Second-order effective energy-momentum tensor of gravitational scalar perturbations in FRW universe

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Outline

- Introduction to cosmological perturbations
- 2nd order perturbations and back reaction
  :- 2nd order effective Energy-Momentum Tensor (2eEMT)
- Gauge Problem of 2eEMT
- Several Results (3 gauges: Longitudinal, Spatially-flat, Comoving)
- Conclusions

Introduction to cosmological perturbations

Classification of perturbations

* FRW-metric w/ perturbations

$$ds^2 = \left[{}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}(x^\gamma)\right] dx^\alpha dx^\beta \qquad , \quad |\delta g_{\alpha\beta}| \ll |{}^{(0)}g_{\alpha\beta}|$$

$$\Downarrow$$

$${}^{(0)}g_{\alpha\beta}\, dx^\alpha dx^\beta = a^2(\eta)\left[d\eta^2 - \delta_{ij}\, dx^i dx^j\right]$$: use conformally flat metric.

* $\delta g_{\alpha\beta}$: ① scalar ② vector ③ tensor perturbation.

: based on the sym. properties of hom. & iso background

invariant under rotations and translations.

① $\delta g_{00} = 2a^2\phi(x^i)$: scalar under 3-rotations.

② $\delta g_{0i} = a^2(B_{,i} + S_i)$: vector

$S^i{}_{,i} = 0$: zero divergence

③ $\delta g_{ij} = a^2(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij})$: tensor

3-scalar

$F^i{}_{,i} = 0$

3 tensor : $\boxed{h^i{}_i = 0,\ h^i{}_{j,i} = 0 \quad \text{traceless \& transverse}}$

4-cond's

1. Scalar Perturbation

$$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + B_{,i}\,dx^i d\eta - \left[(1-2\psi)\delta_{ij} - 2E_{,ij}\right]dx^i dx^j\right]$$

2. Vector Perturbation

$$ds^2 = a^2\left[\qquad d\eta^2 + 2S_i\,dx^i d\eta - (\delta_{ij} - F_{i,j} - F_{j,i})\,dx^i dx^j\right]$$

3. Tensor Perturbation

$$ds^2 = a^2\left[\qquad d\eta^2 \qquad - (\delta_{ij} - h_{ij})\,dx^i dx^j\right]$$

$$\Rightarrow ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + (B_{,i} + 2S_i)\,dx^i d\eta - \left[(1-2\psi)\delta_{ij} - 2E_{,ij} - F_{i,j} - F_{j,i} - h_{ij}\right]dx^i dx^j\right]$$

Number of independent functions of $\delta g_{\alpha\beta}$

$\boxed{S_i,\ F_i \text{ with 2 divergence-free conditions}}$

(4 for scalar) + (4 for vector) + (2 for tensor)

$\boxed{\phi,\ B,\ \psi,\ E}$

$\boxed{h_{ij} \text{ with 4 gauge conditions}}$

= 10 functions = number of independent components of $\delta g_{\alpha\beta}$

Scalar part :- induced by energy density inhomogeneity
:- exhibit gravitational instability → structure formation
:- most important

Vector part :- rotational motion of fluid
:- decays very quickly as in Newtonian gravity
:- not very interesting cosmologically

Tensor part :- gravity waves → dof. of gravitational field itself
:- do not induce any perturbations in perfect fluid

Gauge Transformations & Gauge Invariant Variables

$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha$ **: infinitesimal coord. transformation**

$$\boxed{\xi^\alpha = (\xi^0, \xi^i) = (\xi^0, \xi^i_\perp + \zeta^{,i})} \qquad \xi^i_{\perp,i} = 0, \qquad \xi_{\perp i} = \xi^i_\perp$$

At a given point, the metric tensor in $\tilde{x}$ coord. system: $\tilde{g}_{\alpha\beta}(\tilde{x}^\rho)$

$$g_{\alpha\beta}(x^\rho) = g^{(0)}_{\alpha\beta}(x^\rho) + \delta g_{\alpha\beta}(x^\rho) \qquad \textbf{: perturbation in } x$$

$$\Rightarrow \quad \tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta}(x^\rho) \qquad \textbf{: tensor transformation}$$

$$\approx g^{(0)}_{\alpha\beta}(x^\rho) + \delta g_{\alpha\beta}(x^\rho) - g^{(0)}_{\alpha\delta}(x^\rho)\xi^\delta_{,\beta} - g^{(0)}_{\gamma\beta}(x^\rho)\xi^\gamma_{,\alpha}$$

$$= g^{(0)}_{\alpha\beta}(\tilde{x}^\rho) + \delta\tilde{g}_{\alpha\beta}(\tilde{x}^\rho) \qquad \textbf{: perturbation in } \tilde{x}$$

Taylor expansion: $g^{(0)}_{\alpha\beta}(x^\rho) = g^{(0)}_{\alpha\beta}(\tilde{x}^\rho - \xi^\rho) \approx g^{(0)}_{\alpha\beta}(\tilde{x}^\rho) - g^{(0)}_{\alpha\beta,\gamma}(\tilde{x}^\rho)\xi^\gamma$

Finally we have

$$\boxed{\delta\tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \delta g_{\alpha\beta}(x^\rho) - g^{(0)}_{\alpha\beta,\gamma}(\tilde{x}^\rho)\xi^\gamma - g^{(0)}_{\alpha\delta}(x^\rho)\xi^\delta_{,\beta} - g^{(0)}_{\gamma\beta}(x^\rho)\xi^\gamma_{,\alpha}}$$

eg) scalar perturbation

$$ds^2 = a^2\left[(1+2\phi)d\eta^2 + B_{,i}dx^i d\eta - [(1-2\psi)\delta_{ij} - 2E_{,ij})]\, dx^i dx^j\right] \qquad \textbf{in } x$$

$$\boxed{\delta\tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \delta g_{\alpha\beta}(x^\rho) - g^{(0)}_{\alpha\beta,\gamma}(\tilde{x}^\rho)\xi^\gamma - g^{(0)}_{\alpha\delta}(x^\rho)\xi^\delta_{,\beta} - g^{(0)}_{\gamma\beta}(x^\rho)\xi^\gamma_{,\alpha}}$$

$$ds^2 = a^2\left[(1+2\tilde{\phi})d\eta^2 + \tilde{B}_{,i}dx^i d\eta - \left[(1-2\tilde{\psi})\delta_{ij} - 2\tilde{E}_{,ij})\right] dx^i dx^j\right] \qquad \textbf{in } \tilde{x}$$

1. Scalar Perturbation

$$\begin{cases} \phi \to \tilde{\phi} = \phi - \frac{1}{a}(a\xi^0)' \\ B \to \tilde{B} = B + \xi' - \xi^0 \\ \psi \to \tilde{\psi} = \psi + \frac{a'}{a}\xi^0 \\ E \to \tilde{E} = E + \xi \end{cases}$$

- only ξ^0 and ξ contribute

$\Rightarrow$ can make two of ϕ, B, ψ, E vanish!

* Gauge invariant variables (perturbations):

$$\begin{cases} \Phi \equiv \phi - \frac{1}{a}\left[a(B-E')\right]' \\ \Psi \equiv \psi + \frac{a'}{a}(B-E') \end{cases}$$

} span 2D space of physical perturbations

Bardeen Variables (1980)

① do not change under coord. transf.

② If $\Phi = 0$, $\Psi = 0$ in one coord syst $\Rightarrow$ $\Phi = 0$, $\Psi = 0$ in all coord. syst's.

①

③ If $\Phi = \Psi = 0$, No physical perturbations : fictitious metric pert's can be removed by coord. transf.

④ Can construct ∞ # of gauge-invariant variables:

e.g.) $\alpha\Phi + \beta\Psi \Rightarrow$ also gauge-inv.

2. Vector Perturbation

$$\begin{cases} S_i \to \tilde{S}_i = S_i + \xi'_{\perp i} \\ F_i \to \tilde{F}_i = F_i + \xi_{\perp i} \end{cases}$$ → only $\xi_{\perp i}$ is involved.

* $\overline{V}_i = S_i - F'_i$: gauge-inv. vector perturbation

① Only 2 fn's characterize physical perturbations ($\overline{V}_i{}^{,i} = 0$)

② spans 2D space of physical perturbations.

③ describes rotational motions

④ Corresponding rotational vel $\delta u_{\perp i}$ ($\delta u_{\perp i}{}^{,i} = 0$) : also gauge invariant.

3. Tensor Perturbation

h_{ij} : no change under coord. transf.

: describes already the G.W. in a gauge invariant manner.

Gauge Fixing

- Gauge freedom: most important in scalar perturbation
- Free to choose ξ^0 and ζ → 2 conditions
- Imposing gauge conditions → Fixing coord. system

e.g.) Longitudinal Gauge

$\boxed{B = E = 0 \quad \Rightarrow \xi^0 = \zeta = 0}$: gauge conditions

$\Rightarrow$ $\boxed{\tilde{\phi} = \phi = \Phi, \quad \tilde{\psi} = \psi = \Psi}$: the other 2 scalars → gauge invariant

$\Rightarrow$ $\boxed{ds^2 = a^2(\eta)[(1 + 2\Phi)\, d\eta^2 - (1 - 2\Psi)\gamma_{ij}\, dx^i\, dx^j]}$

: so called, "Conformal Newtonian" gauge

1st order Einstein's equation

$$\boxed{G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu}}$$: Einstein's equation

$$\boxed{G^{\mu}{}_{\nu} = {}^{(0)}G^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu} + \cdots ,}$$

$$\boxed{\delta G^{\mu}{}_{\nu} = 8\pi G\, \delta T^{\mu}{}_{\nu}}$$: 1st order Einstein's equation

NOT gauge-invariant, individually

Define Gauge-Invariant Quantities:

$$\delta G_0^{(\mathrm{gi})0} = \delta G_0^0 + \boxed{({}^{(0)}G_0^0)'(B - E')}, \qquad \delta G_i^{(\mathrm{gi})0} = \delta G_i^0 + ({}^{(0)}G_0^0 - \tfrac{1}{3}{}^{(0)}G_k^k)(B - E')_{|i} ,$$

$$\delta G_j^{(\mathrm{gi})i} = \delta G_j^i + ({}^{(0)}G_j^i)'(B - E') ,$$

$$\delta T_0^{(\mathrm{gi})0} = \delta T_0^0 + \boxed{({}^{(0)}T_0^0)'(B - E')}, \qquad \delta T_i^{(\mathrm{gi})0} = \delta T_i^0 + ({}^{(0)}T_0^0 - \tfrac{1}{3}{}^{(0)}T_k^k)(B - E')_{|i} ,$$

$$\delta T_j^{(\mathrm{gi})i} = \delta T_j^i + ({}^{(0)}T_j^i)'(B - E') .$$

$$\boxed{\delta G^{\mu}{}_{\nu} = 8\pi G\, \delta T^{\mu}{}_{\nu}} \Rightarrow \boxed{\delta G_{\nu}^{(\mathrm{gi})\mu} = 8\pi G\, \delta T_{\nu}^{(\mathrm{gi})\mu}}$$: can be written in gauge-invariant form

From now, consider Scalar Perturbations

$$\boxed{\delta G^{(\mathrm{gi})\mu}_{\nu} = 8\pi G\, \delta T^{(\mathrm{gi})\mu}_{\nu}}$$

: 1st order equation in gauge-invariant form

$$-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi + 3\mathcal{K}\Psi = 4\pi G a^2\, \delta T^{(\mathrm{gi})0}_{0},$$

$$(\mathcal{H}\Phi + \Psi')_{,i} = 4\pi G a^2\, \delta T^{(\mathrm{gi})0}_{i},$$

$$[(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' - \mathcal{K}\Psi + \tfrac{1}{2}\nabla^2 D]\delta^i{}_j - \tfrac{1}{2}\gamma^{ik}D_{|kj} = -4\pi G a^2\, \delta T^{(\mathrm{gi})i}_{j}$$

:- derived with NO gauge fixing
:- all written in gauge-invariant (Bardeen) variables
:- valid in arbitrary coord. system

Matter Part

Fluid (Hydrodynamical perturbation)

* $T^\alpha_\beta = (\varepsilon + p) u^\alpha u_\beta - p\,\delta^\alpha_\beta$: perfect fluid.
* Gauge-invariant perturbations

contributes to Vector pert. : the others ⇒ scalar part.

$$\left\{\begin{array}{l} \overline{\delta T^0_0} = \overline{\delta\varepsilon} \\ \overline{\delta T^0_i} = \frac{1}{a}(\varepsilon_0 + p_0)\left(\overline{\delta u_{\parallel i}} + \overline{\delta u_{\perp i}}\right) \\ \overline{\delta T^i_j} = -\overline{\delta p}\,\delta^i_j \end{array}\right. \Leftarrow \left\{\begin{array}{l} \overline{\delta\varepsilon} = \delta\varepsilon - \varepsilon_0'(B-E') \\ \overline{\delta p} = \delta p - p_0'(B-E') \\ \overline{\delta u_0} = \delta u_0 - [a(B-E')]' \\ \overline{\delta u_i} = \delta u_i - a(B-E')_{,i} \end{array}\right.$$

Inflaton (Scalar field perturbation)

$\overline{\delta\varphi} \equiv \delta\varphi - \varphi_0'(B-E')$: gauge invariant variable

$\overline{\delta\varphi}'' + 2\mathcal{H}\overline{\delta\varphi}' - \Delta\overline{\delta\varphi} + a^2 V_{,\varphi\varphi}\overline{\delta\varphi} - \varphi_0'(3\Psi + \Phi)' + 2a^2 V_{,\varphi}\Phi = 0$: scalar fied eq.

Gauge Invariance

$$G_{\mu\nu} = G^{(0)}_{\mu\nu} + \epsilon G^{(1)}_{\mu\nu} + \epsilon^2 G^{(2)}_{\mu\nu} + \epsilon^3 G^{(3)}_{\mu\nu} \cdots$$

real equality

$$\begin{array}{cccc} \equiv & \parallel & \parallel & \parallel \end{array}$$

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \epsilon T^{(1)}_{\mu\nu} + \epsilon^2 T^{(2)}_{\mu\nu} + \epsilon^3 T^{(3)}_{\mu\nu} \cdots$$

$$\begin{array}{cccc} \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \text{G.I.} & \text{G.I.} & \text{G.I.} & \text{G.I.??} \end{array}$$

order-by-order equality.

2^{nd} order Perturbations and Back Reaction

Picture

0^{th} order: $G^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu}$ $(8\pi G = 1)$

1^{st} order: $G^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu}$ **: solutions**

plug in

2^{nd} order: $G^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu}$

$\| \quad \boxed{G^{(1)}_{\mu\nu}[g^{(2)}] + G^{(2)}_{\mu\nu}[g^{(1)}]}$

Quadratic terms

linear in $g^{(2)}$

Back-reaction source

$$\Rightarrow \boxed{G^{(1)}_{\mu\nu}[g^{(2)}] = T^{(2)}_{\mu\nu} - G^{(2)}_{\mu\nu}[g^{(1)}] \equiv T^{(2,\text{eff})}_{\mu\nu}}$$

$\Downarrow$

$T^{(2)}_{\mu\nu}[\delta\phi^{(1)}]$: Inflaton

$T^{(2)}_{\mu\nu}[g^{(1)}\delta\rho^{(1)}]$: fluid

$\rightarrow$ quadratic terms only

cf) $\langle\delta\phi^{(2)}\rangle = 0$, $\langle\delta\rho^{(2)}\rangle = 0$: stochastic

$$T^{(2,\mathrm{eff})\mu}{}_{\nu} = \begin{pmatrix} -\rho_{\mathrm{eff}} & & & \\ & p_{\mathrm{eff}} & & \\ & & p_{\mathrm{eff}} & \\ & & & p_{\mathrm{eff}} \end{pmatrix} \Longrightarrow g^{(2)}_{\mu\nu}$$

$$\boxed{G^{(1)}_{\mu\nu}[g^{(2)}] = T^{(2)}_{\mu\nu} - G^{(2)}_{\mu\nu}[g^{(1)}] \equiv T^{(2,\mathrm{eff})}_{\mu\nu}}$$

Fluid Matter & Scalar Perturbations

$$ds^2 \boxed{= a^2(\eta)[-(1+2A)d\eta^2 - 2B_i d\eta dx^i + (\delta_{ij} + 2C_{ij})dx^i dx^j]}$$

$$\boxed{A = \alpha, \qquad B_i = \beta_{,i} + B_i^{(v)}, \qquad C_{ij} = -\psi\delta_{ij} + E_{,ij} + C_{(i,j)}^{(v)} + C_{ij}^{(t)}.}$$: general

$$\boxed{A = \alpha \ , \quad B_i = \beta_{,i} \ , \quad C_{ij} = -\psi\delta_{ij} + E_{,ij}}$$: scalar only

$$\boxed{= a^2(\eta)[-(1+2\alpha)d\eta^2 - 2\beta_{,i}d\eta dx^i + ((1-2\psi)\delta_{ij} + 2E_{,ij})dx^i dx^j]}$$

Einstein Tensor

$$\begin{aligned}
G_{00} = {} & 3\mathcal{H}^2 + 2\mathcal{H}B_{k,k} + 2\mathcal{H}C'_{kk} - \Delta C_{kk} + C_{kl,kl} - (2\mathcal{H}' + \mathcal{H}^2)B_kB_k + \frac{1}{2}B_{k,k}B_{l,l} - \frac{1}{4}B_{k,l}B_{k,l} \\
& -\frac{1}{4}B_{k,l}B_{l,k} + B_kB_{l,lk} - B_k\Delta B_k - \frac{1}{2}C'_{kl}C'_{kl} + \frac{1}{2}C'_{kk}C'_{ll} - 4\mathcal{H}C_{kl}C'_{kl} + \frac{3}{2}C_{kl,m}C_{kl,m} \\
& -C_{kl,m}C_{km,l} - \frac{1}{2}(C_{kk,l} - 2C_{lk,k})(C_{mm,l} - 2C_{lm,m}) + 2C_{kl}(C_{mm,kl} + \Delta C_{kl} - 2C_{mk,ml}) \\
& -4\mathcal{H}A_{,k}B_k + 2B_k(C'_{mm,k} - C'_{km,m}) + 2\mathcal{H}B_k(C_{mm,k} - 2C_{km,m}) - B_{k,l}(C'_{kl} + 4\mathcal{H}C_{kl}) \\
& +B_{k,k}C'_{ll} - 2A(\Delta C_{kk} - C_{kl,kl}) \ ,
\end{aligned} \tag{26}$$

$$\begin{aligned}
G_{0i} = {} & 2\mathcal{H}A_{,i} + B_{[i,k]k} + (2\mathcal{H}' + \mathcal{H}^2)B_i + C'_{ik,k} - C'_{kk,i} - 4\mathcal{H}AA_{,i} - B_k(B'_{(i,k)} + 2\mathcal{H}B_{[i,k]}) \\
& +B_i(B'_{k,k} + 2\mathcal{H}B_{k,k}) + (C_{mm,k} - 2C_{km,m})C'_{ik} + 2C_{kl}(C'_{kl,i} - C'_{ik,l}) + C'_{kl}C_{kl,i} - 2\mathcal{H}A'B_i \\
& +A_{i,k}B_k + A_{,i}B_{k,k} - A_{,k}B_{(i,k)} - 2(2\mathcal{H}' + \mathcal{H}^2)AB_i - (\Delta A)B_i - 2\mathcal{H}B_kC'_{ik} + 2\mathcal{H}B_iC'_{kk} \\
& -2B_{[i,k]l}C_{kl} + 2B_{[i,k]}(C_{ll,k} - 2C_{kl,l}) - 2B_{[k,l]}C_{i[k,l]} - B_i(\Delta C_{kk} - C_{kl,kl}) + A_{,i}C'_{kk} \\
& -A_{,k}C'_{ik} \ ,
\end{aligned} \tag{27}$$

$$
\begin{aligned}
G_{ij} = \delta_{ij}\{ & -(2\mathcal{H}'+\mathcal{H}^2) + 2\mathcal{H}A' + 2(2\mathcal{H}'+\mathcal{H}^2)A + \Delta A - B'_{k,k} - 2\mathcal{H}B_{k,k} - C''_{kk} - 2\mathcal{H}C'_{kk} \\
& +\Delta C_{kk} - C_{kl,kl} - 8\mathcal{H}AA' - (\nabla A)^2 - 2A\Delta A - 4(2\mathcal{H}'+\mathcal{H}^2)A^2 + 2\mathcal{H}B_kB'_k \\
& +(2\mathcal{H}'+\mathcal{H}^2)B_kB_k + \frac{3}{4}B_{k,l}B_{k,l} - \frac{1}{4}B_{k,l}B_{l,k} - \frac{1}{2}B_{k,k}B_{l,l} - B_lB_{k,kl} + B_k\Delta B_k \\
& +2C_{kl}C''_{kl} + \frac{3}{2}C'_{kl}C'_{kl} - \frac{1}{2}C'_{kk}C'_{ll} + 4\mathcal{H}C_{kl}C'_{kl} + \frac{1}{2}(C_{kk,l} - 2C_{lk,k})(C_{mm,l} - 2C_{lm,m}) \\
& -\frac{3}{2}C_{lm,k}C_{lm,k} + C_{lm,k}C_{lk,m} - 2C_{kl}(C_{mm,kl} + \Delta C_{kl} - 2C_{mk,ml}) + A'B_{k,k} + 2AB'_{k,k} \\
& +2\mathcal{H}A_{,k}B_k + 4\mathcal{H}AB_{k,k} + 2B'_{k,l}C_{kl} - B'_k(C_{ll,k} - 2C_{kl,l}) - 2B_k(C'_{ll,k} - C'_{kl,l}) - B_{k,k}C'_{ll} \\
& +B_{k,l}C'_{kl} + 4\mathcal{H}B_{k,l}C_{kl} - 2\mathcal{H}B_l(C_{kk,l} - 2C_{lk,k}) + 2AC''_{kk} + A'C'_{kk} + 4\mathcal{H}AC'_{kk} \\
& -2A_{,kl}C_{kl} + A_{,k}(C_{ll,k} - 2C_{kl,l})\} \\
& -A_{,ij} + B'_{(i,j)} + 2\mathcal{H}B_{(i,j)} + C''_{ij} + 2\mathcal{H}C'_{ij} - \Delta C_{ij} - 2(2\mathcal{H}'+\mathcal{H}^2))C_{ij} + 2C_{k(i,j)k} - C_{kk,ij} \\
& +A_{,i}A_{,j} + 2AA_{,ij} + B_{k,k}B_{(i,j)} + B_kB_{(i,j)k} - \frac{1}{2}B_{i,k}B_{j,k} - \frac{1}{2}B_{k,i}B_{k,j} - B_kB_{k,ij} - 2C''_{kk}C_{ij} \\
& +C'_{kk}C'_{ij} - 2C'_{ik}C'_{jk} - 4\mathcal{H}C'_{kk}C_{ij} + 2(\Delta C_{kk} - C_{kl,kl})C_{ij} + 2C_{kl}(C_{ij,kl} + C_{kl,ij} - 2C_{k(i,j)l}) \\
& -(C_{kk,l} - 2C_{lk,k})(C_{ij,l} - 2C_{l(i,j)}) + C_{kl,i}C_{kl,j} + 2C_{ik,l}(C_{jk,l} - C_{jl,k}) - A'B_{(i,j)} \\
& -2A(B'_{(i,j)} + 2\mathcal{H}B_{(i,j)}) - 2B_k(C'_{k(i,j)} - C'_{ij,k}) - (B'_k + 2\mathcal{H}B_k)(2C_{k(i,j)} - C_{ij,k}) \\
& +B_{k,k}(C'_{ij} - 4\mathcal{H}C_{ij}) - 2B'_{k,k}C_{ij} + B_{(i,j)}C'_{kk} - B_{i,k}C'_{jk} - B_{j,k}C'_{ik} - 2A(C''_{ij} + 2\mathcal{H}C'_{ij}) \\
& -A'(C'_{ij} - 4\mathcal{H}C_{ij}) + 4(2\mathcal{H}'+\mathcal{H}^2)AC_{ij} + A_{,k}(2C_{k(i,j)} - C_{ij,k}) + 2(\Delta A)C_{ij}\ .
\end{aligned}
\tag{28}
$$

Energy-Momentum Tensor (Fluid)

$$
\begin{aligned}
T^{(\mathrm{PF})}_{00} &= a^2\left[\rho_0 + \delta\rho + 2A\rho_0 + 2A\delta\rho + (\rho_0 + p_0)v_kv_k\right]\ , \\
T^{(\mathrm{PF})}_{0i} &= a^2\left[\rho_0(B_i - v_i) - p_0v_i + \delta\rho(B_i - v_i) - \delta p v_i - 2(\rho_0 + p_0)C_{ik}v_k\right]\ , \\
T^{(\mathrm{PF})}_{ij} &= a^2\left[p_0\delta_{ij} + 2p_0C_{ij} + \delta p\delta_{ij} + 2C_{ij}\delta p + (\rho_0 + p_0)(B_i - v_i)(B_j - v_j)\right]\ .
\end{aligned}
$$

0^{th} order (Background Friedmann Eqs.)

$$3\mathcal{H}^2 = 8\pi G a^2 \rho_0 \ ,$$
$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G a^2 p_0$$
$$\rho_0' + 3\mathcal{H}(\rho_0 + p_0) = 0 \ .$$

1st order → Obtain solutions

$$6\mathcal{H}^2 A - 2\mathcal{H} B_{k,k} - 2\mathcal{H} C'_{kk} + \Delta C_{kk} - C_{kl,kl} = -8\pi G a^2 \delta\rho \ ,$$
$$2\mathcal{H} A_{,i} + B_{[i,k]k} + 2(\mathcal{H}' - \mathcal{H}^2) B_i + C'_{ik,k} - C'_{kk,i} = 2(\mathcal{H}' - \mathcal{H}^2) v_i \ ,$$
$$\delta_{ij}\left[2\mathcal{H}A' + 2(2\mathcal{H}' + \mathcal{H}^2)A + \Delta A - B'_{k,k} - 2\mathcal{H}B_{k,k} - C''_{kk} - 2\mathcal{H}C'_{kk} + \Delta C_{kk} - C_{kl,kl}\right] - A_{,ij}$$
$$+ B'_{(i,j)} + 2\mathcal{H}B_{(i,j)} + C''_{ij} + 2\mathcal{H}C'_{ij} - \Delta C_{ij} + 2C_{k(i,j)k} - C_{kk,ij} = 8\pi G a^2 \delta p \ \delta_{ij} \ ,$$

1st order in Gauge Invariant Variables

$$3\mathcal{H}\Psi' + 3\mathcal{H}^2\Psi - \Delta\Psi = -4\pi G a^2 \overline{\delta\rho} \ ,$$
$$\left(\Psi' + \mathcal{H}\Psi\right)_{,i} = \left(\mathcal{H}' - \mathcal{H}^2\right) \overline{v}_i \ ,$$
$$\Psi'' + 3\mathcal{H}\Psi' + (2\mathcal{H}' + \mathcal{H}^2)\Psi = 4\pi G a^2 \overline{\delta p} \ ,$$
$$\Phi = \Psi \ .$$

$$\Phi = \alpha - Q' - \mathcal{H}Q \ ,$$
$$\Psi = \psi + \mathcal{H}Q \ ,$$
$$\overline{\delta\rho} = \delta\rho - \rho_0' Q \ ,$$
$$\overline{\delta p} = \delta p - p_0' Q \ ,$$
$$\overline{v}_i = v_i + E'_{,i} \ ,$$

$$\boxed{Q = \beta + E' \ .}$$: gauge variable

2eEMT (2nd order effective Energy-Momentum Tensor)

$$G^{(1)}_{\mu\nu}[g^{(2)}] = 8\pi G T^{(2)}_{\mu\nu} - G^{(2)}_{\mu\nu}[g^{(1)}] \equiv 8\pi G \boxed{T^{(2,\mathrm{eff})}_{\mu\nu}}$$

$$\boxed{\tau_{\mu\nu} \equiv \langle T^{(2,\mathrm{eff})}_{\mu\nu} \rangle}$$

: integrated over several wave lengths

2eEMT

$\boxed{Q = \beta + E'\,.}$ $\boxed{E}$: gauge variables

$$
\begin{aligned}
\tau_{00} &= \frac{1}{8\pi G}\Big[-\frac{2}{\mathcal{H}'-\mathcal{H}^2}\langle(\nabla\Psi')^2\rangle-\frac{4\mathcal{H}}{\mathcal{H}'-\mathcal{H}^2}\langle\nabla\Psi'\cdot\nabla\Psi\rangle+\frac{5\mathcal{H}'-7\mathcal{H}^2}{\mathcal{H}'-\mathcal{H}^2}\langle(\nabla\Psi)^2\rangle-3\langle(\Psi')^2\rangle \\
&\quad -12\mathcal{H}^2\langle\Psi^2\rangle+6\langle(\mathcal{H}'Q-\mathcal{H}Q'-4\mathcal{H}^2Q)\Psi'\rangle+2\langle\nabla Q\cdot\nabla\Psi'\rangle-4\mathcal{H}\langle\nabla Q\cdot\nabla\Psi\rangle \\
&\quad -24\mathcal{H}^2\langle(Q'+\mathcal{H}Q)\Psi\rangle-2\mathcal{H}\langle\nabla Q'\cdot\nabla Q\rangle-3\langle(\mathcal{H}'Q-\mathcal{H}Q')^2\rangle+12\mathcal{H}^2(2\mathcal{H}'-\mathcal{H}^2)\langle Q^2\rangle \\
&\quad +\langle\Delta E'(2\mathcal{H}\Psi+4\mathcal{H}^2Q-3\mathcal{H}^2E'+2\mathcal{H}\Delta E)\rangle \\
&\quad -2\langle\Delta E(2\mathcal{H}\Psi'-\Delta\Psi-2\mathcal{H}^2Q'-2\mathcal{H}\mathcal{H}'Q)\rangle\Big], \\
\tau_{0i} &= 0, \\
\tau_{ij} &= \frac{1}{8\pi G}\delta_{ij}\Big[-\frac{2}{3(\mathcal{H}'-\mathcal{H}^2)}\langle(\nabla\Psi')^2\rangle-\frac{4\mathcal{H}}{3(\mathcal{H}'-\mathcal{H}^2)}\langle\nabla\Psi'\cdot\nabla\Psi\rangle-\frac{\mathcal{H}'+\mathcal{H}^2}{3(\mathcal{H}'-\mathcal{H}^2)}\langle(\nabla\Psi)^2\rangle+\langle(\Psi')^2\rangle \\
&\quad +8\mathcal{H}\langle\Psi\Psi'\rangle+4(2\mathcal{H}'+\mathcal{H}^2)\langle\Psi^2\rangle+4\langle(Q'+2\mathcal{H}Q)\Psi''\rangle+2\langle\{Q''+9\mathcal{H}Q'+(\mathcal{H}'+12\mathcal{H}^2)Q\}\Psi'\rangle \\
&\quad +4\langle\{2\mathcal{H}Q''+4(\mathcal{H}'+\mathcal{H}^2)Q'+2\mathcal{H}(3\mathcal{H}'+\mathcal{H}^2)Q\}\Psi\rangle+\frac{2}{3}\langle\nabla Q''\cdot\nabla Q\rangle+\frac{2}{3}\langle(\nabla Q')^2\rangle \\
&\quad +\frac{4}{3}\mathcal{H}\langle\nabla Q'\cdot\nabla Q\rangle+2\mathcal{H}\langle Q''Q'\rangle-2\mathcal{H}'\langle Q''Q\rangle+\mathcal{H}^2\langle Q'^2\rangle-2(2\mathcal{H}''+3\mathcal{H}\mathcal{H}')\langle Q'Q\rangle \\
&\quad -(8\mathcal{H}\mathcal{H}''+3\mathcal{H}'^2-4\mathcal{H}^4)\langle Q^2\rangle-\frac{2}{3}\langle\Delta E''(\Psi+2\mathcal{H}Q-3\mathcal{H}E'+2\Delta E)\rangle \\
&\quad -\frac{1}{3}\langle\Delta E'(4\Psi'+10\mathcal{H}\Psi+8\mathcal{H}Q'+8(\mathcal{H}'+\mathcal{H}^2)Q-3(2\mathcal{H}'+\mathcal{H}^2)E'+2\Delta E'+8\mathcal{H}\Delta E)\rangle \\
&\quad +\frac{2}{3}\langle\Delta E\{2\Psi''+4\mathcal{H}\Psi'-2\mathcal{H}Q''-4(\mathcal{H}'+\mathcal{H}^2)Q'-2(\mathcal{H}''+2\mathcal{H}'\mathcal{H})Q+\Delta(E''+2\mathcal{H}E')\}\rangle\Big].
\end{aligned}
$$

Therefore, 2eEMT is gauge dependent

Gauge Problem

INFLATON: scalar field

Abramo-Brandenberger-Mukhanov (1997)

2eEMT: gauge invariant

Unruh (1998), Ishibashi-Wald (2006)

2eEMT: gauge dependent = Our Result for FUILD

Several Results with Gauge Choices

Longitudinal Gauge

$$\boxed{\beta = 0 \,, \qquad E = 0 \qquad \Longrightarrow \qquad Q = 0}$$

$$\begin{aligned}
\tau_{00} &= \frac{1}{8\pi G}\Big[-\frac{2}{\mathcal{H}' - \mathcal{H}^2}\big\langle (\nabla\Psi')^2 \big\rangle - \frac{4\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2}\big\langle \nabla\Psi' \cdot \nabla\Psi \big\rangle + \frac{5\mathcal{H}' - 7\mathcal{H}^2}{\mathcal{H}' - \mathcal{H}^2}\big\langle (\nabla\Psi)^2 \big\rangle \\
&\qquad -3\big\langle (\Psi')^2 \big\rangle - 12\mathcal{H}^2\big\langle \Psi^2 \big\rangle \Big] \,, \\
\tau_{0i} &= 0 \,, \\
\tau_{ij} &= \frac{1}{8\pi G}\delta_{ij}\Big[-\frac{2}{3(\mathcal{H}' - \mathcal{H}^2)}\big\langle (\nabla\Psi')^2 \big\rangle - \frac{4\mathcal{H}}{3(\mathcal{H}' - \mathcal{H}^2)}\big\langle \nabla\Psi' \cdot \nabla\Psi \big\rangle - \frac{\mathcal{H}' + \mathcal{H}^2}{3(\mathcal{H}' - \mathcal{H}^2)}\big\langle (\nabla\Psi)^2 \big\rangle \\
&\qquad +8\mathcal{H}\big\langle \Psi\Psi' \big\rangle + \big\langle (\Psi')^2 \big\rangle + 4(2\mathcal{H}' + \mathcal{H}^2)\big\langle \Psi^2 \big\rangle \Big] \,.
\end{aligned}$$

Spatially-flat Gauge

$$\boxed{\psi = 0, \qquad E = 0}$$

$$\boxed{\Psi = \alpha - \beta' - \mathcal{H}\beta = \mathcal{H}\beta = \mathcal{H}Q.}$$

$$\begin{aligned}
\tau_{00} = \frac{1}{8\pi G}\Big[&-\frac{2}{\mathcal{H}'-\mathcal{H}^2}\langle(\nabla\Psi')^2\rangle - \frac{4\mathcal{H}}{\mathcal{H}'-\mathcal{H}^2}\langle\nabla\Psi'\cdot\nabla\Psi\rangle + \frac{(2\mathcal{H}'-3\mathcal{H}^2)(\mathcal{H}'+\mathcal{H}^2)}{\mathcal{H}^2(\mathcal{H}'-\mathcal{H}^2)}\langle(\nabla\Psi)^2\rangle \\
&-12\langle(\Psi')^2\rangle + \frac{24(\mathcal{H}'-2\mathcal{H}^2)}{\mathcal{H}}\langle\Psi'\Psi\rangle - \frac{12(\mathcal{H}'-2\mathcal{H}^2)^2}{\mathcal{H}^2}\langle\Psi^2\rangle\Big] , \qquad (80)
\end{aligned}$$

$$\tau_{0i} = 0 \ , \qquad (81)$$

$$\begin{aligned}
\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[&\frac{2}{3\mathcal{H}^2}\langle\nabla\Psi''\cdot\nabla\Psi\rangle + \frac{2(\mathcal{H}'-2\mathcal{H}^2)}{3\mathcal{H}^2(\mathcal{H}'-\mathcal{H}^2)}\langle(\nabla\Psi')^2\rangle - \frac{4(2\mathcal{H}'^2-3\mathcal{H}'\mathcal{H}^2+2\mathcal{H}^4)}{3\mathcal{H}^3(\mathcal{H}'-\mathcal{H}^2)}\langle\nabla\Psi'\cdot\nabla\Psi\rangle \\
&+\frac{8}{\mathcal{H}}\langle\Psi''\Psi'\rangle - \frac{8(\mathcal{H}'-2\mathcal{H}^2)}{\mathcal{H}^2}\langle\Psi''\Psi\rangle - \frac{4(2\mathcal{H}'-5\mathcal{H}^2)}{\mathcal{H}^2}\langle(\Psi')^2\rangle \\
&-\frac{2\mathcal{H}(\mathcal{H}'-\mathcal{H}^2)\mathcal{H}''-6\mathcal{H}'^3+10\mathcal{H}'^2\mathcal{H}^2-3\mathcal{H}'\mathcal{H}^4+\mathcal{H}^6}{3\mathcal{H}^4(\mathcal{H}'-\mathcal{H}^2)}\langle(\nabla\Psi)^2\rangle \\
&-\frac{8(\mathcal{H}\mathcal{H}''-2\mathcal{H}'^2+3\mathcal{H}'\mathcal{H}^2-6\mathcal{H}^4)}{\mathcal{H}^3}\langle\Psi'\Psi\rangle \\
&+\frac{4(\mathcal{H}'-2\mathcal{H}^2)(2\mathcal{H}\mathcal{H}''-2\mathcal{H}'^2-3\mathcal{H}'\mathcal{H}^2-2\mathcal{H}^4)}{\mathcal{H}^4}\langle\Psi^2\rangle\Big] . \qquad (82)
\end{aligned}$$

Comoving Gauge

$$\boxed{\beta_{,i} = v_i \ , \qquad E = 0 \ ,}$$

$$\boxed{Q = \beta = \frac{\Psi' + \mathcal{H}\Psi}{\mathcal{H}' - \mathcal{H}^2}.}$$

$$\begin{aligned}
\tau_{00} = \frac{1}{8\pi G}\Big[&-\frac{2\mathcal{H}}{(\mathcal{H}'-\mathcal{H}^2)^2}\langle\nabla\Psi''\cdot\nabla\Psi'\rangle - \frac{2\mathcal{H}^2}{(\mathcal{H}'-\mathcal{H}^2)^2}\langle\nabla\Psi''\cdot\nabla\Psi\rangle \\
&+\frac{2\mathcal{H}(\mathcal{H}''-3\mathcal{H}\mathcal{H}'+\mathcal{H}^3)}{(\mathcal{H}'-\mathcal{H}^2)^3}\langle(\nabla\Psi')^2\rangle + \frac{4\mathcal{H}(\mathcal{H}\mathcal{H}''-2\mathcal{H}'^2+\mathcal{H}^2\mathcal{H}'-\mathcal{H}^4)}{(\mathcal{H}'-\mathcal{H}^2)^3}\langle\nabla\Psi'\cdot\nabla\Psi\rangle \\
&+\frac{5\mathcal{H}'^3+2\mathcal{H}^3\mathcal{H}''-23\mathcal{H}^2\mathcal{H}'^2+25\mathcal{H}^4\mathcal{H}'-11\mathcal{H}^6}{(\mathcal{H}'-\mathcal{H}^2)^3}\langle(\nabla\Psi)^2\rangle - \frac{3\mathcal{H}^2}{(\mathcal{H}'-\mathcal{H}^2)^2}\langle\Psi''^2\rangle \\
&+\frac{6\mathcal{H}^2(\mathcal{H}''-2\mathcal{H}\mathcal{H}')}{(\mathcal{H}'-\mathcal{H}^2)^3}\langle\Psi''\Psi'\rangle + \frac{6\mathcal{H}^2(\mathcal{H}\mathcal{H}''-4\mathcal{H}'^2+6\mathcal{H}^2\mathcal{H}'-4\mathcal{H}^4)}{(\mathcal{H}'-\mathcal{H}^2)^3}\langle\Psi''\Psi\rangle \\
&-\frac{3\mathcal{H}^2(\mathcal{H}''-4\mathcal{H}\mathcal{H}'+2\mathcal{H}^3)(\mathcal{H}''-2\mathcal{H}^3)}{(\mathcal{H}'-\mathcal{H}^2)^4}\langle(\Psi')^2\rangle \\
&-\frac{6\mathcal{H}^2\{\mathcal{H}\mathcal{H}''^2-4\mathcal{H}''(\mathcal{H}'^2-\mathcal{H}^2\mathcal{H}'+\mathcal{H}^4)+12\mathcal{H}\mathcal{H}'^3-28\mathcal{H}^3\mathcal{H}'^2+28\mathcal{H}^5\mathcal{H}'-8\mathcal{H}^7\}}{(\mathcal{H}'-\mathcal{H}^2)^4}\langle\Psi'\Psi\rangle \\
&-\frac{3\mathcal{H}^2(\mathcal{H}\mathcal{H}''-6\mathcal{H}'^2+12\mathcal{H}^2\mathcal{H}'-8\mathcal{H}^4)(\mathcal{H}\mathcal{H}''-2\mathcal{H}'^2)}{(\mathcal{H}'-\mathcal{H}^2)^4}\langle\Psi^2\rangle\Big] \ ,
\end{aligned}$$

$$
\begin{aligned}
\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[& \frac{2}{3(\mathcal{H}'-\mathcal{H}^2)^2}\big\langle \nabla\Psi'''\cdot\nabla\Psi'\big\rangle + \frac{2\mathcal{H}}{3(\mathcal{H}'-\mathcal{H}^2)^2}\big\langle \nabla\Psi'''\cdot\nabla\Psi\big\rangle + \frac{2}{3(\mathcal{H}'-\mathcal{H}^2)^2}\big\langle (\nabla\Psi'')^2\big\rangle \\
& - \frac{2F_1}{3(\mathcal{H}'-\mathcal{H}^2)^3}\big\langle \nabla\Psi''\cdot\nabla\Psi'\big\rangle - \frac{2F_2}{3(\mathcal{H}'-\mathcal{H}^2)^3}\big\langle \nabla\Psi''\cdot\nabla\Psi\big\rangle - \frac{2F_3}{3(\mathcal{H}'-\mathcal{H}^2)^4}\big\langle (\nabla\Psi')^2\big\rangle \\
& - \frac{2F_4}{3(\mathcal{H}'-\mathcal{H}^2)^4}\big\langle \nabla\Psi'\cdot\nabla\Psi\big\rangle - \frac{F_5}{3(\mathcal{H}'-\mathcal{H}^2)^4}\big\langle (\nabla\Psi)^2\big\rangle + \frac{2\mathcal{H}}{(\mathcal{H}'-\mathcal{H}^2)^2}\big\langle \Psi'''\Psi''\big\rangle \\
& - \frac{2F_6}{(\mathcal{H}'-\mathcal{H}^2)^3}\big\langle \Psi'''\Psi'\big\rangle - \frac{2F_7}{(\mathcal{H}'-\mathcal{H}^2)^3}\big\langle \Psi'''\Psi\big\rangle - \frac{F_8}{(\mathcal{H}'-\mathcal{H}^2)^3}\big\langle \Psi''^2\big\rangle + \frac{2F_9}{(\mathcal{H}'-\mathcal{H}^2)^4}\big\langle \Psi''\Psi'\big\rangle \\
& - \frac{2F_{10}}{(\mathcal{H}'-\mathcal{H}^2)^4}\big\langle \Psi''\Psi\big\rangle + \frac{F_{11}}{(\mathcal{H}'-\mathcal{H}^2)^5}\big\langle (\Psi')^2\big\rangle - \frac{2F_{12}}{(\mathcal{H}'-\mathcal{H}^2)^5}\big\langle \Psi'\Psi\big\rangle + \frac{F_{13}}{(\mathcal{H}'-\mathcal{H}^2)^5}\big\langle \Psi^2\big\rangle\Big] ,
\end{aligned}
$$

$$
\begin{aligned}
F_1 &= 4\mathcal{H}'' - 13\mathcal{H}\mathcal{H}' + 5\mathcal{H}^3 \ , \\
F_2 &= 4\mathcal{H}\mathcal{H}'' - 2\mathcal{H}'^2 - 9\mathcal{H}^2\mathcal{H}' + 3\mathcal{H}^4 \ , \\
F_3 &= (\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' - 3\mathcal{H}''^2 - 3\mathcal{H}'^3 + 4\mathcal{H}(4\mathcal{H}' - \mathcal{H}^2)\mathcal{H}'' - 24\mathcal{H}^2\mathcal{H}'^2 + 19\mathcal{H}^4\mathcal{H}' - 4\mathcal{H}^6 \ , \\
F_4 &= 2\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' - (6\mathcal{H}\mathcal{H}'' - 3\mathcal{H}'^2 - 26\mathcal{H}^2\mathcal{H}' + 5\mathcal{H}^4)\mathcal{H}'' - 4\mathcal{H}(4\mathcal{H}'^3 + 6\mathcal{H}^2\mathcal{H}'^2 \\
&\quad -5\mathcal{H}^4\mathcal{H}' + \mathcal{H}^6) \ , \\
F_5 &= 2\mathcal{H}^2(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' + 2\mathcal{H}(3\mathcal{H}'^2 + 10\mathcal{H}^2\mathcal{H}' - \mathcal{H}^4)\mathcal{H}'' - 6\mathcal{H}^2\mathcal{H}''^2 - \mathcal{H}'^4 - 22\mathcal{H}^2\mathcal{H}'^3 \\
&\quad -6\mathcal{H}^4\mathcal{H}'^2 + 6\mathcal{H}^6\mathcal{H}' - \mathcal{H}^8 \ , \\
F_6 &= \mathcal{H}(\mathcal{H}'' - 2\mathcal{H}\mathcal{H}') \ , \\
F_7 &= \mathcal{H}(\mathcal{H}\mathcal{H}'' - 4\mathcal{H}'^2 + 6\mathcal{H}^2\mathcal{H}' - 4\mathcal{H}^4) \ , \\
F_8 &= 4\mathcal{H}\mathcal{H}'' - 4\mathcal{H}'^2 - 3\mathcal{H}^2\mathcal{H}' - \mathcal{H}^4 \ , \\
F_9 &= 4\mathcal{H}\mathcal{H}''^2 - \mathcal{H}(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' + 20\mathcal{H}\mathcal{H}'^3 - 2(2\mathcal{H}'^2 + 5\mathcal{H}^2\mathcal{H}' + \mathcal{H}^4)\mathcal{H}'' - 28\mathcal{H}^3\mathcal{H}'^2 \\
&\quad +38\mathcal{H}^5\mathcal{H}' - 14\mathcal{H}^7 \ , \\
F_{10} &= \mathcal{H}^2(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' - 4\mathcal{H}^2\mathcal{H}''^2 + \mathcal{H}(13\mathcal{H}'^2 - 8\mathcal{H}^2\mathcal{H}' + 11\mathcal{H}^4)\mathcal{H}'' - 10\mathcal{H}'^4 - 10\mathcal{H}^2\mathcal{H}'^3 \\
&\quad -52\mathcal{H}^6\mathcal{H}' + 40\mathcal{H}^4\mathcal{H}'^2 + 16\mathcal{H}^8 \ , \\
F_{11} &= 2\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)(\mathcal{H}'' - 2\mathcal{H}\mathcal{H}')\mathcal{H}''' - 4\mathcal{H}\mathcal{H}''^3 + (4\mathcal{H}'^2 + 17\mathcal{H}^2\mathcal{H}' + 3\mathcal{H}^4)\mathcal{H}''^2 \\
&\quad -4\mathcal{H}(10\mathcal{H}'^3 - 11\mathcal{H}^2\mathcal{H}'^2 + 20\mathcal{H}^4\mathcal{H}' - 7\mathcal{H}^6)\mathcal{H}'' + 4\mathcal{H}^2(18\mathcal{H}'^4 - 46\mathcal{H}^2\mathcal{H}'^3 + 70\mathcal{H}^4\mathcal{H}'^2 \\
&\quad -43\mathcal{H}^6\mathcal{H}' + 9\mathcal{H}^8) \ , \\
F_{12} &= 4\mathcal{H}^2\mathcal{H}''^3 - 2\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)(\mathcal{H}\mathcal{H}'' - 2\mathcal{H}'^2 + 2\mathcal{H}^2\mathcal{H}' - 2\mathcal{H}^4)\mathcal{H}''' \\
&\quad -\mathcal{H}(13\mathcal{H}'^2 - \mathcal{H}^2\mathcal{H}' + 12\mathcal{H}^4)\mathcal{H}''^2 + 2(5\mathcal{H}'^4 + 22\mathcal{H}^2\mathcal{H}'^3 - 40\mathcal{H}^4\mathcal{H}'^2 + 50\mathcal{H}^6\mathcal{H}' - 13\mathcal{H}^8)\mathcal{H}'' \\
&\quad -4\mathcal{H}(15\mathcal{H}'^5 - 28\mathcal{H}^2\mathcal{H}'^4 + 16\mathcal{H}^4\mathcal{H}'^3 + 19\mathcal{H}^6\mathcal{H}'^2 - 18\mathcal{H}^8\mathcal{H}' + 4\mathcal{H}^{10}) \ , \\
F_{13} &= 2\mathcal{H}^2(\mathcal{H}' - \mathcal{H}^2)(\mathcal{H}\mathcal{H}'' - 4\mathcal{H}'^2 + 6\mathcal{H}^2\mathcal{H}' - 4\mathcal{H}^4)\mathcal{H}''' - 4\mathcal{H}^3\mathcal{H}''^3 + (22\mathcal{H}^2\mathcal{H}'^2 - 19\mathcal{H}^4\mathcal{H}' \\
&\quad +21\mathcal{H}^6)\mathcal{H}''^2 - 4\mathcal{H}(7\mathcal{H}'^4 + 4\mathcal{H}^2\mathcal{H}'^3 - 17\mathcal{H}^4\mathcal{H}'^2 + 22\mathcal{H}^6\mathcal{H}' - 4\mathcal{H}^8)\mathcal{H}'' \\
&\quad +4(6\mathcal{H}'^4 + 3\mathcal{H}^2\mathcal{H}'^3 - 25\mathcal{H}^4\mathcal{H}'^2 + 32\mathcal{H}^6\mathcal{H}' - 8\mathcal{H}^8)\mathcal{H}'^2 \ .
\end{aligned}
$$

Field Equation

$$\Psi'' + 3(1+w)\mathcal{H}\Psi' - w\Delta\Psi + [2\mathcal{H}' + (1+3w)\mathcal{H}^2]\Psi = 0$$

$a(\eta) \propto \eta^2$ in MDE
$a(\eta) \propto \eta$ in RDE

Fourier Mode Expansion

$$\Psi(\vec{x},\eta) = \sum_{\vec{k}} \tilde{\Psi}_k(\eta) e^{i\vec{k}\cdot\vec{x}}$$

Solutions

$$\tilde{\Psi}_k(\eta) = \tilde{C}_1(k) + \frac{\tilde{C}_2(k)}{\eta^5} \qquad \textbf{for dust } (w=0)$$

$$\tilde{\Psi}_k(\eta) = \frac{\tilde{D}_1(k)}{\eta^3}\left[\frac{k\eta}{\sqrt{3}}\cos\left(\frac{k\eta}{\sqrt{3}}\right) - \sin\left(\frac{k\eta}{\sqrt{3}}\right)\right] + \frac{\tilde{D}_2(k)}{\eta^3}\left[\frac{k\eta}{\sqrt{3}}\sin\left(\frac{k\eta}{\sqrt{3}}\right) + \cos\left(\frac{k\eta}{\sqrt{3}}\right)\right] \qquad \textbf{for radiation } (w=\tfrac{1}{3})$$

I. DUST ($w = 0$)

$$\boxed{\rho \propto \frac{1}{a^2}, \quad p \propto \frac{1}{a^2}}$$

(i) Longitudinal Gauge

$$\tau_{00} = \frac{1}{8\pi G}\left[\frac{19k^2}{3}|\tilde{C}_1|^2 - \frac{48}{\eta^2}|\tilde{C}_1|^2 + \cdots\right.$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\left[\frac{k^2}{9}|\tilde{C}_1|^2 - \frac{k^2}{\eta^5}(\tilde{C}_1, \tilde{C}_2) + \cdots\right.$$

$$\rightarrow p \approx 57\rho$$

(ii) Flat Gauge

$$\tau_{00} = \frac{1}{8\pi G}\left[\frac{4k^2}{3}|\tilde{C}_1|^2 - \frac{300}{\eta^2}|\tilde{C}_1|^2 \cdots\right.$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\left[\left(4 + \frac{17k^2}{18}\right)|\tilde{C}_1|^2 - \frac{16}{\eta^2}|\tilde{C}_1|^2 \cdots\right.$$

$$\rightarrow p \approx \frac{4 + 17k^2/18}{4k^2/3}\rho$$

$= p \approx 4\rho$ ($k \ll$: **long wave**)

$= p \approx \frac{17}{24}\rho$ ($k \gg$: **short wave**)

(iii) Comoving Gauge

$$\tau_{00} = \frac{1}{8\pi G}\left[\frac{77k^2}{9}|\tilde{C}_1|^2 + \cdots\right.$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\left[\left(\frac{16}{9} + \frac{13k^2}{27}\right)|\tilde{C}_1|^2 - \frac{64}{9\eta^2}|\tilde{C}_1|^2 + \cdots\right.$$

$$\rightarrow p \approx \frac{16 + 13k^2/3}{77k^2}\rho$$

$= p \approx 16\rho$ ($k \ll$: **long wave**)

$= p \approx \frac{13}{231}\rho$ ($k \gg$: **short wave**)

II. RADIATION ($w = \frac{1}{3}$)

(A) Long-wave limit ($\sqrt{w}k\eta \ll 1$)

(i) Longitudinal Gauge

$$\tau_{00} = \frac{1}{8\pi G}\Big[-\frac{39}{\eta^8}|\tilde{D}_2|^2 + \frac{6}{\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$
$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[-\frac{19}{\eta^8}|\tilde{D}_2|^2 - \frac{14}{\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$

$$\boxed{\rho \propto \frac{1}{a^{10}}, \quad p \propto \frac{1}{a^{10}}}$$

(ii) Flat Gauge

$$\tau_{00} = \frac{1}{8\pi G}\Big[\frac{9}{\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 \cdots$$
$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[4\Big(\frac{1}{\eta^6} - \frac{1}{\eta^8}\Big)|\tilde{D}_2|^2 + \Big(\frac{4}{\eta^6} + \frac{11}{\eta^8}\Big)|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 \cdots$$

$$\boxed{\rho \propto \frac{k^2}{a^8}, \ll p \propto \frac{1}{a^8}}$$

(iii) Comoving Gauge

$$\tau_{00} = \frac{1}{8\pi G}\Big[\frac{45}{2\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$
$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[\Big(\frac{1}{\eta^6} - \frac{1}{\eta^8}\Big)|\tilde{D}_2|^2 + \Big(\frac{1}{\eta^6} + \frac{1}{2\eta^8}\Big)|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$

(B) Short-wave limit ($\sqrt{w}k\eta \gg 1$)

(i) Longitudinal Gauge

$$\tau_{00} = \frac{2}{8\pi G\eta^8}\left[3\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

$$\tau_{ij} = \frac{2}{8\pi G\eta^8}\delta_{ij}\left[\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

Dominant terms: $\tau_{00} = 3\tau_{ii}$

$$\boxed{\rho \propto \frac{1}{a^4}, \quad p = \frac{1}{3}\rho}$$

: behaves as radiation !

(ii) Flat Gauge

$$\tau_{00} = \frac{2}{8\pi G\eta^8}\left[3\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

$$\tau_{ij} = \frac{2}{8\pi G\eta^8}\delta_{ij}\left[\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

Dominant terms: same with in Longitudinal Gauge

(iii) Comoving Gauge

$$\tau_{00} = \frac{2}{8\pi G\eta^8}\left[\frac{45}{2}\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^4 + \cdots\right.$$

$$\tau_{ij} = \frac{2}{8\pi G}\delta_{ij}\left[\frac{3}{2\eta^8}\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^4 + \frac{1}{\eta^6}\left(\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)\right)(k\eta/\sqrt{3})^2 + \cdots\right.$$

$$\boxed{\rho \propto \frac{k^4}{a^6}, \quad p = \frac{1}{15}\rho}$$

Behaves in a different way

Conclusions

- We evaluated the 2nd order effective energy-momentum tensor (2eEMT) for FLUID
- 2eEMT is gauge dependent
- 2nd order Einstein's equation: can be written in gauge invariant way??
- We have shown the results of 2eEMT

  in Longitudinal, Spatially-flat, Comoving Gauges

  ➔ 2eEMT is strictly gauge dependent in ALL LIMITS