Gravitating cosmic superconducting tubes in the Einstein gauged non-linear σ-model in (3+1)-dimensions

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Outline

Introduction

Skyrmion

- The first example of stable topological soliton in three space dim.
- Fermionic excitations from an SU(2)-valued scalar field.
- At leading order in large N expansion: low energy QCD.
- The Skyrme model = NLSM(low energy pion) + Skyrme term.
- The Skyrme term: added to avoid Derricks no-go theorem.
- Fermionic excitations: based only on the existence of stable solitons with non-trivial 3rd homotopy class (the Skyrme term not used).
- Another mechanism of "Fermion out of boson"?

Fermion out of boson

- Instead of Skyrme term, what kind of term shall we add?
- NLSM + Gravity? Or NLSM + Gravity + Maxwell? $\rightarrow$ Successful!
- Exact solns of self-gravitating NLSM with non-trivial topological charge have been found!
- t-dependent SU(2)-valued field with t-independent $T_{\mu\nu}$.

Non-Abelian regularization of monopoles

- Dirac monopole in U(1): singular at origin.
- t Hooft-Polyakov monopole in SU(2) Yang-Mills-Higgs:
  $\longrightarrow$ at large distances looks like a Dirac monopole, but regular.
- Non-Abelian internal symmetry group regularizes" the singularity.
- We found a similar effect in the case of gravitating hadronic tubes.

Topological defect in Cosmology

- Grand unified theories: the actual symmetry group of the standard model is a result of a series of spontaneous symmetry breaking of a larger symmetry group.
- Topological defects play a fundamental role from microscopic scale to cosmology
- Universe expanded and cooled down, and went through several phase transitions $\longrightarrow$ topological defects have formed.
- The most relevant interaction is gravity in cosmology, the topological defects must be studied in the context of field theories coupled to gravity.

Cosmic strings: a simple example

- The simplest exact cosmic string solution is given by an energy-momentum tensor concentrated in a line

$$T_{\alpha\beta} = \mu\delta(x)\delta(y)\mathrm{diag}(1,0,0,1) \ . \quad (1)$$

- The exact solution is locally but not globally flat

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \ , \quad (2)$$

with

$$0 \leq \theta \leq 2\pi(1 - 4G\mu) \ . \quad (3)$$

- It has a conical defect, an angular deficit $\Delta = 8G\mu$ and has a curvature singularity on the z axis.

Cosmic strings: problems

- How to regularize this singularity?
- To smear $T_{\mu\nu}$ on a cylinder of finite radius: exact solutions found.
- Still they have one of the following drawbacks:

  a sharp boundary whose radius is arbitrary,

  $T_{\mu\nu}$ not derived from some fundamental action principle, $\cdots$.

Cosmic strings: cosmological and astrophysical implications

- a role in the galaxy formation as a source of density perturbations.
- gravitational lensing, double images.
- superconducting effect (Witten, 1985).

- Superconducting strings act as sources of synchrotron radiation or high energy cosmic rays.
- Superconducting strings moving in a magnetized plasma can be a mechanism for the production of gamma ray bursts.
- It is of great interest to find analytic non-singular solutions which can be derived from some fundamental action principle which leaves no arbitrariness in the choice of fields and their potentials.

What we will see are

- To construct the first examples of analytic and singularity free cosmic tube solutions for the self-gravitating SU(2)-NLSM.
- At large distance: a cosmic string boosted in the axis and has an angular defect related to the parameter of the theory
- Near the axis: free of singularities and without angular defect.
- Thus, free of singularities everywhere and the angular defect depends on the distance from the axis
- the matter field do not have a sharp boundary and the curvature reaches its maximum on a tube around the axis rather than on the axis itself. (Reason why we call it tube.)

What we will see are

- Dirac VS 't Hooft-Polyakov = cosmic string VS NLSM cosmic tube.
- NLSM regularizes the global string keeping a similar behavior at large distances!
- Possess non-trivial topological charge (the third homotopy class)
- It can be promoted to full solutions of Einstein + Maxwell + NLSM.
- These gauged solutions carry a persistent current even when the U(1) gauge field is zero. $\longrightarrow$ Witten's superconducting effect!
- Superconducting currents are tied to the topological charge so that they cannot be deformed continuously to zero (thus, persistent).

Gravitating SU(2) Nonlinear σ-model

- The Einstein-NLSM theory is described by the action

$$I[g, U] = \int d^4x \sqrt{-g} \left(\frac{\mathcal{R}}{2\kappa} + \frac{K}{4} \mathrm{Tr}[L^\mu L_\mu] \right) , \qquad (4)$$

- $\mathcal{R}$ is the Ricci scalar,
- L_μ are the Maurer-Cartan form components $L_\mu = U^{-1}\nabla_\mu U$ for $U \in SU(2)$
- κ is the gravitational constant
- Coupling K is fixed by experimental data.
- $c = \hbar = 1$.

Gravitating SU(2) Nonlinear σ-model

- The complete Einstein-NLSM equations read

$$\nabla^{\mu} L_{\mu} = 0 \ , \qquad G_{\mu\nu} = \kappa T_{\mu\nu} \ , \tag{5}$$

- the energy-momentum tensor of the NLSM given by

$$T_{\mu\nu} = -\frac{K}{2} \mathrm{Tr} \left[L_{\mu} L_{\nu} - \frac{1}{2} g_{\mu\nu} L^{\alpha} L_{\alpha} \right] .$$

- The winding number of the configurations reads

$$w_{\mathrm{B}} = \frac{1}{24\pi^2} \int \rho_{\mathrm{B}} \ , \quad \rho_{\mathrm{B}} = \mathrm{Tr}[\epsilon^{ijk} L_i L_j L_k] \ . \tag{6}$$

- We consider cases with

$$d\alpha \wedge d\Theta \wedge d\Phi \neq 0 \tag{7}$$

Ansatz and Field equations

- Consider a spacetime with a metric given by

$$ds^2 = -\frac{B_0^2\omega_s}{e^{f_0}}\left(\frac{2e^{f_0}}{B_0} - \omega_s G\right)dt^2 - \frac{2B_0^2}{e^{f_0}}\left(\frac{e^{f_0}}{B_0} - \omega_s G\right)dtdz + \frac{B_0^2}{e^{f_0}}Gdz^2 + e^{-2R}\left(dr^2 + d\theta^2\right) \,, \tag{8}$$

where $G = G(r,\theta)$, $R = R(r,\theta)$, B_0, ω_s, f_0 are constants.

- Choose the following range of coordinates

$$-\infty < r < \infty \,, \qquad \theta \in [0, 2\pi]. \tag{9}$$

- Space-time with this metric is always Lorentzian regardless of the metric components, since

$$\begin{vmatrix} g_{tt} & g_{tz} \\ g_{zt} & g_{zz} \end{vmatrix} = -B_0^2 < 0 \,, \tag{10}$$

Ansatz and Field equations

- Take generalized hedgehog ansatz ($t_i \equiv i\sigma_i$),

$$U^{\pm 1}(x^\mu) = \cos(\alpha)\,\mathbf{1}_2 \pm \sin(\alpha)\,n^i t_i \ , \qquad n^i n_i = 1 \ , \tag{11}$$
$$n^1 = \sin\Theta\cos\Phi \ , \quad n^2 = \sin\Theta\sin\Phi \ , \quad n^3 = \cos\Theta \ ,$$

- Assume that

$$\alpha = \alpha(r) \ , \qquad \Theta = q\theta \ , \qquad \Phi = \omega_s t + z \ , \tag{12}$$

and

$$\nabla_\mu\Phi\nabla^\mu\Phi = 0 \ , \qquad \nabla_\mu\Theta\nabla^\mu\Phi = 0 \ ,$$

- The NLSF eqns reduce to a single equation

$$\alpha'' - \frac{q^2}{2}\sin(2\alpha) = 0 \,. \tag{13}$$

- This equation is integrated to

$$(\alpha')^2 - q^2\sin^2\alpha = E_0 \,, \tag{14}$$

where E_0 is an integration constant.
- The compatibility with the Einstein equation requires $E_0 = 0$.
- The Einstein's eqns are

$$R'' - K\kappa q^2\sin^2\alpha = 0 \,, \tag{15}$$

$$(\partial_r^2 + \partial_\theta^2)G + 2C_0\sin^2(q\theta)e^{-2R}\sin^2\alpha = 0 \,, \tag{16}$$

where $C_0 = K\kappa e^{f_0}/B_0^2$.

Gravitating Tubes

Gravitating Tubes

- The eqns for $\alpha(r)$, $R(r)$ are easily solved;

$$\alpha(r) = 2 \arctan \exp \left(qr + C_1 \right) \ , \tag{17}$$

$$R(r) = K\kappa \ln \Big(\cosh(qr + C_1) \Big) + C_2 r + C_3 \ , \tag{18}$$

where C_1, C_2 and C_3 are integration constants.

- With these, the eqn for $G(r,\theta)$ becomes

$$\partial_r^2 G + \partial_\theta^2 G + \frac{2K\kappa}{B_0^2} e^{-2(C_2 r + C_3) + f_0} \cosh^{-2(K\kappa+1)}(qr + C_1) \sin^2(q\theta) \ = \ 0 \ . \tag{19}$$

- Impose the range of the parameter C_2 by

$$-(1-K\kappa)\ |q| < C_2 < -K\kappa\ |q|\ . \tag{20}$$

- Then, the energy density becomes finite;

$$T_{\hat{0}\hat{0}} = K\left[q^2 e^{2C_2 r + C_3} sech^{2(1-K\kappa)}(qr + C_1) + \frac{\omega_s e^{f_0}}{B_0(2e^{f_0} - B_0\omega_s G)} \sin^2(q\theta) sech^2(qr + C_1)\right] \tag{21}$$

- Regardless of G, our spacetime is regular, (S: Ricci Scalar)

$$S = 2K\kappa q^2 e^{2(C_2 r + C_3)} \cosh^{2(K\kappa - 1)}(qr + C_1)\ , \qquad R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = S^2\ ,$$
$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{4K^2\kappa^2}{3}\left[q e^{C_2 r + C_3} \cosh^{K\kappa - 1}(qr + C_1)\right]^4 .$$

Gravitating Tubes

- Let us introduce a radial coordinate

$$X(r) = \int_{-\infty}^{r} e^{-R(y)} dy = \int_{-\infty}^{r} \frac{e^{-C_2 y - C_3}}{\left(\cosh(qy + C_1)\right)^{K\kappa}} dy \ . \qquad (22)$$

- With the same range of C_2, the range of X becomes,

$$-\infty < r < \infty \qquad \Longrightarrow \qquad 0 < X < \infty \ . \qquad (23)$$

- Our metric becomes,

$$ds^2 = -\frac{B_0^2 \omega_s}{e^{f_0}} \left(\frac{2e^{f_0}}{B_0} - \omega_s G \right) dt^2 - \frac{2B_0^2}{e^{f_0}} \left(\frac{e^{f_0}}{B_0} - \omega_s G \right) dt dz + \frac{B_0^2}{e^{f_0}} G dz^2 + dX^2 + e^{-2\tilde{R}(X)} d\theta^2 \ , \qquad (24)$$

Gravitating Tubes

- In the limit of r, the component $g_{\theta\theta}$ tends to

$$\begin{aligned} & r \longrightarrow \pm\infty \\ \Longrightarrow \quad & e^{-R} \longrightarrow 2^{K\kappa} e^{\mp K\kappa C_1 - C_3} e^{-(C_2 \pm K\kappa|q|)r} \approx (\text{const.}) \times X \ , \end{aligned} \tag{25}$$

so that

$$\frac{g_{\theta\theta}(r=\infty)}{g_{\theta\theta}(r=-\infty)} = \frac{g_{\theta\theta}(X=\infty)}{g_{\theta\theta}(X=0)} = \left(\frac{K\kappa|q| + C_2}{K\kappa|q| - C_2}\right)^2 < 1 \ . \tag{26}$$

- We can choose C_3 such that the angular deficit is 1 near the axis!

Gauged Gravitating Tubes

Gauged Gravitating Tubes

- The previous solution can be generalized to U(1)-gauged system. The action is,

$$I[g,U,A] = \int d^4x\sqrt{-g}\left[\frac{\mathcal{R}}{2\kappa} + \frac{K}{4}\mathrm{Tr}\left(L^\mu L_\mu\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \ ,$$

$$L_\mu = U^{-1}D_\mu U \ , \quad D_\mu = \nabla_\mu + A_\mu\left[t_3, \ . \ \right] \ , \ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ,$$

- The field equations are

$$D_\mu L^\mu = 0 \ , \qquad G_{\mu\nu} = \kappa(T_{\mu\nu} + \bar{T}_{\mu\nu}) \ , \qquad \nabla_\mu F^{\mu\nu} = J^\nu \ , \quad (27)$$

where the current J^μ is given by

$$J^\mu = \frac{K}{2}\mathrm{Tr}\left[\widehat{O}L^\mu\right] \ , \qquad \widehat{O} = U^{-1}t_3U - t_3 \ , \qquad (28)$$

and the electromagnetic energy-momentum tensor is

$$\bar{T}_{\mu\nu} = F_{\mu\alpha}F_\nu^{\ \alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} \ . \qquad (29)$$

Gauged Gravitating Tubes

- The topological number density is

$$w_{\rm B} = \frac{1}{24\pi^2}\int_\Sigma \rho_{\rm B} \ , \tag{30}$$

where

$$\rho_{\rm B} = \epsilon^{ijk}{\rm Tr}\Big[\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right) - \partial_i\left[3A_j t_3\left(U^{-1}\partial_k U + \left(\partial_k U\right)U^{-1}\right)\right]\Big] \ . \tag{31}$$

- Assume that the gauge potential has the form of

$$A_\mu = \left(u, 0, 0, \frac{1}{\omega_s}u\right) \ , \qquad u = u(r,\theta) \ , \tag{32}$$

Gauged Gravitating Tubes

- The equation for $\alpha(r)$ and $R(r)$ are the same as the non-gauged case;

$$\alpha'' - \frac{q^2}{2}\sin 2\alpha = 0 \ , \qquad R'' - \frac{1}{2}K\kappa(\alpha'^2 + q^2\sin^2\alpha) = 0 \ ,$$

  so that we have the same solution

$$\alpha(r) = 2\arctan(e^{qr+C_1}) \ , \qquad R(r) = K\kappa\log(\cosh(qr+C_1))+C_2 r+C_3 \ .$$

- The Maxwell eqns reduce to a single equation

$$\Delta u - 2Ke^{-2R}(\omega_s - 2u)\sin^2\alpha\sin^2 q\theta = 0 \ . \qquad (33)$$

- The eqn for G modifies to

$$\Delta G + \frac{2\kappa}{B_0^2\omega_s^2} e^{f_0 - 2R}\left(K \sin^2\alpha \sin^2 q\theta (\omega_s - 2u)^2 + e^{2R}(\nabla u)^2 \right) = 0 \ .$$

- The energy density, topological number density, and current are

$$T_{\hat{0}\hat{0}} = \frac{K e^{f_0}}{2\omega_s B_0 (2e^{f_0} - B_0\omega_s G)} \Big[2\sin^2(q\theta) sech^2(qr + C_1)(2u - \omega_s)^2 + e^{2C_2 r + C_3}\cosh^{2K\kappa}(qr + C_1)\Big\{(\partial_r u)^2 + (\partial_\theta u)^2\Big\}\Big] + K q^2 e^{2C_2 r + C_3} sech^{2(1-K\kappa)}(qr + C_1) \ ,$$

$$\rho_{\rm B} = \partial_r\Big(6q\big(\alpha - \sin\alpha\cos\alpha\big)\sin(q\theta) + \frac{12q}{\omega_s}\sin\alpha\cos\alpha\sin(q\theta)\cdot u\Big) + \partial_\theta\Big(\frac{12}{\omega_s}\cos(q\theta)\cdot u\partial_r\alpha\Big) \ ,$$

$$J_\mu = 2K\sin^2\alpha\sin^2 q\theta(\partial_\mu\Phi - 2A_\mu) \ .$$

Gauged Gravitating Tubes

- We find the following plots for

$$C_1 = 0 \ , \quad C_2 = -\frac{1}{50} \ , \quad C_3 = 0 \ , \quad K = \frac{1}{10} \ , \quad \kappa = \frac{1}{4} \ ,$$
$$q = \frac{1}{2} \ , \quad \omega = 1 \ , \quad f_0 = 0 \ , \quad B_0 = \frac{1}{2} \ .$$

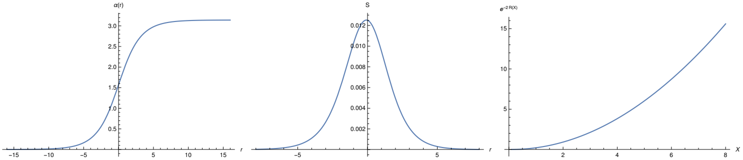

Figure: From left to right, the α profile and the Ricci scalar S as functions of the r coordinate and metric function $e^{-2R(X)}$ as a function of the X coordinate.

Gauged Gravitating Tubes

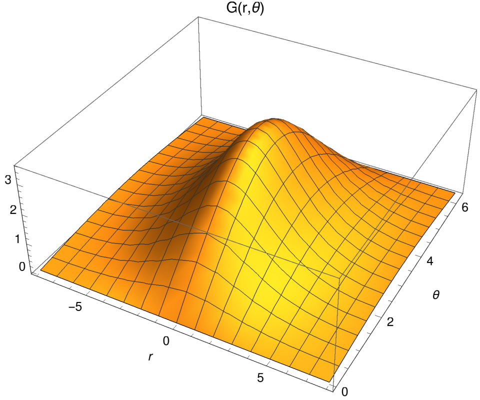

Figure: The metric function G as a function of r and θ.

Gauged Gravitating Tubes

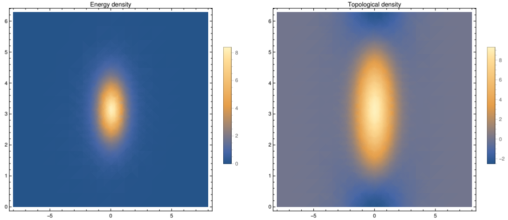

Figure: The Energy density $T_{\hat{0}\hat{0}}$ and topological density ρ_B as functions of r and θ.

Gauged Gravitating Tubes

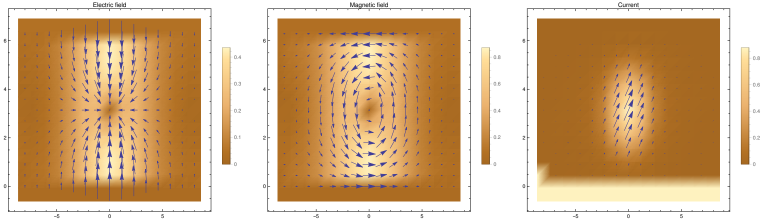

Figure: From left to right, the vector density plots of the Electric field E, the magnetic field B and of the current J_{μ}.

Superconducting Effect

Superconducting Effect

- Begin with

$$S_{\text{tot}} = \int d^4x \sqrt{g} \left(L_{\text{kin}} + V(\sigma, \psi) \right) \ ,$$

$$L_{\text{kin}} = -\frac{1}{4} \left(F^2 + B^2 \right) + |D\sigma|^2 + |D\psi|^2 \ ,$$

$$V(\sigma, \psi) = \frac{\lambda}{8} \left(|\psi|^2 - \mu^2 \right)^2 + \frac{\widetilde{\lambda}}{4} |\sigma|^4 + f |\sigma|^2 |\psi|^2 - m^2 |\sigma|^2 \ ,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \quad B_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu \ ,$$
$$D_\mu \sigma = (\partial_\mu + ieA_\mu) \sigma \ , \ D_\mu \psi = (\partial_\mu + ieS_\mu) \psi \ .$$

- $U(1) \times U(1)$ gauge symmetry: A_μ and S_μ.

Superconducting Effect

- Breakdown of U(1) corresponding to $S_\mu \longrightarrow \exists$ vortices.
- Assume Higgs field $\psi = \psi(r, \theta)$ in the core of vortex(along z-axis)!
- Choose the parameters such that the potential energy favors $\langle\sigma\rangle \neq 0$ within the core, and $\langle\sigma\rangle = 0$ asymptotically.
- If $\sigma_0\,(r, \theta)$ minimizes the energy of the string, then the superconducting current is associated with the (slowly varying) phase Θ of $\sigma_0\,(r, \theta)$:

$$\sigma\,(r, \theta, z, t) = \sigma_0\,(r, \theta)\exp\left[i\Theta\,(z, t)\right]\ . \tag{34}$$

- The current is

$$\overrightarrow{J} \approx 2e\sigma\left(\overrightarrow{\partial}\Theta + e\overrightarrow{A}\right)\ , \tag{35}$$

Superconducting Effect: mechanism

- Θ is only defined modulo 2π, the integral over a close loop of the above current will not vanish in general even when one turns off the electromagnetic field. $\longrightarrow$ Superconducting!
- Such a current cannot relax when the topological invariant associated to Θ is non-zero.
- Shifman(2013): How to get superconduction?

1. bulk theory with unbroken global non-Abelian symmetry such as $SU(2)$. (allowing $\exists$ string-like configurations)
2. Break $SU(2)$ down to a subgroup $U(1)$.
3. Introduce suitable potentials for the scalars.

Superconducting Effect: our tube

- Einstein-Maxwell-NLSM has all the above ingredients already!
- No need to introduce any potential; interactions among the Maurer-Cartan forms associated to the Isospin d.o.f. do the job!
- The current does not vanish even when the electromagnetic potential vanishes ($u = 0$).
- Such a "current left over" $J_{(0)\mu}$:

$$J^{(0)}_{\mu} = 2K \sin^2(\alpha) \sin^2(q\theta) \partial_\mu \Phi \ , \tag{36}$$

- It is a persistent current which cannot vanish as it is topologically protected.
- Superconducting current supported by the present gauged tubes!

Discussion: gravitational lensing

Discussion: gravitational lensing

- In many situations of high interest in astrophysics and cosmology, the analysis of light-like geodesics is already enough to get relevant information (avoiding the analysis of the full Maxwell eqns in the space-times of interest which is considerably more difficult).
- In the usual cases one analyzes the Maxwell equations

$$\nabla^{\mu} F_{\mu\nu} = 0 \ ,$$

  within the eikonal approximation: null geodesic argument from massless photon!
- Our tube: Maxwell field couple to gravity + NLSM + current!

Discussion: gravitational lensing

- The Maxwell eqns within the background of gravitating tube read

$$\begin{aligned} \nabla^{\mu} f_{\mu\nu} &= -4K \sin^2(q\theta) \sin^2(\alpha(r)) (a_\nu + \partial_\nu \Omega) \ , \\ 0 &= \nabla^{\nu} \left[\sin^2(q\theta) \sin^2(\alpha(r)) (a_\nu + \partial_\nu \Omega) \right] \ , \\ f_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu \ , \end{aligned}$$

  where a_ν is the electromagnetic perturbation.
- Gauge transformation Ω was introduced since we are considering electromagnetic perturbations keeping fixed the $SU(2)$ valued field.
- As it happens in the usual Ginzburg-Landau description of superconductors, the mass-like term arises from the terms quadratic in A_μ in the action.

Discussion: gravitational lensing

- In order to analyze the propagation of light-rays in these gravitating tubes, it is mandatory to develop the geometrical optics approximation corresponding to the system in Eqns. above.
- A very interesting but rather difficult topic!
- Physical effects of the mass-like term are very small far from the peaks in the energy density and topological density of the gravitating soliton.
- Thus, the light rays which propagate very far from the position of the tube do not feel the presence of the gravitating soliton itself and follow light-like geodesics.
- Close to the peaks the light rays will deviate considerably from light-like geodesics.

Thank you for your attention!