# Black Hole Thermodynamics: General Relativity & Beyond

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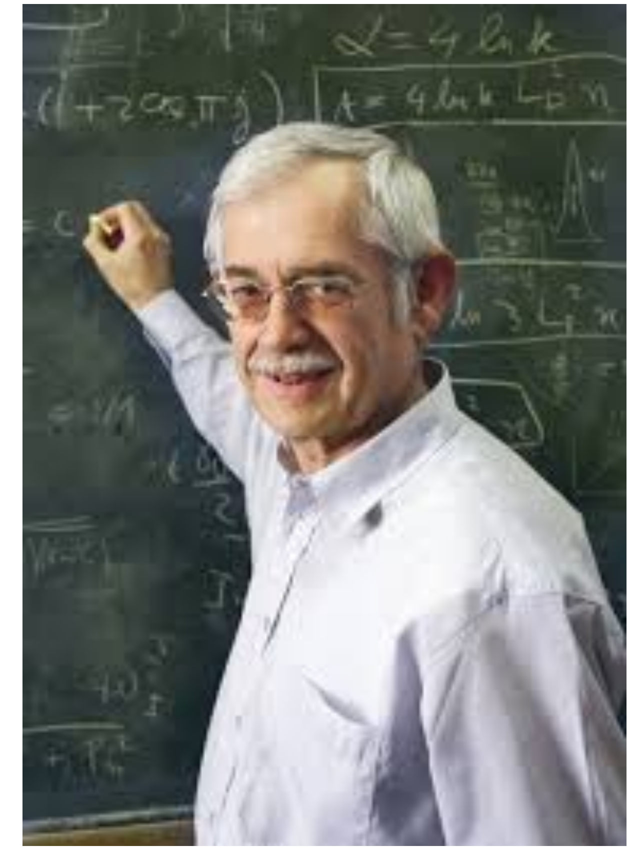
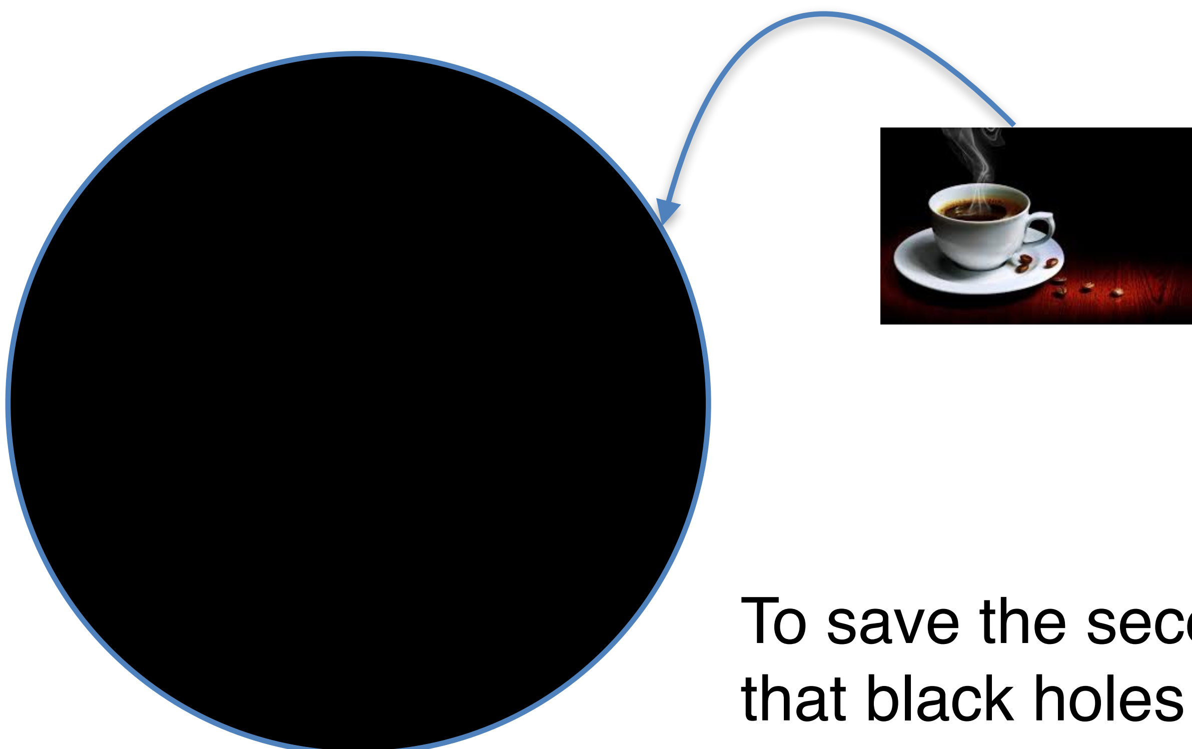
Black hole entropy: Bekenstein's great idea!



Violation of second law outside a black hole.



To save the second law, Bekenstein proposed that black holes must have entropy proportional to the area of the horizon.

$$S \sim r_h^2 \sim M^2$$

The argument does not determine the proportionality constant!

This is a good news because in classical GR, we have the area theorem:

Area of the black hole event horizon can not decrease as long as:

1. Matter energy is positive (Null Energy Condition).
2. Some form of cosmic censorship is valid.

The area theorem ensures that when a entropic object falls into the black hole, the outside world looses entropy but the black hole area increases to compensate the loss and thereby saves the second law.

If black holes have entropy, it should have a temperature.

The Hawking Temperature:

$$T_{BH} = \frac{\hbar c^3}{8\pi G K_B M}$$

The origin of the Hawking temperature lies in the deep mathematical structure of Quantum Field Theory; the choice of the correct vacuum state.

Check out The Bisognano-Wichmann Theorem.

Hawking's result and the validity of the Clausius Theorem: T dS = dM fixes the proportionality Constant between entropy and area.

$$S = \frac{K_B c^3 A}{4G\hbar}$$

The Bekenstein Entropy Formula; One of the most Beautiful equation we know so far. It contains all the Fundamental constants of nature we know so far.

The Microscopic derivation of the Black Hole entropy. $\Omega \sim Exp(S)$

The Physics of D-Branes in String Theory accurately produces the Bekenstein entropy of certain 5 dimensional black hole solutions. (Strominger & Vafa, 1996)

Black Hole Thermodynamics:

An intriguing analogy between the laws of black hole mechanics & the laws of thermodynamics!

A Thermodynamic system in equilibrium with Temperature and Entropy!

A stationary black hole with Hawking Temperature and Benkenstein Entropy.

Assuming some reasonable energy conditions on matter:

*Surface Gravity is constant on the stationary event horizon*

*In a quasi static process, when a black hole of mass M and area A is taken to a new black hole with parameters M + dM and A + dA, we have*

$$\left(\frac{\kappa}{2\pi}\right) d\left(\frac{A}{4G}\right) = dQ$$

*The area of a black hole can never decreases!* $\Delta A \geq 0$

Temperature ~ Surface Gravity; Entropy ~ Area

This does not fix the temp: $\left(\frac{\eta\kappa}{2\pi}\right) d\left(\frac{A}{4\eta G}\right) = dQ$

What we want to discuss:

How to derive these laws, particularly the zeroth and the first law.

How to extend these laws beyond general relativity and to modified gravity theories.

What are the crucial assumptions and open problems.

**References:**


"The thermodynamics of Black Holes", Robert Wald (gr-qc/9912119)

"Black Holes" Online lecture notes by Paul Townsend

"Introductory Lectures on Black Hole Thermodynamics" Online Lecture notes by Ted Jacobson

"A Survey of Black Hole Thermodynamics" Aron Wall, arXiv:1804.10610

"Black Hole Thermodynamics: General Relativity and Beyond" Sudipta Sarkar, arXiv:1905.04466

"Relativist Toolkit" Eric Poisson

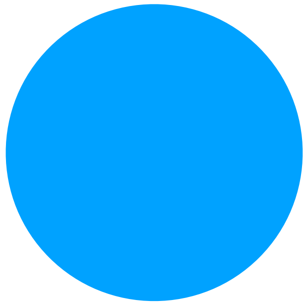
We start with Einstein's equations

Gravity = Curved Space time.

The dynamics of gravity is described by a curved space time with a metric.
The matter energy momentum tensor is related to the curvature by:

$$R_{ab} - \frac{1}{2} R \, g_{ab} = 8\pi G T_{ab}$$

The simplest spherically symmetric solution:

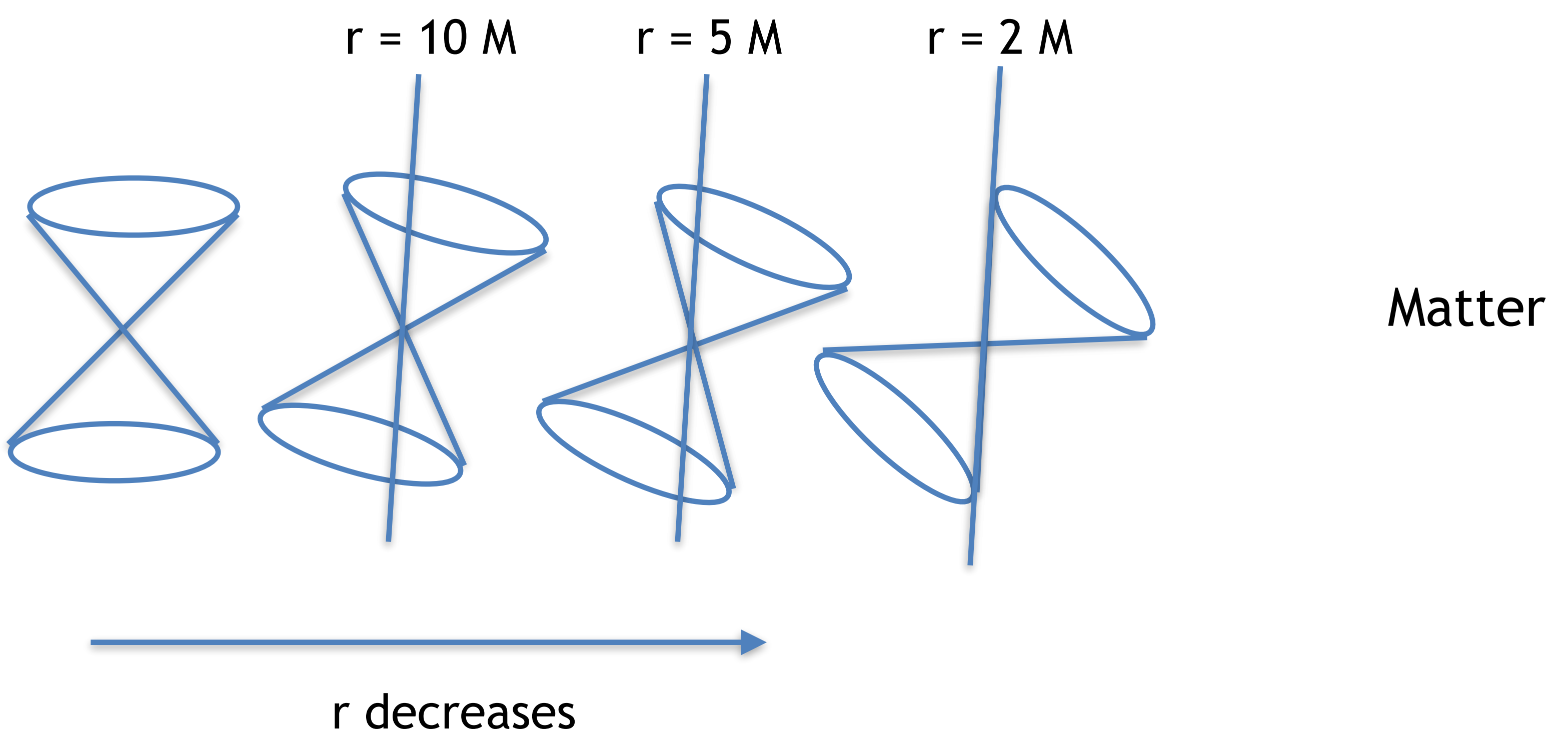


Determine the gravitational field outside a spherical star!

Schwarzschild solution: $$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2$$

The location of the event horizon is at r = 2GM

Matter affects the causal structure, and therefore the behaviour of the light cones:



The Schwarzschild space time has a time like Killing vector $\xi^a = (1,0,0,0)$

It is easy to check that: $\xi^a \xi_a = -\left(1 - \frac{2GM}{r}\right)$

The time like vector field becomes null on the surface r = 2M.

Also, a simple calculation reveals that at r = 2M: $\xi^a \nabla_a \xi^b = \kappa \ \xi^b$

Where: $\boxed{\kappa = \frac{1}{4GM}}$ Surface Gravity!

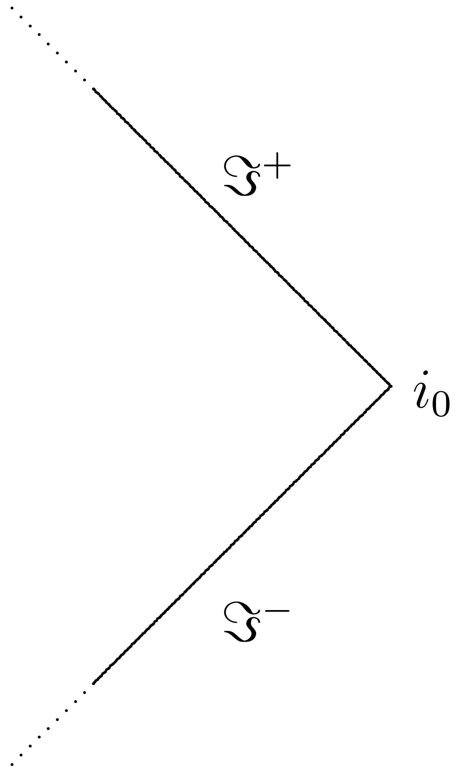
The r = 2GM surface has following properties:

1. It is an event horizon.
2. It is a Killing horizon.

It is possible to generalise these concepts without the use of any specific solution.

Event Horizon:

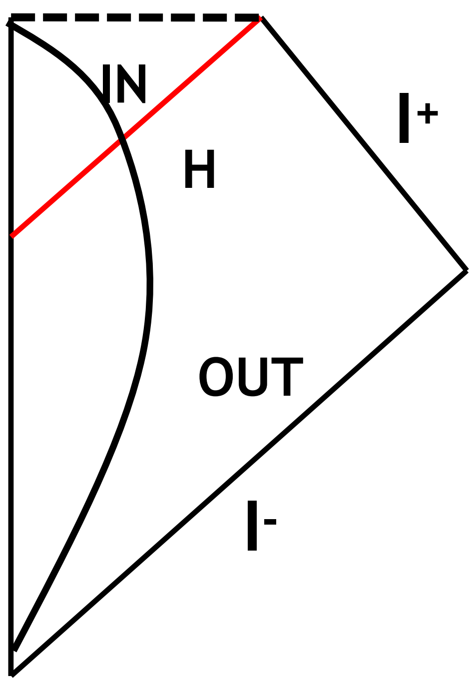
Let us assume asymptotically flat space time:




A space time has a black hole region, if there are outgoing null geodesics which never reaches the future null infinity!

We may ``*Loosely*" define the black hole as the set of all points which are not in the causal past of the future null infinity.

$$B = M - J^-\left(I^+\right) \qquad H = \partial B$$




Black hole: Compliment of the past of future null infinity.
Event Horizon: Boundary of the black hole region.

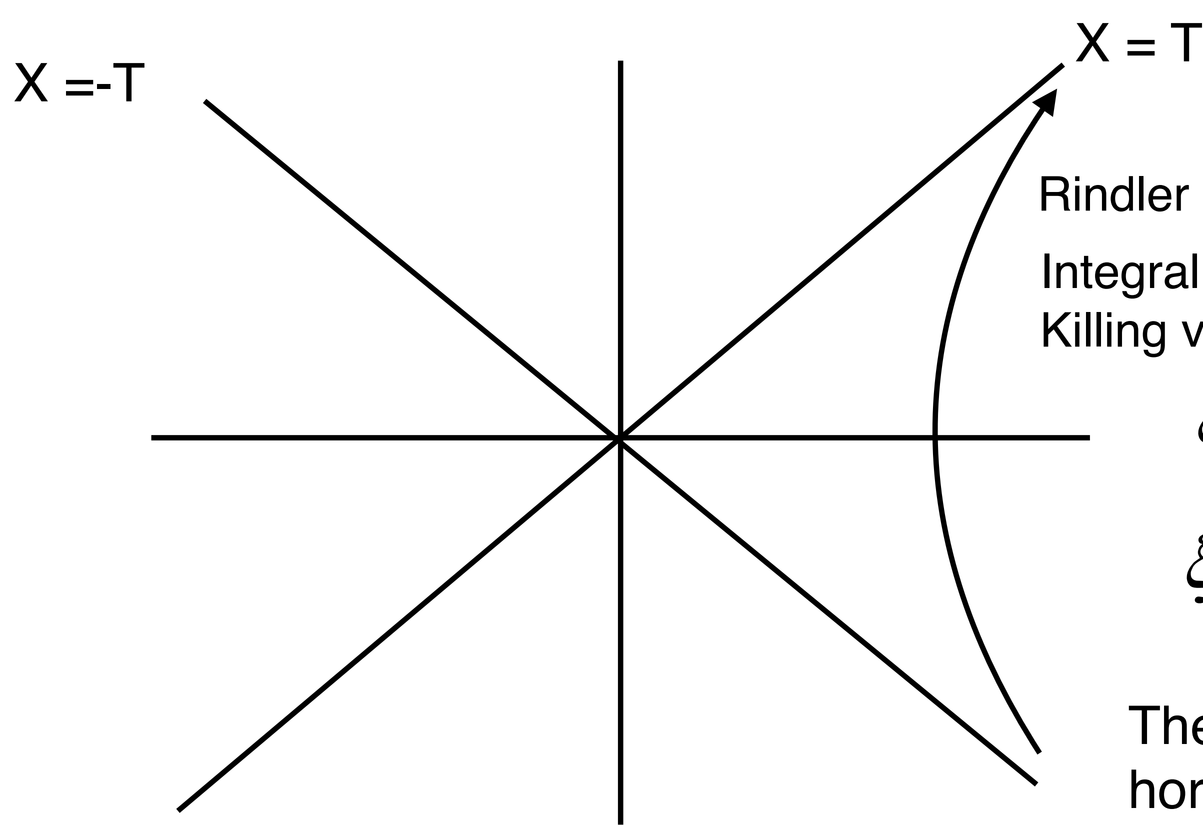
Event horizon is also a null hyper surface generated by null geodesics which is future complete.
*Penrose 1970*

Killing Horizon:

Consider a stationary space time with a timelike Killing vector, if there is a hyper-surface on which, the Killing vector becomes null, it is called a Killing horizon.

There may not be any relationship between these two concepts:

Consider the Rindler Horizon in flat space time, it is a Killing horizon, but there is no event horizon in flat space time:




Rindler Observer

Integral curves of the boost Killing vector.

$$\xi^a = (x, t, 0, 0)$$

$$\xi^a \xi_a = -x^2 + t^2$$

The surface X = T is a Killing horizon!

An event horizon which is not a Killing horizon!

Consider a ingoing Vaidya space time, which is a dynamical space time with NO time like Killing vector.

$$ds^2 = -\left(1 - \frac{2m(v)}{r(v)}\right)dv^2 + 2dvdr + r(v)^2 d\Omega^2$$

There is an event horizon determined from the equation of null geodesic with appropriate boundary conditions:

$$\frac{dr_h(v)}{dv} = -\frac{1}{2}\left(1 - \frac{2m(v)}{r_h(v)}\right)$$

For inflating matter, the event horizon grows with time, this is not a Killing horizon!

## Strong rigidity theorems:

In stationary space time, the event horizon must be a Killing horizon!

Remember the case of r=2M surface of Schwarzschild geometry, also the Kerr space time has the outer event horizon where a time like Killing vector becomes null.

So a stationary event horizon is also a Killing horizon. we can unambiguously associate a surface gravity with the event horizon.

$$\boxed{\xi^a \nabla_a \xi^b = \kappa \ \xi^b}$$

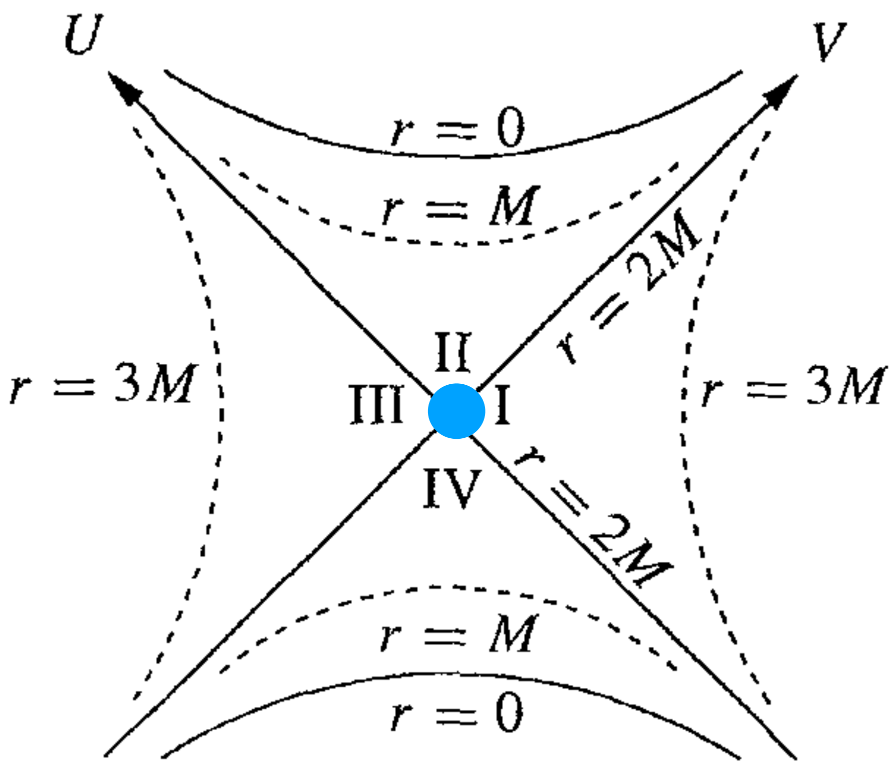
For Kerr Outer horizon: $$\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}$$

So, let us consider a stationary event horizon which is also a Killing horizon

*So far rigidity theorems are only proven using Einstein's equation*

The bifurcation surface:

Consider the Kruskal extension of the Schwarzschild Geometry:




The bifurcation surface is a codimension 2 surface, on which the Killing field vanishes.

In (U,V) coordinates, the Killing field is

$$\xi^a = U\frac{\partial}{\partial V} - V\frac{\partial}{\partial U}$$

At U = V = 0, the Killing field vanishes. This is a 2-sphere where the past and the future horizons meet.

The bifurcation surface exists in the past, only for a eternal solution. A Killing horizon with bifurcation surface is called a bifurcate Killing horizon.

*The zeroth law of black hole mechanics:*

*The surface gravity must be constant on the (stationary) event horizon.*

To prove these, we would require one of these:

1. Einstein's equation and dominant energy condition!
2. Some form of symmetry but no field equation!

Event horizon is a codimension 1 surface:

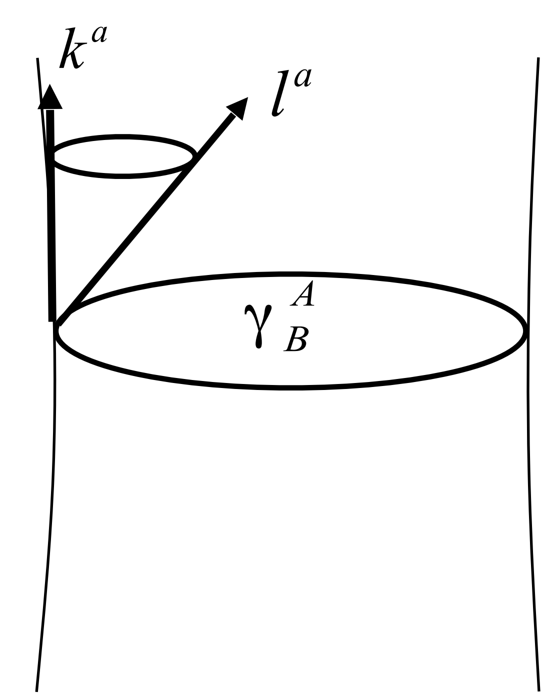
A choice of Horizon basis: $\{k^a, l^a, e^a_A\}$

$$k^a k_a = l^a l_a = 0 \; ; \; k^a l_a = -1$$

$$g_{ab} = -2k_{(a}l_{b)} + \gamma_{ab}$$

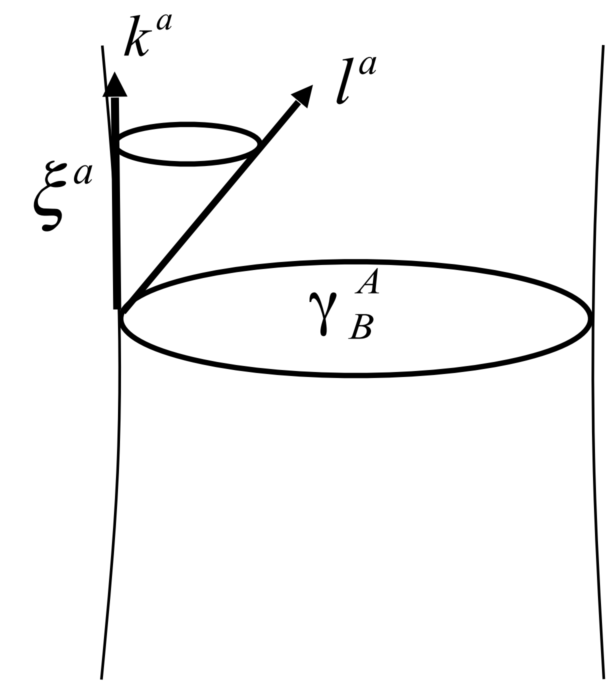
2 + (D - 2) decomposition

$$k^a \nabla_a k^b = 0$$



So what we want to prove are:

$$k^a \nabla_a \kappa = 0 \qquad \gamma^{ab} \nabla_a \kappa = 0$$




The first equation is easy to prove, note the Killing vector is related to the null generator of the horizon as:

$$\xi^a = \lambda \kappa k^a$$

The location of the bifurcation surface becomes:

$$\boxed{\lambda = 0}$$

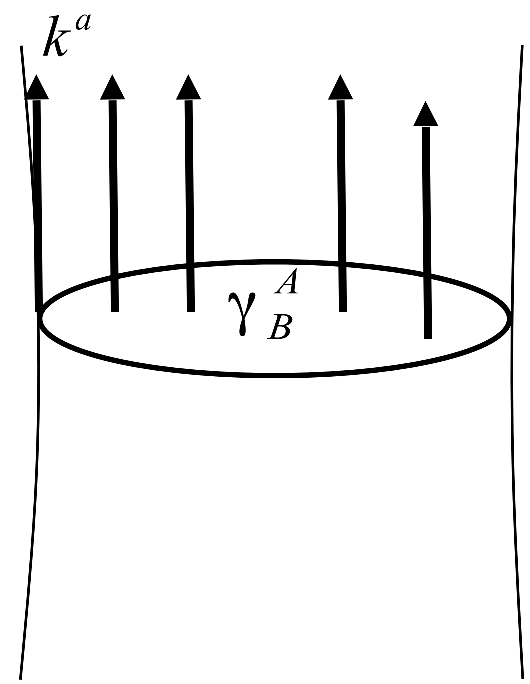
$$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \quad \text{:Non Affine}$$

$$k^a = \left(\frac{\partial}{\partial \lambda}\right)^a \quad \text{:Affine}$$

Then, we immediately obtain:

$$\boxed{2\kappa \kappa_{,\alpha} = -\xi^{\mu;\nu} R_{\mu\nu\alpha\beta} \xi^{\beta}}$$

This gives: $\xi^a \nabla_a \kappa = 0$ Surface gravity is constant along a generator.




$$k^a \nabla_a \kappa = 0$$

So, by definition, surface gravity is constant along a generator, so it can only vary across a generator.

One could show: $\gamma^a_b \nabla_a \kappa = -\xi^a R_{ac} \gamma^c_b.$

Note: if bifurcation surface exists, then $\gamma^{ab} \nabla_a \kappa = 0$, on the bifurcation surface, but surface gravity can not change along the generator, this will immediately establish that $\gamma^{ab} \nabla_a \kappa = 0$ on every cross section of the horizon.

If the bifurcation surface exists, the zeroth law is just a geometric fact, independent of the field equations!

Similarly, it is possible show that the zeroth law holds true for all static black holes, where a hyper surface orthogonal time like Killing field is available.

But, to establish the zeroth law for a general stationary black hole, we will require Einstein's equations.

The Einstein equations give: $\gamma_b^a \nabla_a \kappa = -8\pi \, \xi^a T_{ac} \gamma_b^c$

Now the dominant energy condition implies $T_{ab}\xi^b$, is a non space like vector.

Also, stationarity implies: $T_{ab}\xi^a\xi^b = 0$

This gives: $T_{ab}\xi^b \sim \xi_a$ and therefore: $\gamma^{ab}\nabla_a\kappa = 0$

If the field equation changes, we need to redo the derivation.

As an example, if there are higher curvature corrections, we expect the Einstein equation to change.

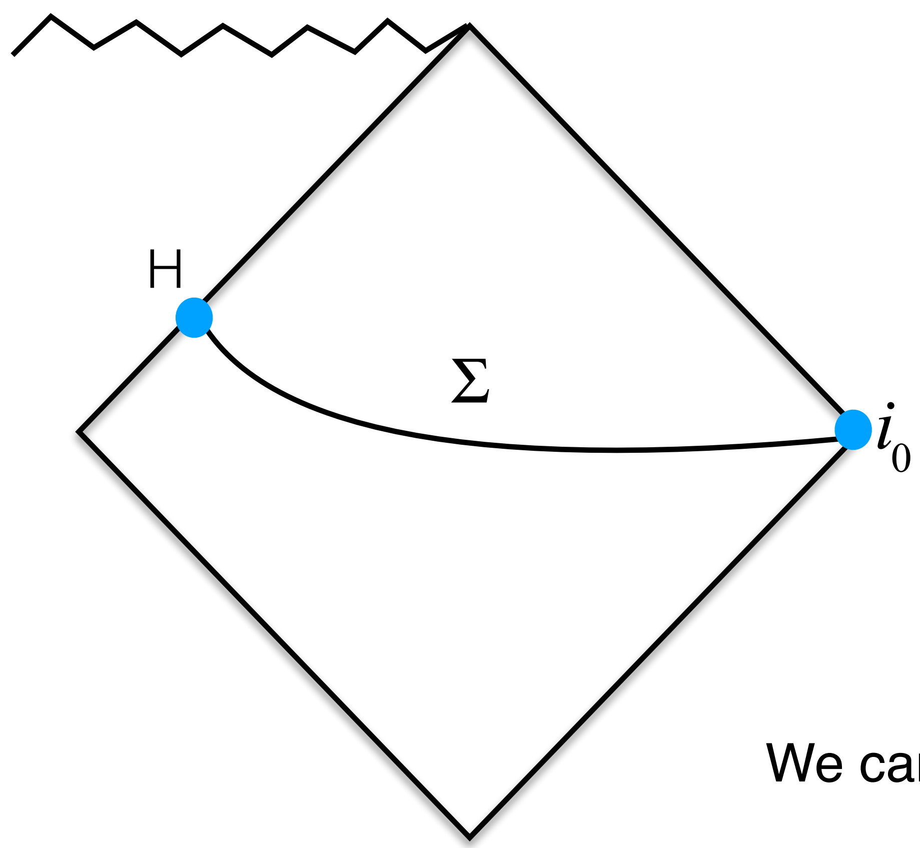
$$G_{ab} + \alpha H_{ab} = 8\pi G T_{ab}$$

The derivation of the zeroth law of stationary lack holes in arbitrary higher curvature gravity is an open problem!

***Swastik Bhattacharjee, Sudipta Sarkar PRD 2012***

Now the first Law

Consider a static asymptotically flat vacuum black hole solution of GR: $\xi^a = (1,0,0,0)$




The ADM/Komar mass is defined as,

$$\boxed{M = -\frac{1}{8\pi}\int_S \nabla^a \xi^b \, dS_{ab}}$$

The integration is at the special infinity!

We can use Stokes theorem to write this as $M = \int_\Sigma + M_H$

$$M = -2\int_\Sigma \left( T^a_b \xi^b - \frac{1}{D-2} T \xi^a \right) d\Sigma_a + \frac{1}{8\pi}\int_H \nabla^a \xi^b \, dS_{ab},$$

For vacuum solutions, the surface integral vanishes and we only have the boundary term at the horizon.

$$\boxed{M_{ADM} = M_H = \left(\frac{D-2}{D-3}\right) T_H S} \quad T_{BH} = \frac{\hbar c^3}{8\pi G K_B M} \quad S = \frac{K_B c^3 A}{4G\hbar}$$

Let us consider two such asymptotically flat vacuum solutions slightly differing in ADM mass in the space to solutions.

Variation on the space of solutions: $dM_{ADM} = \left(\frac{D-2}{D-3}\right)\left(T_H dS + S dT_H\right)$

Consider the metric: $ds^2 = -\left(1 - \frac{C}{r^{D-3}}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{C}{r^{D-3}}\right)} + r^2 d\Omega^2$

Location of the horizon: $r_h = C^{\frac{1}{D-3}}$ ; Surface Gravity: $\kappa = ((D-3)/2)C^{-\frac{1}{D-3}}$

$$T_H = \frac{\kappa}{2\pi} = \frac{D-3}{4\pi}C^{-\frac{1}{D-3}}; \qquad S = \frac{A_{D-2}C^{\frac{D-2}{D-3}}}{4}.$$

Using all these expressions, one can show:

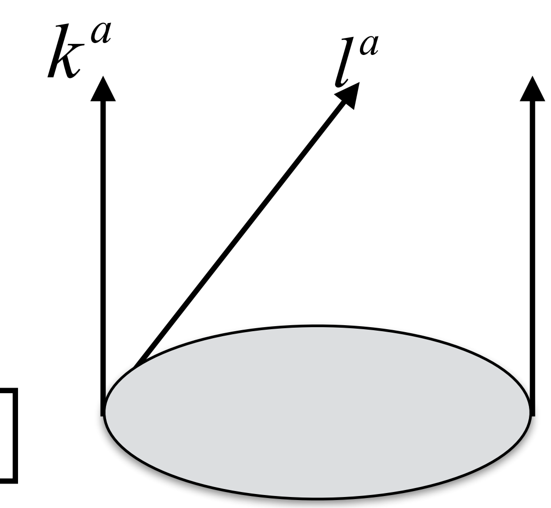
$$dM_{ADM} = T_H dS$$

Stationary comparison version of the first law.

This simplified derivation can be easily generalised to a stationary black hole solutions in any diffeomorphism invariant theory of gravity.

The first law takes the form: $$\frac{\kappa}{2\pi} dS = dM - \Omega dJ$$

But, the entropy is not in general proportional to the area, but a local function of the horizon geometry.




$$S = -2\pi \int_B Q^{ab}(\chi)\varepsilon_{ab} dA \qquad Q^{ab} = \frac{\partial L}{\partial R_{abcd}} \nabla_c \xi_d$$

*Black hole entropy is the Noether charge of Killing isometry generating the horizon.*

$$\varepsilon_{cd} = k_a l_b - k_b l_a$$

Assumptions:

The theory admits a stationary black hole solution with a regular bifurcation surface.

On the bifurcation surface: $\nabla_c \xi_d = \varepsilon_{cd} = k_a l_b - k_b l_a$

$$S_W = -2\pi \int_B \frac{\partial L}{\partial R_{abcd}} \varepsilon_{ab} \varepsilon_{cd} dA$$

The Wald Entropy!

Wald Entropy has several ambiguities. These do not contribute for stationary black holes with a regular bifurcation surface.

***Jacobson, Kang, Myers, PRD 1993***

What are these ambiguities?

They are related to the Noether charge construction, given a Lagrangian of a gravity theory, the Noether potential $Q^{ab}$ is not uniquely determined.

The stationary comparison law is free from these ambiguities.

Example: Gauss Bonnet Gravity: $L_{GB} = R + \alpha\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)$

$$S_W = \frac{1}{4}\int_B \left(1 + \alpha\left[R + 4R_{ab}k^a l^b + R_{abcd}k^a l^b k^c l^d\right]\right) dA \doteq \frac{1}{4}\int \left(1 + 2\alpha \; {}^{D-2}R\right) dA$$

But, we will focus on a simpler version of the first law, which is may be more physically relevant!

The Physical Process Version

Physical Process Law:

How the area/entropy of the horizon changes when the black hole is perturbed.

The answer is in the use of the Null Raychaudhuri Equation which governs the evolution of the horizon area:

Expansion: $\theta = \frac{1}{\delta A}\frac{d(\delta A)}{d\lambda}$ Shear = $\sigma_{ab}$

Raychaudhuri equation describes the evolution of the expansion of the null event horizon

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{D-2} - \sigma^2 - R_{ab}k^a k^b$$

Let us write the black hole entropy as: $S = \frac{1}{4}\int (1+\rho)\; dA$

We set $\rho = 0$ for general relativity, otherwise it is a local function of horizon geometry, not necessarily intrinsic.

Consider the field equation as: $G_{ab} + \alpha H_{ab} = 8\pi T_{ab}$

We want to calculate the change of the entropy due to matter perturbing the horizon.

Remember the GR limit: $\rho \to 0$ and $\alpha \to 0$

The change of the entropy (due to an accretion of matter/ energy):

$$\Delta S = \frac{1}{4}\int\left(\theta + \rho\theta + \frac{d\rho}{d\lambda}\right) dA\, d\lambda$$

Define: $\Theta = \theta + \rho\theta + \frac{d\rho}{d\lambda}$

Integrating by parts, we obtain,

$$\Delta S = -\frac{1}{4}\int \lambda\, \Theta\Big|_{\lambda_1}^{\lambda_2}\, dA - \frac{1}{4}\int \lambda\, d\lambda\, dA\, \frac{d\Theta}{d\lambda}$$

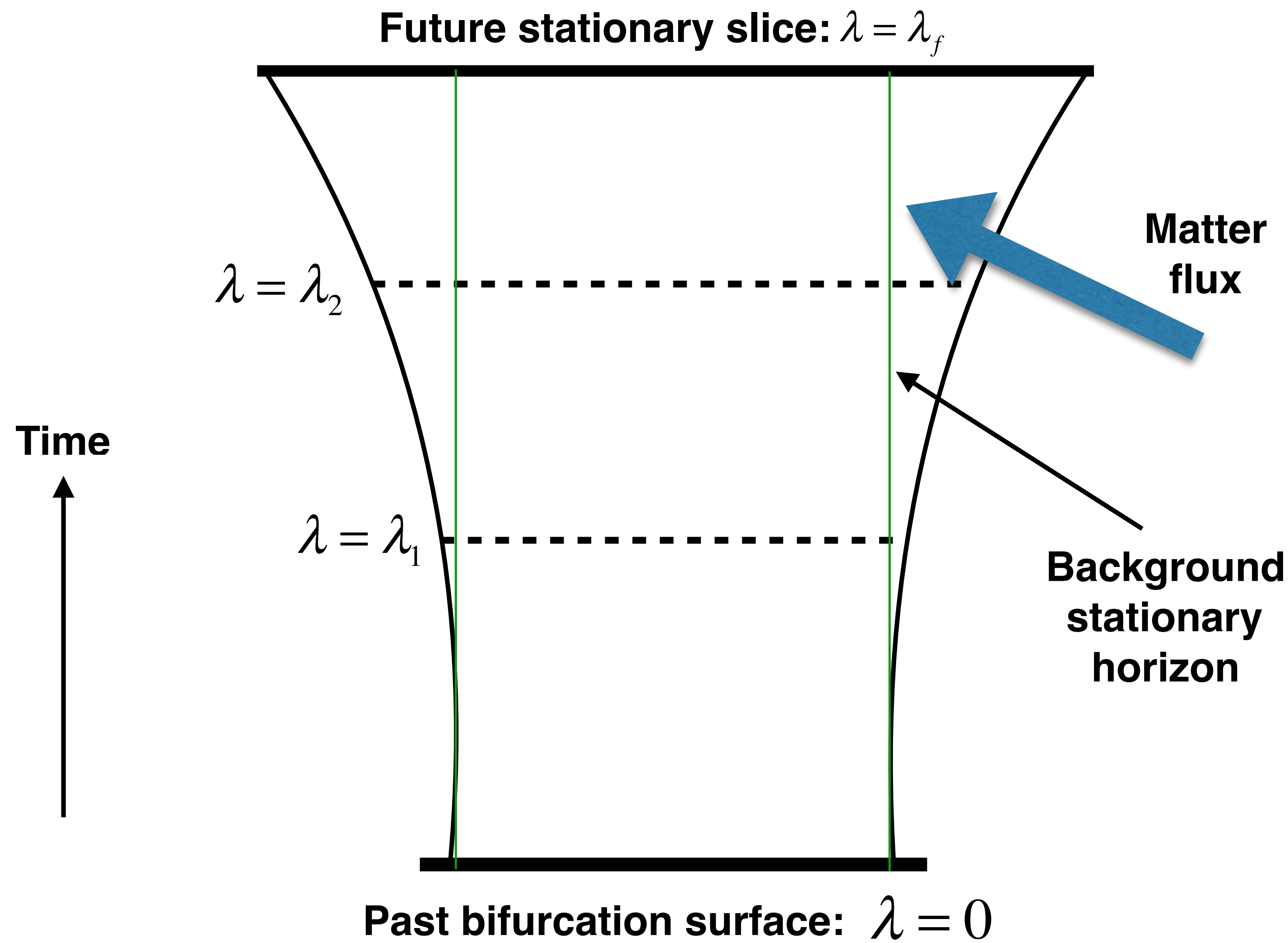
Assumptions:

Existence of Bifurcation surface in the past.

The asymptotic future state is stationary again. All perturbations die down in the future.

$$\Theta(\lambda) \to 0 \text{ as } \lambda \to \lambda_f$$

So the upper limit of the first term vanishes!




$$\Delta S = \boxed{-\frac{1}{4}\int \lambda\ \Theta\Big|_{\lambda_1}^{\lambda_2}\ dA} - \frac{1}{4}\int \lambda\ d\lambda\, dA \frac{d\Theta}{d\lambda}$$

The first term vanishes if we integrate from the past bifurcation surface to the final stationary cross section.

For an interpretation of this terms check: *Akash, Sumanta, Avirup, SS JHEP 2017*

Final Expression for Entropy change between BF surface and final stationary slice:

$$\Delta S = -\frac{1}{4}\int_0^{\lambda_f} \lambda \; d\lambda \; dA \; \frac{d\Theta}{d\lambda} \qquad \Theta = \theta + \rho \; \theta + \frac{d\rho}{d\lambda}$$

The field equation as: $G_{ab} + \alpha H_{ab} = 8\pi T_{ab}$

The final expression of the entropy change becomes:

$$\Delta S = 2\pi \int \lambda \; d\lambda \, dA \; T_{ab} k^a k^b - \frac{1}{4}\int \lambda \; d\lambda \, dA \left( \frac{d^2\rho}{d\lambda^2} - \rho R_{kk} + \alpha H_{kk} \right)$$
$$+ \frac{1}{4}\int \lambda \; d\lambda \, dA \left[ -\left( \frac{D-3}{D-2} \right)(1+\rho)\theta_k^2 + (1+\rho)\sigma_k^2 \right]$$

Now, let us assume that the perturbation is small, we can neglect the second order terms.

The linearized version is:

$$\Delta S = 2\pi \int \lambda \ d\lambda dA \ T_{ab} k^a k^b - \frac{1}{4} \int \lambda \ d\lambda dA \left( \frac{d^2\rho}{d\lambda^2} - \rho R_{kk} + \alpha H_{kk} \right)$$

Let us now make a crucial assumption, given a theory we always choose the entropy density such that,

$$\boxed{\int \lambda \ d\lambda dA \left( \frac{d^2\rho}{d\lambda^2} - \rho R_{kk} + \alpha H_{kk} \right) = \mathrm{O}\left(\varepsilon^2\right)}$$

Note: Though this is trivially true for general relativity, it is highly nontrivial constraint for any modified gravity theory. This needs to be checked case by case.

If it is true. $\boxed{\Delta S = 2\pi \int \lambda \ d\lambda dA \ T_{ab} k^a k^b}$

Consider background Killing vector as $\xi^a = \lambda\kappa k^a$ $\quad d\Sigma_a = k_a \, \mathrm{d}\lambda \, dA$

Then we have the linearised version of the physical process law:

$$\frac{\kappa}{2\pi}\Delta S = \int T_{ab}\xi^a \, d\Sigma^b \quad \Rightarrow \quad T_H \Delta S = \Delta Q$$

The First Law

Assumptions: Existence of the bifurcation surface and stability of the black hole.

Validity of the condition: $$\int \lambda \; d\lambda \, dA\left(\frac{d^2\rho}{d\lambda^2} - \rho R_{kk} + \alpha H_{kk}\right) = \mathrm{O}\left(\varepsilon^2\right)$$

The total entropy change between past bifurcation surface and the future stationary cross section is related to the flux of the inflating matter.

Validity of the condition: $$\boxed{\int \lambda \; d\lambda \, dA \left( \frac{d^2\rho}{d\lambda^2} - \rho R_{kk} + \alpha H_{kk} \right) = \mathrm{O}\left(\varepsilon^2\right)}$$

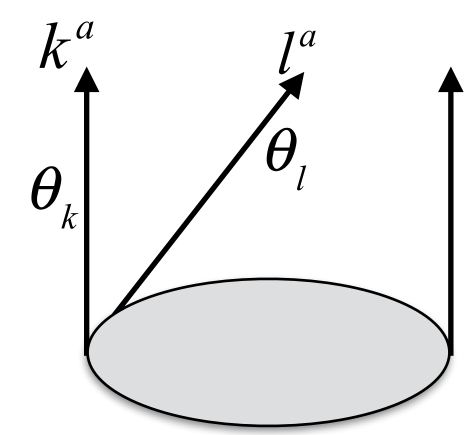
Consider that there is a candidate entropy density, namely the one given by the Wald's formula for a given theory which obeys this condition

$$\int \lambda \; d\lambda \, dA \left( \frac{d^2\rho_w}{d\lambda^2} - \rho_w R_{kk} + \alpha H_{kk} \right) = \mathrm{O}\left(\varepsilon^2\right)$$

This is a non stationary situation, the ambiguities of the Wald entropy need to be considered.

Does the ambiguities effects this condition?

The ambiguities in the Wald's construction can be expressed as




Let us write: $\rho = \rho_w + \Omega\left(\theta_{(k)}\theta_{(l)}, \sigma_{(k)}\sigma_{(l)}\right)$

Let us calculate entropy change with both densities. $\Delta S(\rho)$ and $\Delta S(\rho_w)$

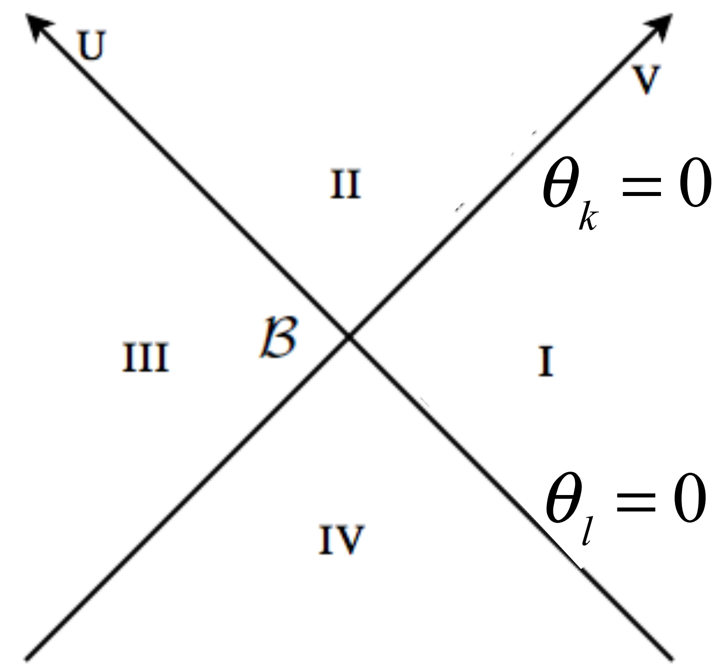
We aim to show: $\Delta S(\rho) = \Delta S(\rho_w)$

Then, up to linearized order, the difference is:

$$\Delta S(\rho) - \Delta S(\rho_w) = -\frac{1}{4}\int dA\, d\lambda \left( \lambda \frac{d^2\Omega}{d\lambda^2} \right) = \frac{1}{4}\int dA\ \Omega\Big|_{\lambda=0}^{\lambda=\lambda_f}$$

Since the final slice is stationary, $\Omega(\lambda = \lambda_f) = 0$

We need to carefully analyse the term: $\Omega(\lambda = 0)$




Consider the background stationary horizon:

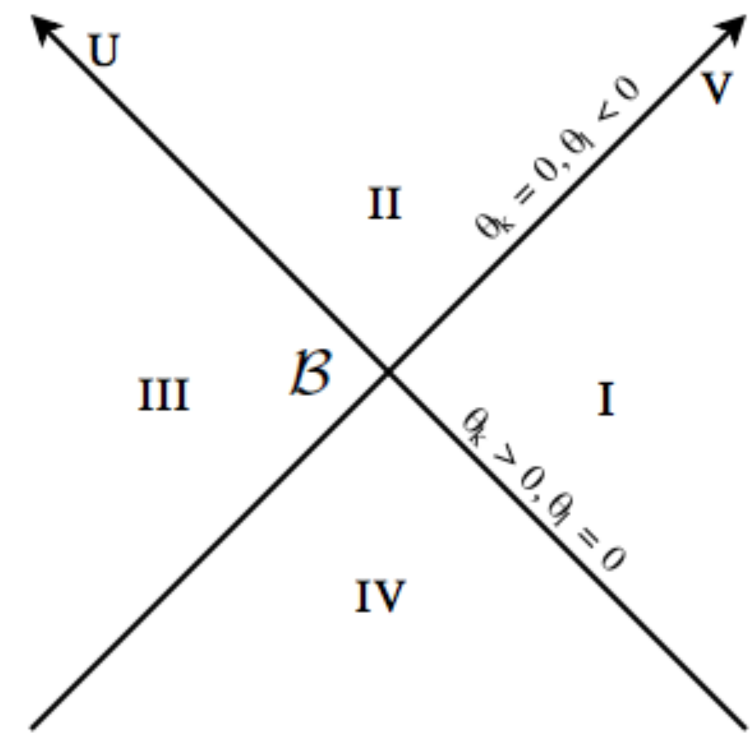
Since the bifurcation surface exists, we can consider a Kruskal extension.

$\theta_k = 0$ On Future Horizon

$\theta_l = 0$ On Past Horizon

Both the expansions must vanish on the bifurcation surface:

For example, Schwarzschild space time in Kruskal coordinates:




$$\theta_{(k)} \sim -U \qquad \theta_{(l)} \sim -V$$

Since both the expansions must vanish on the bifurcation surface:

Therefore, on the bifurcation surface

$$\Omega(\lambda = 0) \sim \theta_k \theta_l \sim \mathrm{O}(\varepsilon^2)$$

This gives: $\Delta S(\rho) - \Delta S(\rho_w) = O(\varepsilon^2)$

The physical process first law is independent of the ambiguities in the Wald's construction

Remember: We are working only with an integrated version!

Validity of the condition: $$\int \lambda \; d\lambda \, dA \left( \frac{d^2 \rho}{d\lambda^2} - \rho R_{kk} + \alpha H_{kk} \right) = \mathrm{O}\left(\varepsilon^2\right)$$

This condition is proven for Wald entropy (+ the ambiguities) for:

General Relativity (Trivially True)

f(R) Gravity

Lovelock gravity

A general Lagrangian quadratic in curvatures.

arXiv:*1905.04466*

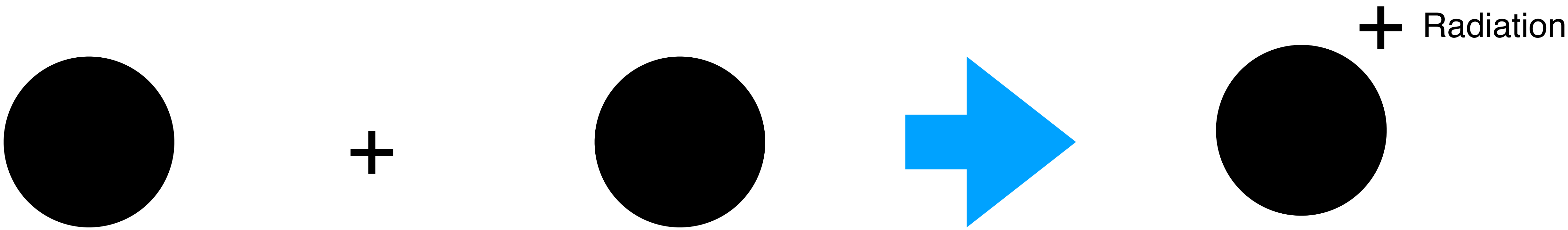
A general understanding of this condition is still lacking

# The second law:

Hawking's area theorem

*The area of a black hole can never decreases!* $\Delta A \geq 0$

Consider two black holes of masses $M_1$ and $M_2$ merge to create a bigger black hole, the area theorem insists that the area of the final black hole must be larger than the sum of areas of the individual ones.



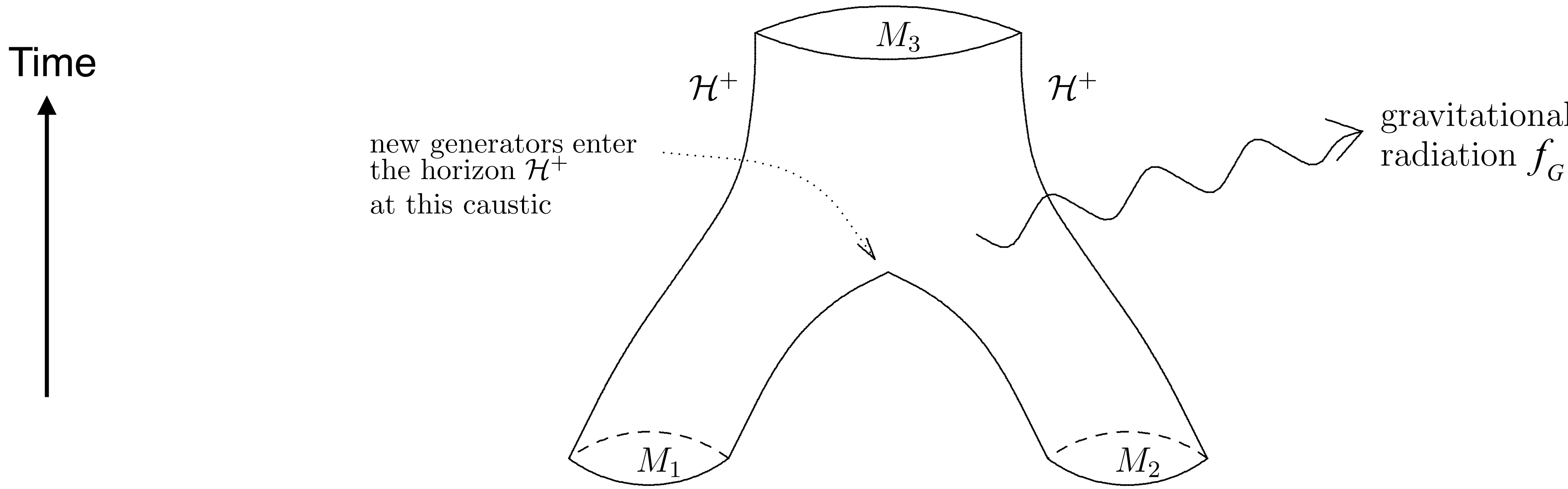

$A_1 = 4\pi(2M_1)^2 = 16\pi M_1^2$ $A_2 = 4\pi(2M_2)^2 = 16\pi M_2^2$ $A_3 = 4\pi(2M_3)^2 = 16\pi M_3^2$

Energy Conservation: $M_3 = M_1 + M_2 - f_G$

Area theorem demands: $A_3 > A_1 + A_2 \rightarrow 16\pi M_3^2 > 16\pi(M_1^2 + M_2^2)$

Now, this limits how much gravitational energy can be radiated away in the process.

In fact, the area theorem limits the efficiency of the energy conversion; akin to the second law of Thermodynamics




$$16\pi M_3^{\,2} > 16\pi(M_1^{\,2} + M_2^{\,2}) \qquad M_3 = M_1 + M_2 - f_G$$

Define the efficiency: $\eta = 1 - \dfrac{M_3}{M_1 + M_2}$

The area theorem limits the efficiency as $\eta \leq 1 - \dfrac{1}{\sqrt{2}}$

Proof of second law in General Relativity: Start with Raychaudhuri Equation

$$\theta = \frac{1}{\delta A}\frac{d(\delta A)}{d\lambda}$$

Shear = $\sigma_{ab}$

Raychaudhuri eq. $$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{(D-2)} - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b$$

Use Einstein's eq and Null energy Condition $T_{ab}k^a k^b \geq 0$

$$\boxed{\frac{d\theta}{d\lambda} < 0}$$

Stability demands ultimately the black hole settle down to a stationary state: $\theta_f = 0$

This is a non trivial assertion. If we do not assume this, there will be caustic in the future

$$\boxed{\theta \to -\infty}$$

Check the Wald's General Relativity Books, chapter 12

So what we have: $$\frac{d\theta}{d\lambda} < 0; \quad \theta_f = 0$$

Therefore, $\theta(\lambda) > 0$

The area always increases:

↓

The black hole entropy always increases:

Note: This is more than what we want.

In usual thermodynamics, entropy is defined only for the equilibrium states, the total entropy change between initial and final equilibrium states must be positive.

Area theorem does more, it gives you a local increase law, not only total increase is positive, the black entropy increases at every instant.

Does second law holds beyond general relativity?

Consider the entropy: $S = \frac{1}{4}\int\left(1+\rho\right) dA$

The field equation is: $G_{ab} + \alpha H_{ab} = 8\pi T_{ab}$

Instantaneous change of the entropy: $\Delta S = \frac{1}{4}\int \Theta \, dA \, d\lambda$

$\Theta = \left(\theta + \rho\theta + \frac{d\rho}{d\lambda}\right)$ Change in entropy per unit area! We call this the generalised entropy!

We want to establish that this is positive at any arbitrary cross section of the horizon.

We start with calculating the change of the generalised expansion $\Theta = \left( \theta + \rho\theta + \frac{d\rho}{d\lambda} \right)$

$$\frac{d\Theta}{d\lambda} = \frac{d\theta}{d\lambda} + \theta \frac{d\rho}{d\lambda} + \rho \frac{d\theta}{d\lambda} + \frac{d^2\rho}{d\lambda^2}$$

In case of GR: $\frac{d\Theta}{d\lambda} = \frac{d\theta}{d\lambda} < 0$

We demand the same for our theory of gravity: Choose the entropy such that with NEC

$$\boxed{\frac{d\Theta}{d\lambda} < 0}$$

We require a thermodynamic generalisation of the Raychaudhuri equation

Except f(R) type theories of gravity, this is not easy to establish

***Kang, Jacobson, Myers PRD 1995***

Doing this for a general quadratic curvature theory $L = R + \left(\alpha R^2 + \beta R_{ab} R^{ab} + \gamma L_{GB}\right)$

Here are question of ambiguities are important: we need to resolve the ambiguities to obtain the second law even for the linearised case.

The validity of the second law demands there are corrections to the Wald formula. These corrections vanish for a stationary black hole, but contributes to the entropy change:

Written in terms of entropy density: $\rho_{BH} = \rho_{Wald} + a\ \theta_{(k)}\theta_{(l)} + \mathrm{b}\ \sigma_{ab(k)}\sigma^{ab(l)}$

The requirement that second law is true for linearised perturbation will give us values of the unknown coefficients.

The correct entropy formula for the quadratic curvature theory:

$$S = \frac{1}{4}\int \left[1 + 2\alpha R - 2\beta\left(R_{kl} - \frac{1}{2}\theta_{(k)}\theta_{(l)}\right) + 2\gamma\ {}^{D-2}R\right] dA \ = \mathrm{Wald} + \mathrm{Corrections}$$

***Srijit Bhattacharjee, SS, Wall PRD 2016***

$$L = R + \left(\alpha R^2 + \beta R_{ab} R^{ab} + \gamma L_{GB}\right)$$

$$S = \frac{1}{4}\int \left[ 1 + 2\alpha R - 2\beta \left( R_{kl} - \frac{1}{2}\theta_{(k)}\theta_{(l)} \right) + 2\gamma \ {}^{D-2}R \right] dA$$

This expression **exactly** matches with the Holographic Entanglement Entropy of the Corresponding theory if this black hole is in Anti-de sitter space.

***Srijit B, SS, Aron Wall PRD 2016***

Demanding that this entropy also obeys second law at higher orders leads to interesting bounds on the higher curvature couplings.
Identical bounds are previously obtained from the consistency of the boundary theory.

***Arpan B, Srijit B, SS, Aninda Sinha PRD 2016***

***Fairoos, Avirup, SS PRD 2018***

***An intriguing connection between holography and black hole thermodynamics***

How to prove second law beyond linearised perturbations:

The Einstein GB case: $L_{GB} = R + \alpha\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)$ $\rho = \frac{1}{4}\left(1 + 2\alpha^{D-2}R\right)$

$$4G\frac{d\Theta}{d\lambda} = -\frac{\theta^{(k)2}}{D-2} - \sigma^{(k)ab}\sigma^{(k)}_{ab} - 6\alpha\frac{(D-4)\theta^{(k)2}\mathcal{R}}{(D-2)^2} - 2\alpha\sigma^{(k)ab}\sigma^{(k)}_{ab}\mathcal{R} - 4\alpha\frac{(D-8)\theta^{(k)}\sigma^{(k)ab}\mathcal{R}_{ab}}{(D-2)}$$
$$+8\alpha\sigma^{(k)a}_{c}\sigma^{(k)cb}\mathcal{R}_{ab} - 4\alpha\mathcal{R}_{fabp}\,\sigma^{(k)ab}\sigma^{(k)pf}$$
$$+2\alpha\left[2\left(D_c\beta^c\right)\left(K^{(k)}_{ab}K^{(k)ab}\right) - 4\left(D_c\beta^b\right)\left(K^{(k)}_{ab}K^{(k)ac}\right) + 2\beta^c\beta_cK^{(k)}_{ab}K^{(k)ab} - 4\beta_cK^{(k)}_{ab}\beta^bK^{(k)ac}\right]$$
$$+4\alpha\left[2\left(D^b\beta^f\right)\left(K^{(k)}K^{(k)}_{bf}\right) - 2\left(D_a\beta^a\right)\left(K^{(k)}\right)^2 + 2h^{ab}\beta^cK^{(k)}_{ac}\beta_bK^{(k)} - h^{ab}\beta_a\beta_b(K^{(k)})^2\right]$$
$$+4\alpha R_{kk}\frac{(D-3)(D-4)\theta^{(n)}\theta^{(k)}}{(D-2)^2} - 4\alpha h^{ac}h^{bd}R_{kckd}\frac{(D-4)\theta^{(k)}\sigma^{(n)}_{ab}}{D-2} - 4\alpha h^{ac}h^{bd}R_{kckd}\frac{(D-4)\theta^{(n)}\sigma^{(k)}_{ab}}{D-2}$$
$$+8\alpha h^{ac}h^{bd}R_{kckd}\sigma^{(k)}_{af}\sigma^{(n)f}_{b} - 4\alpha R_{kk}\sigma^{(k)}_{ab}\sigma^{(n)ab} - 8\pi G\,T_{kk} + \alpha\,(\text{total derivatives})$$

***Fairoos, Avirup, SS PRD 2018***

It is hard to make any general conclusion, but an order by order calculation is possible!

**The validity of full second law beyond general relativity is still an open issue:**

Going beyond Classical Second law:

The sum of black hole entropy and the entropy of everything outside can not decrease

$$S_{\mathrm{gen}} = \frac{\langle A \rangle}{4G\hbar} + S_{\mathrm{out}},$$

where $S_{\mathrm{out}}$ is the von Neumann entropy $-\mathrm{tr}\,(\rho \ln \rho)$ of the density matrix $\rho$ of the matter outside the horizon.

**Generalized second law** $\Delta S_{gen} \geq 0$

This is proven for semiclassical quantum fields falling across a causal horizon with some reasonable assumptions related to the nature of quantum field theory.

***Aron Wall PRD 2012***

감사합니다