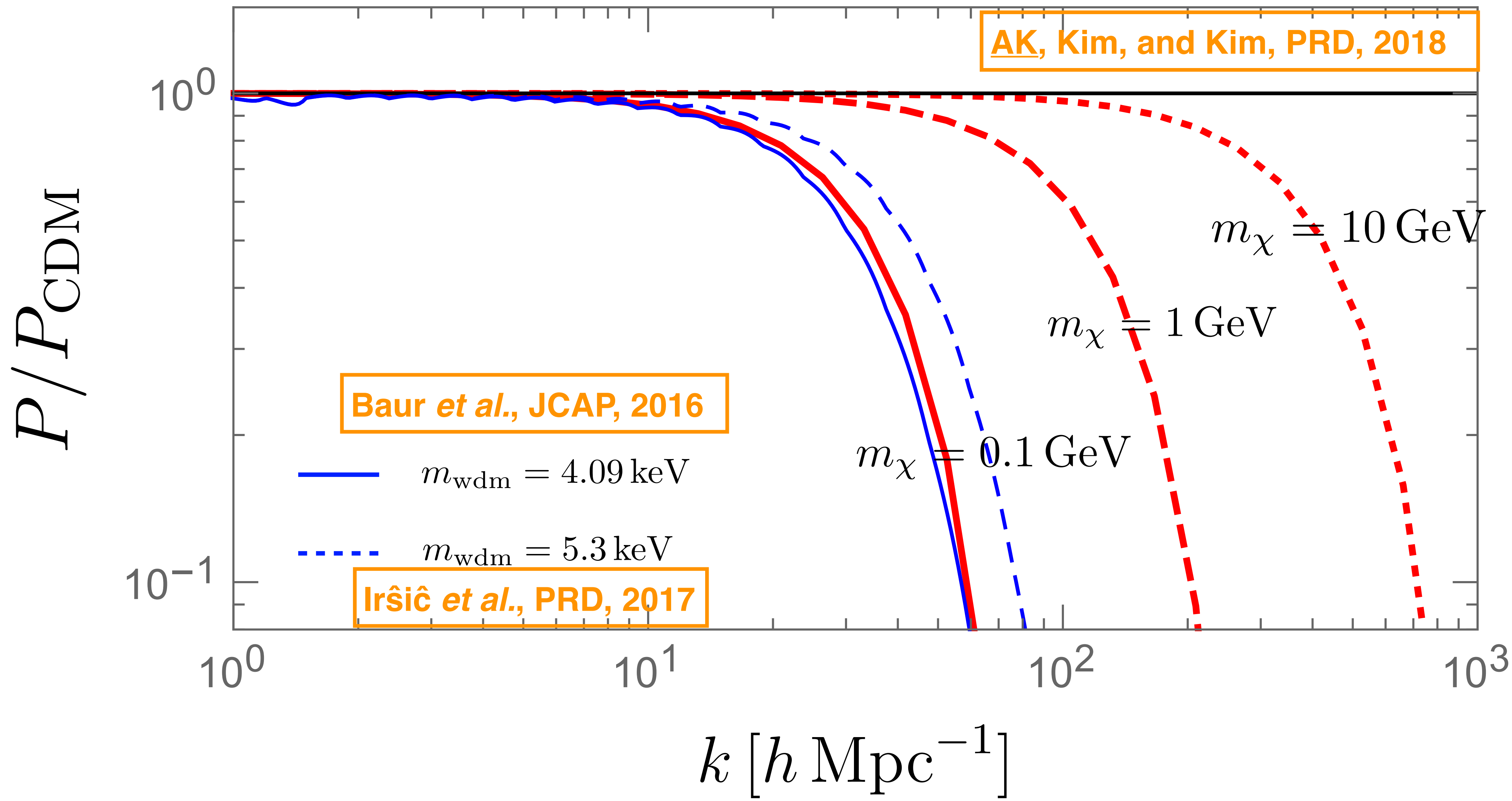


- mass deficit converted into the kinetic energy → self-heating $T_\chi \propto 1/a$



AK, Kim, Kim, and Sekiguchi, PRL, 2018

Linear matter power spectrum



Self-heating ceases when self-scattering becomes inefficient $T_{\rm self}$

Warmness (WDM) is related with self-scattering (SIDM)

$$\frac{m_{\rm wdm}}{5.3\,\mathrm{keV}} \simeq 3\left(\frac{m_\chi}{1\,\mathrm{GeV}}\right)^{3/8} \max\left(1,\sqrt{\frac{T_{\rm self}}{T_{\rm eq}}}\right)^{3/4}\left(\frac{T_\chi}{T}\right)^{-3/8}_{\rm asy}$$

Summary

Velocity-dependent (pure) SIDM is an interesting possibility

- $\sigma_{\rm self}/m \sim 0.1\,{\rm cm^2/g}$ at galaxy clusters $M \sim 10^{14\text{-}15}\,M_\odot$ $v \sim 10^3\,{\rm km/s}$
- $\sigma_{\rm self}/m \sim 1\,{\rm cm^2/g}$ at dwarf/LSB galaxies $M \sim 10^{10\text{-}11}\,M_\odot$ $v \sim 10^2\,{\rm km/s}$

Self-heating (or generically exothermic) dark matter

- velocity dependent cross section + heating
- sharply increasing impact on a smaller-size halo
  - ⭕ galaxy clusters, dwarf/LSB galaxies $\sigma_{\rm self}/m \sim 0.1\,{\rm cm^2/g}$
  - ⚠️ (or ❌) MW satellites $M \sim 10^9\,M_\odot$ $m \sim O(1)\,{\rm MeV}$
    - worth studying in more detail $\langle \sigma_{\rm semi} v_{\rm rel} \rangle = 6 \times 10^{-26}\,{\rm cm^3/s}$
- suppress the linear matter power spectrum

Structure formation of the Universe provides a good (although indirect) bottom-up test on the nature of DM

Thank you for your attention

Missing satellite problem

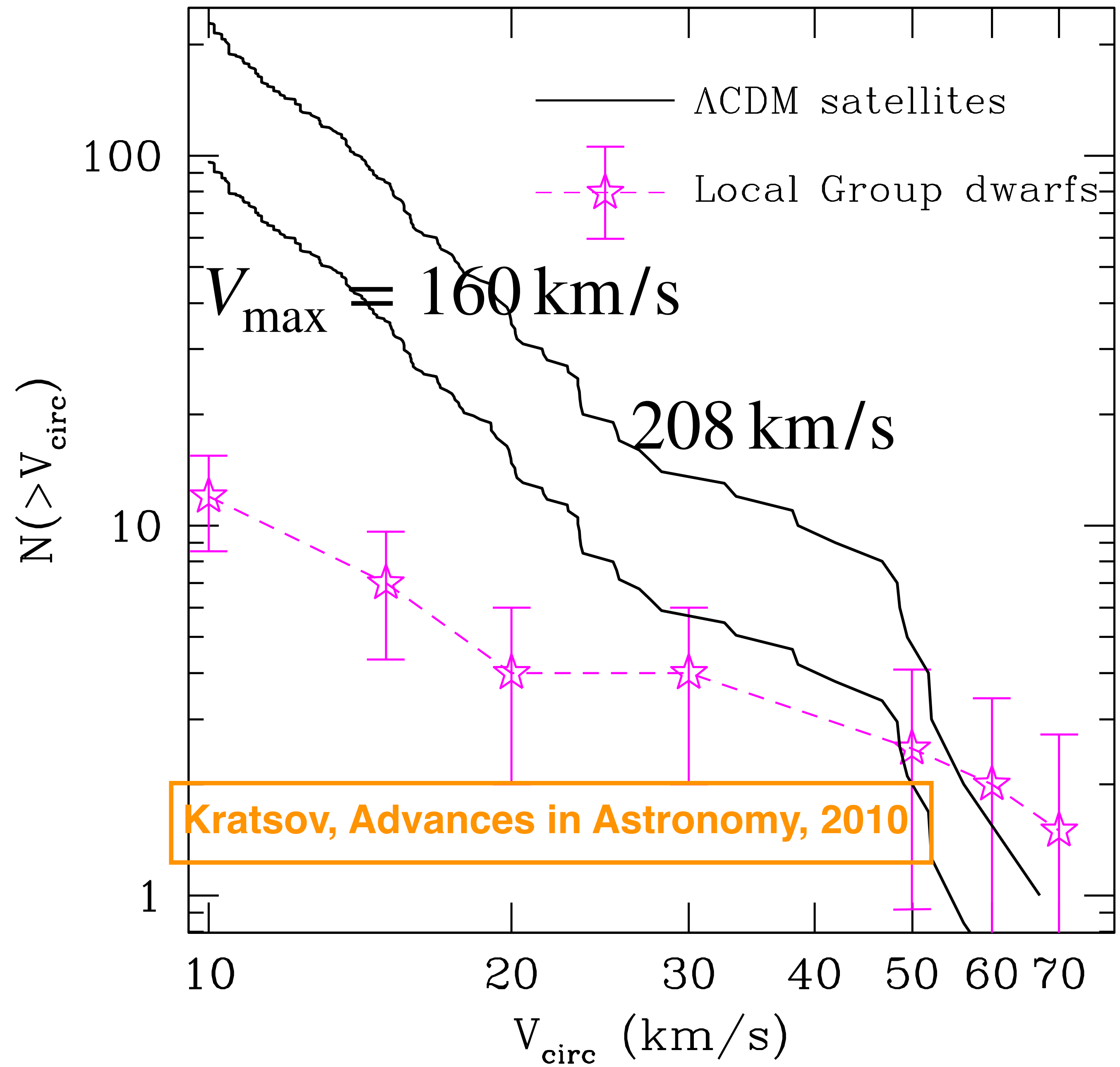
N-body (DM-only) simulations in the ΛCDM model → a MW-size halo hosts a $\mathcal{O}(10)$ times larger number of subhalos than that of observed dwarf spheroidal galaxies

(maximum) circular velocity:

$$V_{\rm circ}^2 = \frac{GM(<r)}{r}$$

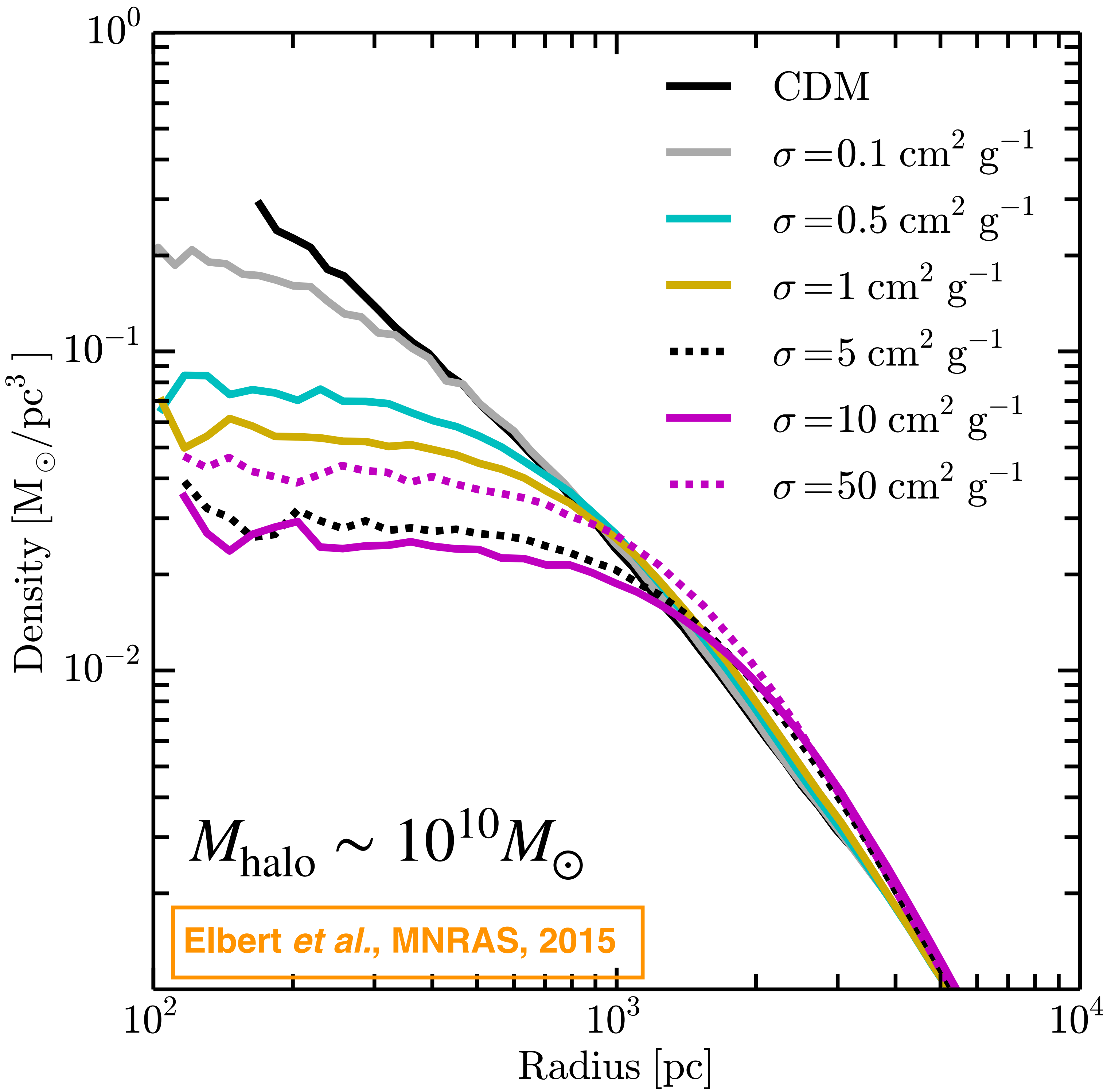
$$V_{\rm max} = \max_r \left(V_{\rm circ}(r)\right)$$

cumulative number of subhalos



maximal circular velocity of subhalo

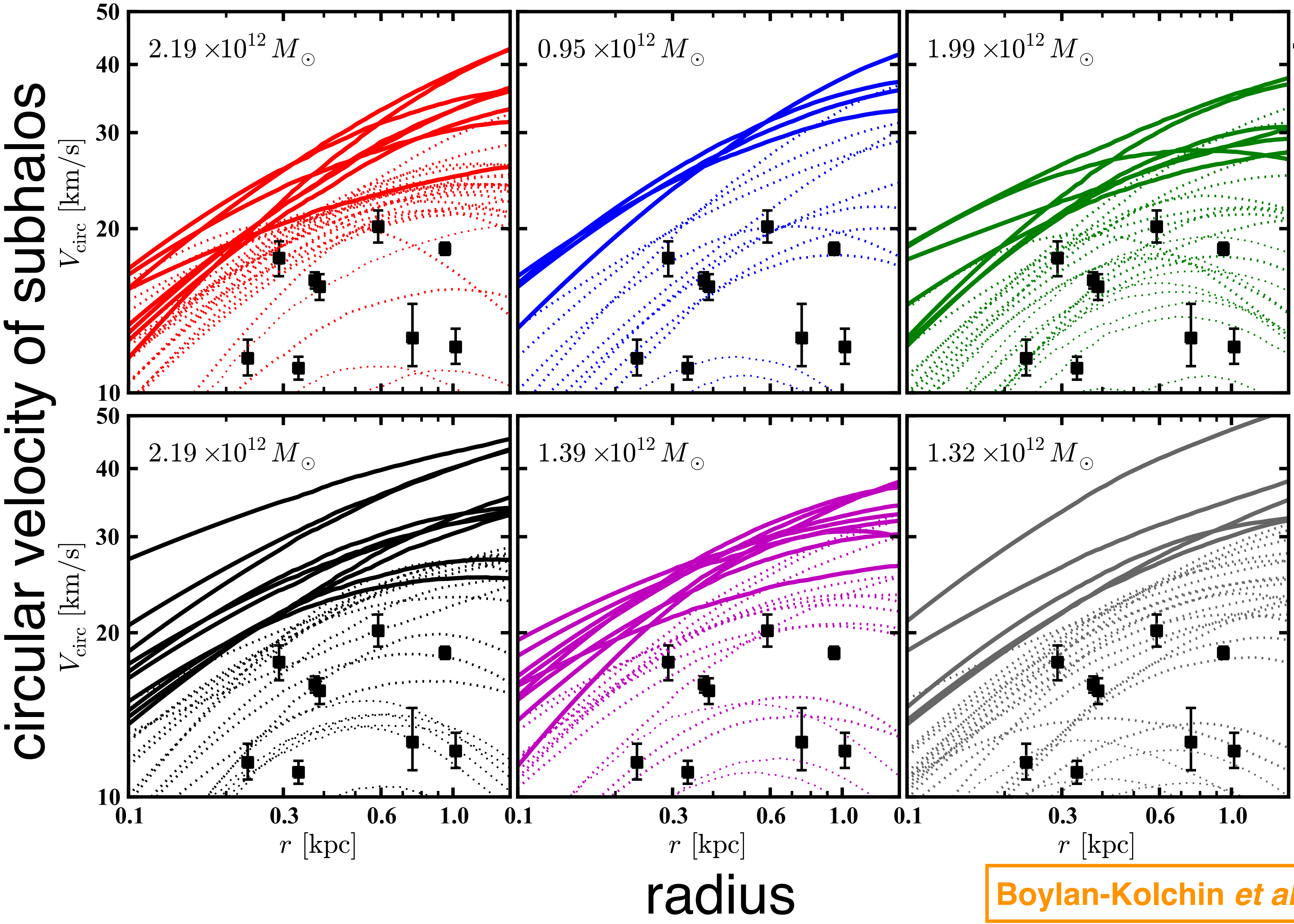
Cusp vs core problem w/ SIDM



SIDM reduces the central mass density

Too-big-to-fail problem

N-body (DM-only) simulations in the ΛCDM model →
~ 10 missing galaxies are the biggest subhalos in simulations
(to big to fail to be detected)



- MW-like halos

Boylan-Kolchin *et al.*, MNRAS, 2011

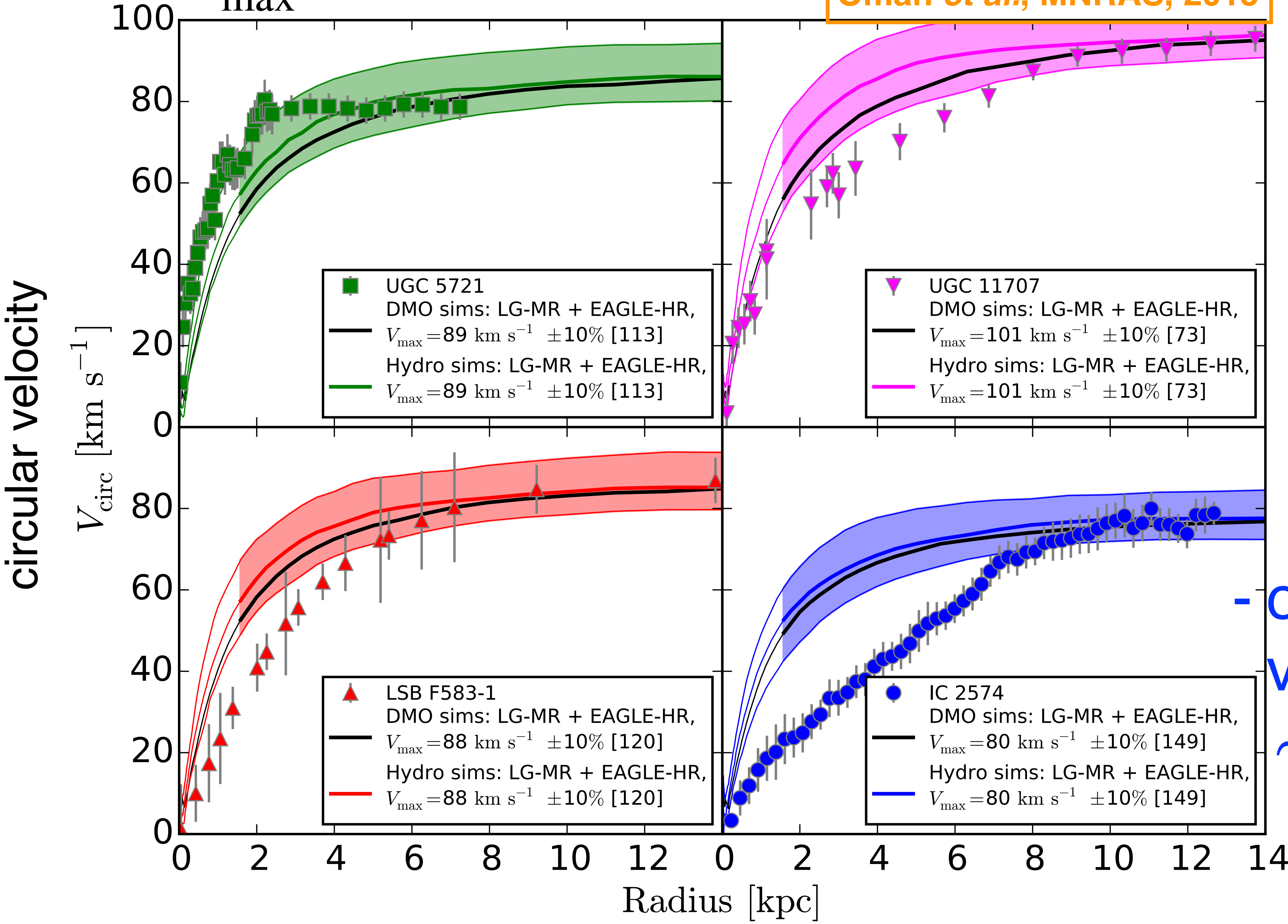
Diversity of inner rotation curves

Collisionless dark matter prediction: inner circular velocity is almost uniquely determined by outer circular velocity

↔ observations show diversity

$V_{max} = 80\text{-}100\,\mathrm{km/s}$

Oman *et al.*, MNRAS, 2015



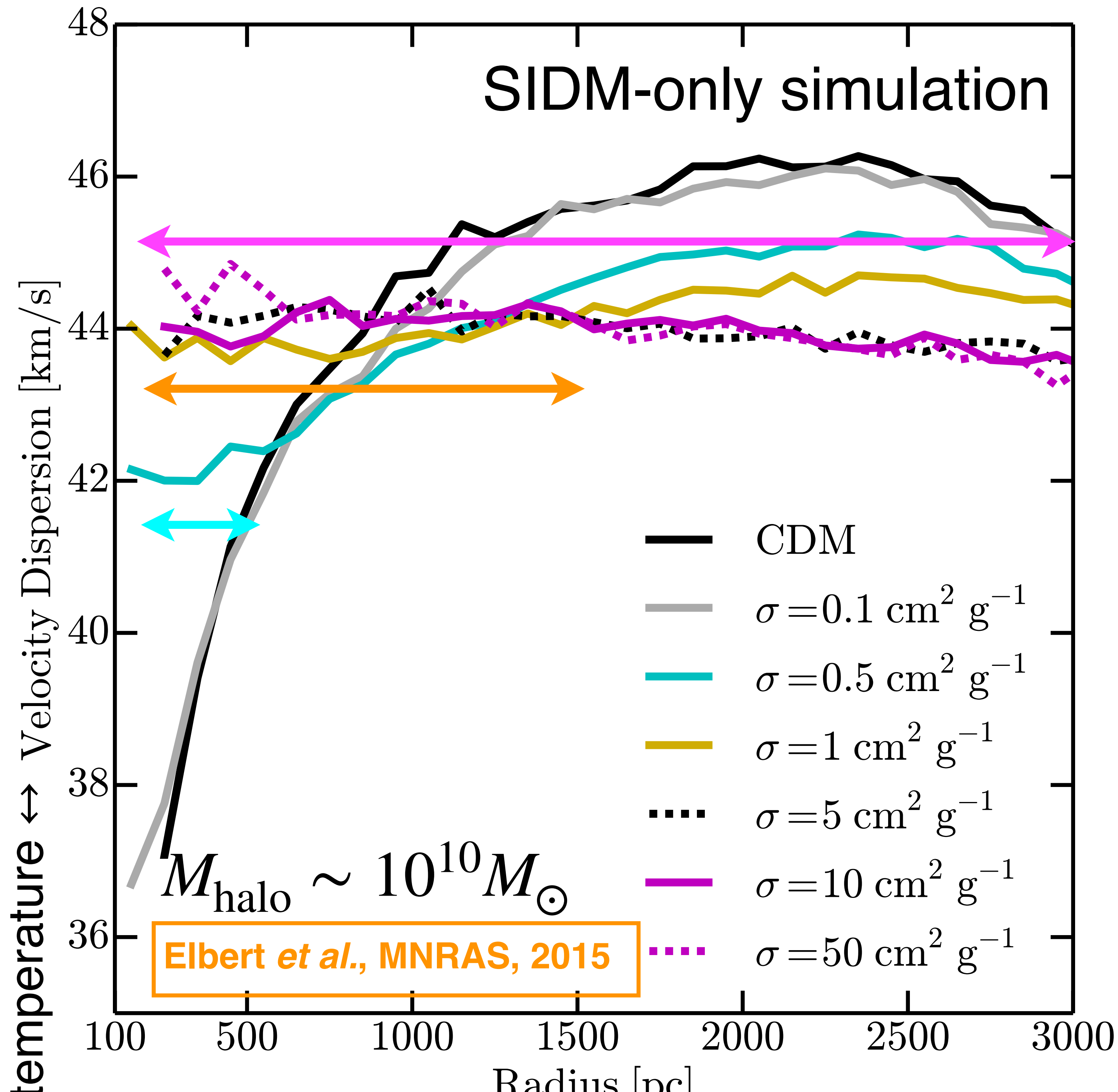
✳ unique prediction is related with the concentration-mass relation

- overpredict the circular velocity by a factor of ~ 2 (~ 4 in mass)

Iso-thermal halo

Self-scattering leads to thermalization of DM halos at $r < r_1$
where self-scattering happens at least one time until now

$$\sigma/m\,\rho(r_1)v(r_1)\,t_{\mathrm{age}} = 1$$



Key observation

Iso-thermal → Boltzmann distribution

$$\rho_{\rm DM}(\vec{x}) = \rho^0_{\rm DM} \exp(-\phi(\vec{x})/\sigma^2)$$

$$\Delta\phi = 4\pi G(\rho_{\rm DM} + \rho_{\rm baryon})$$

- inner profile is exponentially sensitive to baryon distribution

Baryons form complex objects, which show a large diversity

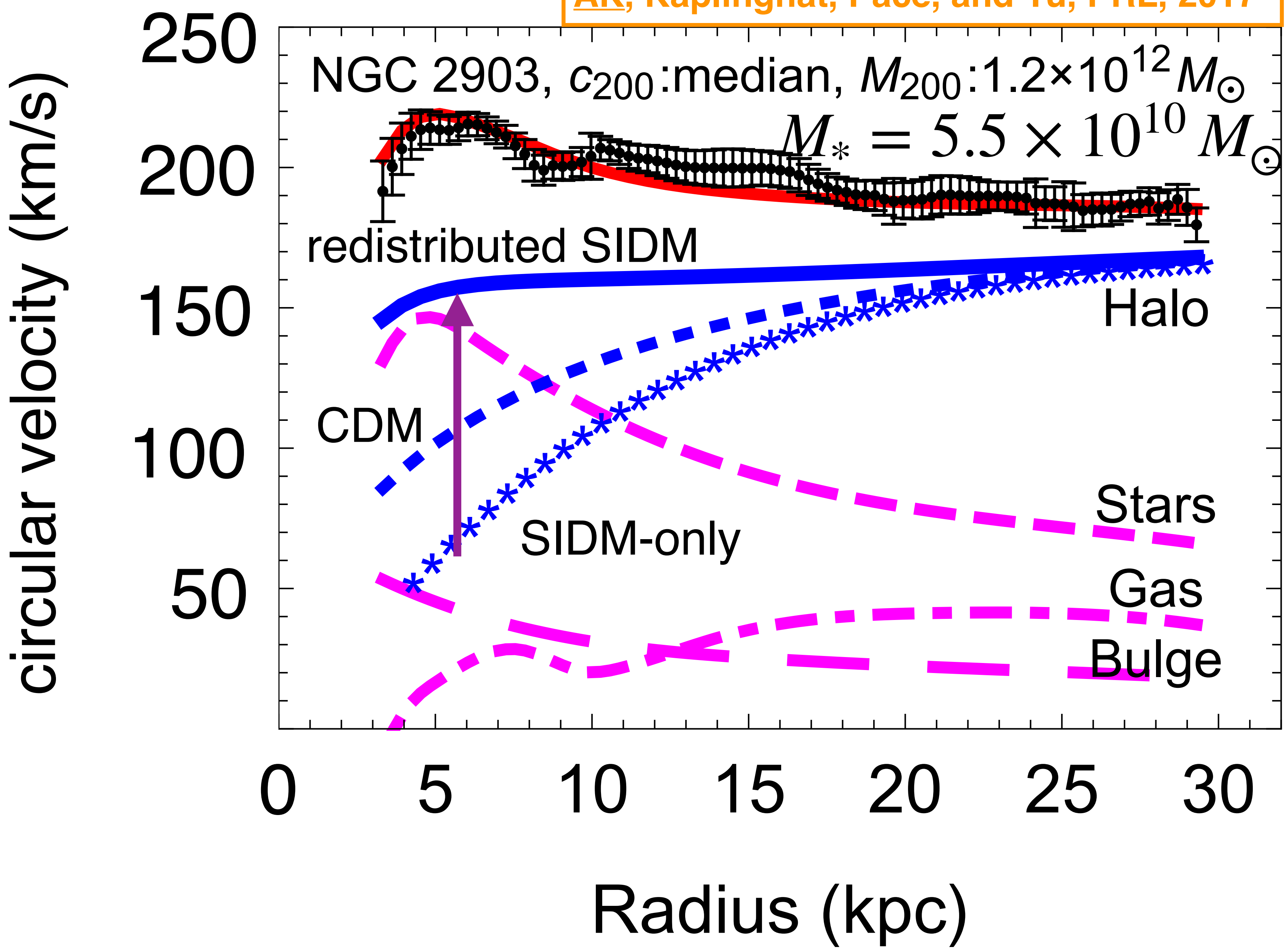
→ SIDM particles, redistributed according to formed baryonic objects, can show a diversity

✳ do not rely on unconstrained subgrid astrophysical processes
take into account observed baryon distribution

Impacts in observed galaxies

✴ Hereafter $\sigma/m = 3\,\mathrm{cm}^2/\mathrm{g}$

AK, Kaplinghat, Pace, and Yu, PRL, 2017



- Observed stellar disk makes SIDM inner circular velocity ~ 3 times higher

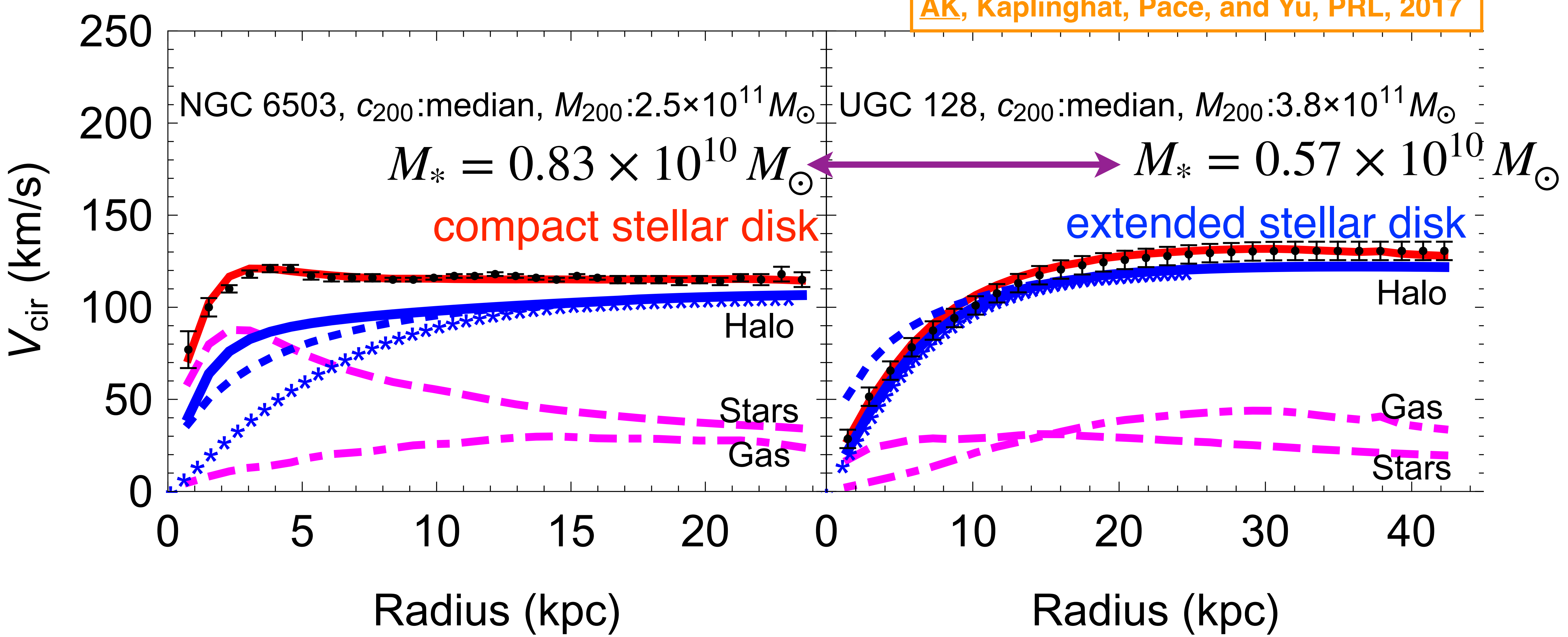
→ reproducing flat circular velocity at $10\text{-}20\,\mathrm{kpc}$

Diversity in stellar distribution

Similar outer circular velocity and stellar mass,
but different stellar distribution

- compact → redistribute SIDM significantly
- extended → unchange SIDM distribution

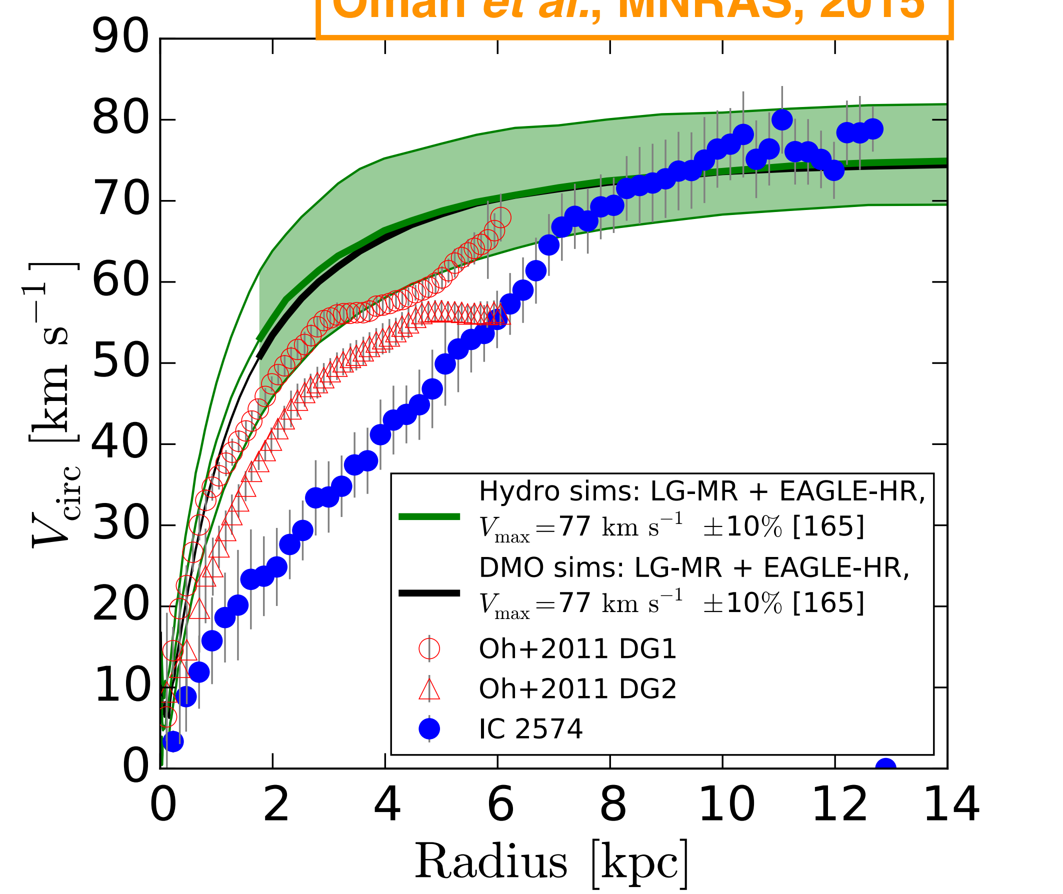
AK, Kaplinghat, Pace, and Yu, PRL, 2017



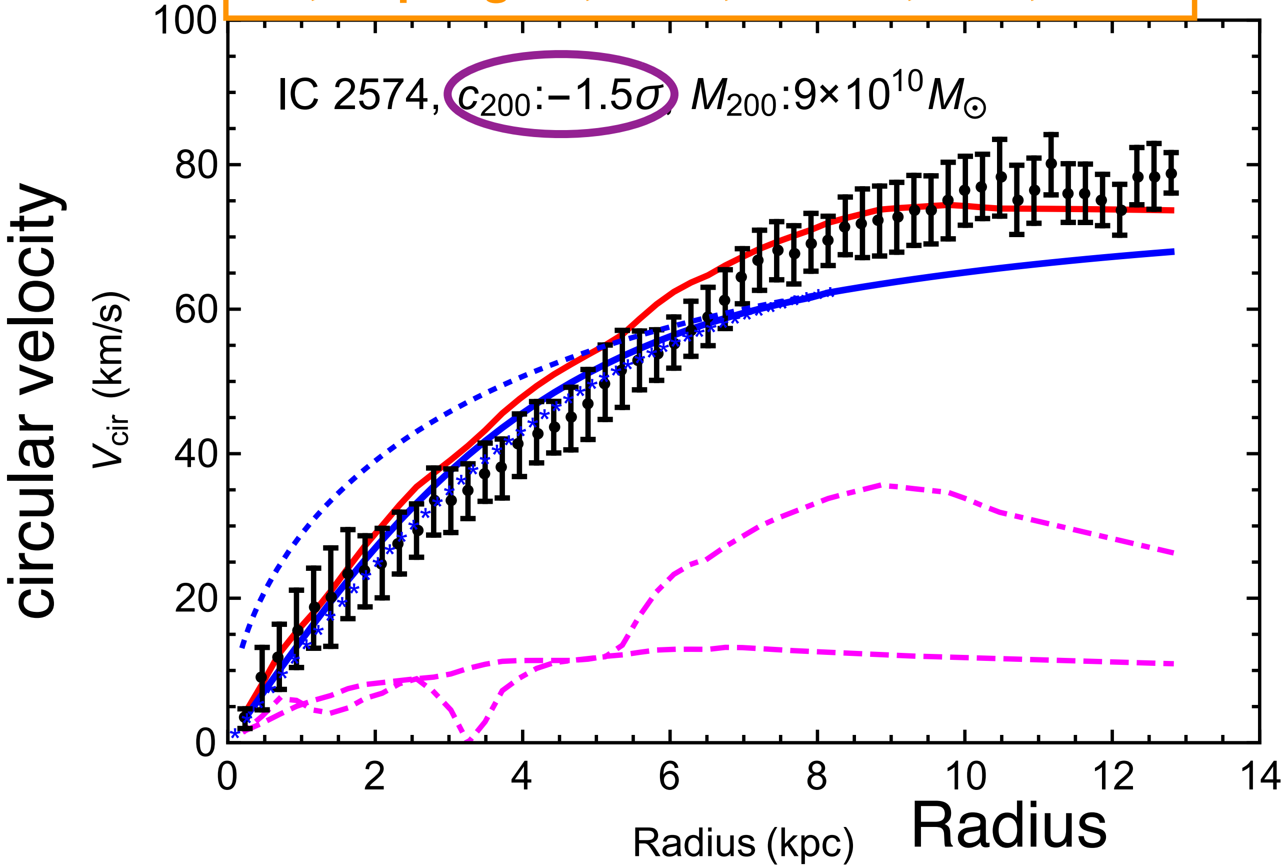
Intrinsic scatter

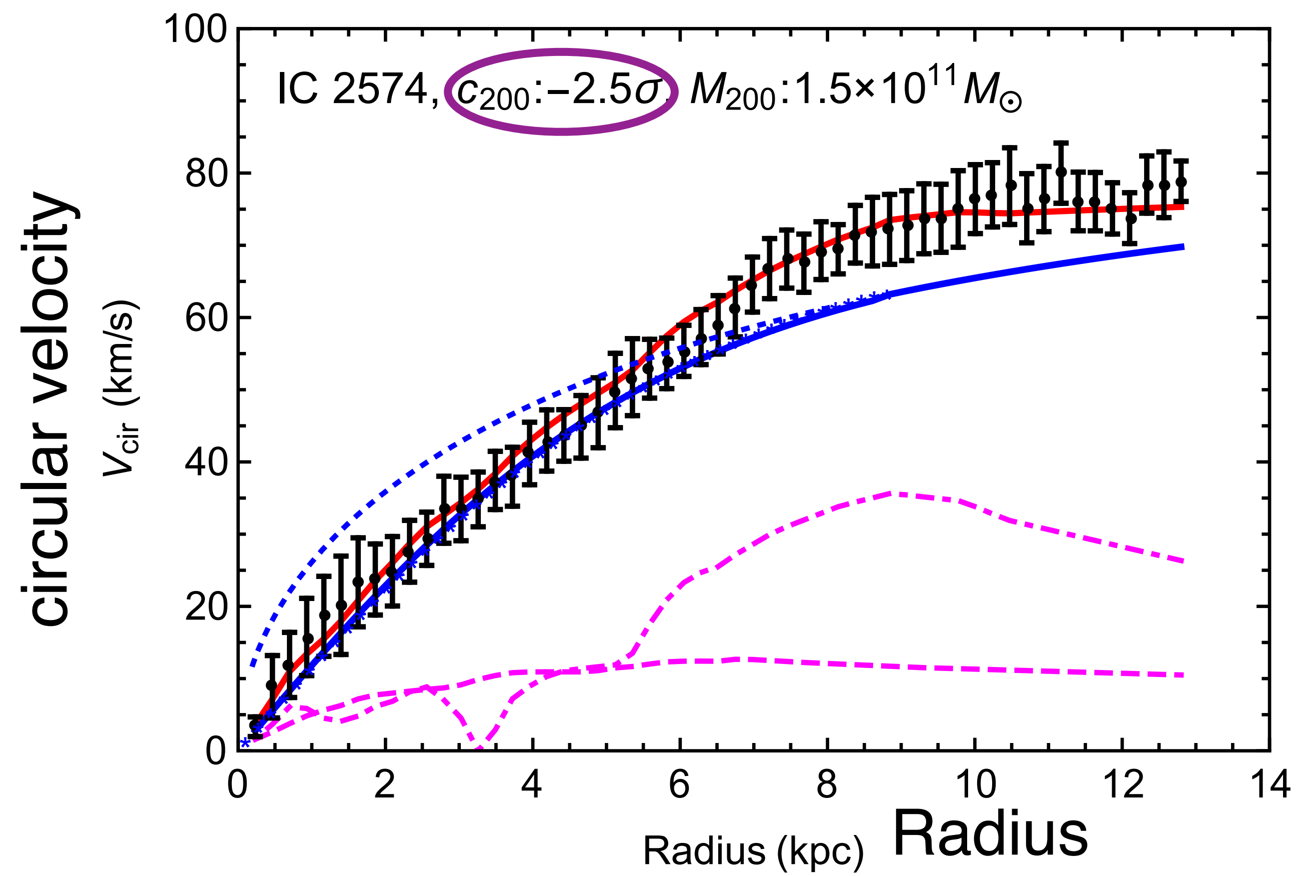
Intrinsic diversity of DM halos should be taken into account to explain the observed diversity

Oman *et al.*, MNRAS, 2015



AK, Kaplinghat, Pace, and Yu, PRL, 2017





Lagrangian vs Eulerian

DM fluid: Eulerian picture

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho V_r\right) = 0$$

- equation of continuity

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} = -\frac{\partial \Phi}{\partial r} - \frac{1}{\rho}\frac{\partial\left(\rho \nu^2\right)}{\partial r}$$

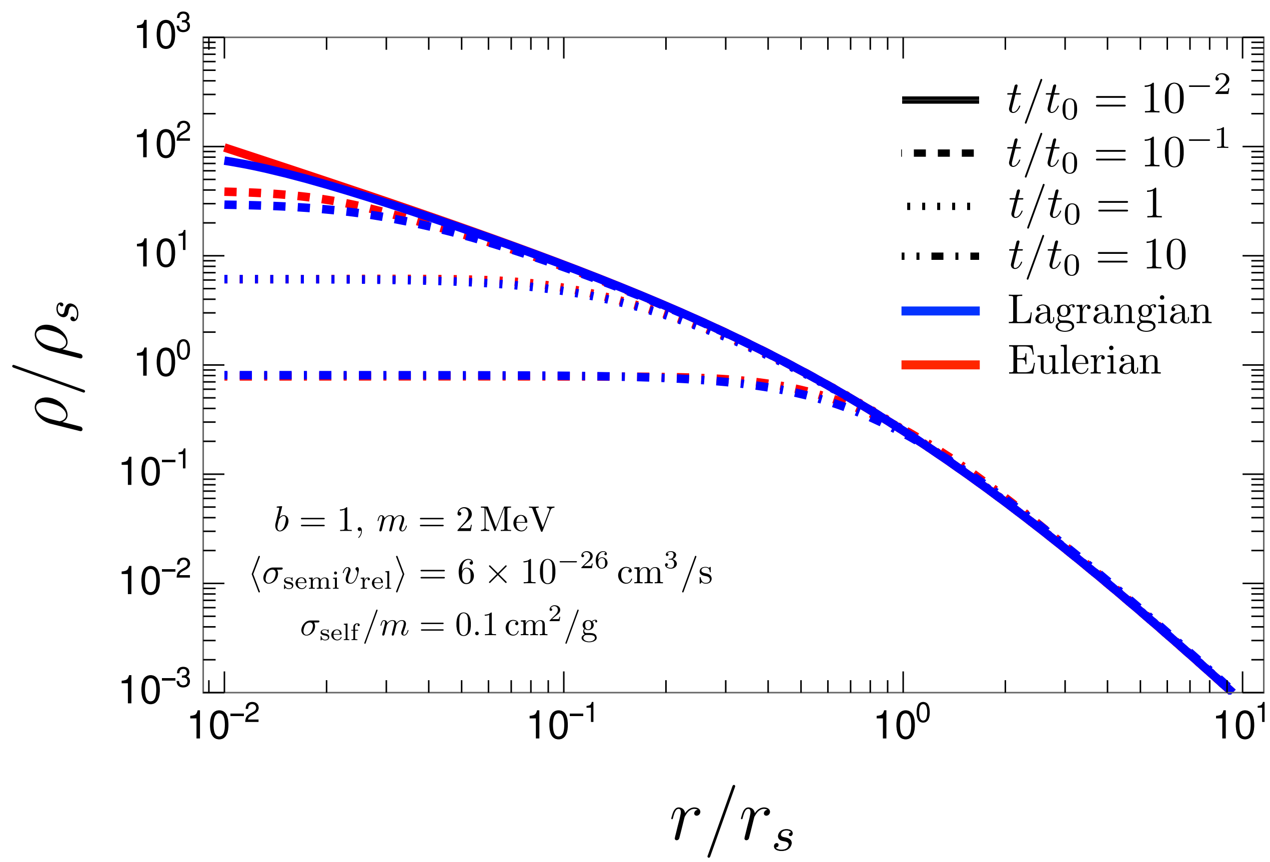
- Euler equation

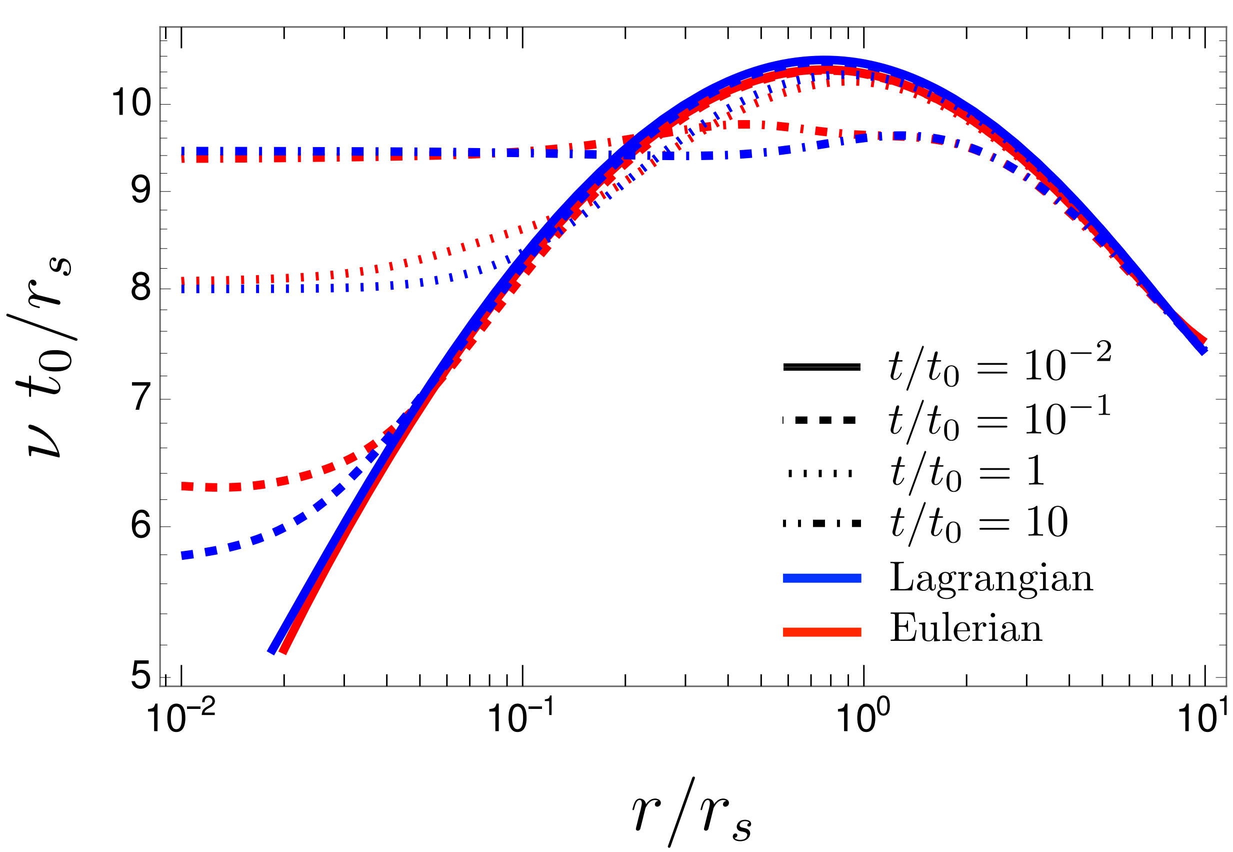
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G \rho$$

- Poisson equation

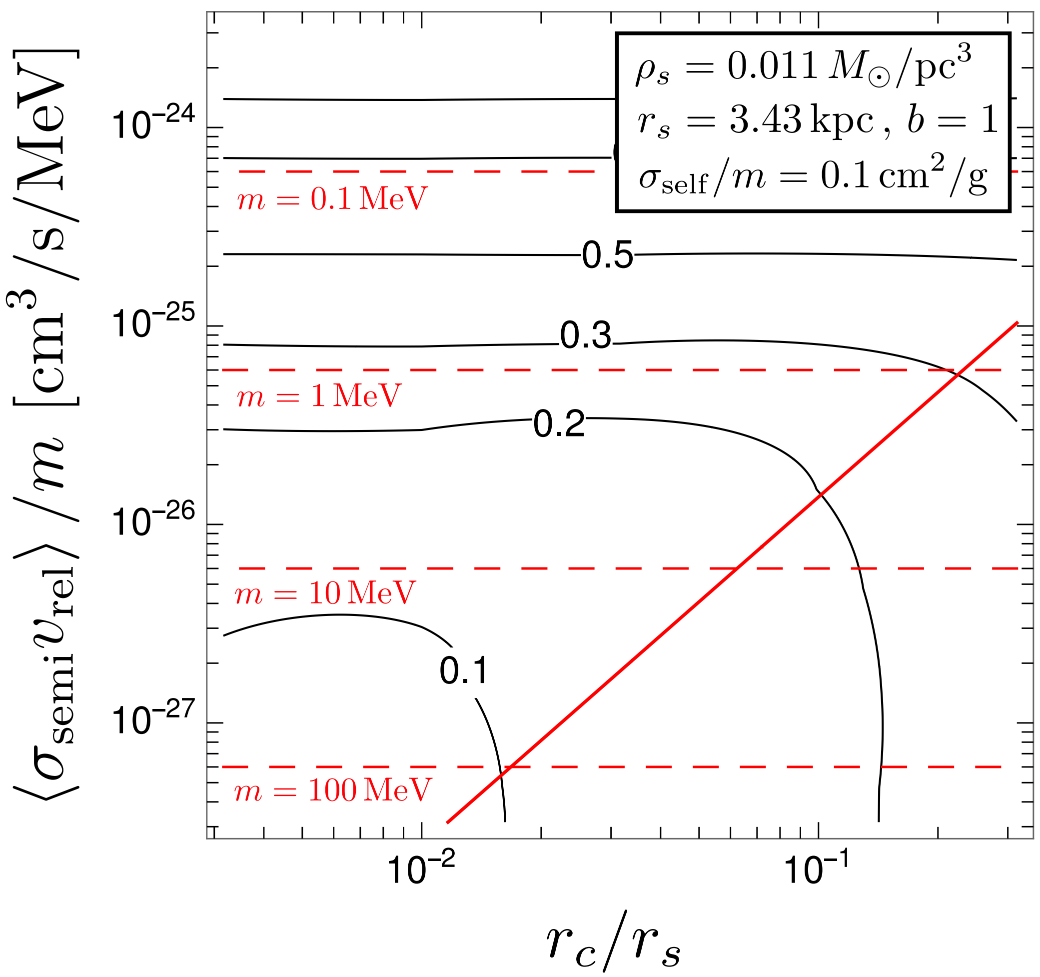
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{3}{\nu}\left[\frac{\partial \nu}{\partial t} + V_r \frac{\partial \nu}{\partial r}\right] = \frac{1}{\nu^2}\frac{\delta u}{\delta t}$$

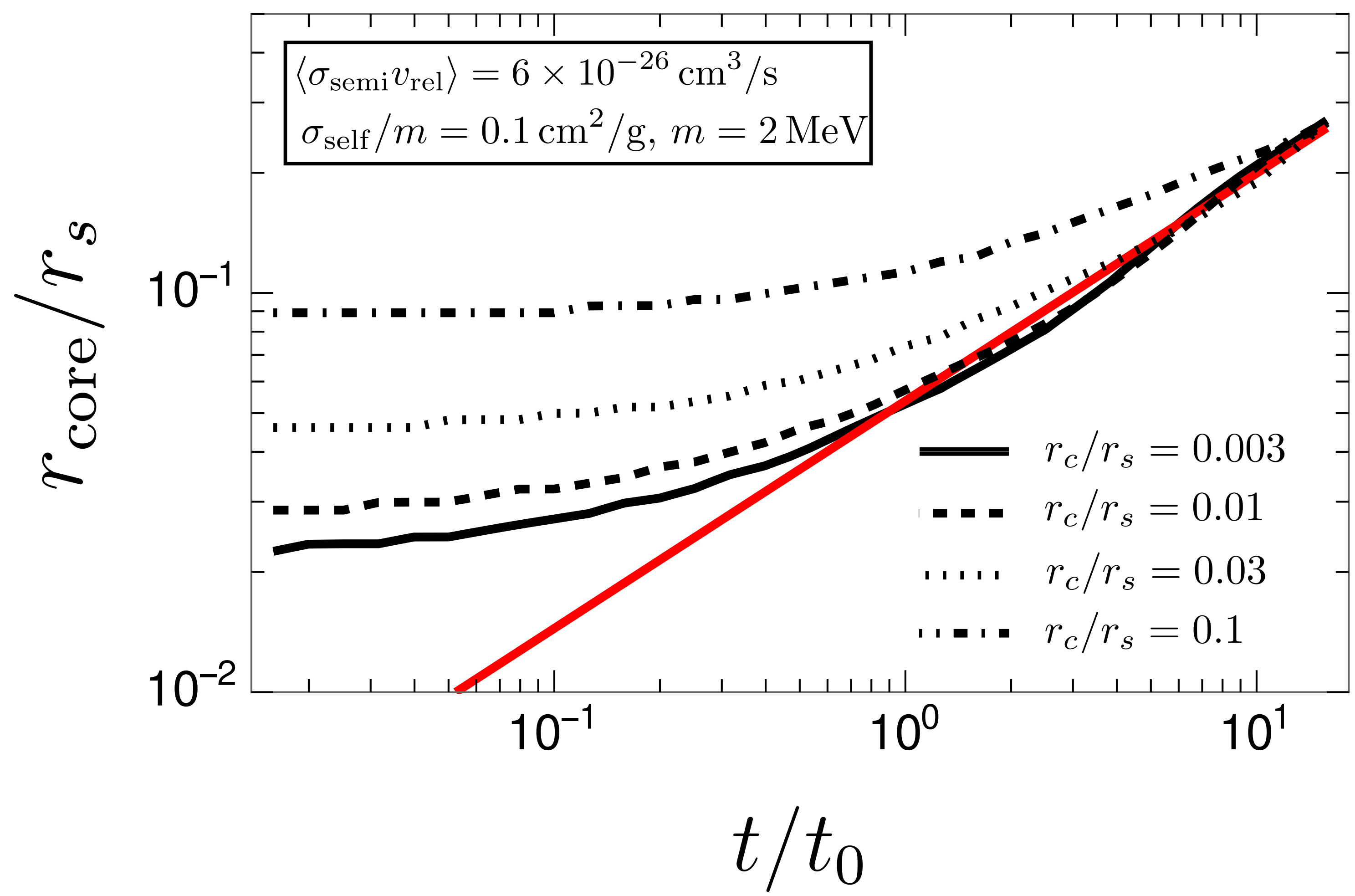
- energy conservation





Initial condition dependence





Different initial profiles converge into the attractor solution

$r_{\rm core}(t) \simeq r_s \cdot \left(t/t_{\mathcal{J}}\right)^{0.57}$ as long as $r_c \ll r_{\rm core}$

- initial core size

Hidden sector in an ALP-extended SIMP model

Lagrangian density w/ $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ and $H = \mathrm{SU}(N_f)_V$:

$$\mathcal{L}_{\rm hid} = \mathcal{L}_0 + \mathcal{L}_{\rm CP} + \mathcal{L}_{\rm CPV} + \mathcal{L}_{\rm WZW}$$

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \pi^a\right)^2 + \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_\pi^2 \left(\pi^a\right)^2 - \frac{1}{2} m_\phi^2 \phi^2$$

$$\begin{aligned}\mathcal{L}_{\rm CP} = {} & \frac{m_\pi^2}{4N_f^2 f^2}\left(\pi^a\right)^2 \phi^2 - \frac{1}{6f_\pi^2} r_{abcd}\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^b\right)\pi^c \pi^d \\ & + \frac{m_\pi^2}{6N_f f_\pi f} d_{abc}\pi^a \pi^b \pi^c \phi + \frac{m_\pi^2}{12 f_\pi^2} c_{abcd}\pi^a \pi^b \pi^c \pi^d\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{\rm CPV} = {} & \tan\left(\theta_H / N_f\right)\left[\frac{m_\pi^2}{2N_f f}\phi(\pi^a)^2\right. \\ & \left. + \frac{m_\pi^2}{6f_\pi} d_{abc}\pi^a \pi^b \pi^c - \frac{m_\pi^2}{30 f_\pi^3}\pi^a \pi^b \pi^c \pi^d \pi^e c_{abcde}\right]\end{aligned}$$

Co-evolution equations

$$\dot{n}_\chi + 3Hn_\chi = -\, n_\chi \langle \sigma_{\text{semi}} v \rangle_{T_\chi T_\chi} \left[n_\chi - \mathcal{J}(T_\chi, T_\phi) n_\chi^{\text{eq}}(T_\chi) \right]$$

$$\dot{T}_\chi + 3HT_\chi \left(\frac{T_\chi}{\sigma_E} \right)^2 = -\left(\frac{T_\chi}{\sigma_E} \right)^2 \frac{n_\phi^{\text{eq}}(T_\chi)}{n_\chi^{\text{eq}}(T_\chi)} \langle \Delta E \sigma_{\text{inv}} v \rangle_{T_\chi, T_\phi = T_\chi} \times \left[n_\chi - n_\chi^{\text{eq}}(T_\chi) \mathcal{K}(T_\chi, T_\phi) \right]$$

$$\sigma_E^2 = \langle E_\chi^2 \rangle - \langle E_\chi \rangle^2$$

$$\mathcal{J}(T_\chi, T_\phi) = \frac{n_\phi^{\text{eq}}(T_\phi)}{n_\phi^{\text{eq}}(T_\chi)} \frac{\langle \sigma_{\text{inv}} v \rangle_{T_\chi, T_\phi}}{\langle \sigma_{\text{inv}} v \rangle_{T_\chi, T_\phi = T_\chi}}$$

$$\mathcal{K}(T_\chi, T_\phi) = \frac{n_\phi^{\text{eq}}(T_\phi)}{n_\phi^{\text{eq}}(T_\chi)} \frac{\langle \Delta E \sigma_{\text{inv}} v \rangle_{T_\chi, T_\phi}}{\langle \Delta E \sigma_{\text{inv}} v \rangle_{T_\chi, T_\phi = T_\chi}}$$

$$\Delta E = E_\phi - \langle E_\chi \rangle_{T_\chi}$$

relativistic: $\sigma_E^2 = 3T_\chi$

non-relativistic: $\sigma_E^2 = 3/2T_\chi$

$$\mathcal{J}(T_\chi = T_\phi, T_\phi) = 1$$

$$\mathcal{K}(T_\chi = T_\phi, T_\phi) = 1$$