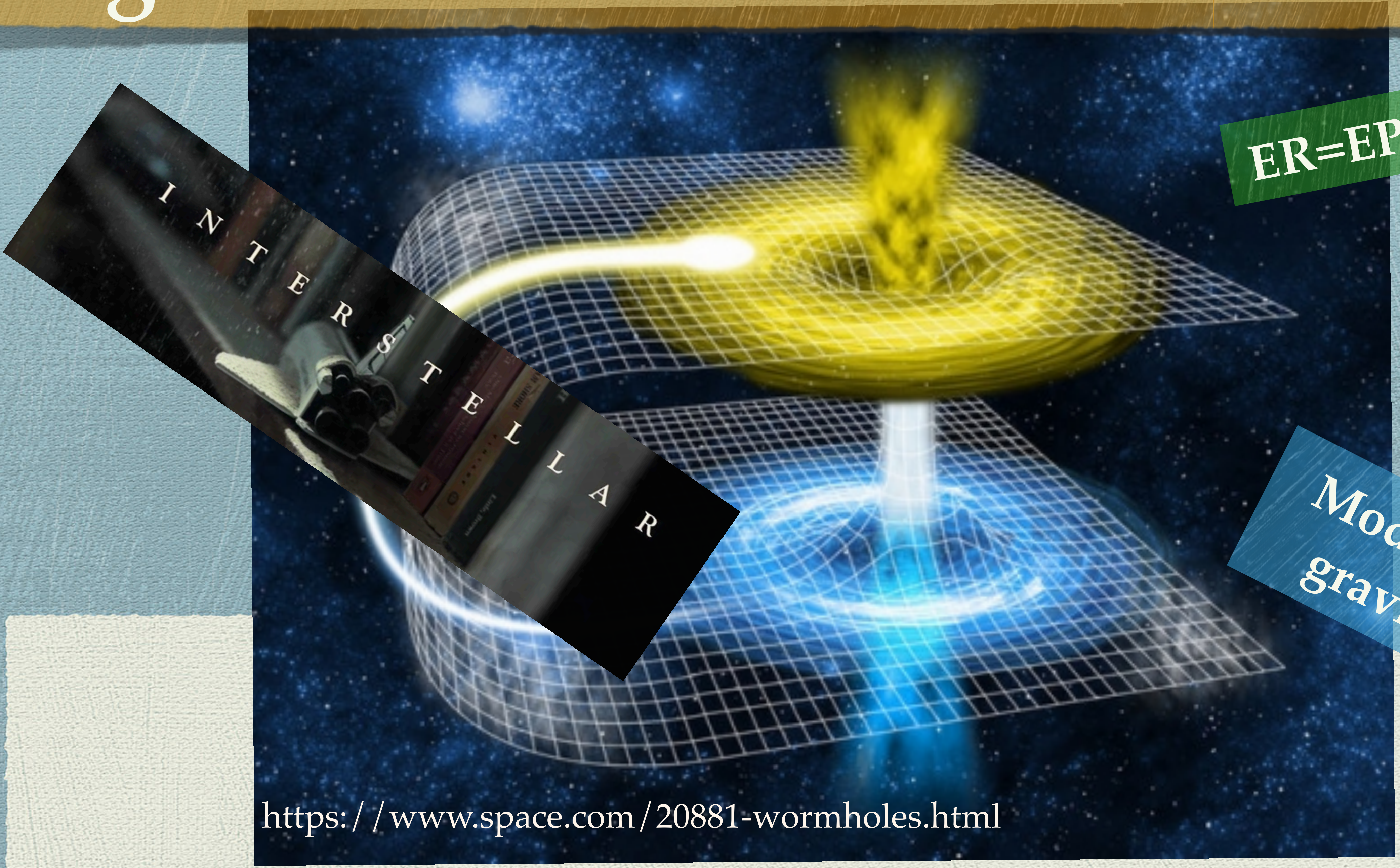
Asia-Pacific School and Workshop on Gravitation and Cosmology 2020,
2월 13일, Daejon, Korea

Regular Wormholes in General Relativity



https://www.space.com/20881-wormholes.html

ER=EPR

Modified gravity

1992-11
1999-22
2010-40
2018-115
2019-134
2020-16

arXiv:1905.10050[JCAP09(2019)001] HC, Y Lee
arXiv:1911.00425 HC

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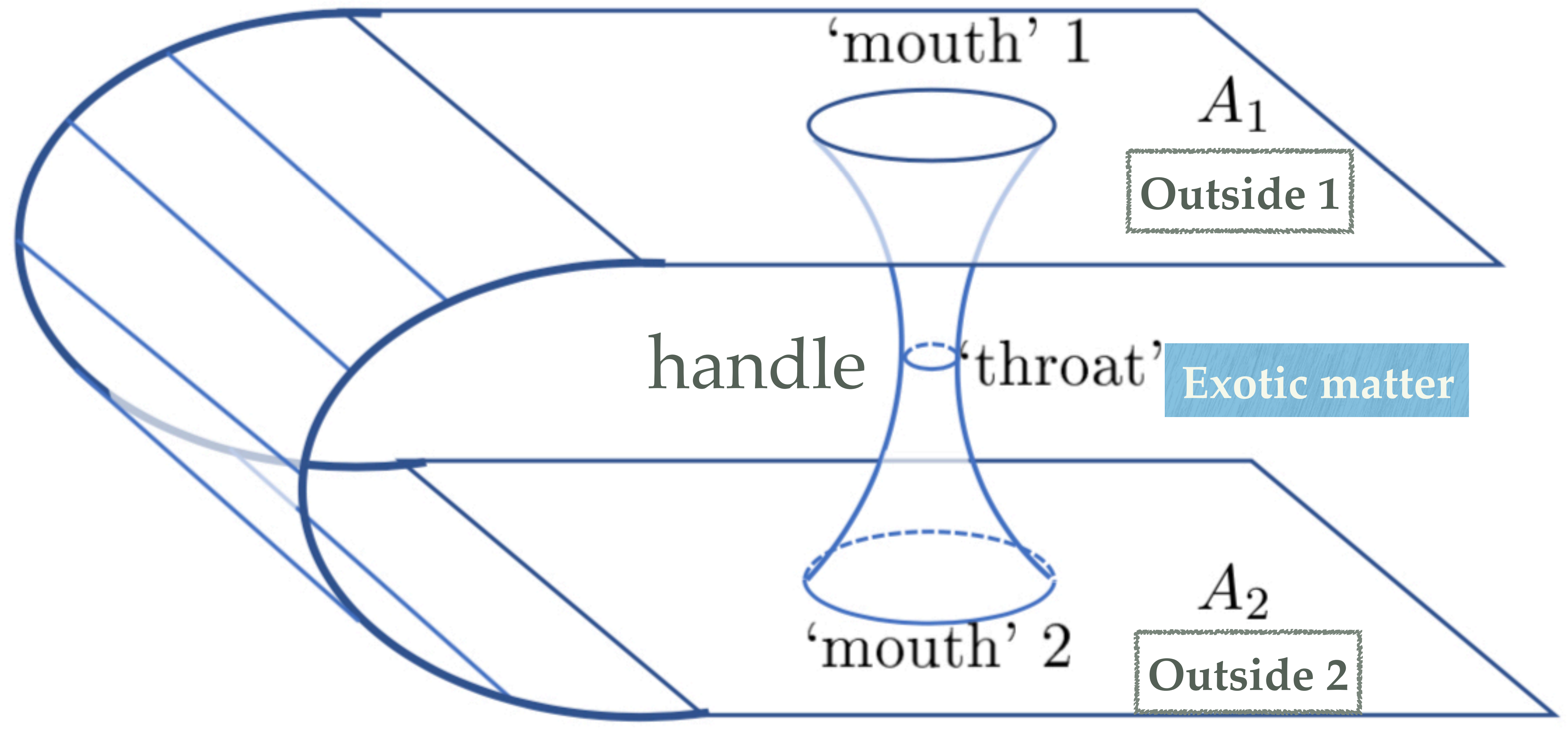
Contents

Traversable wormhole, terminology

A spacelike section of wormhole spacetime

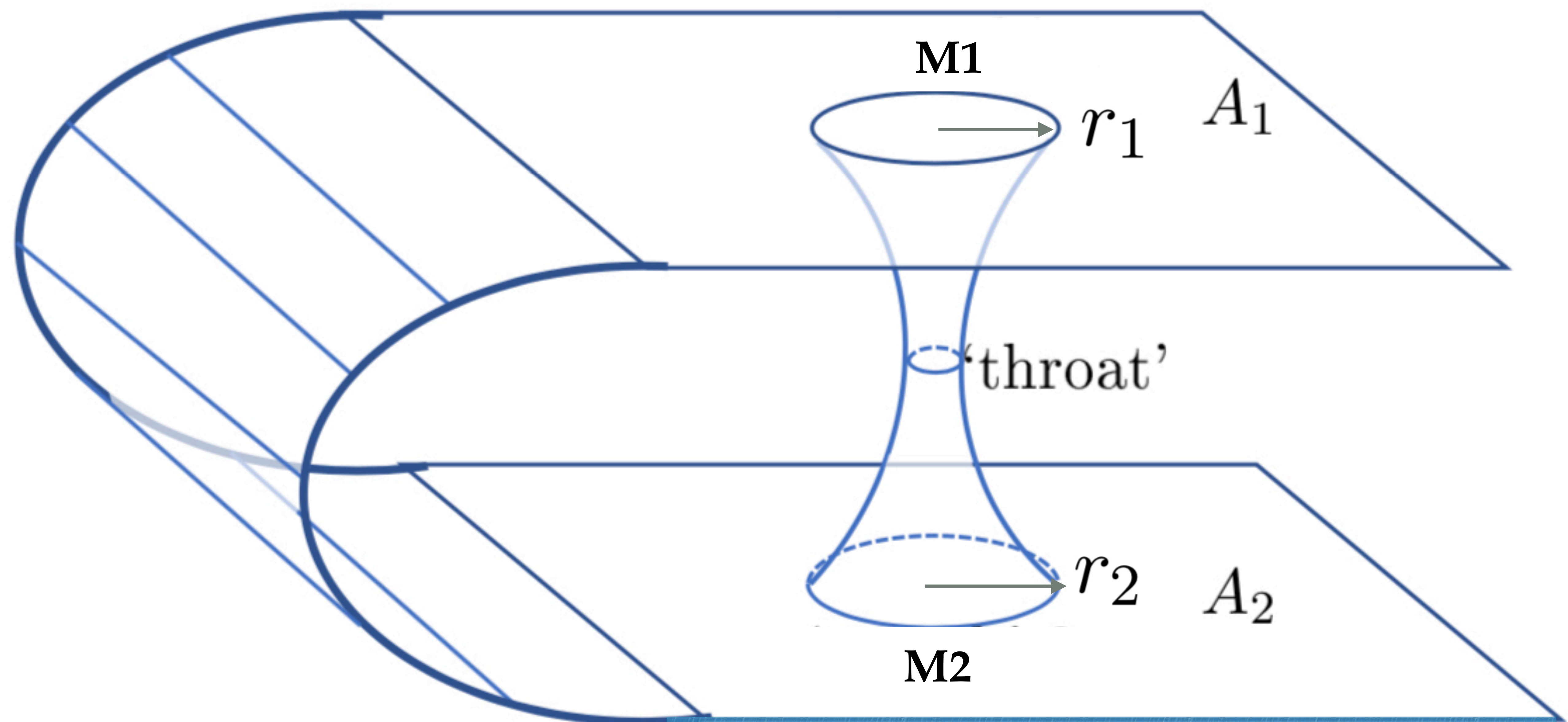


Misner, Wheler (1957), Morris, Thorne, (1988)

Length of the handle does not depend on the distance btw the mouths in external space.

—> Space travel.

A few odd things in wormhole spacetime



Upper/Lower half are not symmetrical.

For an observer who travel through the wormhole, <u>the speed of proper time flow</u> is not the same for the upper/lower half.

-> Timelike Killing vector can be absent when upper/lower are adjoined.

Why do we search for new wormhole solutions?

Traditional wormhole solutions were obtained by using a thin(thick)-shell method. (Matter is confined to a narrow region around the throat.)

1. This requires **lids** which confines `exotic' matter in the throat.

→ Search for solution without lids.

Traditional way of finding a wormhole solution:

1) Guess metric satisfying the flare out condition.
2) Find the EOS for the corresponding matter by calculating the Einstein tensor. Then,
   - The EOS of the matter must be position dependent.
   - It is not easy to find out an appropriate matter satisfying the EOS.
   - The solution space cannot be complete.

→ 1. Specify the matter first then solve the Einstein equation.

2. Categorize all the wormhole solutions.

The work is based on

1. General relativity without cosmological constant

$$I = \int_M \sqrt{-g} d^4x \left(\frac{1}{16\pi} R + L_m \right).$$

2. Spherical symmetry

$$ds^2 = -f(r)dt^2 + \frac{1}{1 - 2m(r)/r} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$m(r) = 4\pi \int^r r'^2 \rho(r') dr'$$

3. Anisotropic matter with linear equation of states

$$T^{\mu\nu} = \rho u^\mu u^\nu + p_1 r^\mu r^\nu + p_2 (\theta^\mu \theta^\nu + \phi^\mu \phi^\nu),$$

$$p_1 = w_1 \rho, \quad p_2 = w_2 \rho$$

Einstein equation and its reduction

Combining the relation $G_{tt} = 8\pi T_{tt}$ and $G_{rr} = 8\pi T_{rr}$,

$$\frac{f'}{f} = \frac{2(m + 4\pi r^3 p_1)}{4(r-2m)}. \quad \rightarrow \quad f(r) = \tilde{f}_0 \frac{(r-2m)^{-w_1}}{r} \exp\left[(1+w_1)\int_{r_0}^{r} \frac{1}{r-2m(r)} dr\right],$$

Anisotropic TOV equation $\nabla^a T_{ab} = 0$ +above Eq.

$$p_1' = -(\rho + p_1)\frac{m + 4\pi r^3 p_1}{r(r-2m)} + \frac{2(p_2 - p_1)}{r}.$$

$$\rightarrow \quad f(r) = f_b \left(\frac{r}{b}\right)^{\frac{4(w_2 - w_1)}{1+w_1}} \left(\frac{\rho}{\rho_b}\right)^{-\frac{2w_1}{1+w_1}},$$

The radius of the throat

First order autonomous equation on the (u,v) plane:

$$\frac{m''}{m'} = -\frac{1+w_1}{2w_1}\frac{1+2w_1 m'}{r-2m} + \frac{1+w_1+4w_2}{2w_1 r},$$

$$\rightarrow \quad \frac{du}{dv} = \frac{-1}{1+w_1}\frac{(1-u)(2v-u)}{v\left[v - v_b + s(1-u)\right]},$$

$$u \equiv \frac{2m(r)}{r}, \qquad v \equiv \frac{dm(r)}{dr} = 4\pi r^2 \rho,$$

$$v_b \equiv -\frac{1}{2w_1}, \qquad s \equiv -\frac{1+w_1+4w_2}{2w_1(1+w_1)}.$$

throat value

slope

Existence condition for a wormhole

At the wormhole throat, g_{rr} must have a coordinate singularity.
Therefore, around the throat, assume $g_{rr} \approx g(1-b/r)^{-1}$ with $g > 0$

$$f(r) \propto \frac{g^{w_1}}{r}(r-b)^{(1+w_1)g-w_1} \text{ must be finite.}$$

This determines $g \equiv \lim_{r\to b}\left(1-\frac{b}{r}\right) g_{rr} = \frac{w_1}{1+w_1} > 0.$ $w_1 < -1$ or $w_1 > 0.$

From the definition of g_{rr}, we also get around the throat,

$$m(r) \simeq \frac{b}{2w_1}\left(1 + w_1 - \frac{r}{b}\right) \quad \Rightarrow \quad m'(b) = -\frac{1}{2w_1}.$$

$v_b = -\frac{1}{2w_1}$

Two types of wormhole throats:

1. $w_1 > 0$ $(v_b < 0)$: the mass function decreases from $b/2$ with r. $\rho(b) = v_b/4\pi b^2 < 0$. **Negative energy density**

2. $w_1 < -1$ $(0 < v_b < 1/2)$: the mass function increases with r. The density takes a positive value.

Phantom-like matter

General analysis

$$\frac{du}{dv} = \frac{-1}{1+w_1}\frac{(1-u)(2v-u)}{v\left[v-v_b+s(1-u)\right]},$$

s is the slope of R2.

Four interesting lines and points on the (u,v) plane:

1. $\frac{dv}{du} = 0$ on R1: $v = 0$, R2: $v - v_b = s(u-1)$.
2. $\frac{du}{dv} = 0$ on B1: $u = 1$, B2: $u = 2v$.

On R2, Solution curve passes horizontally. On B2, solution curve passes vertically.

$$\mathcal{O}(0,0), \quad P_{\mathrm{E}}(1,0), \quad P_b(1,v_b), \quad P_{\mathrm{R}}(2v_R, v_R),$$

B2 meets R2.

On R1, v=0, dv=0: Solution curve can touch R1 only at special points $\mathcal{O}(0,0)$, $\mathrm{P_E}(1,0)$

On B1, u=1, du=0: Solution curve can touch B1 only at special points $\mathrm{P_E}(1,0)$, $\mathrm{P_T}(1,v_b)$

$$\frac{dr}{r} = \frac{du}{2v-u} = \frac{-1}{1+w_1}\frac{1-u}{v[v-v_b+s(1-u)]}dv,$$

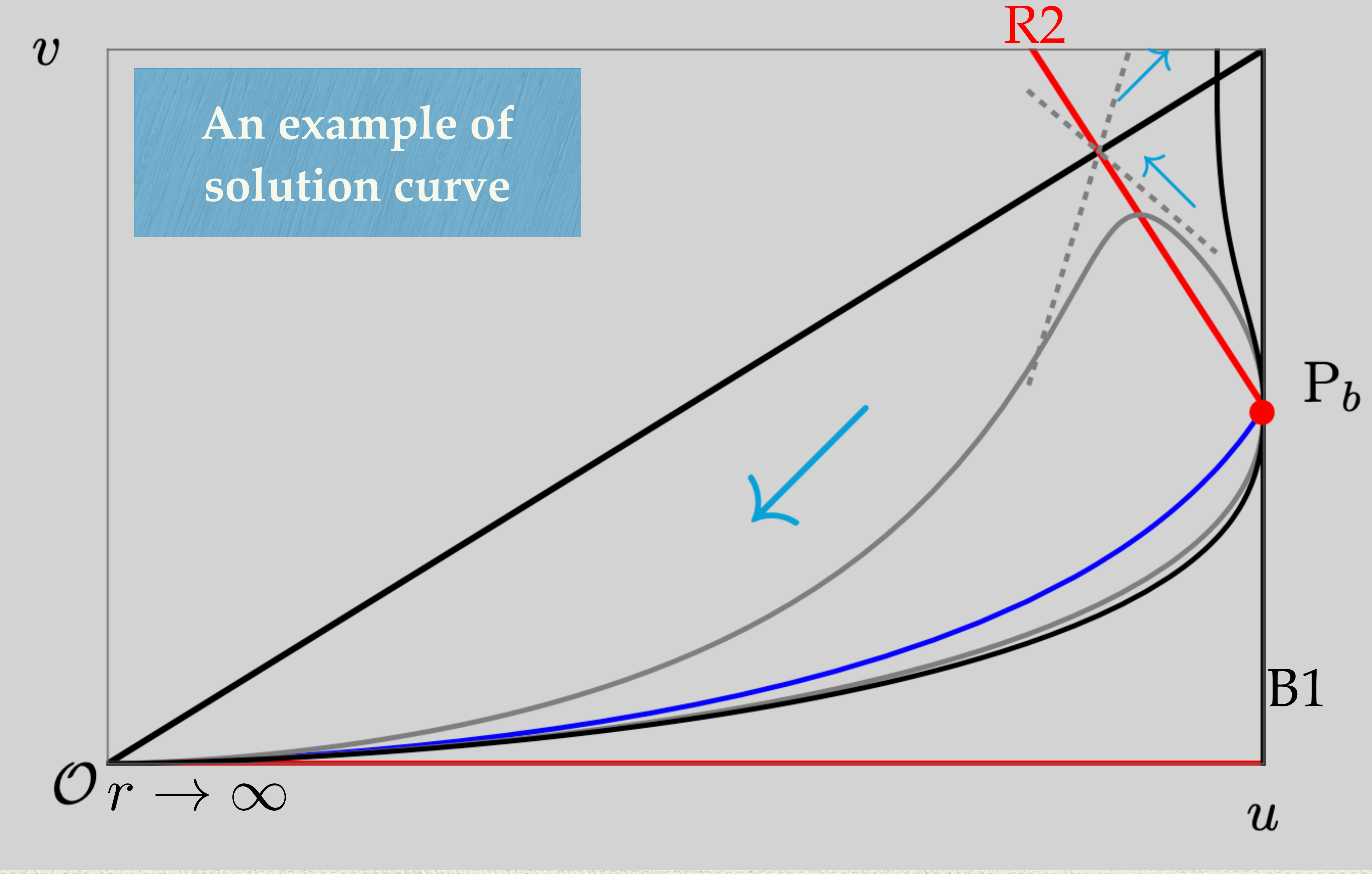
The radius increases/decreases with *u* when the curve is above/below B2.

The radius increases/decreases with *v* depending on many factors.

:Signature of $1+w_1$, whether the curve is above or below R1, R2.

We would like to represent solution as a curve in (u,v) plane

$$ds^2 = f_b\left(\frac{r}{b}\right)^{\frac{4w_2}{1+w_1}}\left(\frac{v}{v_b}\right)^{-\frac{2w_1}{1+w_1}}dt^2 + \frac{1}{1-u}dr^2 + r^2 d\Omega^2$$



The Cyan arrows denote the increasing direction of the radius.

The asymptotic behavior around `$\mathcal{O}$'

$$|u|, |v| \ll 1 \qquad u'(v) = \alpha\left(\frac{u}{v} - 2\right), \quad \alpha = -\frac{w_1}{2w_2} > 0.$$

$$u = \begin{cases} \frac{2\alpha}{\alpha-1} v + q\left(\frac{v}{v_b}\right)^{\alpha} & (\alpha \neq 1), \\ -2v \log|v| + q'\ v & (\alpha = 1), \end{cases}$$

$\mathcal{O}$ plays the role of asymptotic infinity:

$$r = r_0 \left(\frac{v}{v_b}\right)^{-\alpha},$$

mass function: $$m(r) = \frac{ur}{2} = \frac{r_0}{2}\left[q - \frac{2\alpha}{1-\alpha}\left(\frac{r}{r_0}\right)^{(\alpha-1)/\alpha}\right].$$

When $\alpha \geq 1 \ \ (0 < w_2 \leq -w_1/2)$, the mass function diverges.

When $0 < \alpha < 1 (-2w_2 < w_1 < 0$ or $0 < w_1 < -2w_2)$, it approaches a finite value. Forms **asymptotically flat** geometries.

$$\begin{aligned} f(r) &= f_0 \left(\frac{r}{r_0}\right)^{\frac{4w_2}{1+w_1}} \left(\frac{v}{v_b}\right)^{-\frac{2w_1}{1+w_1}} \\ &= f_0\left(1 - \frac{2m_\infty}{r}\right) + \mathcal{O}\left(\frac{1}{r^2}\right), \end{aligned}$$

Behavior around the `wormhole throat' P_b

i) The symmetric solution $\beta = 1, \qquad u = 1 - s^{-1}(v - v_b),$

ii) The asymmetric solution $\beta = 2, \qquad u = 1 - \kappa(v - v_b)^2,$

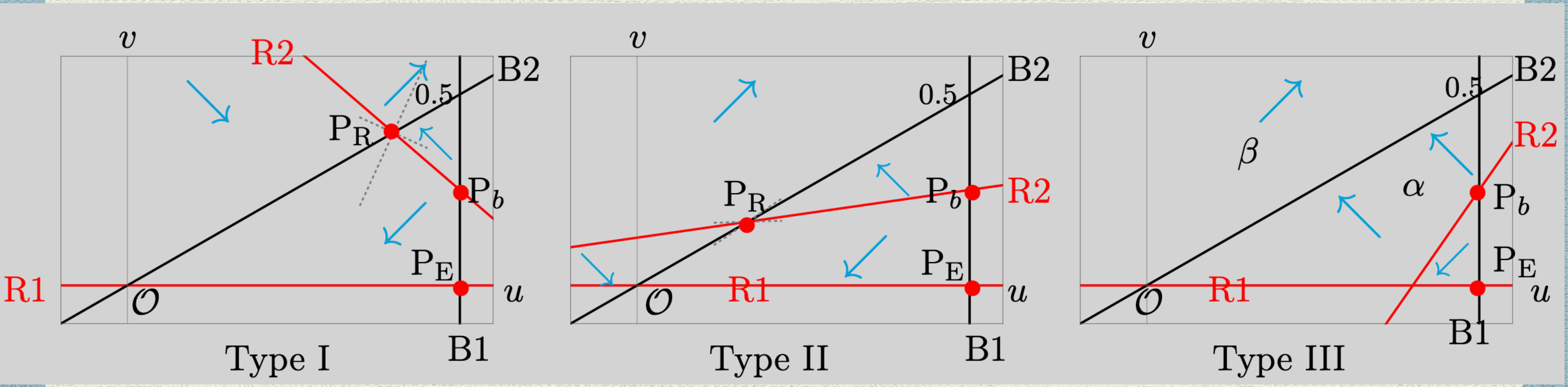
Radius: $$r \approx b\left[1 + \frac{w_1}{1 + w_1}(1 - u)\right].$$

Radius takes its minimum at P_b.

The symmetric solution appears in the *large* \kappa *limit* of the asymmetric solution.

$w_1 < -1$ Case: the Phantom-like matter

I: $s < 0 \quad \left(w_2 > -\frac{1+w_1}{4}\right)$, II: $0 \le s \le v_b \quad \left(0 \le w_2 \le -\frac{1+w_1}{4}\right)$, III: $s > v_b \quad (w_2 < 0)$.



The Cyan arrows denote the increasing direction of the radius.

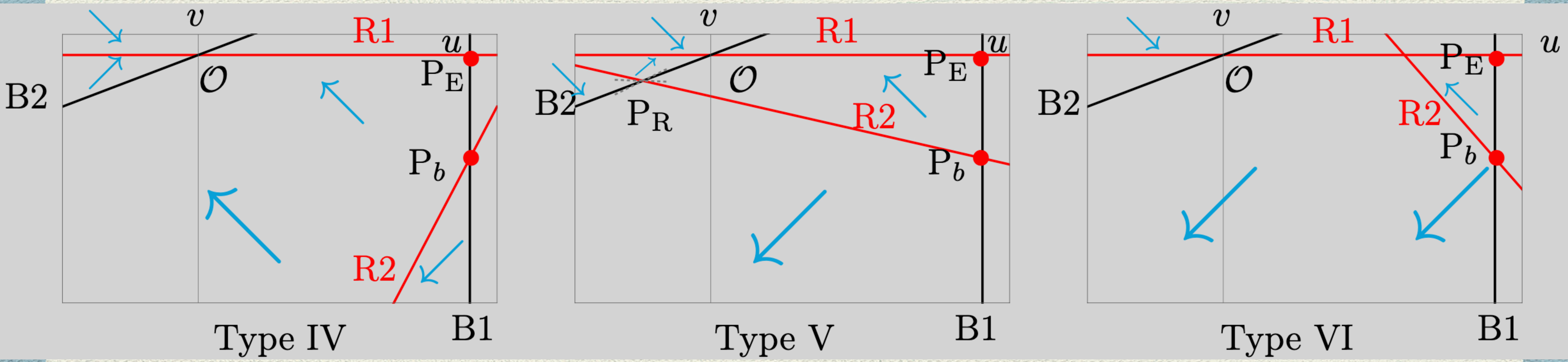
Type III does not have a regular asymptotic region:

Begin at the wormhole throat at P_b and follow the increasing direction of *r.*

Then, we find that *both the upper and the lower solution curves will reach the point (u,v)= (1,\infty).*

$w_1 > 0$ and $\rho < 0$ Case : negative energy density

IV: $s > 0 \quad \left(w_2 < -\frac{1+w_1}{4}\right)$, V: $v_b \le s \le 0 \quad \left(-\frac{1+w_1}{4} \le w_2 \le 0\right)$, VI: $s < v_b \quad (w_2 \ge 0)$.



Type VI does not allow a wormhole solution with a regular asymptotic region:

Both the upper and the lower solution curves approach (u,v)= (-\infty,-\infty).

Type IV: two regular asymptotic region ending at $\mathcal{O}$.

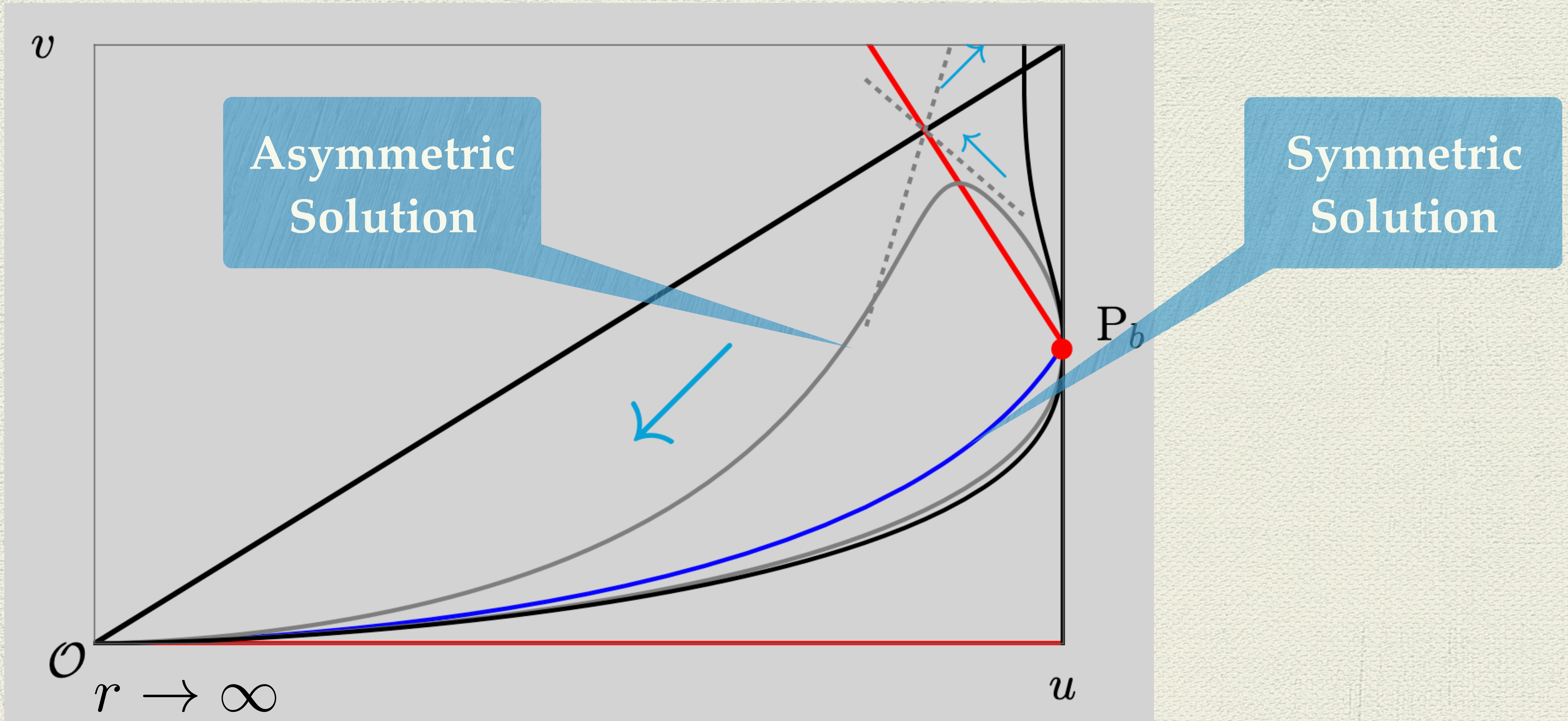
Type V: One regular asymptotic region ending at $\mathcal{O}$
and one singular end.

To summarize, we are interested in the cases with Type I, II, IV, V.
For all the cases, $w_1 w_2 < 0$ is satisfied. < No isotropic fluid!>

Numerical solutions for Phantom-like matter

$w_1 < -1$ Case

Type I solution:

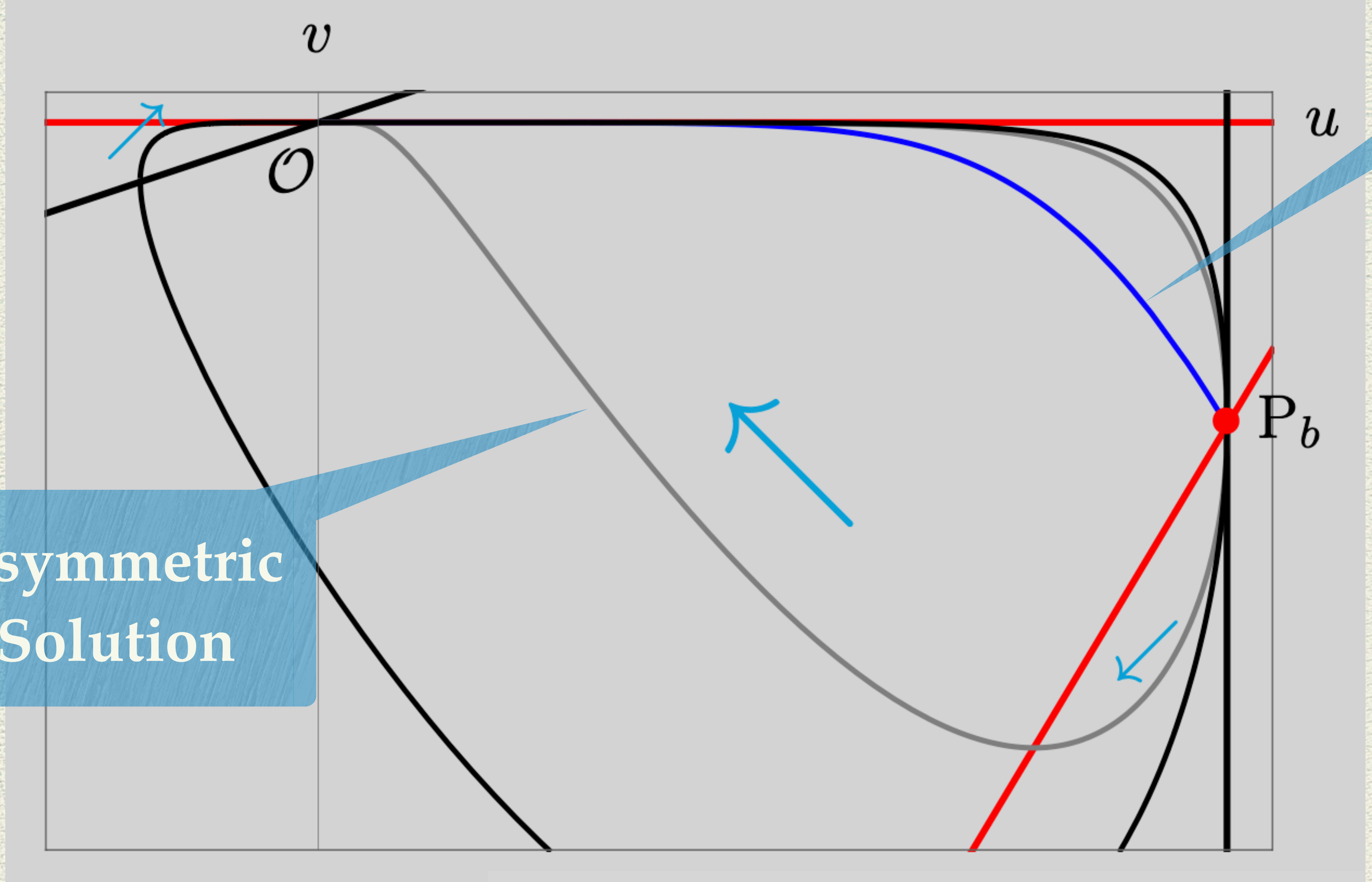


Numerical solutions for the Type I with $w_1 = -2$ and $w_2 = 1.5$.

Numerical solutions with negative energy density

$w_1 > 0$ and $\rho < 0$ Case

Type IV solution:



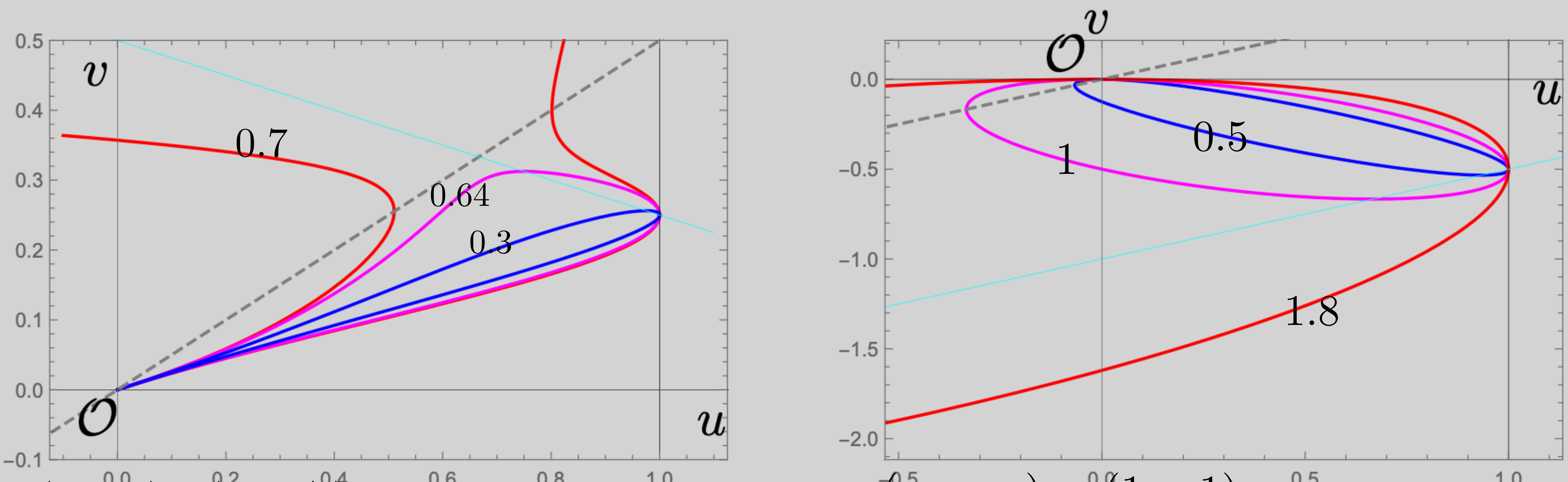
type IV with $w_1 = 2$ and $w_2 = -3$.

Analytic solution for the case $1 + w_1 + 2w_2 = 0$.

$$\rho + \sum_i p_i = (1 + w_1 + 2w_2)\rho = 0.$$

$$\kappa = (v_b \lambda)^{-2}$$

TOV Eq. → $$\left|\frac{v}{v_b} - u\right| \left(\frac{v}{v_b}\right)^{-\frac{w_1}{1+w_1}} = \lambda\,(1-u)^{1/2},$$



$(w_1, w_2) = (-2, 1/2)$ $\lambda = 0.3, 0.64,$ and 0.7

$(w_1, w_2) = (1, -1)$

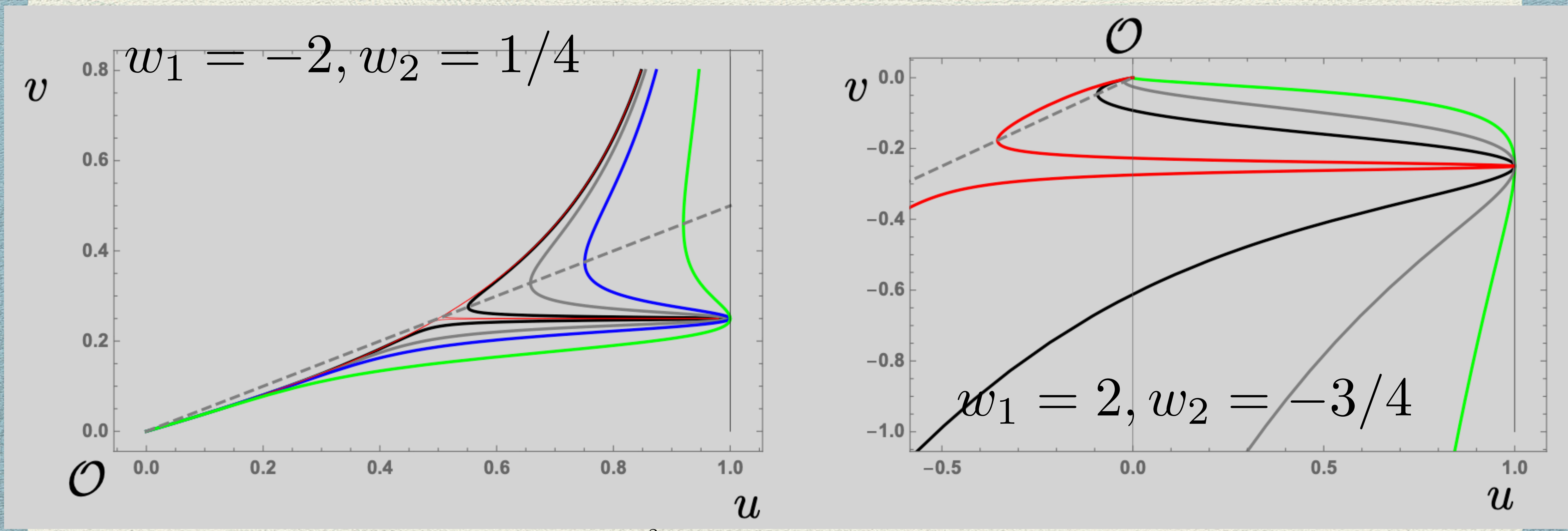
Simplest known solution: $\lambda \to 0$ limit gives $v = v_b u$

$$r = b\left(\frac{v}{v_b}\right)^{-\frac{w_1}{1+w_1}}, \qquad \rho = \frac{v}{4\pi r^2} = \frac{v_b}{4\pi b^2}\left(\frac{b}{r}\right)^{3+\frac{1}{w_1}}.$$

$$g_{tt}(r) = -f_0,$$
$$g_{rr}(r) = \frac{1}{1 - \left(\frac{b}{r}\right)^{(1+w_1)/w_1}}.$$

Analytic solutions for the case $1 + w_1 + 4w_2 = 0$

TOV eq: ➔ $$u = 1 - \frac{\left(1 - \frac{v}{v_b}\right)^2}{1 + \frac{2w_1}{1-w_1}\left(\frac{v}{v_b}\right) - c_1 \frac{1+w_1}{1-w_1}\left(\frac{v}{v_b}\right)^{\frac{2w_1}{1+w_1}}}, \qquad \kappa = \frac{4w_1^2}{1-c_1}\frac{1-w_1}{1+w_1}.$$



Radius: $$r = b\left(\frac{v}{v_b}\right)^{-\frac{2w_1}{1+w_1}} \frac{1-w_1}{(1+w_1)(1-c_1)}\left(1 + \frac{2w_1}{1-w_1}\frac{v}{v_b} - c_1\frac{1+w_1}{1-w_1}\left(\frac{v}{v_b}\right)^{\frac{2w_1}{1+w_1}}\right).$$

metric components

$$f(r) = f_0\left(\frac{r}{r_0}\right)^{\frac{4(w_2-w_1)}{1+w_1}}\left(\frac{\rho}{\rho_0}\right)^{-\frac{2w_1}{1+w_1}} = f_0\left(\frac{r}{r_0}\right)^{\frac{4w_2}{1+w_1}}\left(\frac{v(r)}{v_0}\right)^{-\frac{2w_1}{1+w_1}}. \qquad g_{rr}(r) = \frac{1 + \frac{2w_1}{1-w_1}\left(\frac{v(r)}{v_b}\right) - c_1\frac{1+w_1}{1-w_1}\left(\frac{v(r)}{v_b}\right)^{\frac{2w_1}{1+w_1}}}{\left(1 - \frac{v(r)}{v_b}\right)^2}.$$

Summary and Discussions

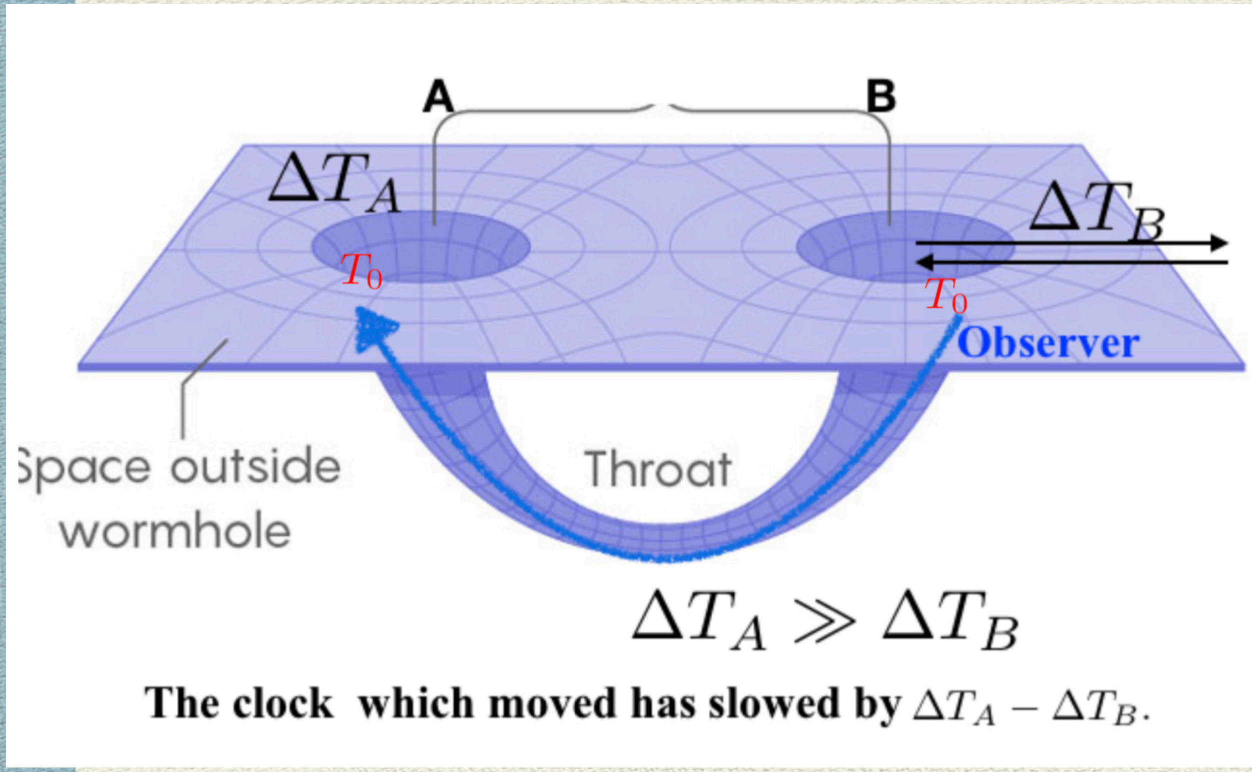
1) We have classified spherically symmetric wormholes in GR which do not have lids constraining exotic matter.
2) For the wormhole spacetime to be nonsingular, it should belong to Type I, IV.

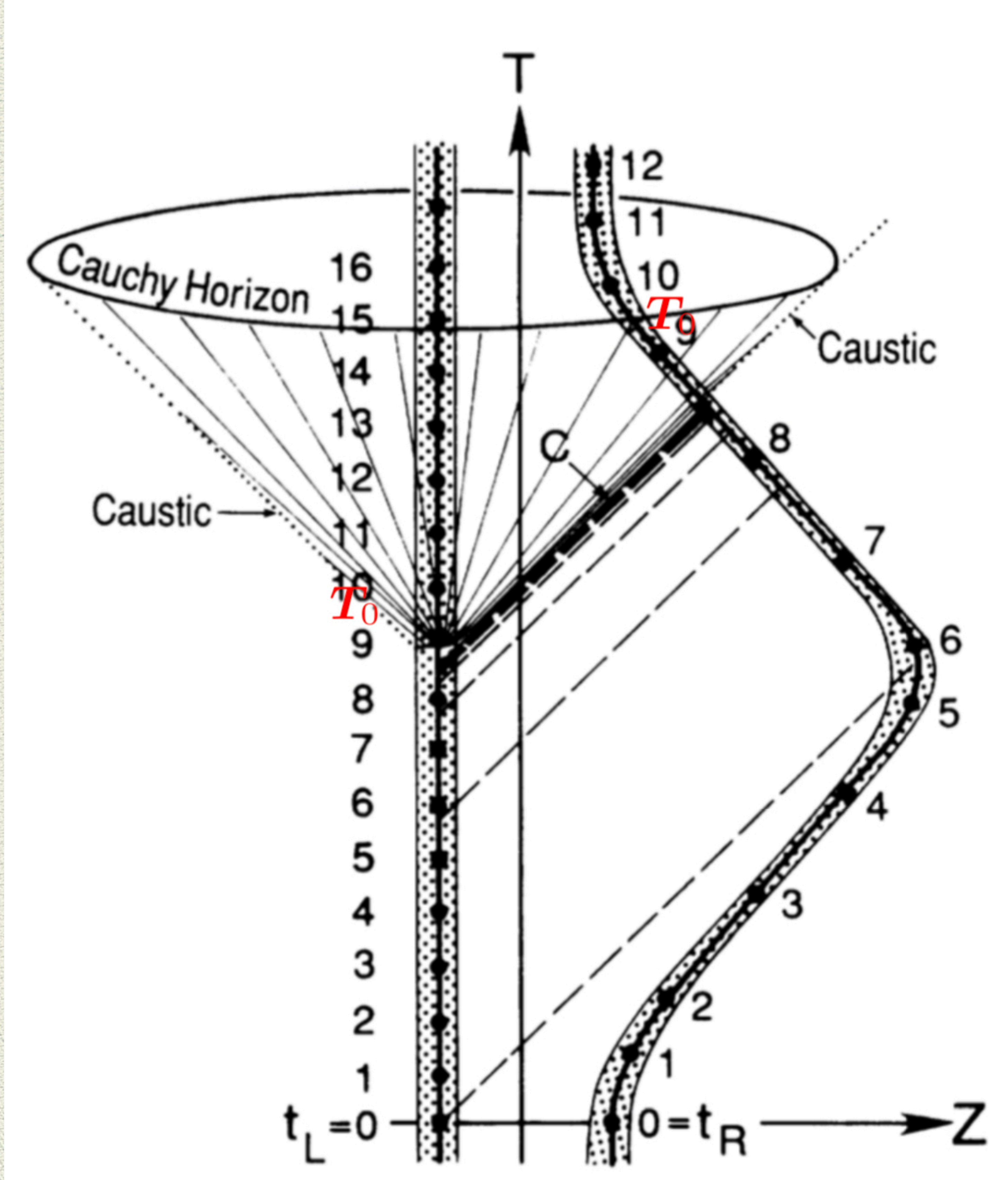
$$\left(\rho > 0, w_1 < 0, w_2 > -\frac{1+w_1}{4}\right), \text{ or } \left(\rho < 0, w_1 > 0, w_2 < -\frac{1+w_1}{4}\right)$$

3) Exact analytic solutions and numerical solutions were found.
4) Throat geometry is insensitive to the angular pressure.
5) If one use minimal amount of exotic matter, one can use truncated solution.
6) Stability issues?


Thanks you very much.

Time-travel through a wormhole





After the movement, a closed timelike curve forms.

See #9 in the Right figure.

Morris, Thorne, Yurtsever, (1988), Frolov, Novikov (1990)