

*Results  
&  
Discussion*

*Derivation*

# *Deformation of the GW spectrum by density perturbations*

*Ryusuke Jinno (DESY)*

*PASCOS 2021, June 18th*

*Main idea*

*Introduction*



# GRAVITATIONAL WAVES: A NEW PROBE TO THE UNIVERSE

## ➤ Gravitational waves:

Transverse-traceless part of the metric

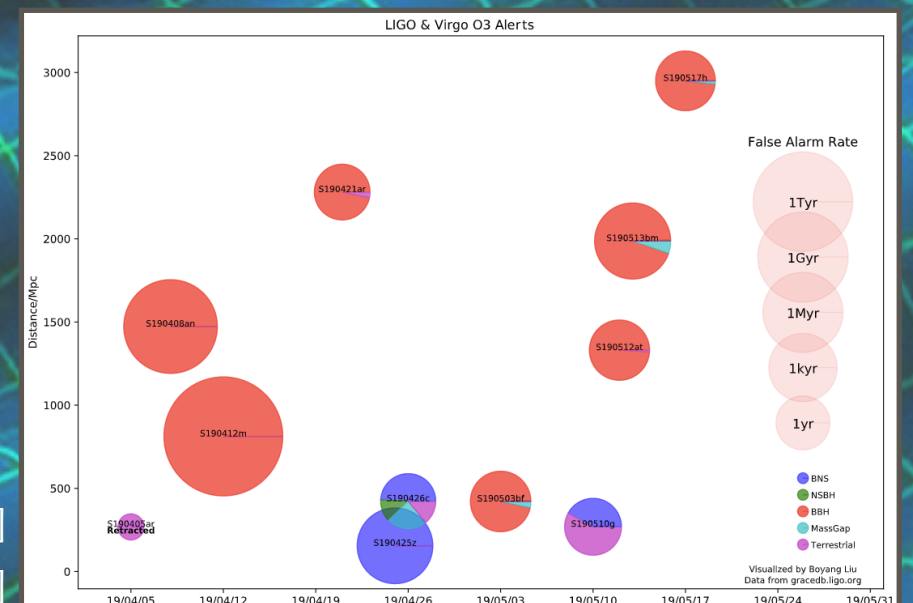
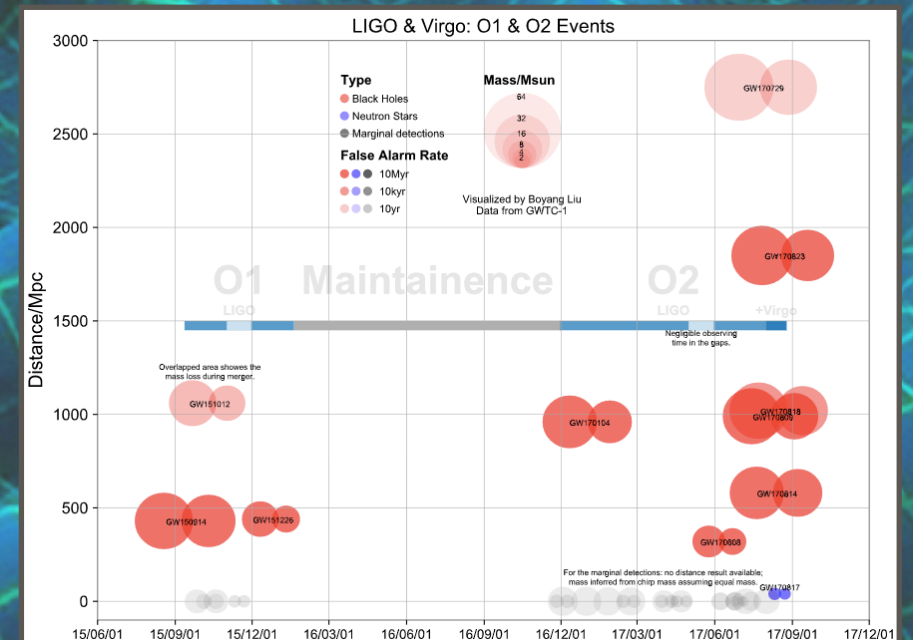
$$ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dx^j$$

obeying the equation of motion sourced by the energy-momentum tensor of the system

$$\square h_{ij} \sim G\Lambda_{ij,kl}T_{kl}$$

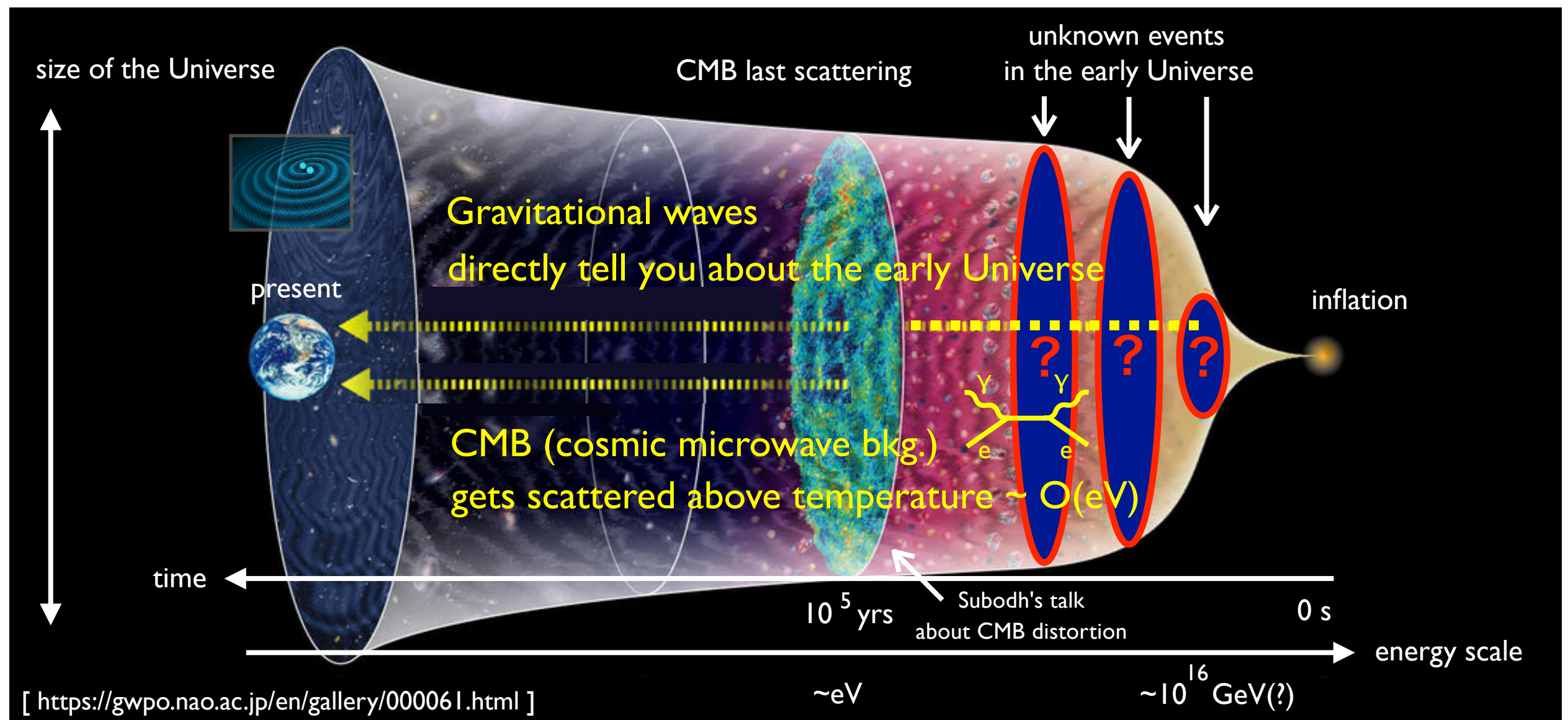
## ➤ Detections by LIGO & Virgo have been exciting us

[ Wikipedia "List of gravitational wave observations" ]  
[ see also <https://gracedb.ligo.org/superevents/public/O3/> ]



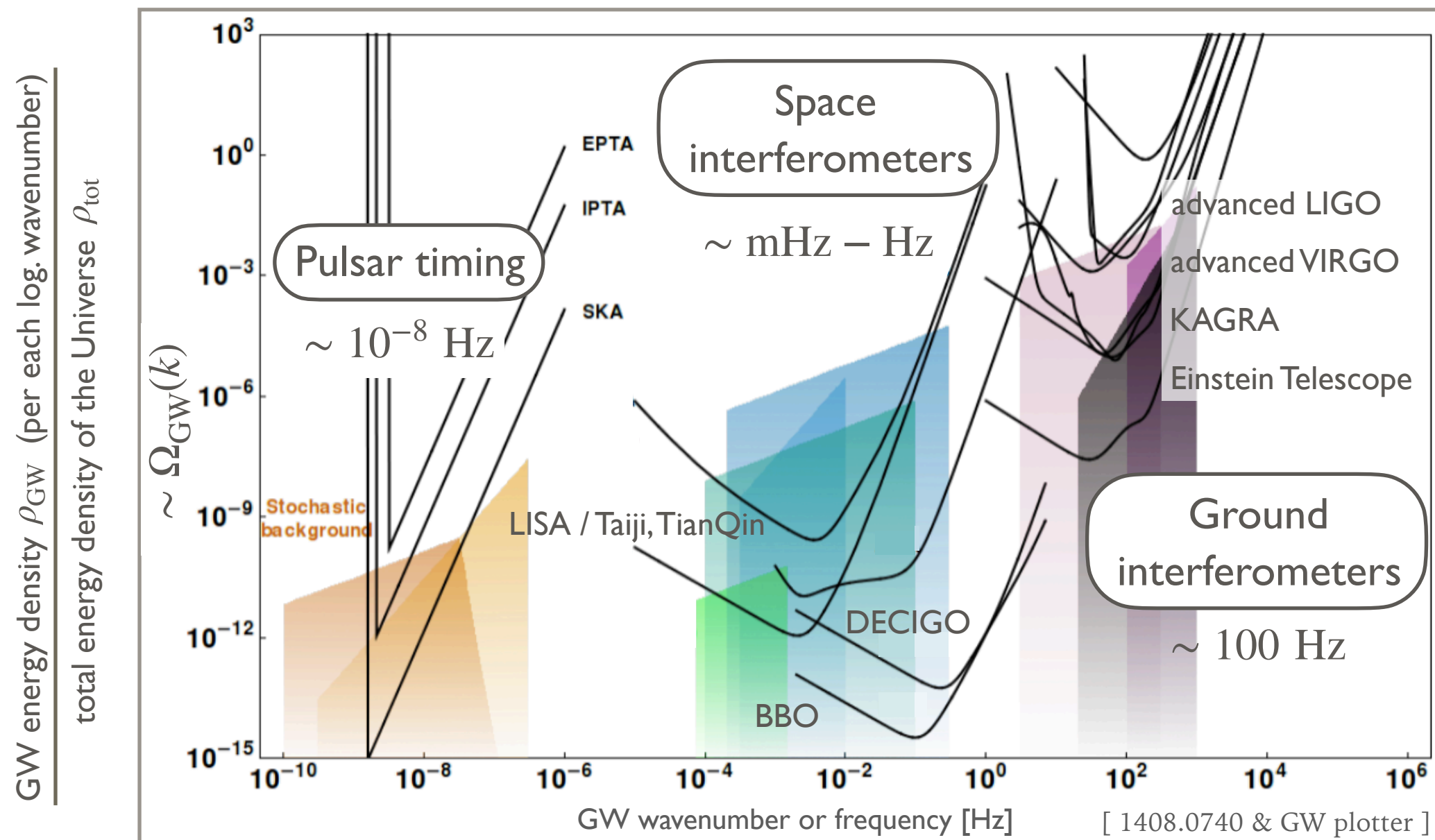
# GWS FROM EARLY UNIVERSE

- What is special about GWs?



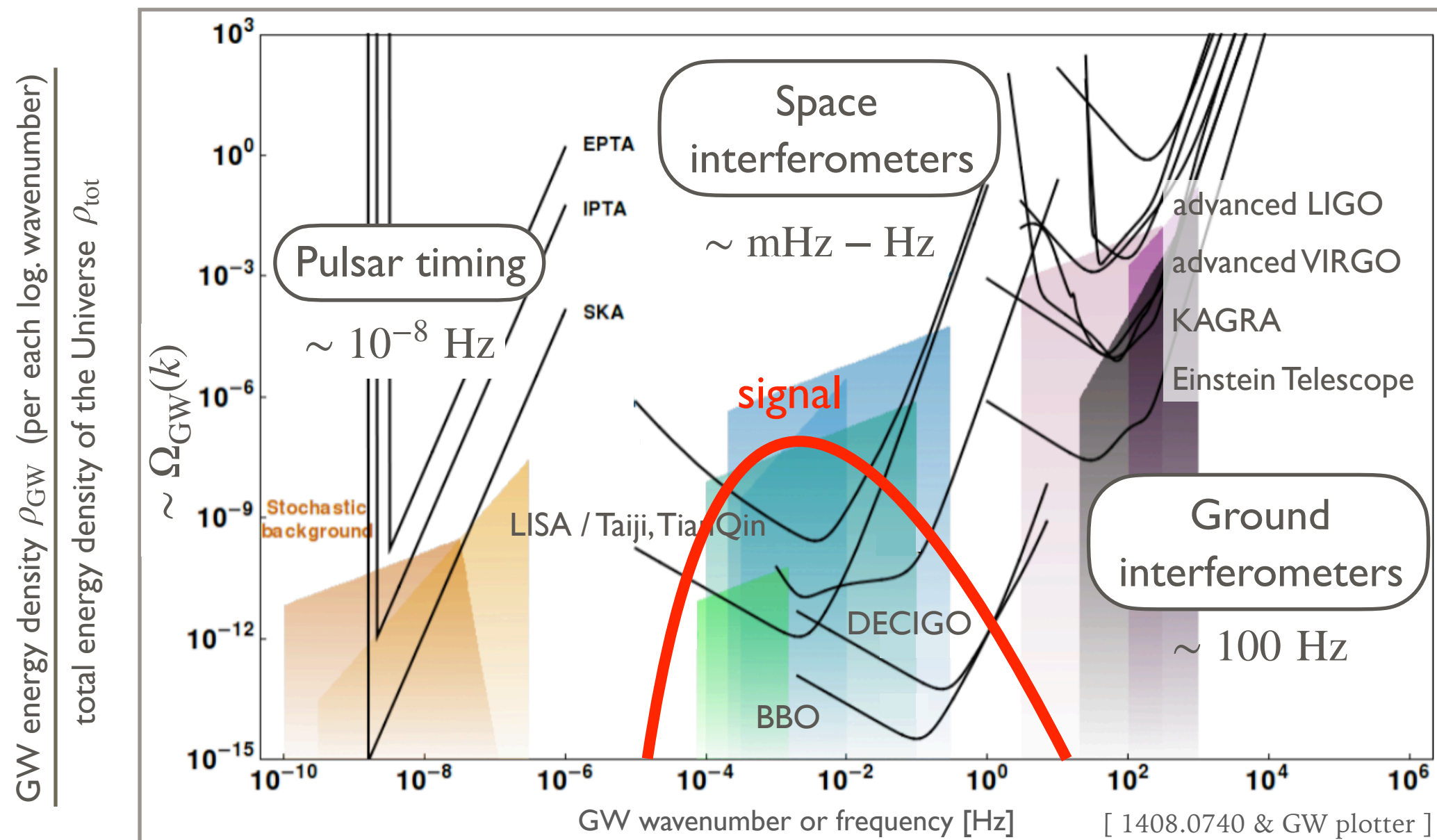
# PRESENT & FUTURE OBSERVATIONS

- Summary of ongoing & future experiments



# PRESENT & FUTURE OBSERVATIONS

- Summary of ongoing & future experiments





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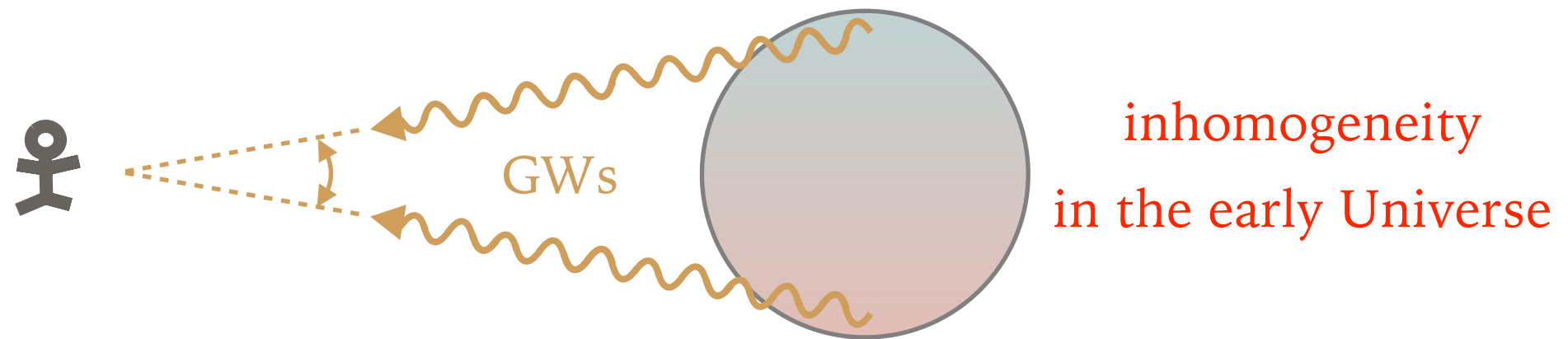
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# THE CENTRAL QUESTION

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- Gravitational waves propagate in an inhomogeneous Universe  
→ *What is the effect on the GW isotropic spectrum?*



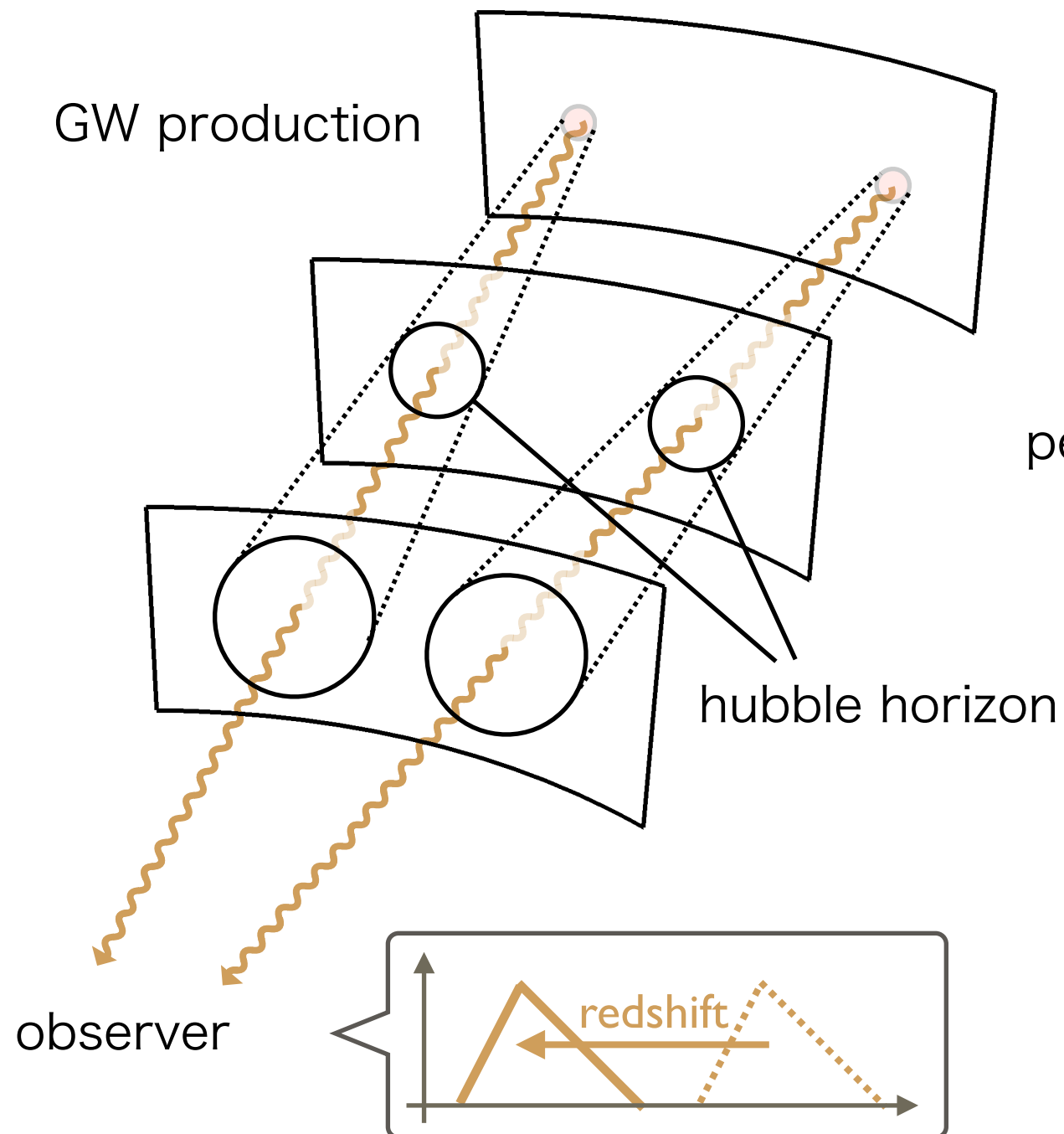
We consider isotropic (= directionally averaged) spectrum  
because typically (angular scale for inhomogeneity)  $\lll$  (detector resolution)

- The same effect also leads to anisotropy of the GW spectrum

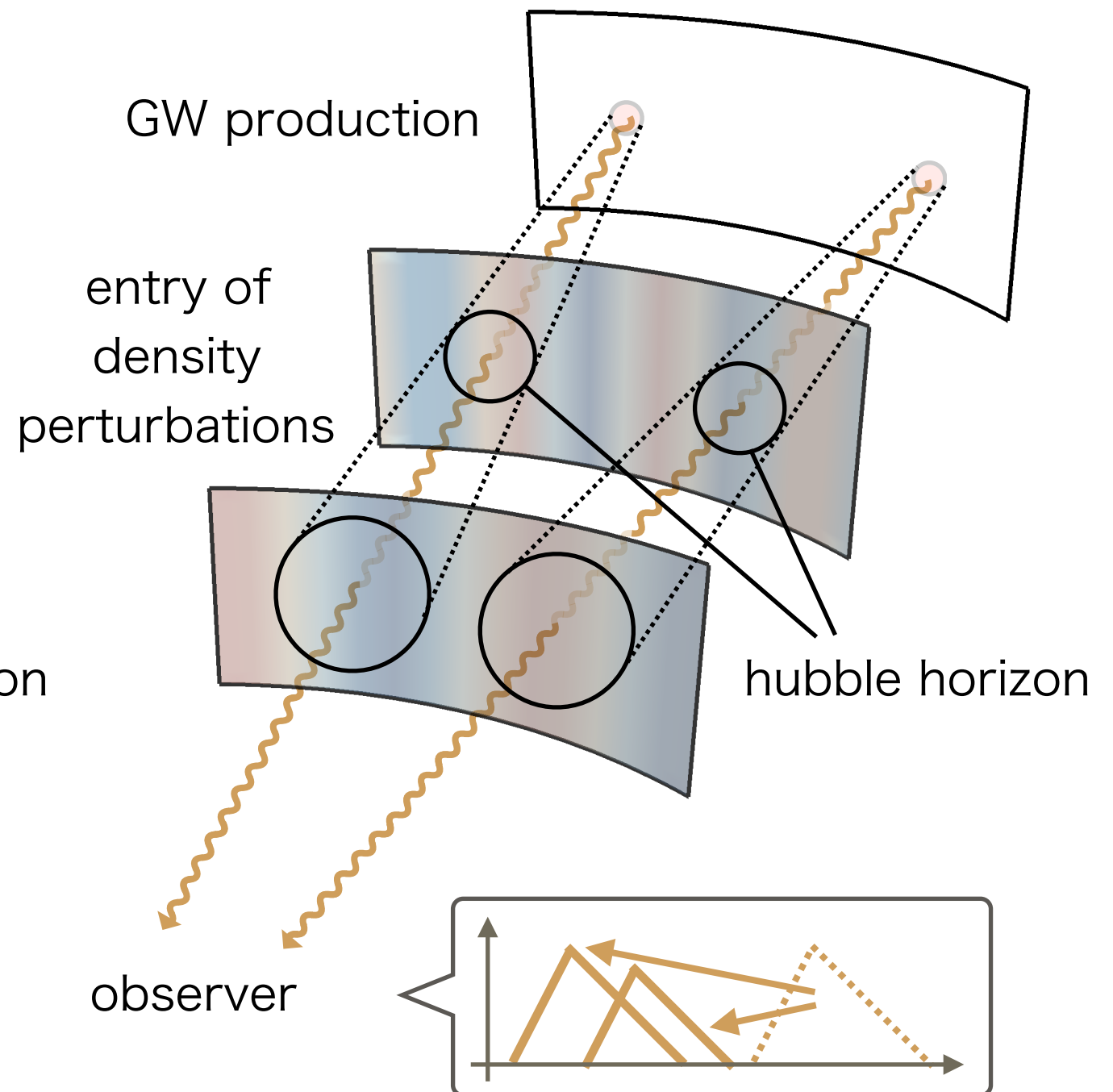
[ Alba, Maldacena '15 ] [ Bartolo, Bertacca, Matarrese, Peloso, Ricciardone, Riotto, Tasinato '19 ]  
[ Adshead, Afshordi, Dimastrogiovanni, Fasiello, Lim, Tasinato '20 ] [ Malhotra, Dimastrogiovanni, Fasiello, Shiraishi '20 ] ...

# SKETCH OF THE MAIN RESULT

FRW case

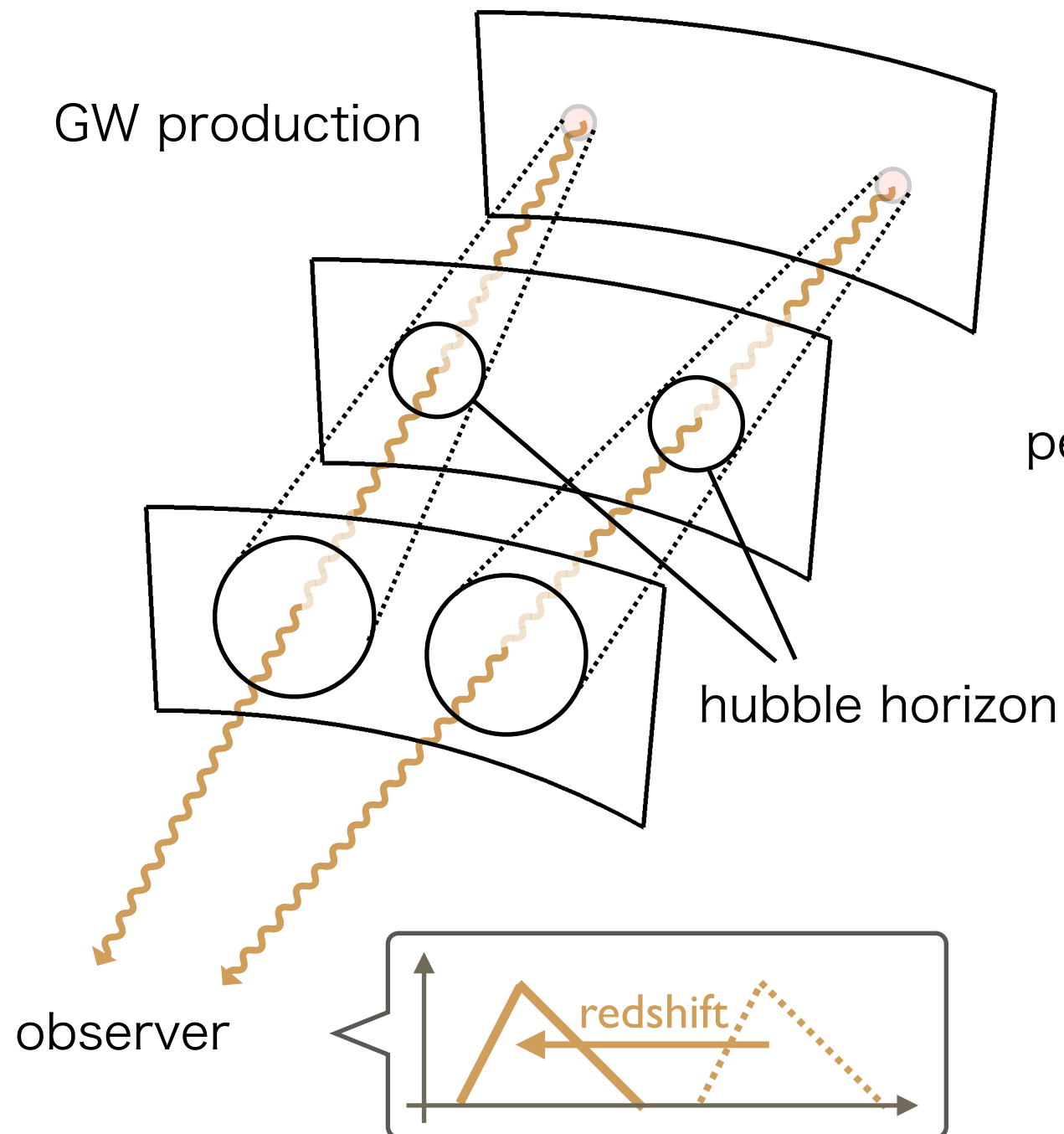


Inhomogeneous case

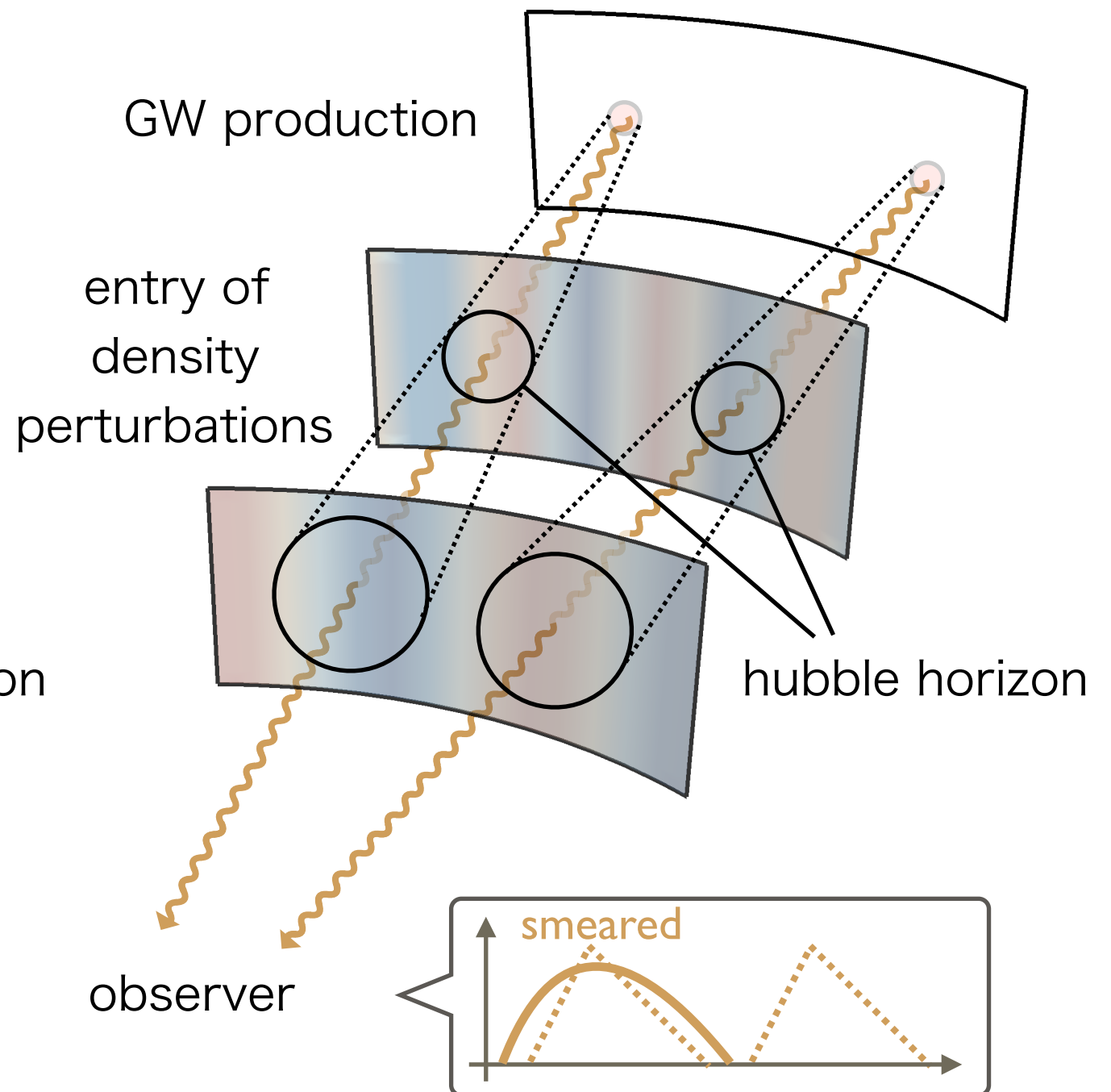


# SKETCH OF THE MAIN RESULT

FRW case



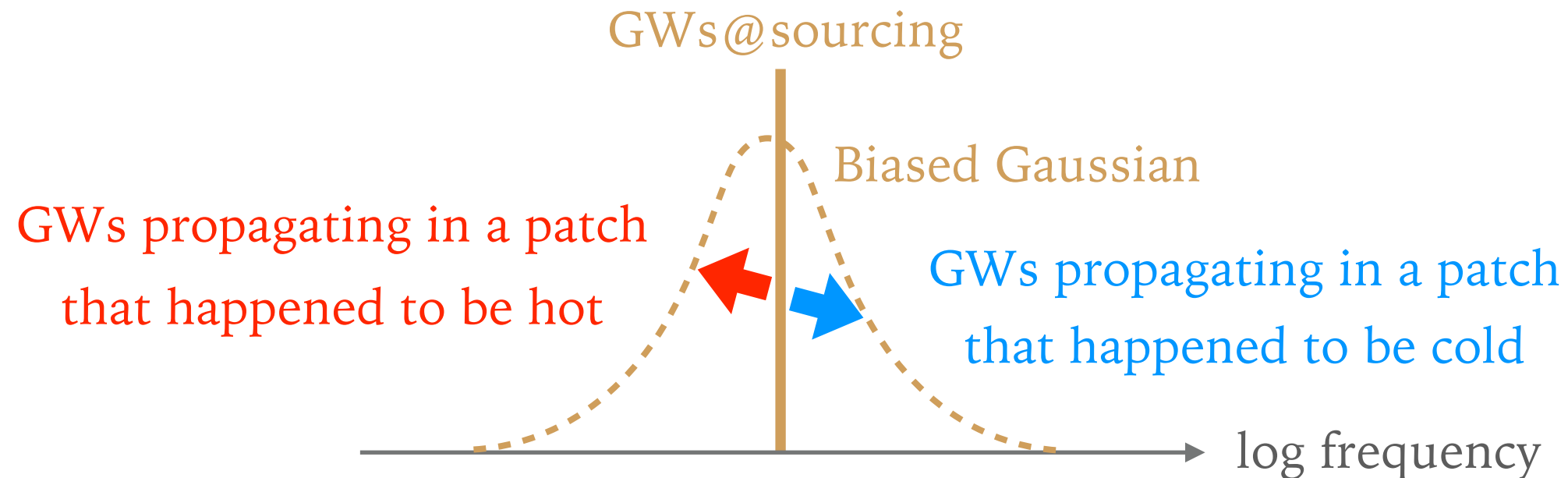
Inhomogeneous case



# SKETCH OF THE MAIN RESULT (CONT'D)

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- Each frequency bin experiences random walk



- For a more generic GW spectrum@sourcing, we convolute the biased Gaussian

$$\Delta_{\text{observer}}^{2,(o)}(\ln f) \simeq \int d \ln f' \Delta_{\text{source}}^{2,(s)}(\ln f') K(f, f') \quad \text{biased Gaussian}$$

- This picture takes into account at least  
part of the relevant contributions (→ later)

# DEFINITION OF THE SETUP

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## ➤ Assume any GW source in very early Universe

- First-order phase transitions e.g. [ Witten '84 ] [ Hogan '86 ] [ Kosowski, Turner, Watkins '92 ]  
[ Kamionkowski, Kosowski, Turner '93 ] [ Hindmarsh, Huber, Rummukainen, Weir '13 ] ...
- Gauge field production e.g. [ Cook, Sorbo '11 ] [ Sorbo '11 ] [ Malek-Nejad, Sheikh-Jabbari '11 ] [ Anber, Sorbo '12 ]  
[ Namba, Peloso, Shiraishi, Sorbo, Unal '15 ] [ Domcke, Pieroni, Binétury '16 ] ...
- Preheating e.g. [ Khlebnikov, Tkachev '97 ]  
[ Easter, Lim '06 ] [ Garcia-Bellido, Figueroa '07 ] [ Dufaux, Bergman, Felder, Kofman, Uzan '07 ] ...
- Topological defects e.g. [ Zeldovich, Kobzarev, Okun '74 ] [ Kibble '76 ] [ Vilenkin '81 ]  
[ Battye, Shellard '93 & '96 ] [ Gleiser, Roberts '98 ] ...

## ➤ Curvature perturbations (from inflation) enter the horizon later

- We assume single clock: each horizon patch is different only by time shift:  
GW production occurs in the same way among all the patches, up to time shift.
- Curvature perturbations are preferably enhanced (→ relation to PBH)

## ➤ We assume hierarchy in scales

(detector resolution)  $\gg \gg$  (length of curvature pert'n)  $\gg \gg$  (GW wavelength)



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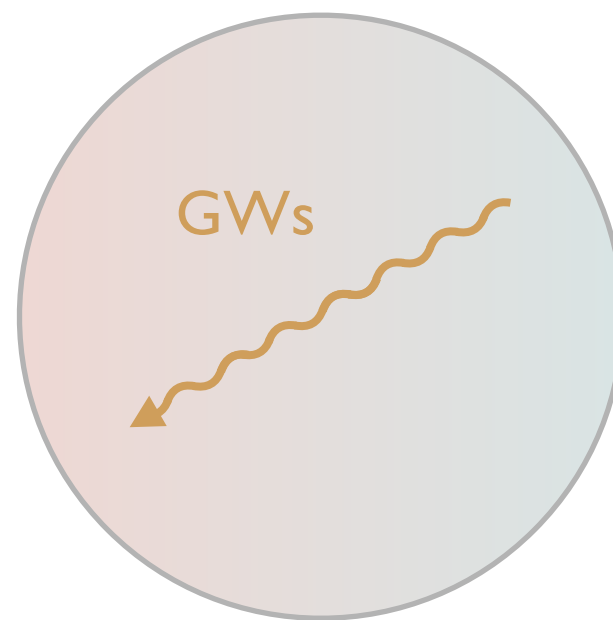
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# QUICK "DERIVATION"

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- Hierarchy in scales justifies using geometric optics [ Laguna, Larson, Spergel, Yunes '10 ]
  - Infinitely short-wave GW rays propagating inside density perturbations



Hubble horizon

- Just like CMB, we have several effects on GW rays

Sachs-Wolfe / Integrated Sachs-Wolfe / Doppler / Lensing

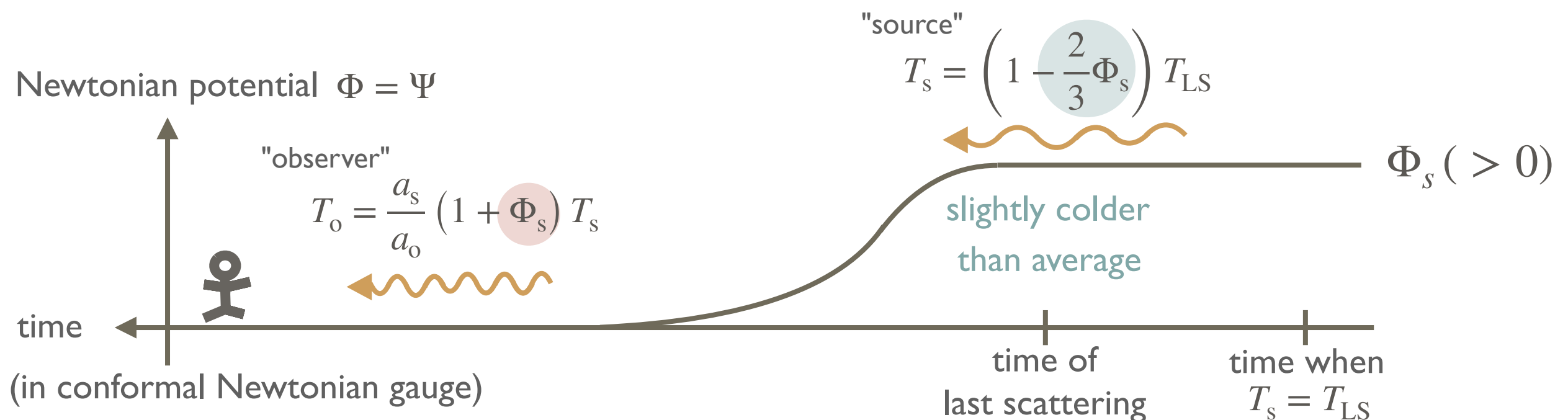
# QUICK "DERIVATION"

- Sachs-Wolfe effect (in CMB context) [ Sachs & Wolfe '67 ] [ Hu & White '97 ]

$$\frac{\Delta T}{T} = \Phi_s - \frac{2}{3}\Phi_s$$

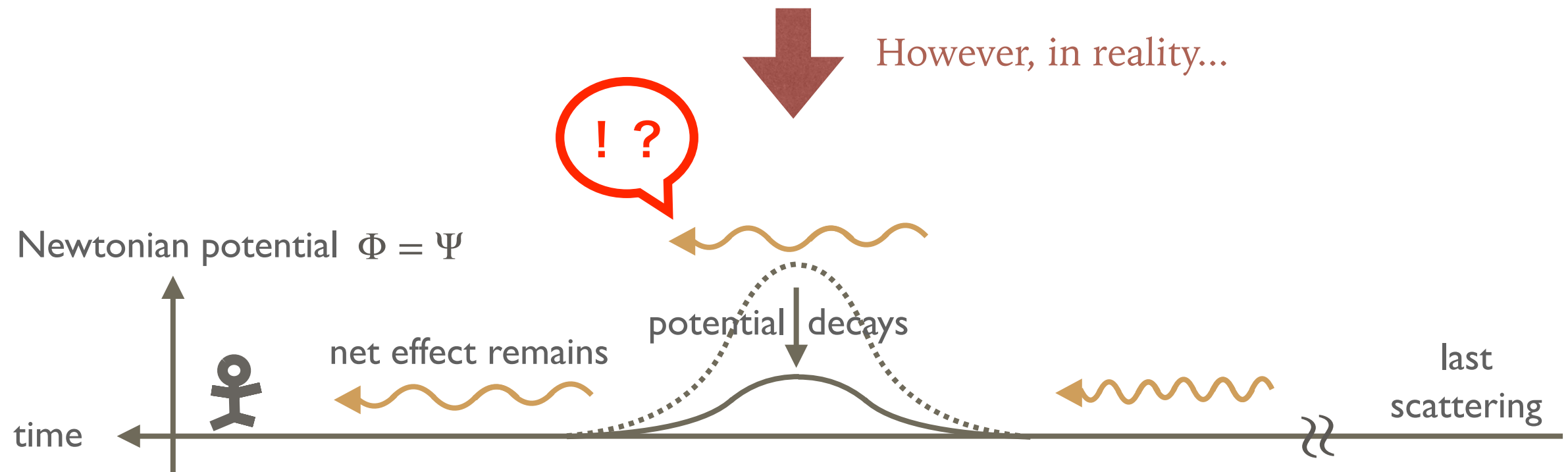
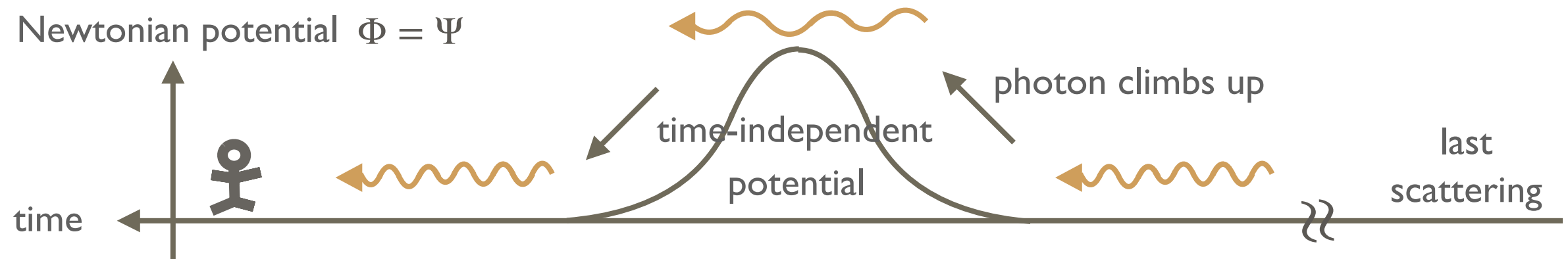
where  $ds^2 = -a^2(1 + 2\Phi) d\tau^2 + a^2\delta_{ij}(1 - 2\Psi) dx^i dx^j$  (conformal Newtonian gauge)

$\Phi = \Psi$  (absense of anisotropic stress)



# QUICK "DERIVATION"

- Integrated Sachs-Wolfe effect (in CMB context) [ Rees & Sciama '68 ]



# QUICK "DERIVATION"

- We average the frequency and amplitude random walk over the curvature perturbations

$$\Delta_h^{2,(o)}(\ln f) \simeq \left\langle \left( 1 + 2 \underline{\Delta \ln A} \right) \Delta_h^{2,(s)}(\ln f - \underline{\Delta \ln f}) \right\rangle_{\text{scalar ens. ave.}}$$

...using linear-order results from geometric optics [Laguna, Larson, Spergel, Yunes '10]

$$\begin{array}{ll} \text{amplitude } \Delta \ln A = & \boxed{-\Psi_s - \frac{1}{2}\Phi_s} + \text{lensing (neglected)} \\ \text{frequency } \Delta \ln f = & \boxed{\Phi_s - \frac{1}{2}\Phi_s} \quad \boxed{+ \int_{\lambda_s}^{\lambda_o} d\lambda \partial_\tau (\Phi + \Psi)} \\ & \text{SW} \qquad \qquad \text{ISW} \end{array}$$

- Then the scalar average is strictly calculable

$$\Delta_h^{2,(o)}(\ln f) \simeq \int d \ln f' \Delta_h^{2,(s)}(\ln f') K(f, f') \quad \begin{array}{l} \text{linearly biased} \\ \text{Gaussian} \end{array} \quad K(f, f') = \frac{1}{\sqrt{2\pi\sigma^2}} [1 + b(\ln f - \ln f')] e^{-\frac{(\ln f - \ln f')^2}{2\sigma^2}} \quad \begin{array}{l} \text{bias } b \simeq -0.52 \\ \text{variance } \sigma^2 \sim \int d \ln k \Delta_{\mathcal{R}}^2 \end{array}$$



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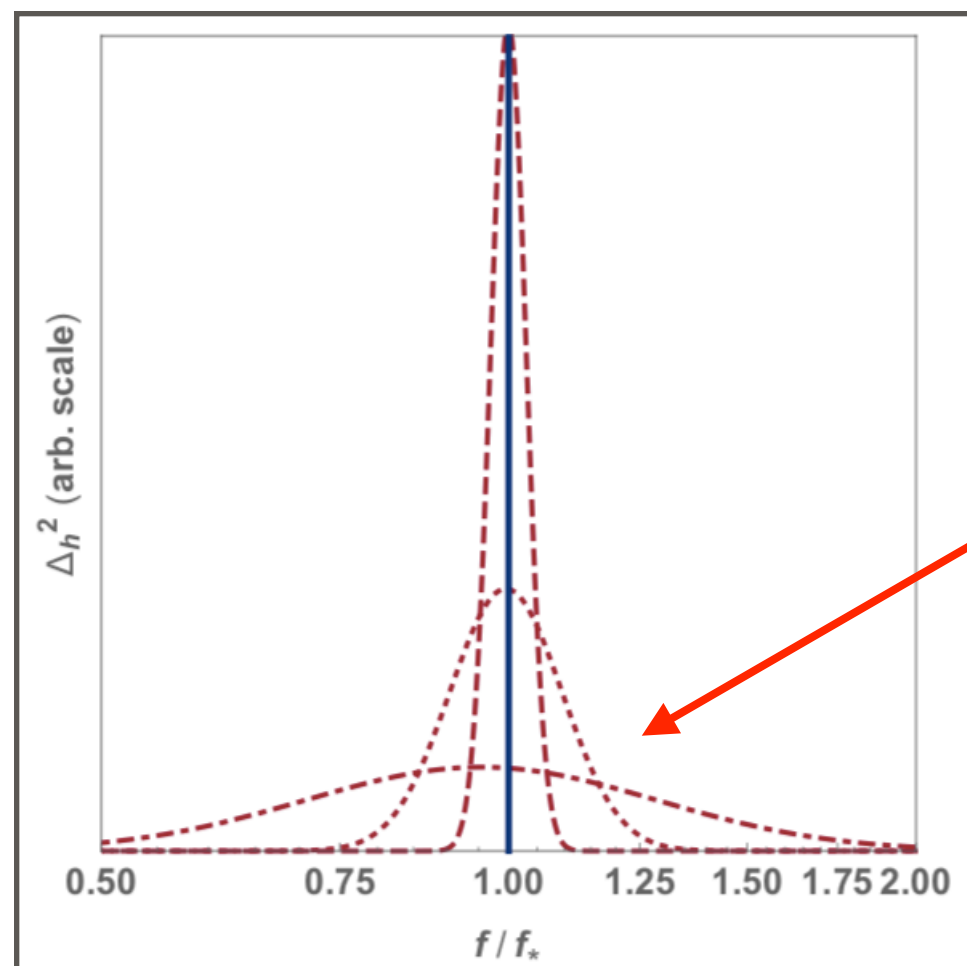
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# RESULTS

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- For a spiky GW spectrum at the sourcing time



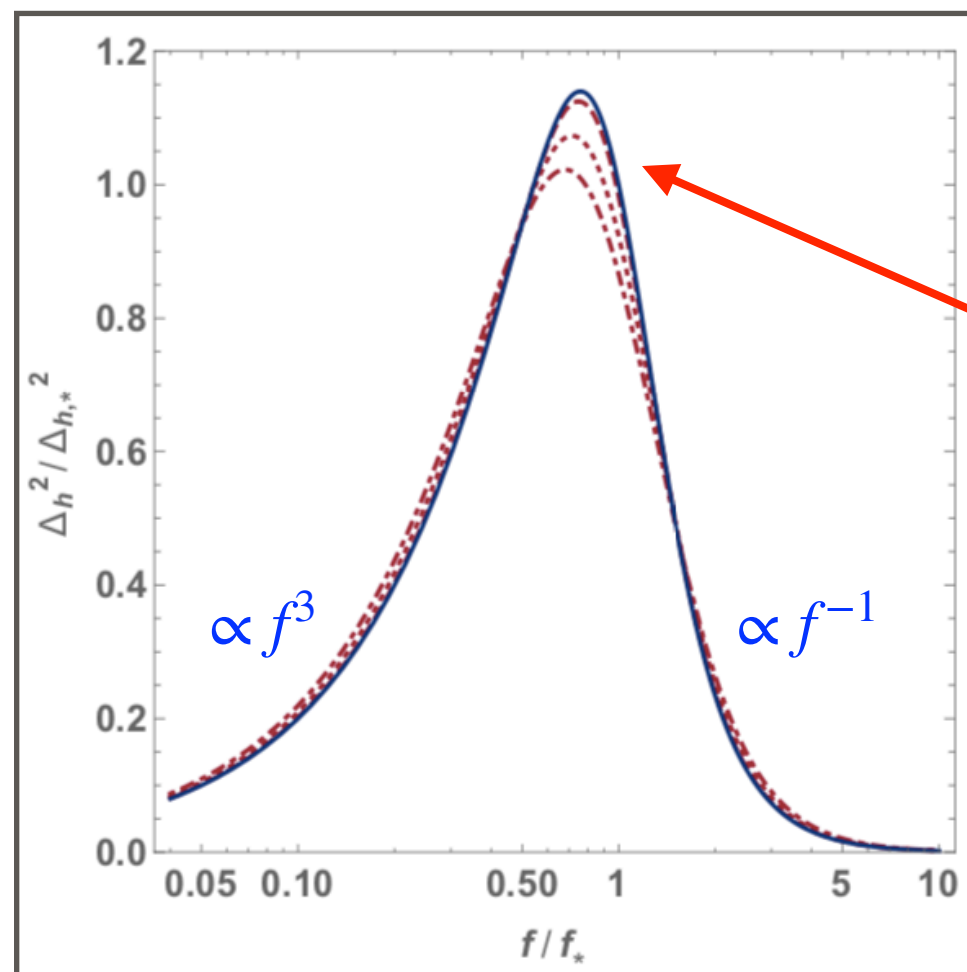
$$\int d \ln k \Delta_{\mathcal{R}}^2$$
$$= 10^{-3}, 10^{-2}, 10^{-1}$$

blue = original (= source) / red = deformed (= observed)

# RESULTS

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- For a smooth GW spectrum at the sourcing time



$$\int d \ln k \Delta_{\mathcal{R}}^2$$
$$= 0.01, 0.05, 0.1$$

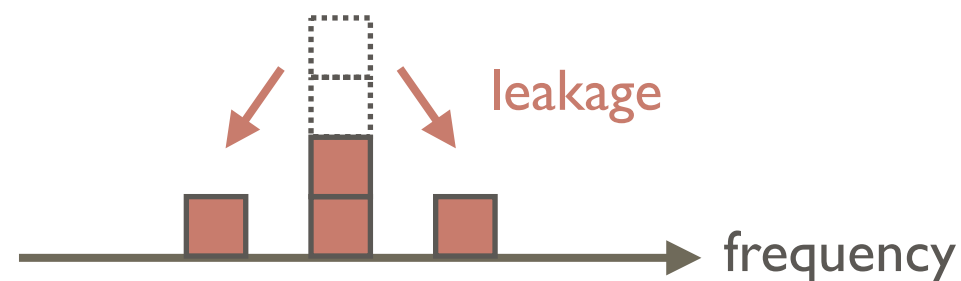
This result should be taken as  
order-of-magnitude estimate at best  
(→ discussion 2)

blue = original (= source) / red = deformed (= observed)

# DISCUSSION 1

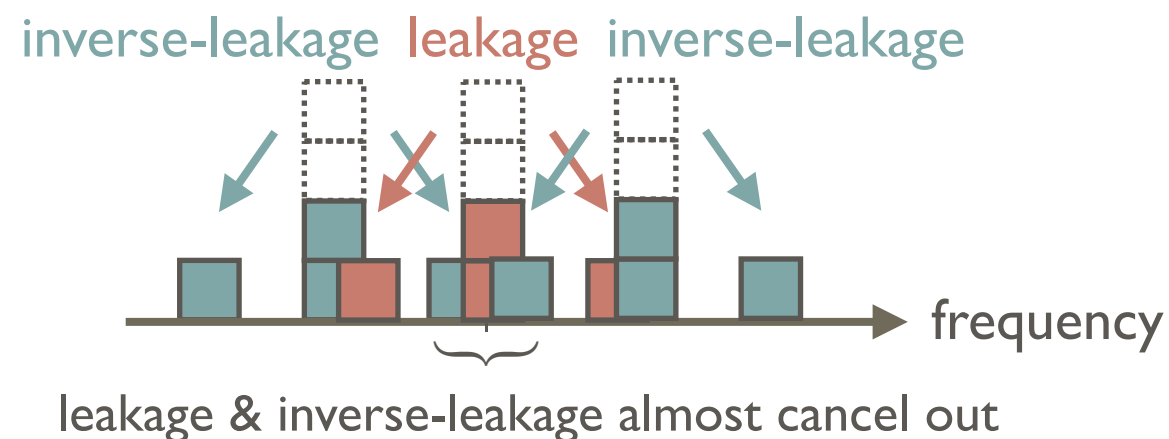
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- For the smooth spectrum, the deformation looks smaller. Why?
  - For the spiky case (= seen bin-by-bin), only leakage occurs



In this case, our prescription is safe (→ backup)

- For the smooth case, leakage & inverse-leakage almost cancel out



In this case, our calculation takes into account part of the whole effect

# DISCUSSION 1 (CONT'D)

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➤ For the smooth case, we had two nontrivial steps:

$$\begin{aligned}
 1. \text{ Taylor exp. } \Delta_h^{2,(o)}(\ln f) &= \left\langle e^{2\Delta \ln A} \Delta_h^{2,(s)}(\ln f - \Delta \ln f) \right\rangle_{\text{scalar ens. ave.}} \\
 &\stackrel{!}{\simeq} \left\langle (1 + 2\Delta \ln A) \Delta_h^{2,(s)}(\ln f - \Delta \ln f) \right\rangle_{\text{scalar ens. ave.}}
 \end{aligned}$$

$$\begin{aligned}
 2. \text{ Linear-order result } \Delta \ln A &\stackrel{!}{=} -\Psi_s - \frac{1}{2}\Phi_s \\
 \Delta \ln f &\stackrel{!}{=} \Phi_s - \frac{1}{2}\Phi_s + \int_{\lambda_s}^{\lambda_o} d\lambda \partial_\tau(\Phi + \Psi)
 \end{aligned}$$

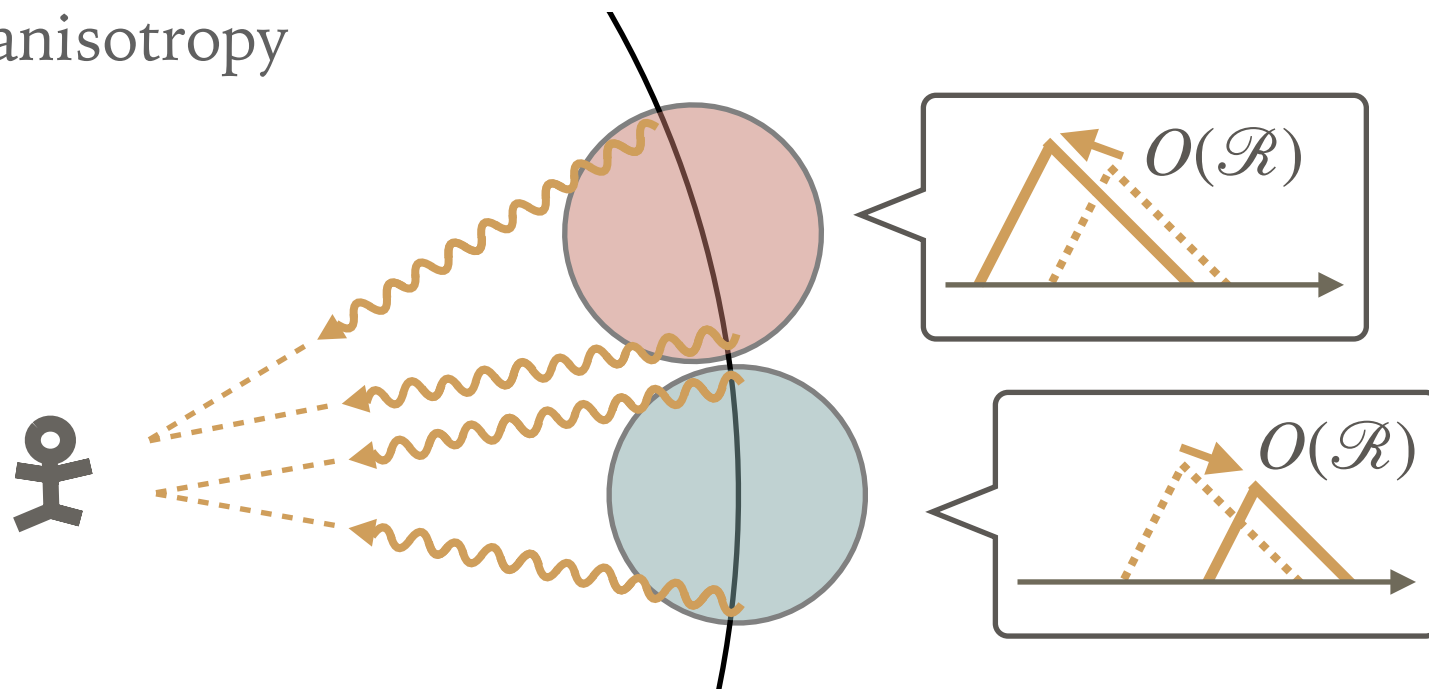
➤ At each step, part of  $\langle (\text{scalar})^2 \rangle_{\text{scalar ens. ave.}}$  terms are neglected

# DISCUSSION 2

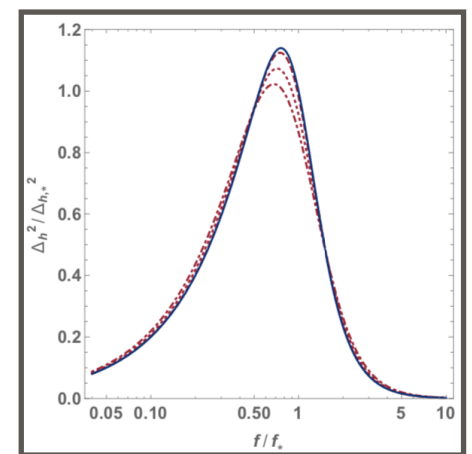
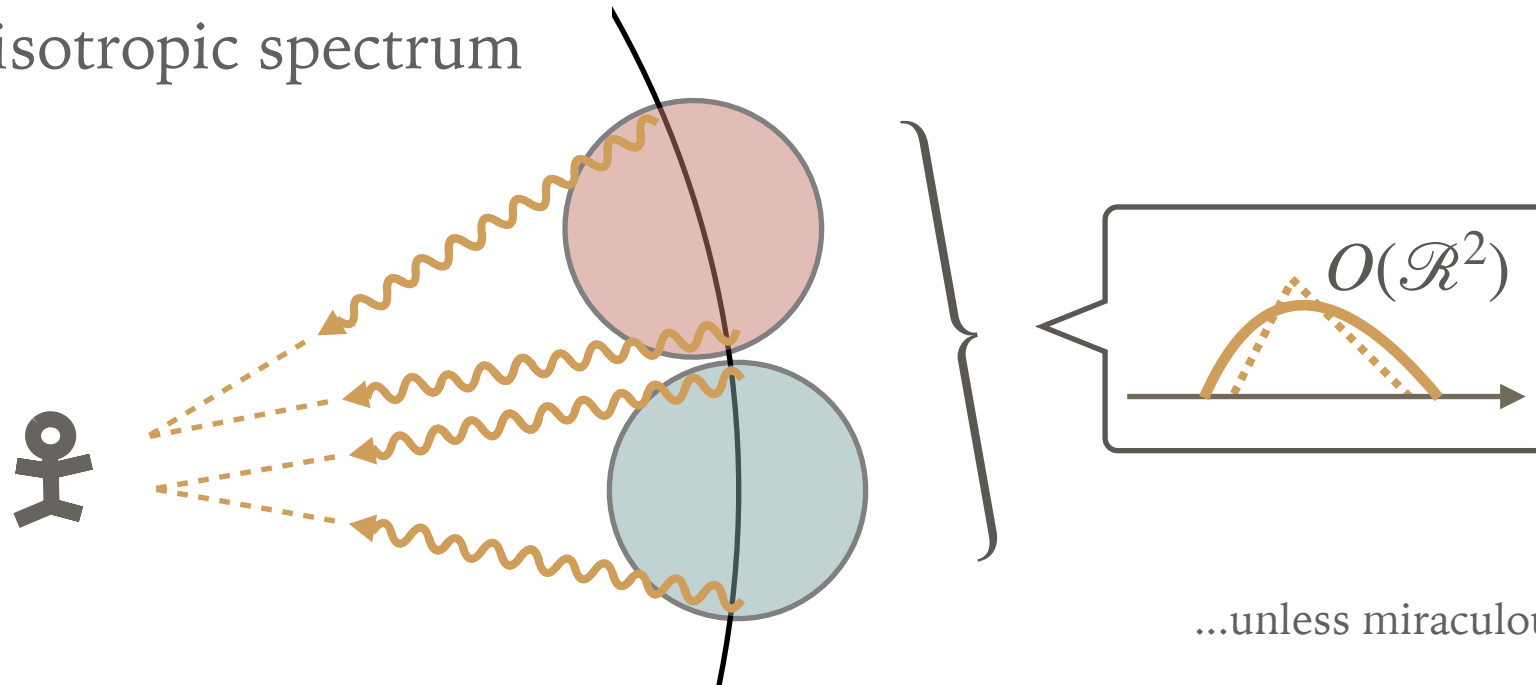
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## ► Consistency with GW anisotropy

- GW anisotropy



- GW isotropic spectrum



...unless miraculous cancellation occurs

# SUMMARY

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- GWs propagate in an inhomogeneous Universe.

The inhomogeneity deforms the original isotropic spectrum.

- If the original GW spectrum is  $\begin{cases} \text{spiky} \rightarrow \text{we can show this effect clearly} \\ \text{smooth} \rightarrow \text{our result is just implicative} \end{cases}$
- GW anisotropy is also important

- Careful comparison btwn. theoretical and observed GW spectra may reveal the intermediate-scale curvature perturbations
- To fully pin down the effect, we need 2nd-order pert'n theory

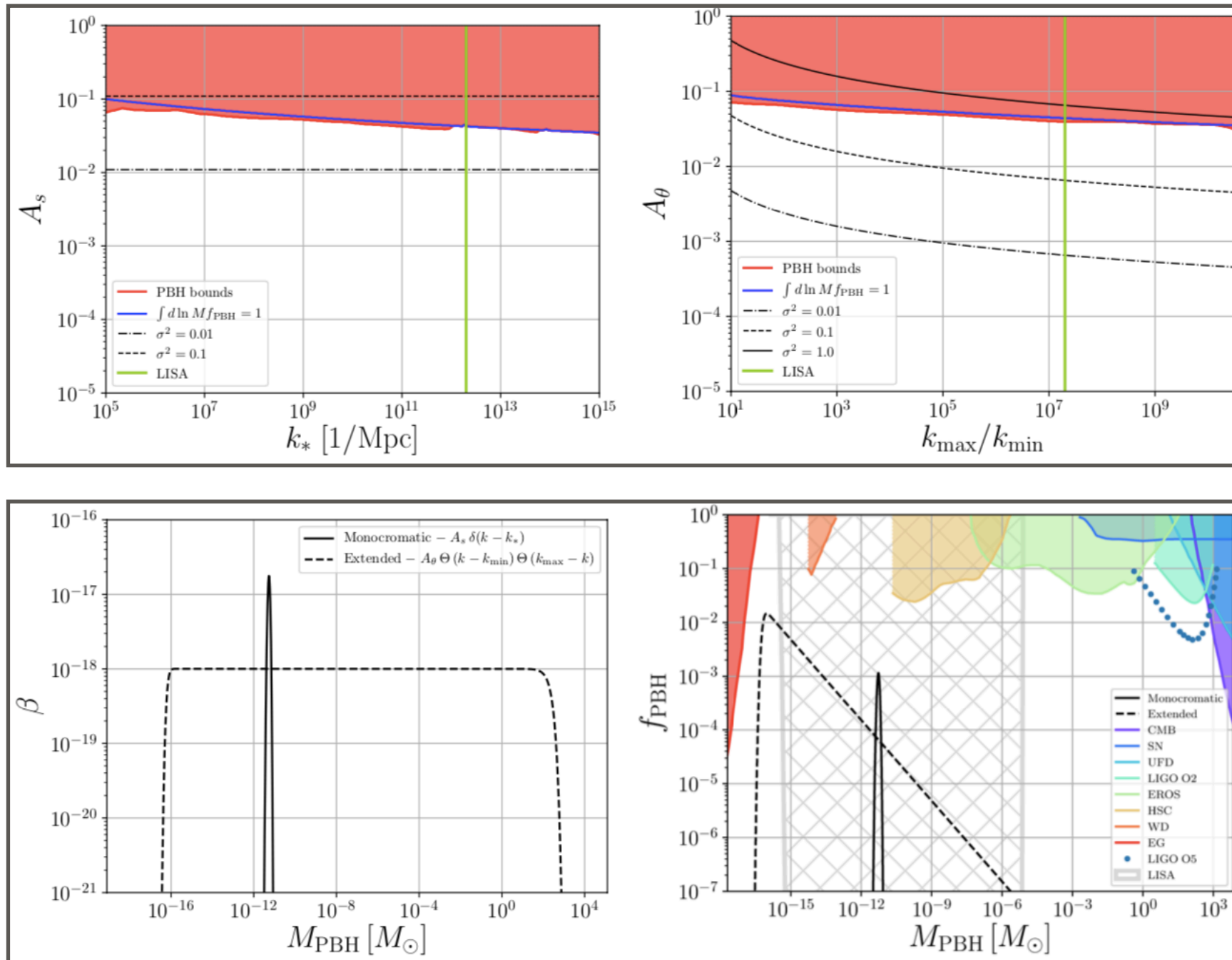
Backup

# PBH CONSTRAINTS

$$\Delta_{\mathcal{R}}^2(k) = A_s k_* \delta(k - k_*)$$

$$\Delta_{\mathcal{R}}^2(k) = A_\theta \Theta(k - k_{\min}) \Theta(k_{\max} - k)$$

$$k_{\min} = 10^5 \text{ Mpc}^{-1}$$



# NEGLECTED TERMS

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## ► Taylor expansion

$$\begin{aligned}\Delta_h^{2(o)}(\ln f) &= \left\langle e^{2\Delta \ln A} \Delta_h^{2(s)} (\ln f - \Delta \ln f) \right\rangle_{\text{ens}(s)} \\ &= \left\langle \left(1 + 2\Delta \ln A^{(1)}\right) \Delta_h^{2(s)} \left(\ln f - \Delta \ln f^{(1)}\right) \right\rangle_{\text{ens}(s)} && : \text{we took this} \\ &\quad + \left\langle 2 \left(\Delta \ln A^{(1)}\right)^2 + 2\Delta \ln A^{(2)} \right\rangle_{\text{ens}(s)} \Delta_h^{2(s)}(\ln f) && : \text{vertical shift} \\ &\quad + \left\langle \Delta_h^{2(s)} \left(\ln f - \Delta \ln f^{(2)}\right) \right\rangle_{\text{ens}(s)} - \Delta_h^{2(s)}(\ln f) && : \mathcal{O}(\sigma^2) \text{ horizontal shift} \\ &\quad + \mathcal{O}(\sigma^3) .\end{aligned}$$

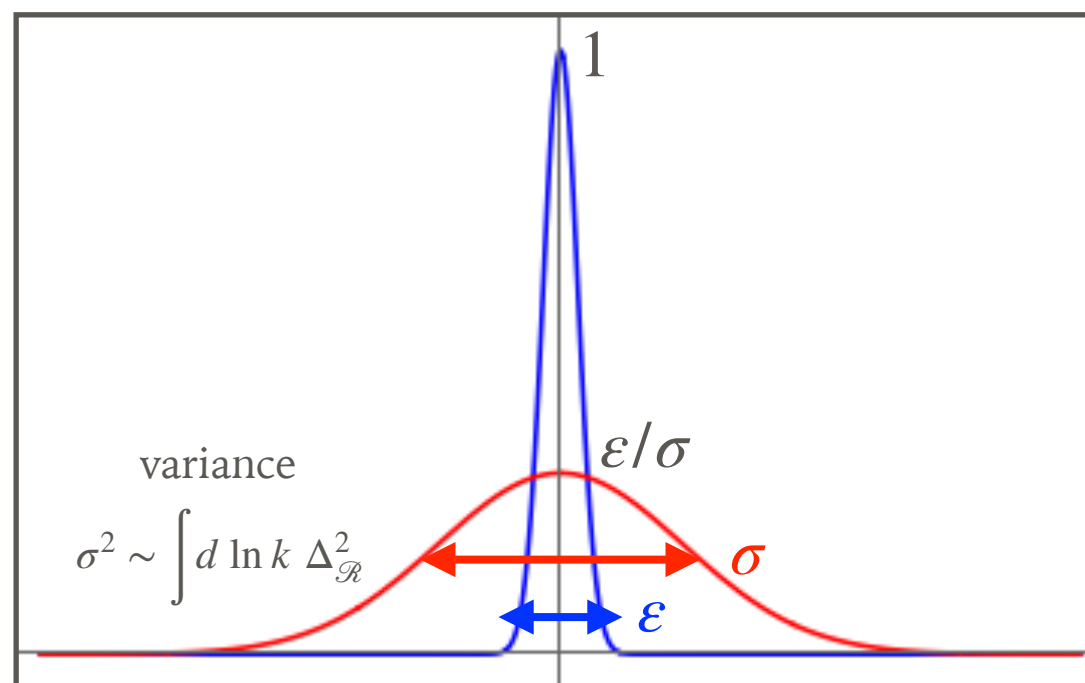
# SPIKY $\rightleftharpoons$ SMOOTH

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➤  $\delta$ -function type source spectrum  $\Delta_h^{2(s)}(f) = \frac{\Delta_{h,*}^2}{\sqrt{2\pi\epsilon^2}} \exp \left[ -\frac{(\ln f - \ln f_*)^2}{2\epsilon^2} \right]$  ( $\delta$ -func. for  $\epsilon \rightarrow 0$ )

➤ Our expression  $\Delta_h^{2(o)}(f) = \int d \ln f' \Delta_h^{2(s)}(f') K(f, f')$  gives

the observed spectrum  $\Delta_h^{2(o)}(f) = \frac{\Delta_{h,*}^2}{\sqrt{2\pi(\epsilon^2 + \sigma^2)}} \left[ 1 + \frac{\sigma^2}{\epsilon^2 + \sigma^2} b(\ln f - \ln f_*) \right] \exp \left[ -\frac{(\ln f - \ln f_*)^2}{2(\epsilon^2 + \sigma^2)} \right]$



# DERIVATION OF THE LINEARLY BIASED GAUSSIAN

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- Imagine a Gaussian variable  $X$  obeying distribution  $P(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$
- Suppose you would like to calculate  $\langle (1 + 2X)f(x - X) \rangle$
- You will get 
$$\begin{aligned}\langle (1 + 2X)f(x - X) \rangle &= \int dX \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} (1 + 2X)f(x - X) \\ &= \int dx' \frac{1}{\sqrt{2\pi\sigma^2}} [1 + 2(x - x')] e^{-\frac{(x - x')^2}{2\sigma^2}} f(x') \\ &\quad (x - X = x')\end{aligned}$$
- In the present case, we have infinitely many Gaussian variables  $X_1, X_2, \dots$  (corresponding to each  $k$  mode), but the calculation is essentially the same

# ANALOGY TO CMB

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