

# Computing Yukawa Textures Using Spectral Data

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Based on 2106.XXXX

PASCOS 2021

# ❖ Introduction

- Low energy limit of the Heterotic string is a 10 dimensional supergravity with  $E_8 \times E_8$  or  $Spin(32)/Z_2$  gauge groups.
- The field content is,

$$\begin{aligned} \text{Gravity Multiplet:} & \quad B_{\mu\nu}, g_{\mu\nu}, \phi, \psi_\mu, \lambda \\ \text{Gauge Multiplet:} & \quad A_\mu, \chi \end{aligned}$$

- In particular if  $H = 0$  and  $d\phi = 0$ , ( $H = dB + \omega_L - \omega_Y$ ), we need to have a covariantly constant spinor on the 6d (or 4d) internal manifold,  $D\eta = 0 \Rightarrow$  Calabi-Yau threefold (twofold)  $X$ .
- The solution for the  $\delta\chi = 0$  gives

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \quad g^{\bar{a}b} F_{\bar{a}b} = 0.$$

- These are satisfied by a holomorphic and stable vector bundle  $V$ .

# ❖ Yukawa Couplings

- Given a 4d Heterotic  $E_8 \times E_8$  model  $(X, V)$ , the (holomorphic) Yukawa couplings in EFT is given by

$$\int_X \Omega \wedge A \wedge A \wedge A \rightarrow \Phi_1 \Phi_2 \Phi_3$$

$$A \wedge A \wedge A \rightarrow \bar{\Omega}$$

- Therefore, in terms of cohomologies,

$$SU(3) \text{ Bundle } V \rightarrow H^1(V) \otimes H^1(V) \otimes H^1(V) \rightarrow H^3(\Lambda^3 V) = H^{3,0}(X) = \mathbb{C}$$

$$\mathbf{27} \mathbf{27} \mathbf{27} \quad E_6$$

$$SU(4) \text{ Bundle } V \rightarrow H^1(V) \otimes H^1(V) \otimes H^1(\Lambda^2 V) \rightarrow H^3(\Lambda^4 V) = H^{3,0}(X) = \mathbb{C}$$

$$\mathbf{16} \mathbf{16} \mathbf{10} \quad SO(10)$$

$$SU(5) \text{ Bundle } V \rightarrow H^1(V) \otimes H^1(V) \otimes H^1(\Lambda^3 V) \rightarrow H^3(\Lambda^5 V) = H^{3,0}(X) = \mathbb{C}$$

$$\mathbf{10} \mathbf{10} \mathbf{5} \quad SU(5)$$

$$SU(5) \text{ Bundle } V \rightarrow H^1(V) \otimes H^1(\Lambda^2 V) \otimes H^1(\Lambda^2 V) \rightarrow H^3(\Lambda^5 V) = H^{3,0}(X) = \mathbb{C}$$

$$\mathbf{10} \bar{\mathbf{5}} \bar{\mathbf{5}} \quad SU(5)$$

## ❖ Goal

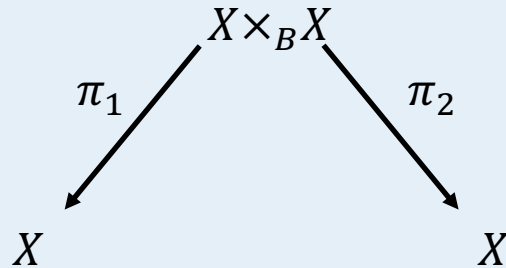
- Direct Computation of these cohomologies can be very hard.

Anderson et.al

- The relation with the Yukawa coupling in other string theory models is obscure.
- Since most of Calabi-Yau threefolds constructed so far are elliptically fibered, we use another approach.

# ❖ Spectral Data

- Let  $\pi: X \rightarrow B_2$  be a Weierstrass elliptically fibered CY3.  $V$  Stable, Holomorphic, Degree Zero bundle over  $X$ .
- Instead of constructing the bundle directly one can use the Fourier-Mukai transform,



$$\begin{aligned}
 \Phi(V) &:= R\pi_{2*}(\pi_1^*V \otimes \mathcal{P}) \\
 \mathcal{P} &:= \mathcal{I}_\Delta \otimes \pi_1^*\mathcal{O}(\sigma) \otimes \pi_2^*\mathcal{O}(\sigma) \otimes K_B^{-1}
 \end{aligned}$$

- Given  $V$  as above with rank  $n$ , its Fourier transform is a Torsion sheaf which is supported over finite cover of  $B_2$ :

$$\Phi(V) = i_{S_n^*} \mathcal{L} [-1]$$

# ❖ Spectral Data

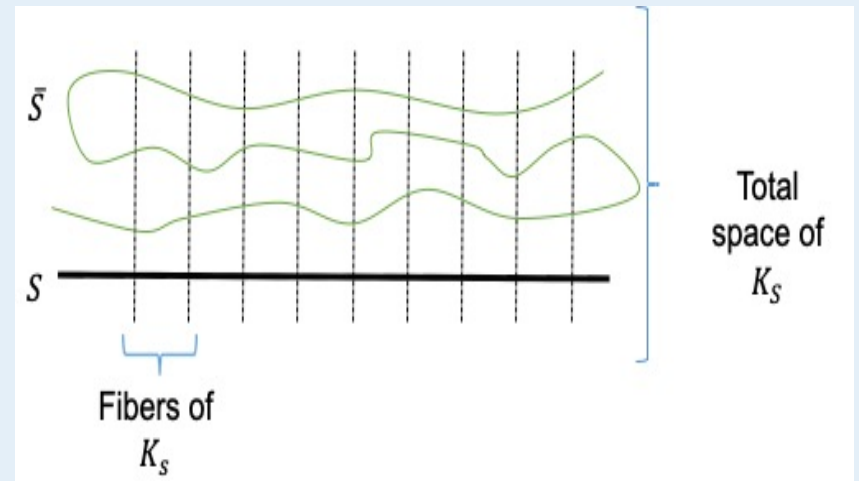
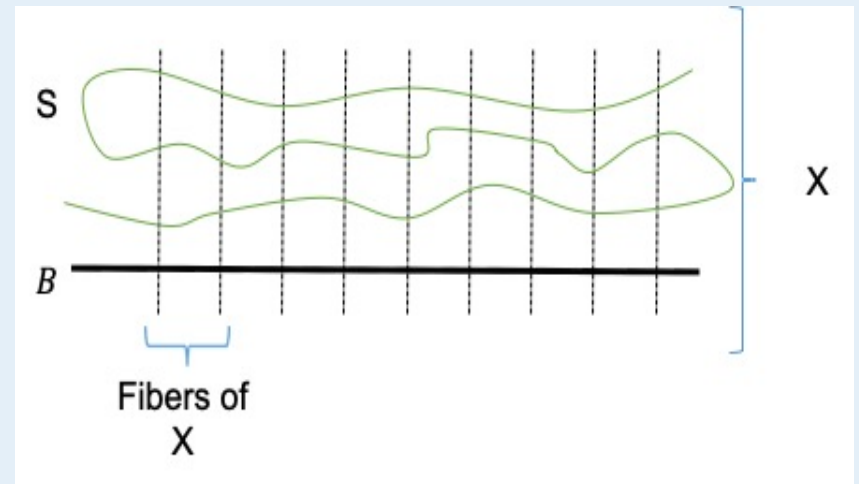
$$\Phi(V) = i_{S_3*} \mathcal{L} [-1]$$

$$i_{S_3}: S_3 \hookrightarrow X$$

$\mathcal{L}$  : A coherent rank one sheaf over  $S_3$ .

$S_3$  : A finite cover of  $B_2$  of degree 3.

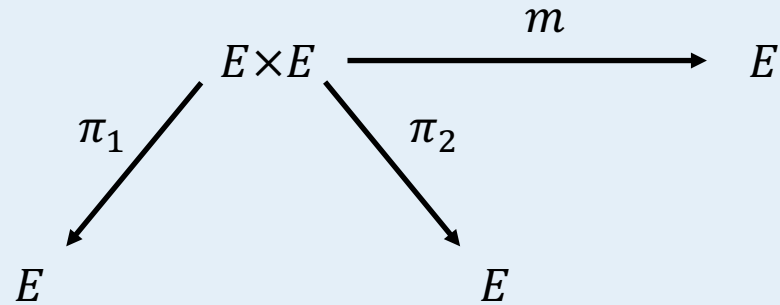
- F-th, 7-brane local data is given by the Hitchin system  $(E, \phi)$  equivalent to the spectral data  $(\bar{S}, \bar{\mathcal{L}})$ .
- The neighborhood of the zero-section of  $X$  is mapped to  $Tot(K_S)$ .



# ❖ Pontrjagin Product

$E$ : Smooth elliptic curve.

$$m(p, q) := p \boxplus q$$



$$\mathcal{F} \star \mathcal{G} := Rm_*(\pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G})$$

$$\Phi(\mathcal{F} \otimes \mathcal{G}) = \Phi(\mathcal{F}) \star \Phi(\mathcal{G})[+1]$$

- Naively for an elliptic fibration this formula generalizes to,

$$\Phi(\mathcal{F} \otimes \mathcal{G}) = \Phi(\mathcal{F}) \star \Phi(\mathcal{G}) \otimes K_B^{-1}[+1]$$

# ❖ Cohomologies

- Pontjagin product cannot be used over singular fibers of  $X$ . More precisely the group law is only valid for the smooth points of the singular elliptic curve.
- This is not too bad,
  1. We only need anti-symmetrized product.
  2. Only the restriction of the spectral sheaf over the zero section  $\sigma$  contributes to the Cohomology of the bundles.

$$E_2^{p,q} = H^p(B_2, R^q \pi_* V) \Rightarrow H^{p+q}(X, V)$$

$$H^0(X, V) = H^0(\pi_* V)$$

$$0 \rightarrow H^1(\pi_* V) \rightarrow H^1(V) \rightarrow H^0(R^1 \pi_* V) \rightarrow H^2(\pi_* V) \rightarrow H^2(V) \rightarrow H^1(R^1 \pi_* V) \rightarrow 0$$

$$H^3(V) = H^2(R^1 \pi_* V)$$

$$R^* \pi_* V = Li_\sigma^* \Phi(V) = Li_\sigma^* i_{S_2*} \mathcal{L}[-1]$$

$$i_\sigma: \sigma \hookrightarrow X$$



# ❖ Cohomologies

$$\Phi(\Lambda^2 V) = (i_{S_2^*} \mathcal{L}) \star_A (i_{S_2^*} \mathcal{L}) \otimes K_B^{-1}[-1]$$

$$R\pi_* \Lambda^2 V = Li_\sigma^*(i_{S_2^*} \mathcal{L}) \star_A (i_{S_2^*} \mathcal{L}) \otimes K_B^{-1}[-1]$$

$$E_2^{p,q} = H^p(B_2, R^q \pi_* \Lambda^2 V) \Rightarrow H^{p+q}(X, \Lambda^2 V)$$

- The idea is to compute  $H^1(V)$  and  $H^i(\Lambda^2 V)$  in terms of the spectral data using the Leray spectral sequence and the pushforward formulas.

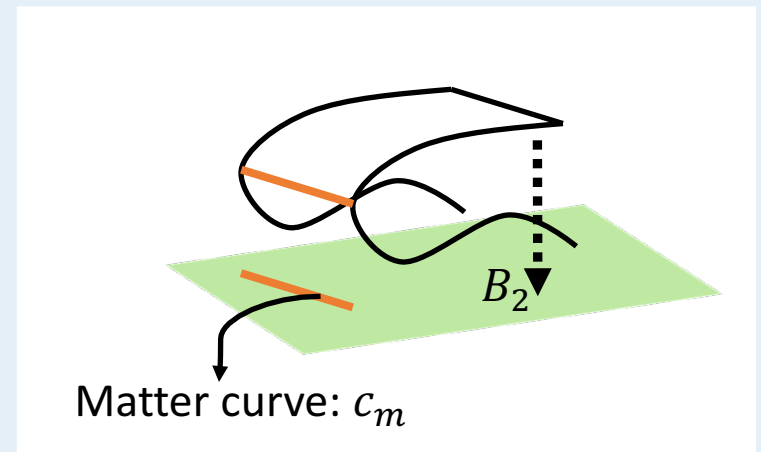
# ❖ Smooth Spectral Cover

- Consider the  $E_6$  model ( $SU(3)$  bundle  $V$ ),

$$S_3 = a_3 Y + a_2 X Z + a_0 Z^3$$

$$R^1 \pi_* V = i_{S_3 \cap \sigma^*} \mathcal{L}$$

$$\Phi(V) = i_{S_3^*} \mathcal{L}[-1]$$

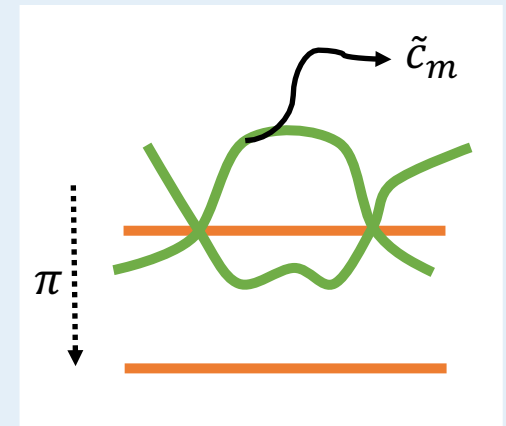


- If  $S_3$  intersects a generic fiber at  $P_1, P_2, P_3$  S.T  $P_1 + P_2 + P_3 = 0$ , then  $S_{\Lambda^2 V}$  intersect at,

$$P_1 + P_2, \quad P_1 + P_3, \quad P_2 + P_3. \quad P_1 + P_2 = 0 \Rightarrow P_3 = 0$$

$$R^1 \pi_* \Lambda^2 V = \text{Det} \left( \pi_* \mathcal{L} \Big|_{\tilde{c}_m} \right) \otimes K_B^{-1}$$

$$S_{\Lambda^2 V} = -a_3 Y + a_2 X Z + a_0 Z^3$$



# ❖ Smooth Spectral Cover

$$H^1(V) \otimes H^1(V) \otimes H^1(V) \rightarrow \mathbb{C} \quad \text{equivalently} \quad H^1(V) \otimes H^1(V) \rightarrow H^2(\Lambda^2 V)$$

This map cannot be non-zero  $\Rightarrow$  Vanishing Yukawa coupling **27 27 27**

- Similarly for  $SU(4)$  bundle one can do a similar analysis to show there are non-vanishing **16 16 10** Couplings on the intersection of the curves

$$S_4 = a_4 X^2 + a_3 YZ + a_2 XZ^2 + a_0 Z^4 \quad a_4 = a_3 = 0$$

- For  $SU(5)$  models **10 10 5** Yukawa couplings are zero for similar reasons as in  $SU(3)$  but **10  $\bar{5}$   $\bar{5}$**  will be non-zero over the intersections of the following curves

$$S_5 = a_5 XY + a_4 XZ + a_3 YZ^2 + a_2 XZ^3 + a_0 Z^5 \quad a_5 = a_4 a_3^2 - a_2 a_3 a_5 + a_0 a_5^2 = 0$$

# ❖ Singular Spectral Covers

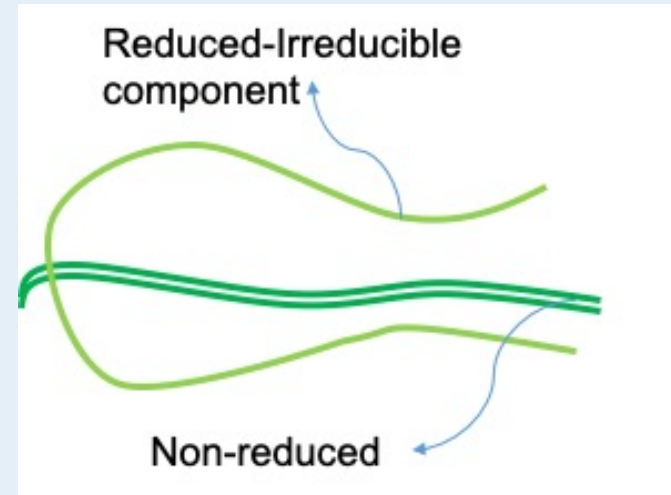
- $S_n$  can be reducible and even have non-reduced components.
- In such cases we can always express the spectral sheaf as an extension of simpler parts.

$$S_n = Z S_{n-1}$$

$$0 \rightarrow i_{\sigma^*} \mathcal{L}_1 \rightarrow i_{S_n^*} \mathcal{L}_n \rightarrow i_{S_{n-1}^*} \mathcal{L}_{n-1} \rightarrow 0$$

$$0 \rightarrow i_{\sigma^*} \mathcal{L}_1 \star i_{S_{n-1}^*} \mathcal{L}_{n-1} \rightarrow i_{S_n^*} \mathcal{L}_n \star_A i_{S_n^*} \mathcal{L}_n \rightarrow i_{S_{n-1}^*} \mathcal{L}_{n-1} \star_A i_{S_{n-1}^*} \mathcal{L}_{n-1} \rightarrow 0.$$

- The idea is to construct all spectral data iteratively using such relations.
- At the intersection of the components of  $S_n$  there can be singularities, generically they do not contribute in the Yukawa couplings, therefore we assume from now on the spectral sheaves over each components is a smooth line bundle.



# ❖ Singular $SU(3)$ Spectral Data

- Reducible  $S_3$  with Reduced components:

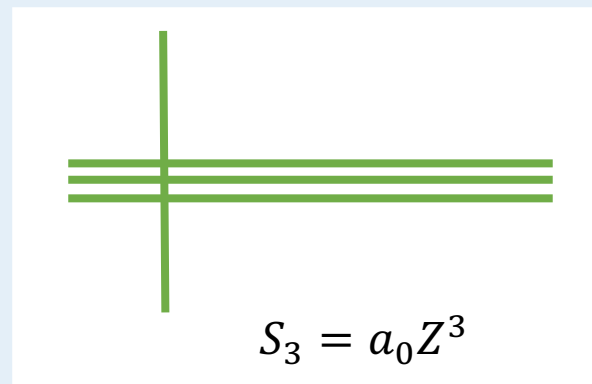
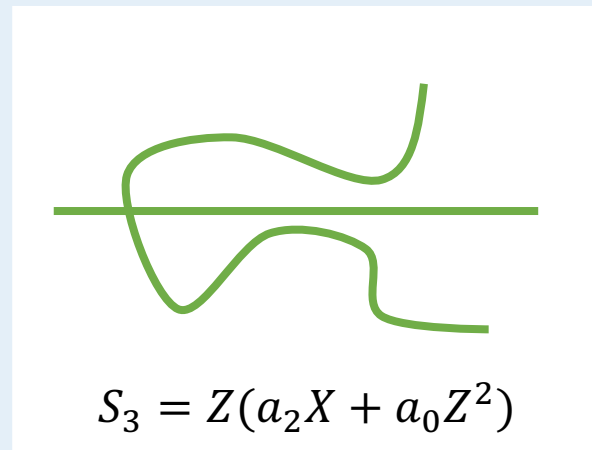
$$0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_3 \rightarrow \mathcal{L}_2 \rightarrow 0$$

$$0 \rightarrow \pi^* \pi_* \mathcal{L}_1 \otimes \mathcal{L}_2 \rightarrow \mathcal{L}_3 \star_A \mathcal{L}_3 \rightarrow i_{\sigma^*} \text{Det}(\pi_* \mathcal{L}_2) \rightarrow 0$$

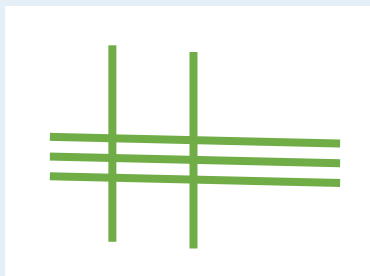
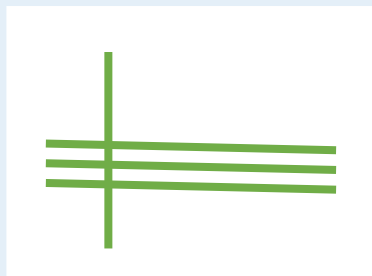
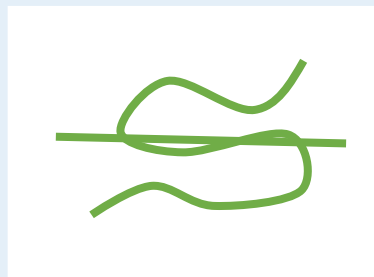
- In this case there are possible **27 27 27** couplings from interaction of Bulk and local zero-modes localized on  $a_2 = 0$  :

$$H^1(B_2, \mathcal{L}_1 \otimes K_B^{-1}) \otimes H^0(B_2, i_{\{a_2=0\}*} \mathcal{L}_2) \rightarrow H^1(B_2, i_{\{a_2=0\}*} \mathcal{L}_2 \otimes \mathcal{L}_1 \otimes K_B^{-1})$$

- For non reduced spectral cover it is possible to have couplings coming from the bulk zero modes.



# ❖ $SU(4)/SU(5)$ Spectral Data



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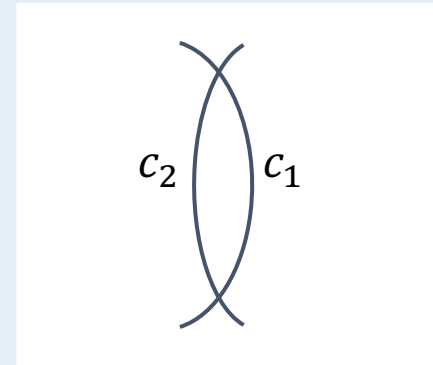
## ❖ Caveats

- The possible non-vanishing couplings means one should compute whether the corresponding Cohomologies are non-zero and whether they contribute in  $H^1(V)$  and/or  $H^i(\Lambda^2 V)$ .
- The analysis so far is restricted to CY3 manifold with at least a holomorphic section.

## ❖ Elliptic CY3 w/ Non-Holomorphic sections.

- Non-holomorphic sections wrap around rational curve(s) in the fiber.
- Over such fibers the restriction Poincaré sheaf doesn't parametrize the (semi)-stable.
- To cure this one can modify the kernel of the integral transform,

$$\bar{\mathcal{P}} := \mathcal{P} \otimes \pi_1^* \Lambda^2 N_{c_1/X}$$



- The effect of this twist is that the restriction of  $\bar{\mathcal{P}}$  on the  $I_2$  fiber is a well defined Poincaré sheaf.



## ❖ Elliptic CY3 w/ Non-Holomorphic sections.

- The Fourier transform of  $V$  with this new kernel will not be sheaf in general,

$$\bar{\Phi}(V) := R\pi_{2*}(\pi_1^*V \otimes \bar{\mathcal{P}}) = \mathcal{L}.$$

$$\mathcal{L}: 0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_2 \rightarrow 0$$

- Where  $\mathcal{L}_1$  can wrap the components of the reducible fibers.