Defects and Quantization

based on ongoing work with N. Nekrasov

arXiv:2103.17186 with N. Lee and N. Nekrasov

arXiv:2007.03660 with N. Nekrasov

arXiv:1806.08270 with N. Nekrasov

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Motivation

For a given classical system, how do we quantize?

$$[\hat{p}^2 + \Lambda^2 \cos \hat{x} - E]\chi(x) = 0$$
$$[\hat{p}, \hat{x}] = \hbar$$

Represent the non-commutative algebra by $\hat{p} = \hbar\partial_x$ acting on $\chi(x)$. Then, with the change of coordiantes $y = e^{ix}$, (also redefine $\chi(x) \to y^{\frac{1}{2}}\chi(y)$)

$$0 = \left[\partial_y + \begin{pmatrix} 0 & \hbar^{-2}\left(\Lambda^2\left(\frac{1}{y}+\frac{1}{y^3}\right) - \frac{2E+\frac{1}{4}\hbar^2}{y^2}\right) \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix}\partial_y\chi(y) \\ \chi(y)\end{pmatrix}$$

Hence the spectral equation defines a meromorphic flat connection on a rank 2 holomorphic vector bundle over a Riemann surface (in this example, the sphere $\mathbb{CP}^1$ with two punctures at 0 and ∞).

The quantization translates into a geometric problem.

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This is a particular example of a large class of integrable systems, called Hitchin integrable systems. They are associated to four-dimensional $\mathcal{N} = 2$ field theories of class S [Gaiotto 09] [Gaiotto, Moore, Neitzke 09].

I will present how the quantization of Hitchin integrable systems can be addressed with the help of half-BPS defects in the class S theory.

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From $\mathcal{N} = 2$ theory of class S to Hitchin sigma model

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- $\mathcal{C}$ is a Riemann surface (possibly with punctures). Our main example is the four-punctured sphere $\mathbb{CP}^1 \setminus \{0, \mathfrak{q}, 1, \infty\}$.

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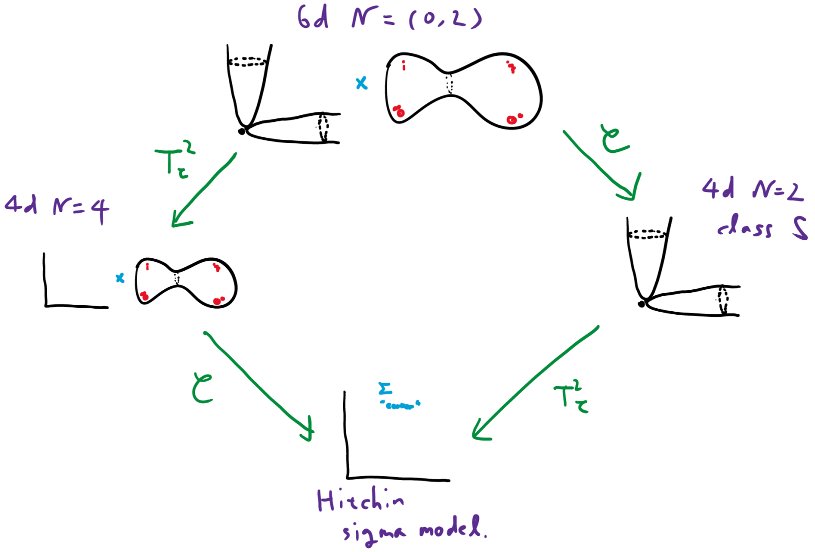

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Turn on the Ω-background on $\mathbb{C}^2$ [Nekrasov 02].

$$S_{\mathcal{T}[\mathfrak{g},\mathfrak{C}]} = \tau \int_{\mathbb{C}^2} \operatorname{Tr} F \wedge F + Q_\varepsilon(\cdots), \quad Q_\varepsilon^2 = \mathcal{L}_{V_\varepsilon}$$

The Ω-background only utilizes the $U(1)_{\varepsilon_1} \times U(1)_{\varepsilon_2}$ isometry V_ε of $\mathbb{C}^2$, so no harm to regard $\mathbb{C}^2 = \Sigma \tilde{\times} T^2_\tau$.

Compactify the theory along the torus T^2_τ. Then we get a topological A-model of maps $\Phi : \Sigma \to \mathcal{M}_H$ (with non-zero B-field) [Nekrasov, Witten 10].

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Forget the boundaries of Σ for a moment and treat it as $\mathbb{R}^2$.

The target $\mathcal{M}_H(G, \mathcal{C})$ of the sigma model is the Hitchin moduli space.

It is defined by the Hitchin equations on $\mathcal{C}$,

$$F_A + [\varphi, \bar{\varphi}] = 0$$
$$\bar{D}_A \varphi = D_A \bar{\varphi} = 0,$$

- A : connection on G-bundle over $\mathcal{C}$
- φ : adjoint-valued (1,0)-form on $\mathcal{C}$, called the Higgs field

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Some geometry of $\mathcal{M}_H$

$\mathcal{M}_H$ is hyper-Kähler; it has $\mathbb{CP}^1$ worth of complex structures. Pick a basis $IJ = K$.

The Hitchin map

$$\pi : \mathcal{M}_H(G, \mathcal{C}) \longrightarrow \mathcal{B} = \bigoplus_{k=2}^{N} H^0\left(K_{\mathcal{C}}^k\right)$$
$$(A, \varphi) \longmapsto \left(\operatorname{Tr} \varphi^k\right)_{k=2}^{N}$$

defines an algebraic integrable system. It is holomorphic in I.

By definition, the holomorphic functions on the base $\mathcal{B}$ are classical Hamiltonians. These are the Coulomb branch chiral operators of the class S theory.

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The complex structure $\pm I$ is special. For all the other complex structures $\mathcal{I}_t$, the Hitchin moduli space is isomorphic to

$$\mathcal{M}_H(G, \mathcal{C}) \simeq \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \mathcal{C}) = \text{Hom}(\pi_1(\mathcal{C}), G_{\mathbb{C}})/G_{\mathbb{C}}$$

(For $G = SU(N)$, $G_{\mathbb{C}} = SL(N, \mathbb{C})$)

Some aspects of the quantization can be realized in a purely geometric manner.

The holomorphic functions on a complex Lagrangian submanifold $\mathrm{Op}(\mathcal{C}) \subset \mathcal{M}_{\mathsf{flat}}$ are the quantum spectra of Hitchin Hamiltonians [Beilinson, Drinfeld 91].

It admits a gauge theoretical account [Nekrasov, Rosly, Shatashvili 11] [SJ, Nekrasov 18] [SJ, Nekrasov], as I will now explain.

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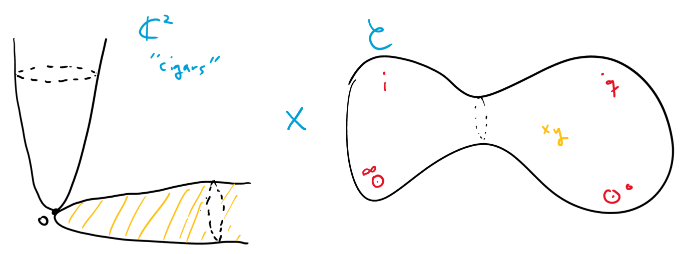
Defects and quantization

The gauge theoretical account for the quantization problem involves half-BPS defects in the class S theory.

Canonical surface defect

Insert an $M2$-brane at $\mathbb{C} \times \{y\}$



In the field theory limit, it defines a half-BPS surface defect by the coupling to a two-dimensional sigma model, with the target

$$\mathcal{O}(-1) \otimes \mathbb{C}^N \to \mathbb{CP}^{N-1},$$

with the complexified Kähler parameter y ($=$ complexified FI parameter).

The partition function of the coupled system (the vacuum expectation value of the defect) is exactly computable [Nekrasov 17] [SJ, Nekrasov 18].

- Without defect : $\mathcal{Z}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) = \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mu_{\boldsymbol{\lambda}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2)$
- With the defect :

$$\begin{aligned} &\mathfrak{X}_\alpha(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}, y) \\ &= \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mu_{\boldsymbol{\lambda}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2) \left(\sum_{n=0}^{\infty} y^{-n} \mathcal{O}_{\alpha,n,\boldsymbol{\lambda}}(\mathbf{a}, \mathbf{m}, \varepsilon_1) \right) \\ &= \langle \mathcal{O}_\alpha(y) \rangle \end{aligned}$$

Here, $\alpha = 1, \cdots, N$ enumerates the choice of the 2d vacuum.

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Note that y-derivatives of $\mathfrak{X}$ compute the vev of the twisted chiral operators of the sigma model.

There is a non-trivial 2d-4d coupled twisted chiral ring relation [Gaiotto, Gukov, Seiberg 13].

This equation can be exactly derived, using analytic constraints (qq-characters and Dyson-Schwinger equations.. [Nekrasov 15] see [SJ, Nekrasov 18] for derivation) on $\mathfrak{X}$:

$$0 = \left[\partial_y^N + t_2(y)\partial_y^{N-2} + \cdots + t_N(y)\right] \mathfrak{X}(\mathfrak{q}, y),$$

where $t_k(y)$ are meromorphic functions in $\mathfrak{q}$ and y with residues determined by $\left\langle \mathrm{Tr}\, \phi^k \right\rangle = \sum_{\alpha=1}^N a_\alpha^k + \cdots$.

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What is its geometric meaning in $\mathcal{M}_H$?

Take the limit $\varepsilon_2 \to 0$. The defect is lying on the z_1-plane, so that it only contributes to the regular part. It is well-approximated by the evaluation at the *limit shape* [Nekrasov, Okounkov 03]:

$$\mathfrak{X}(\mathfrak{q}, y) = \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mu_{\boldsymbol{\lambda}} \mathcal{O}[\boldsymbol{\lambda}](y) = e^{\frac{\tilde{\mathcal{W}}(\mathfrak{q})}{\varepsilon_2}} \chi(\mathfrak{q}, y) + \mathcal{O}(\varepsilon_2^0)$$

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The equation can be reorganized into

$$
\begin{aligned}
0 &= \left[\partial_y + \begin{pmatrix} 0 & t_2 & \cdots & t_N \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -1 & 0 \end{pmatrix}\right] \begin{pmatrix} \partial_y^{N-1}\chi \\ \vdots \\ \partial_y^2\chi \\ \partial_y\chi \\ \chi \end{pmatrix} \\
&= (\partial_y + \mathcal{A}_y)\boldsymbol{\chi}.
\end{aligned}
$$

Namely, it defines a meromorphic flat connection $\mathcal{A} \in \mathcal{M}_{\text{flat}}$, called an *oper*. Recall that t_k are determined by $\left\langle \operatorname{Tr} \phi^k \right\rangle_{\varepsilon_2 \to 0}$. Thus such flat connections form an $N-1$-dimensional vector space, in fact, a complex Lagrangian submanifold $\operatorname{Op}(\mathcal{C}) \subset \mathcal{M}_{\text{flat}}$.

[Beilinson, Drinfeld 91]

Holomorphic functions on $\mathrm{Op}(\mathcal{C})$ are quantum spectra of Hitchin Hamiltonians.

But now, in the gauge theory setting this is a consequence of the Bethe/gauge correspondence [Nekrasov, Shatashvili 09]:

$$\left\langle \mathrm{Tr}\,\phi^k \right\rangle_{\varepsilon_2 \to 0} = E_k = \text{quantum spectrum of } \hat{H}_k$$

Remark:

- The generating function of $\mathrm{Op}(\mathcal{C}) \subset \mathcal{M}_{\text{flat}}$ is identified with the effective twisted superpotential in the limit $\varepsilon_2 \to 0$. See [Nekrasov, Rosly, Shatashvili 11] [Hollands, Kidwai 17] [SJ, Nekrasov 18].

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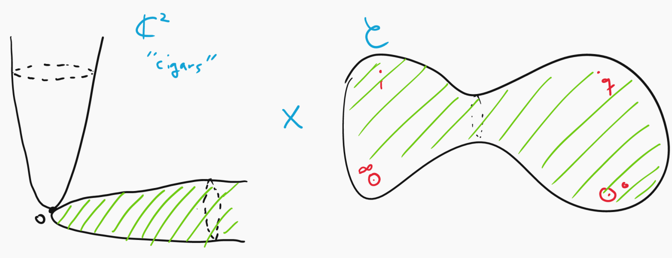
How about the quantum Hamiltonians and the common eigenfunction?

$$0 = (\hat{H}_k - E_k)\psi(\mathbf{z}), \quad k = 2, \cdots, N$$

We need another type of half-BPS surface defect.

Regular surface defect

Insert an $M5$-brane at $\mathbb{C} \times \mathcal{C}$:



In the 4d $\mathcal{N} = 2$ theory, it defines a half-BPS surface defect by the singularity along the surface (monodromy defect) [Gukov, Witten 08]. Breaks the global gauge symmetry to $U(1)^{N-1} \subset SU(N)$.

There are $N-1$ complex parameters $\mathbf{z}$ encoding the singularity of the gauge field

$$A_\mu dx^\mu \sim (\alpha_1, \alpha_2, \cdots, \alpha_N) d\theta$$

and the coupling to the magnetic fluxes

$$\exp\left(i\eta \cdot \mathfrak{m}\right).$$

The vev of the defect is exactly computable:

$$\Psi(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}, \mathbf{z}) = \langle \mathcal{S}(\mathbf{z}) \rangle = \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mu_{\boldsymbol{\lambda}} \mathcal{S}[\boldsymbol{\lambda}](\mathbf{z})$$

There is a dual description of the same surface defect, as coupling to a two-dimensional sigma model on a vector bundle over a flag variety.

In this description, $\mathbf{z}$ parameters are the Kähler parameters.

The analytic constraints on the partition function implies

$$0 = \left(\frac{\varepsilon_2}{\varepsilon_1} \frac{\partial}{\partial \mathfrak{q}} + \hat{H}_2 \right) \Psi(\mathbf{z}),$$

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Thus,

$$0 = \left(\hat{H}_2 + \frac{\partial \tilde{\mathcal{W}}}{\partial \mathfrak{q}} \right) \psi(\mathbf{z}) = (\hat{H}_2 - E_2) \psi(\mathbf{z})$$

$\mathbf{z}$ can be understood as holomorphic coordinates on the moduli space $\mathcal{M}$ of (stable) $G_{\mathbb{C}}$-bundles, where $\mathcal{M} \subset T^*\mathcal{M} \simeq \mathcal{M}_H$.

Is $\psi(\mathbf{z})$ a common eigenfunction of $\hat{H}_k$? How about the Schrödinger equations for other $\hat{H}_k$?

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In the limit $\varepsilon_2 \to 0$, the surface defect only contributes to the regular part since it lies on the z_1-plane:

$$\begin{aligned}\Psi(\mathfrak{q}, \mathbf{z}) = \langle \mathcal{S}(\mathbf{z}) \rangle &= \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mu_{\boldsymbol{\lambda}} \mathcal{S}[\boldsymbol{\lambda}](\mathbf{z}) \\ &= e^{\frac{\tilde{\mathcal{W}}(\mathfrak{q})}{\varepsilon_2}} \psi(\mathfrak{q}, \mathbf{z}) + \mathcal{O}(\varepsilon_2^0)\end{aligned}$$

Thus,

$$0 = \left(\hat{H}_2 + \frac{\partial \tilde{\mathcal{W}}}{\partial \mathfrak{q}} \right) \psi(\mathbf{z}) = (\hat{H}_2 - E_2) \psi(\mathbf{z})$$

$\mathbf{z}$ can be understood as holomorphic coordinates on the moduli space $\mathcal{M}$ of (stable) $G_{\mathbb{C}}$-bundles, where $\mathcal{M} \subset T^*\mathcal{M} \simeq \mathcal{M}_H$.

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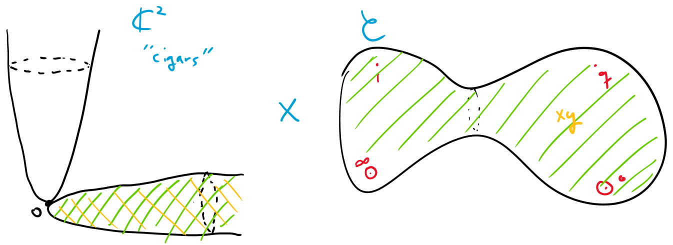
Regular surface defect + canonical surface defect

Insert an $M2$-brane at $\mathbb{C} \times \{y\}$ on top of an $M5$-brane at $\mathbb{C} \times \mathcal{C}$:



In the 4d $\mathcal{N} = 2$ theory, the configuration reduces to surface defects on the z_1-plane parallel to each other.

Aligned surface defects

The combined partition function (the correlation function of the two surface defects) is exactly computable.

$$\Upsilon(\mathfrak{q}, \mathbf{z}, y) = \langle \mathcal{S}(\mathbf{z}) \mathcal{O}(y) \rangle$$

The differential equation in coupling parameters $\mathfrak{q}$, $\mathbf{z}$, and y looks like

$$0 = \left(\partial_y + \frac{\hat{\mathcal{A}}_0}{y} + \frac{\hat{\mathcal{A}}_\mathfrak{q}}{y - \mathfrak{q}} + \frac{\hat{\mathcal{A}}_1}{y - 1} \right) \boldsymbol{\Upsilon}(\mathfrak{q}, \mathbf{z}, y),$$

where $\hat{\mathcal{A}}_{0,\mathfrak{q},1}$ are $N \times N$ matrix of differential operators in $\mathbf{z}$.

In the limit $\varepsilon_2 \to 0$, the partition function as an ensemble average is approximated by the evaluation at the limit shape.

Then the correlation function factorizes:

$$\Upsilon(\mathfrak{q}, \mathbf{z}, y) = \langle \mathcal{S}(\mathbf{z}) \mathcal{O}(y) \rangle = \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mu_{\boldsymbol{\lambda}} \mathcal{S}[\boldsymbol{\lambda}](\mathbf{z}) \mathcal{O}[\boldsymbol{\lambda}](y)$$
$$= e^{\frac{\tilde{\mathcal{W}}(\mathfrak{q})}{\varepsilon_2}} \psi(\mathfrak{q}, \mathbf{z}) \chi(\mathfrak{q}, y) + \mathcal{O}(\varepsilon_2^0)$$

Then we have

$$0 = \left(\partial_y + \hat{\mathcal{A}}_y\right) \psi(\mathbf{z}) \boldsymbol{\chi}(y)$$

where $\hat{\mathcal{A}}$ is 'almost' an oper, with E_k replaced by k-th order differential operators $\hat{H}_k$ in $\mathbf{z}$.

Recall $\chi(y)$ is a solution to the oper equation with given E_k,

$$0 = (\partial_y + \mathcal{A}_y)\chi(y).$$

Thus we get

$$0 = (\hat{H}_k - E_k)\psi(\mathbf{z}), \quad k = 2, \cdots, N.$$

Conversely, for a given quantum spectral problem defined by the Schrödinger equations above, we can obviously reconstruct an oper.

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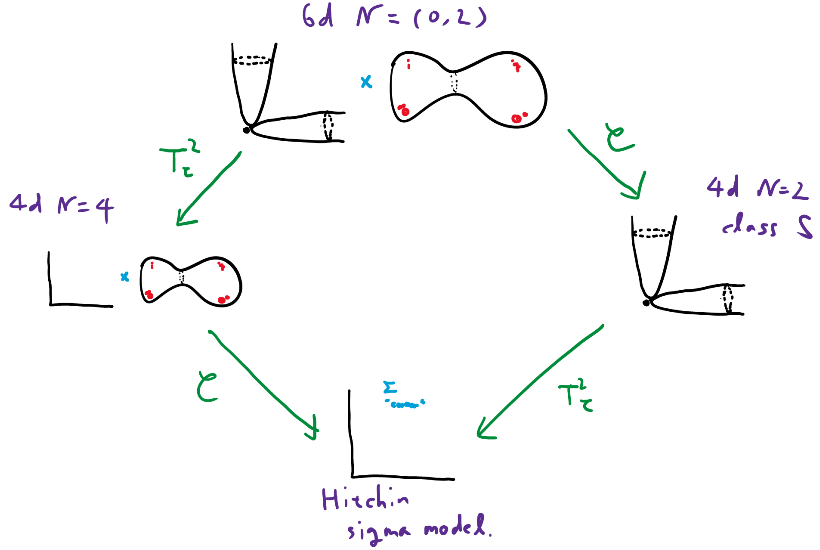
Conversely, for a given quantum spectral problem defined by the Schrödinger equations above, we can obviously reconstruct an oper.

To be precise, we should understand the space where $\psi(\mathbf{z})$ lives in.

An adequate framework to address this is the brane quantization [Gukov, Witten 08] in the topological sigma model. I will now show how the $\mathcal{N} = 2$ theory gets connected to the brane quantization setup. This is also directly related to the GL-twisted $\mathcal{N} = 4$ theory of Kapustin-Witten [Kapustin, Witten 06].

GL-twisted $\mathcal{N} = 4$ gauge theory and Hitchin sigma model



Thank you!