The Yang-Mills duals of small AdS black holes

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This talk is based on

- **SC**, Saebyeok Jeong (Rutgers U.), and Seok Kim (SNU)
  "The Yang-Mills duals of small AdS black holes"
  arXiv:2103.01401 [hep-th]

Microstates of BPS black holes in AdS

- **Microstates of black holes** in $\text{AdS}_5 \times S^5$: $S_{BH} \sim \frac{1}{G_5}$ when $E, Q_I, J_i \sim \frac{1}{G_5}$
- CFT_4 dual on $S^3 \times \mathbb{R}$: ***N*=4 SYM w/ $U(N)$ gauge group** - $\frac{1}{G_5} \sim N^2$

→ When $E, Q_I, J_i \sim N^2$, there should exist a macroscopic number $\sim e^{N^2}$ **of local gauge invariant operators** on $\mathbb{R}^4$ (~ states on $S^3 \times \mathbb{R}$).

- Today, we only focus on $\frac{1}{16}$**-BPS subsector**, in which certain operators are **protected**, i.e. do not receive anomalous dimensions at strong coupling (~ gravity).

→ **Microstates of BPS black holes** in $\text{AdS}_5 \times S^5$

[Gutowski, Reall 04; Chong, Cvetic, Lu, Pope 05; Kunduri, Lucietti, Reall 06]

- **Superconformal index**: counts such **protected operators w/ $(-1)^F$**
[Romelsberger 05] [Kinney, Maldacena, Minwalla, Raju 05]

Outline

- **Index** of radially quantized CFT on $S^3 \times \mathbb{R}$: counting BPS states with $(-\mathbf{1})^F$
  → captures the Bekenstein-Hawking entropy of **BPS black holes in AdS$_5$**
  [Cabo-bizet, Cassani, Martelli, Murthy 18] [**SC**, J.Kim, S.Kim, Nahmgoong 18] [Benini, Milan 18] …

- Matrix model for the index [Romelsberger 05] [Kinney, Maldacena, Minwalla, Raju 05]
  - saddle points in the large N limit ~ classical solutions of the general relativity

- **Truncated versions** of the matrix model
  [Copetti, Grassi, Komargodski, Tizzano 20] [Aharony, Marsano, Minwalla, Papadodimas, Raamsdonk 03]
  - deconfinement transitions ~ Hawking-Page transition
  - various saddle points ~ AdS black holes

- Exact analytic saddle points for **small black holes** (sizes $\ll$ AdS radius)
  - small black holes ~ **asymptotically flat black holes** [Strominger, Vafa 96] [Breckenridge, Myers, Peet, Vafa 96]
  - microstates of small black holes ~ D-branes (cf. large black holes ~ deconfined quark-gluon plasma)
    [Kinney, Maldacena, Minwalla, Raju 05] [**SC**, J.Kim, S.Kim, Nahmgoong 18]

Index

- **Index** of the $\mathcal{N}$=4 SYM on $S^3 \times \mathbb{R}$: [Romelsberger 05] [Kinney, Maldacena, Minwalla, Raju 05]

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^3 R_I \Delta_I - \sum_{i=1}^2 J_i \omega_i}\right] \qquad \Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2=0.$$

  - R: Cartan charges of $SO(6)$ R-symmetry ~ electric charges of the black hole
  - J: Cartan charges of $SO(4)$ rotation symmetry on S^3 ~ angular momenta of the black hole
  - BPS condition: $E = Q_1 + Q_2 + Q_3 + J_1 + J_2$

- **Unitary matrix integral** representation

$$Z(\Delta_I, \omega_i) = \frac{1}{N!}\prod_{a=1}^N \int_0^{2\pi} \frac{d\alpha_a}{2\pi} \cdot \prod_{a<b}\left(2\sin\frac{\alpha_{ab}}{2}\right)^2 \cdot \exp\left[\sum_{a,b=1}^N \sum_{n=1}^\infty \frac{a_n(\Delta_I,\omega_i)}{n} e^{in\alpha_{ab}}\right]$$

$$a_n \equiv 1 - \frac{\prod_{I=1}^3 (1-e^{-n\Delta_I})}{(1-e^{-n\omega_1})(1-e^{-n\omega_2})}$$

- Unrefined index $e^{-\Delta_I} = x^2, e^{-\omega_i} = x^3$

$$Z(x) = \mathrm{Tr}\left[(-1)^F x^{6\left(\frac{R_1+R_2+R_3}{3}+\frac{J_1+J_2}{2}\right)}\right] \equiv \mathrm{Tr}\left[(-1)^F x^{6(R+J_+)}\right]$$

Matrix model

- Matrix integral: $$Z(\Delta_I,\omega_i)=\frac{1}{N!}\prod_{a=1}^{N}\int_0^{2\pi}\frac{d\alpha_a}{2\pi}\exp\left[N\sum_{n=1}^{\infty}\frac{a_n}{n}\right]\exp\left[-\sum_{a<b}V(\alpha_a-\alpha_b)\right]\qquad V(\theta)\equiv-\log\left[4\sin^2\frac{\theta}{2}\right]-\sum_{n=1}^{\infty}\frac{a_n}{n}(e^{in\theta}+e^{-in\theta})$$

- Saddle point equation: $$\sum_{b(\neq a)}V'(\alpha_a-\alpha_b)=0$$

- Continuum limit → $$\int_{-\theta_0}^{\theta_0}d\theta\rho(\theta)V'(\alpha-\theta)=0\qquad\rho(\alpha_a)\equiv\frac{1}{N}\frac{\Delta a}{\Delta\alpha_a}$$ eigenvalues lie on $s(\alpha)=\int\rho(\alpha)d\alpha+\text{constant}$ is real

→ $$\int_{-\theta_0}^{\theta_0}\cot\left(\frac{\alpha-\theta}{2}\right)\rho(\theta)d\theta-2\sum_{n=1}^{\infty}a_n\rho_n\sin(n\alpha)=0\qquad\rho_n\equiv\int_{-\theta_0}^{\theta_0}d\theta\rho(\theta)\cos(n\theta)\quad,\quad n=1,2,\cdots$$

- Strategy: first solve the equation treating ρ_n as **independent** variables [Aharony, Marsano, Minwalla, Papadodimas, Raamsdonk 03]

[Jurkiewicz, Zalewski 83]

then **relate** $\rho(\theta)$ and ρ_n $$\rho(\theta)=\frac{1}{\pi}\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}\sum_{n=1}^{\infty}Q_n\cos\left[(n-\tfrac{1}{2})\theta\right]$$

$$Q_n\equiv2\sum_{l=0}^{\infty}a_{n+l}\rho_{n+l}P_l(\cos\theta_0)\ ,\quad Q_1=Q_0+2$$

Truncated matrix model

- Matrix integral: $$Z(\Delta_I,\omega_i)=\frac{1}{N!}\prod_{a=1}^{N}\int_0^{2\pi}\frac{d\alpha_a}{2\pi}\exp\left[N\sum_{n=1}^{\infty}\frac{a_n}{n}\right]\exp\left[-\sum_{a<b}V(\alpha_a-\alpha_b)\right]\qquad V(\theta)\equiv-\log\left[4\sin^2\frac{\theta}{2}\right]-\sum_{n=1}^{p}\frac{a_n}{n}(e^{in\theta}+e^{-in\theta})$$

[Aharony, Marsano, Minwalla, Papadodimas, Raamsdonk 03] [Copetti, Grassi, Komargodski, Tizzano 20]

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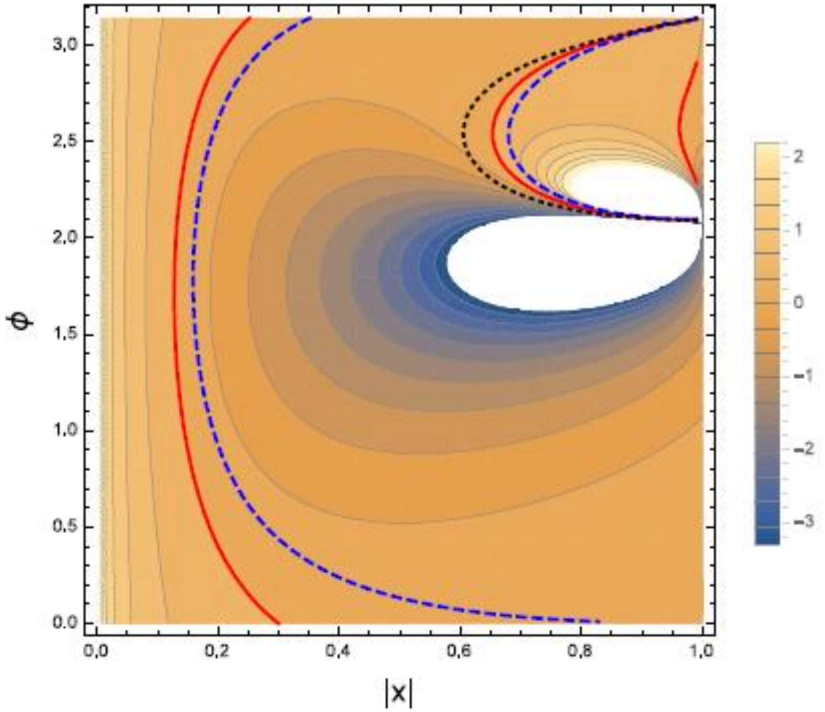
The p=1 model

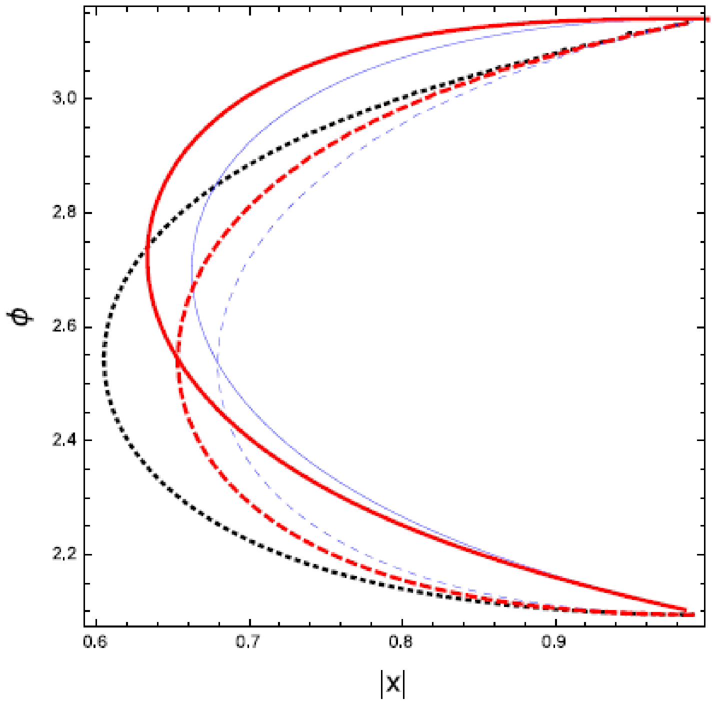
- Deconfinement transition [Copetti, Grassi, Komargodski, Tizzano 20]



(a) $\frac{1}{N^2}\mathrm{Re}(\log \mathrm{Z}_+)$ for the $p = 1$ model

$$\log Z_{\mathrm{BH}} = \frac{N^2}{2}\frac{\Delta^3}{\omega^2} \quad , \quad x = e^{-\frac{\omega}{3}+\frac{2\pi i}{3}} = -e^{-\frac{\Delta}{2}}$$

- Saddles of Legendre transformation

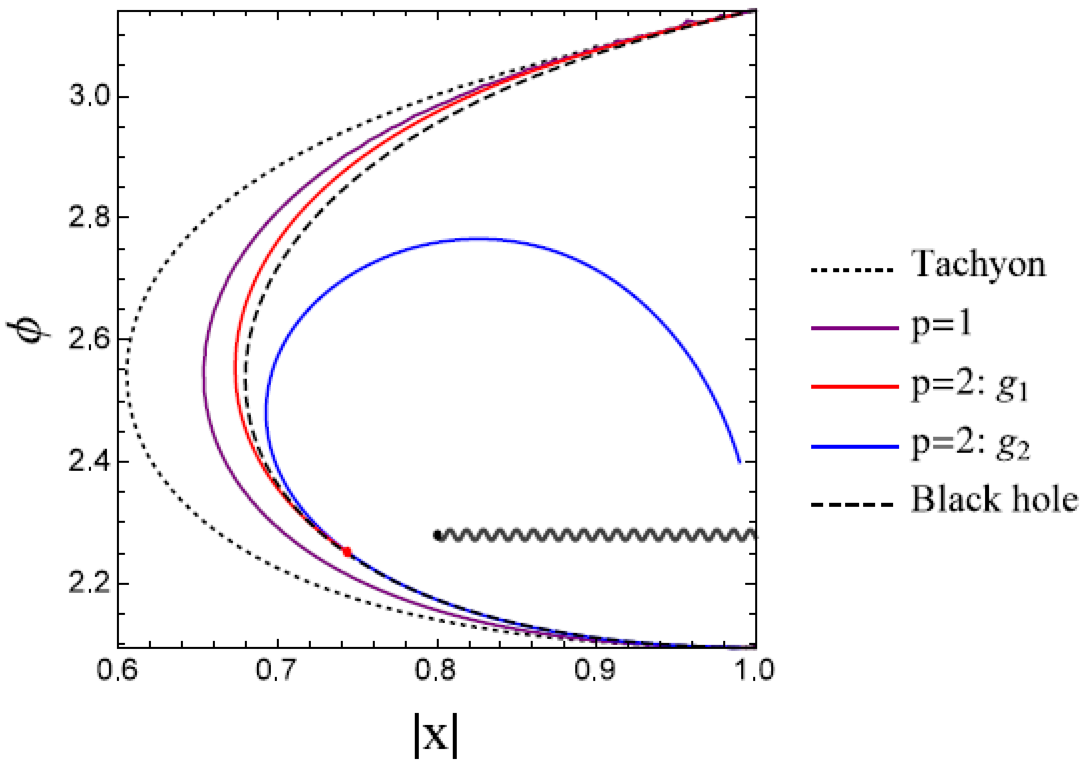


- tachyon threshold $\mathrm{Re}(a_1(x) - 1) = 0$.

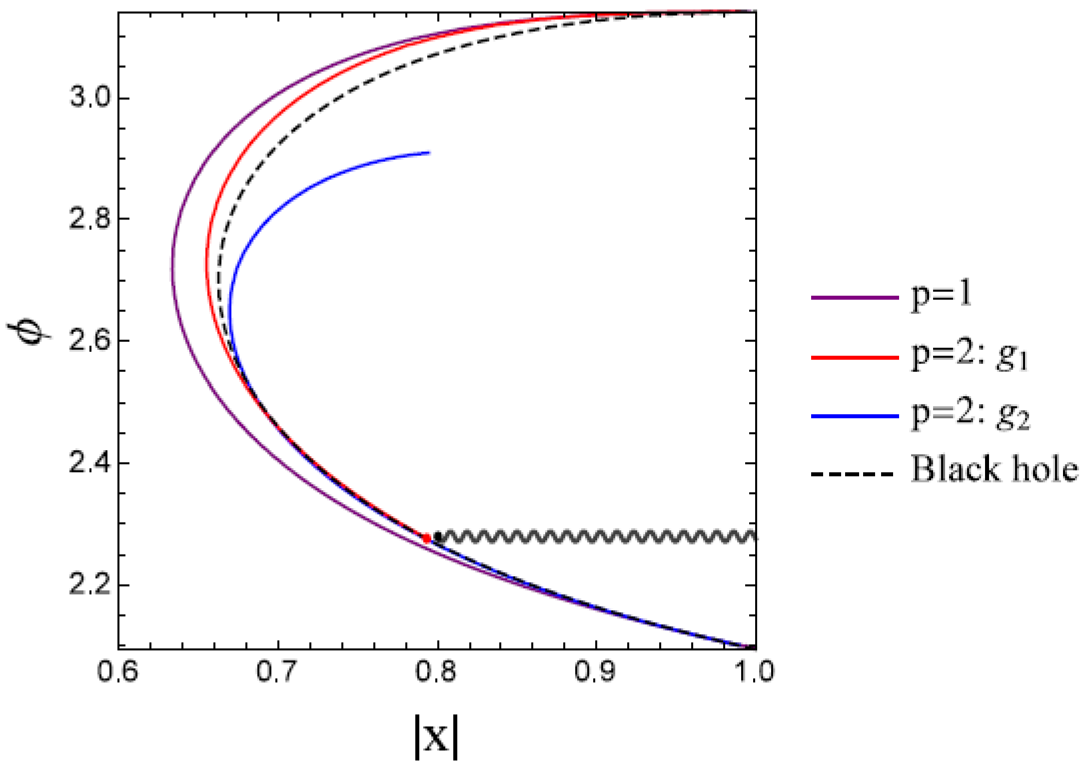
$$S_{\mathrm{eff}} = N^2 \sum_{n=1}^{\infty} \frac{1-a_n}{n} |\rho_n|^2$$

The p=2 model

- Deconfinement transition

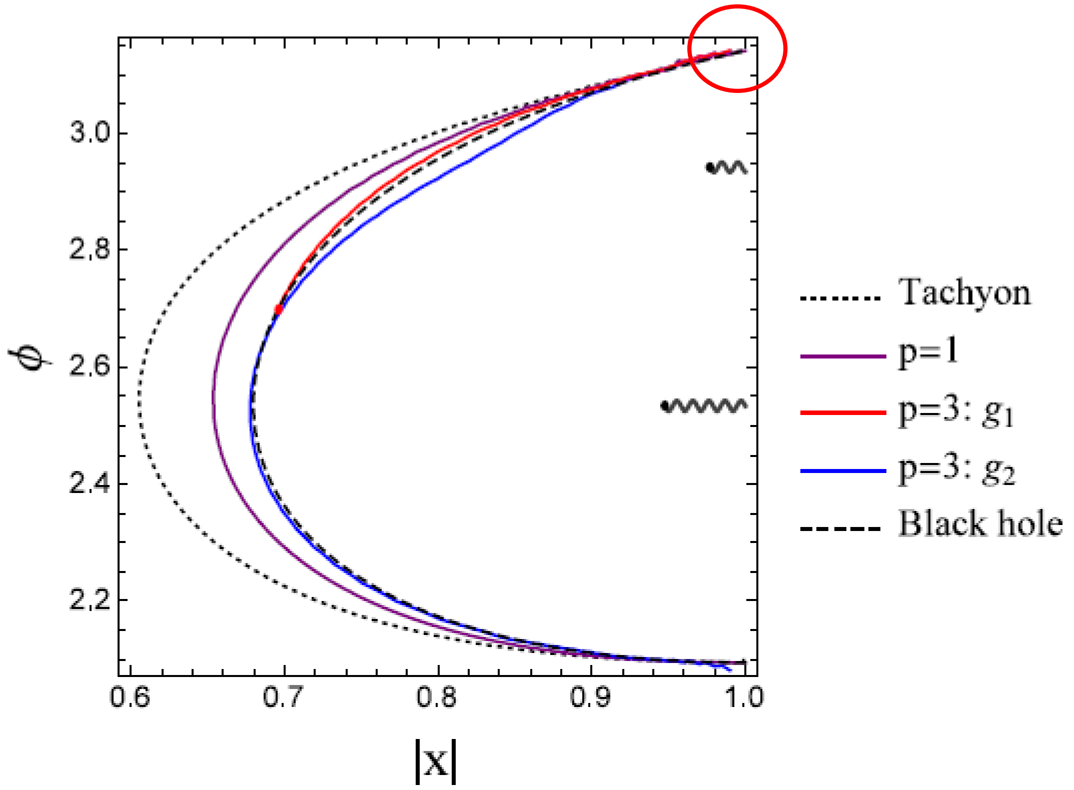


- Saddles of Legendre transformation

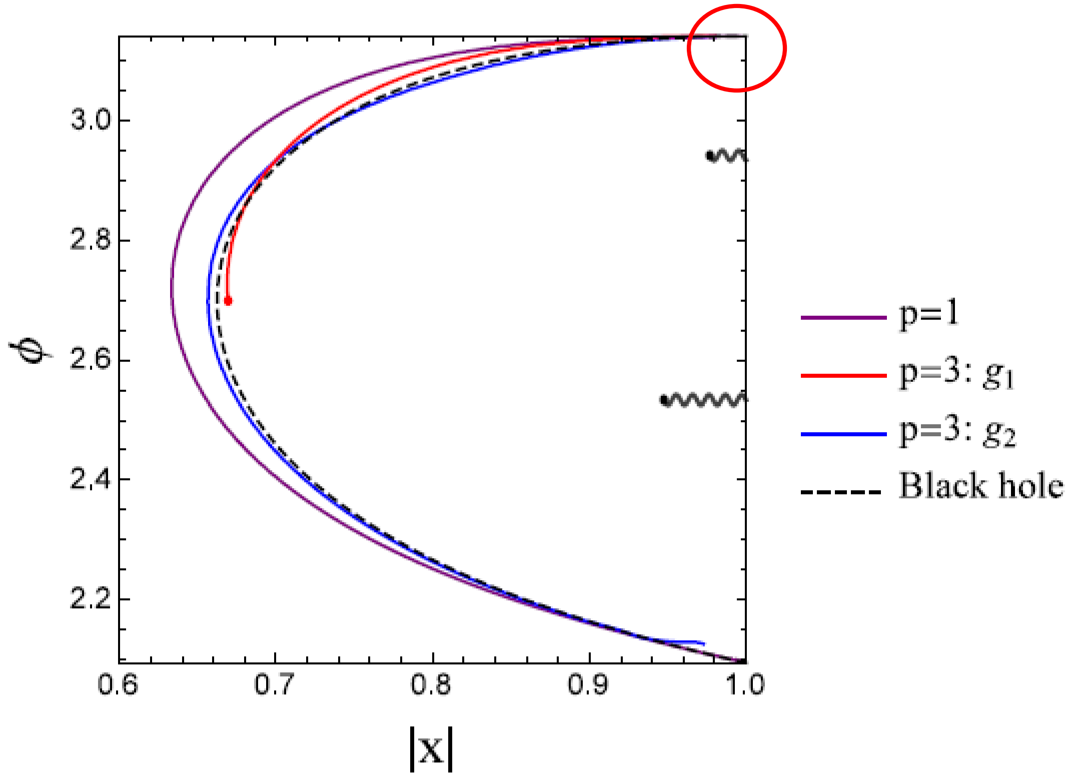


The p=3 model

- Deconfinement transition



- Saddles of Legendre transformation



Small black hole saddle points

- Truncated **2-body potential** $V(\alpha) = -\log\left[4\sin^2\frac{\alpha}{2}\right] - 2\sum_{n=1}^{p}\frac{a_n(x)}{n}\cos(n\alpha)$

$$a_n(-e^{-\beta}) = \begin{cases} 1 - \frac{2\sinh^3(n\beta)}{\sinh^2\frac{3n\beta}{2}} & \text{for even } n \\ 1 - \frac{2\sinh^3(n\beta)}{\cosh^2\frac{3n\beta}{2}} & \text{for odd } n \end{cases} \qquad x = -e^{-\beta}.$$

- Solutions to the matrix model at $\beta \to 0$: $\rho_{2n-1} = \dfrac{\frac{1}{2(2n-1)^2}}{\sum_{l=1}^{\lceil\frac{p}{2}\rceil}\frac{1}{(2l-1)^2}}$ $\qquad \rho_{2n} \sim \mathcal{O}(\beta^2)$

- **Full matrix model** at $p \to \infty$: $\rho_{2n-1} = \dfrac{\frac{1}{2(2n-1)^2}}{\sum_{l=1}^{\infty}\frac{1}{(2l-1)^2}} = \dfrac{4}{\pi^2}\dfrac{1}{(2n-1)^2}$

- **Eigenvalue distribution** : $\rho(\theta) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\rho_n\cos(n\theta) = \dfrac{1}{2\pi}\left[1 + \dfrac{8}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)\theta}{(2n-1)^2}\right] = \dfrac{1}{\pi^2}(\pi - |\theta|)$

Small black hole free energy

- Large N free energy:

$$\log Z = -\frac{N^2}{2}\int_{-\theta_0}^{\theta_0} d\theta_1 d\theta_2 V(\theta_1-\theta_2)\rho(\theta_1)\rho(\theta_2)$$

$$= N^2\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\theta_0}^{\theta_0} d\theta_1 d\theta_2\,[a_n-1]\,e^{in(\theta_1-\theta_2)}\rho(\theta_1)\rho(\theta_2) = N^2\sum_{n=1}^{\infty}\frac{a_n-1}{n}(\rho_n)^2$$

$$a_n(\beta)-1 \approx \begin{cases} -2n^3\beta^3 & \text{for odd } n \\ -\frac{8n\beta}{9} & \text{for even } n \end{cases}$$

$$\log Z \approx -2N^2\beta^3\sum_{n=1}^{\infty}(2n-1)^2\cdot\left(\frac{4}{\pi^2}\frac{1}{(2n-1)^2}\right)^2 = -\frac{4N^2\beta^3}{\pi^2}$$

→ **precisely agrees** with the free energy of **small black holes in $\mathrm{AdS}_5\times S^5$** [Gutowski, Reall 04]

- Legendre transformation: $-\frac{4N^2}{\pi^2}\beta^3+\beta q \xrightarrow{\text{extremize}} S(q)=\pi\sqrt{\frac{q^3}{27N^2}}=\pi\sqrt{\frac{8(R+J_+)^3}{N^2}}$

- In the **small black hole** limit : $\frac{J_+}{N^2}\sim\left(\frac{R}{N^2}\right)^2\ll\frac{R}{N^2}$ → $S\approx\pi\sqrt{\frac{8R_1R_2R_3}{N^2}}$

~ entropy of **asymptotically flat black holes** [Strominger, Vafa 96]

[Kinney, Maldacena, Minwalla, Raju 05]

Concluding remarks

- We studied the large N saddle points of the **matrix model for 4d maximal SYM on $S^3 \times \mathbb{R}$**, and investigated the physics of **holographically dual black holes**.

- From the numerical studies on the **truncated models**, we found that multiple branches of saddle points should be patched to describe known AdS black holes.

- We analytically constructed the **exact saddle points for the small black holes**.

 - They perfectly account for the thermodynamics of small AdS black holes. [Gutowski, Reall 04]

 - Small AdS_5 black holes are related to the **5d asymptotically flat black holes** [Strominger, Vafa 96], thus providing a first-principle account for them.

 - For the **BMPV black holes**, we found their **thermodynamic instabilities** when embedded into AdS, both from gravity and QFT sides.

Future directions

- Infinitely many small black hole saddles exhibiting same large N thermodynamics

  → continuum spectrum of saddles ~ **small hairy black holes**?

- New small black holes? ~ multi-cut saddles

  → related to **other asymptotically flat black holes**

- Exact analytic saddle points for finite size black holes?

Thank you for listening.