5D BPS quivers and KK towers

Dongwook Ghim

KIAS School of Physics

Jun. 14th, PASCOS 2021

via ZOOM

Based on joint work (arXiv:2011.04661)
with Zhihao Duan (KIAS) and Piljin Yi (KIAS)

5D BPS quivers and KK towers

[1] Construction of D=5 BPS quivers
(facilitated by brane tiling of toric CY_3)

[2] Witten Index of abelian BPS quivers (single KK)
(L^2 BPS states as L^2 cohomology of quiver Higgs moduli)

[3] Witten Index of non-abelian BPS quivers (higher KK, etc)
(Coulomb branch formula and weak-coupling chamber)

What is BPS quiver?

(4D) BPS quiver

Effective SUSY QM for BPS objects in D=4 SW theories

e.g. BPS quiver of 4D SU(2) theory

$\gamma_1 = (1\,,0)$ $\gamma_2 = (-1\,,2)$

[1] Node for each elementary object (basis)

[2] Arrows from Schwinger product of basis charges (m, e)

NB Stable bound states ⇔ non-zero Witten index of SQM

$$\Gamma = \sum_i N_i \gamma_i$$

[Seiberg, Witten 94]
[Denef 02]

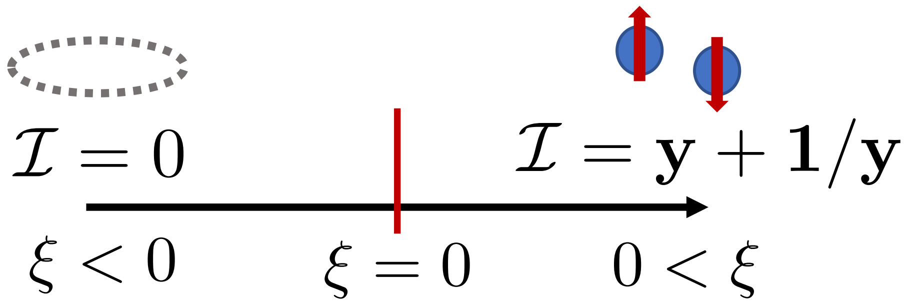
(4D) BPS quiver

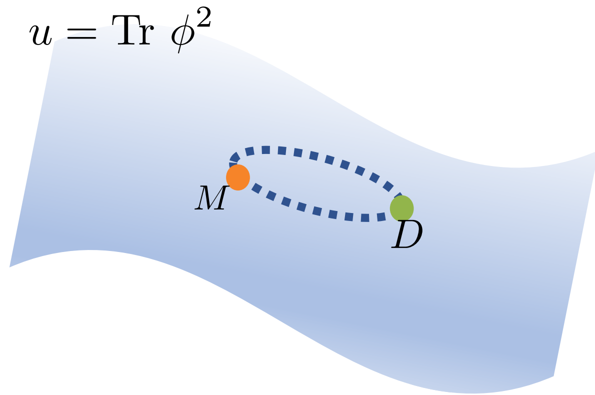
Effective SUSY QM for BPS objects in D=4 SW theories

e.g. stability of W-boson SU(2) SW theory; $\gamma_W = (0\,,2)$

$\gamma_1 = (1\,,0)$ $\gamma_2 = (-1\,,2)$

NB Wall-crossing in moduli space is captured by jump of index under the variation of FI constant.





[Seiberg, Witten 94]
[Denef 02]

5D BPS quivers and KK towers

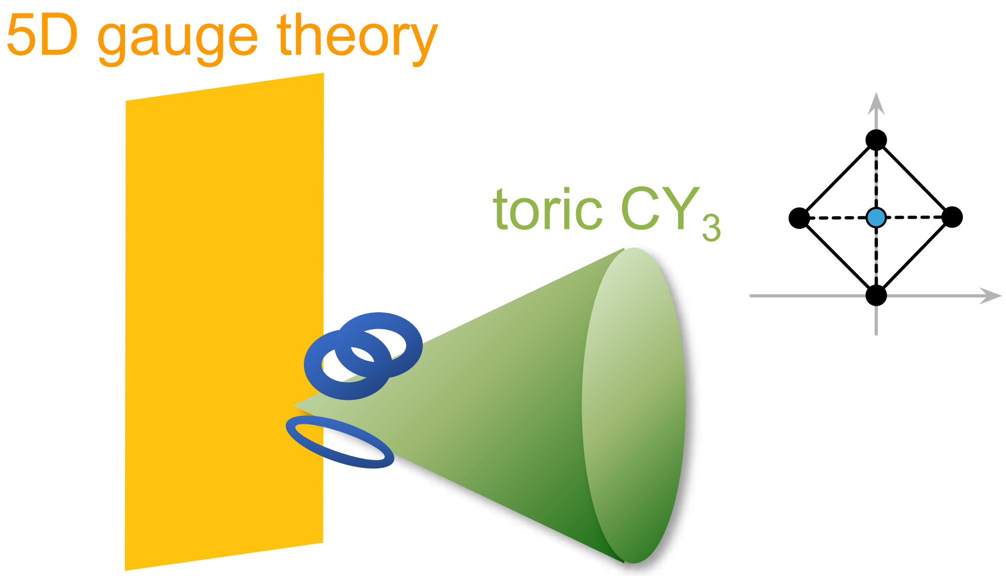
[1] Construction of D=5 BPS quivers [Closset, Del Zotto 19]
(facilitated by brane tiling of toric CY_3)
e.g. [Franco, Hanany, Kennaway, Vegh, Wecht 96]

[2] Witten Index of abelian BPS quivers (single KK)
(L^2 BPS states as L^2 cohomology of quiver Higgs moduli)

[3] Witten Index of non-abelian BPS quivers (higher KK, etc)
(Coulomb branch formula and weak-coupling chamber)

Geometric engineering

D=5 gauge theory can be engineered by
M-theory 'compactified' over toric Calabi-Yau 3-folds.



NB **M-branes** wrapping on internal **2-** and **4-cycles** of CY3 give **BPS objects in 5D gauge theory.**

[Klemm, Lerche, Mayr, Vafa, Warner 96]
[Katz, Klemm, Vafa 96]

5D BPS quiver

Effective SUSY QM for BPS objects
in circle-compactified D=5 gauge theories

[1] Node for each elementary object (… + **instanton** + **KK**)

[2] Arrows from Schwinger products of charges (m, e)

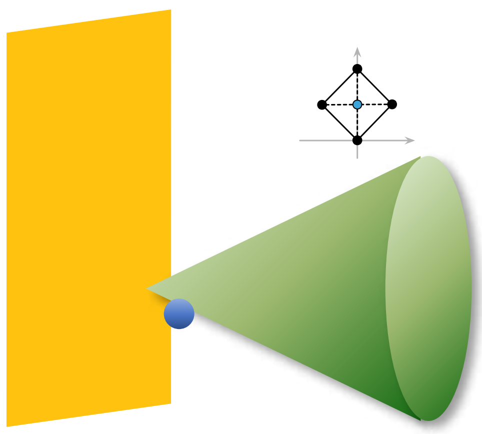
NB Circle compacitification is essential in that we have spatially extended BPS objects like monopole strings in D=5.

[Banerjee, Longhi, Romo 18, 19]
[Closset, Del Zotto 19]

BPS quiver from Probe theory

Upon the reduction to D=4 KK theory,
there is a canonical way to draw 5D BPS quiver due to states attributed to **D0 branes (without D2 or D4)**
⇔ pure **KK** modes of D=4 KK theory

$$\Gamma = \cancel{n_m \hat{m}} + \cancel{n_e \hat{e}} + \cancel{n_I \hat{I}} + n_K \hat{K} = \sum_i N_i \gamma_i$$



4D KK gauge theory

[Closset, Del Zotto 19]

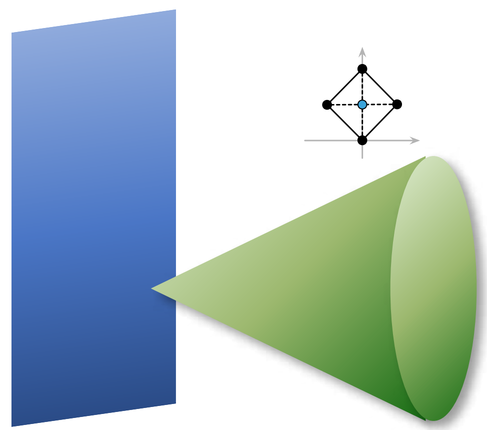
BPS quiver from Probe theory

Upon the reduction to D=4 KK theory,
there is a canonical way to draw 5D BPS quiver due to states attributed to **D0 branes (without D2 or D4)**

⇔ pure **KK** modes of D=4 KK theory

⇔ T-dual^3 of D3 brane probing CY3

$$\Gamma = \cancel{n_m}\hat{m} + \cancel{n_e}\hat{e} + \cancel{n_I}\hat{I} + n_K \hat{K} = \sum_i N_i \gamma_i$$

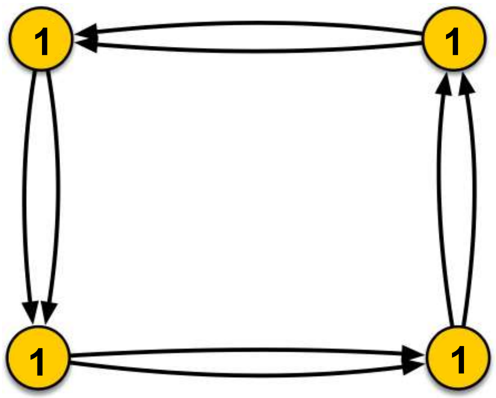


4d gauge theory on D3

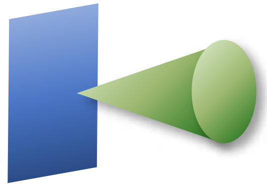
[Closset, Del Zotto 19]

Brane Tiling

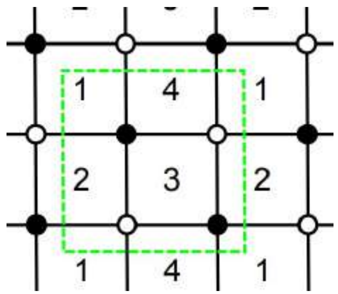
A combinatorial tool exists **to engineer 4d N=1 quiver gauge theories** supported by **D3 probing toric CY 3-folds.**



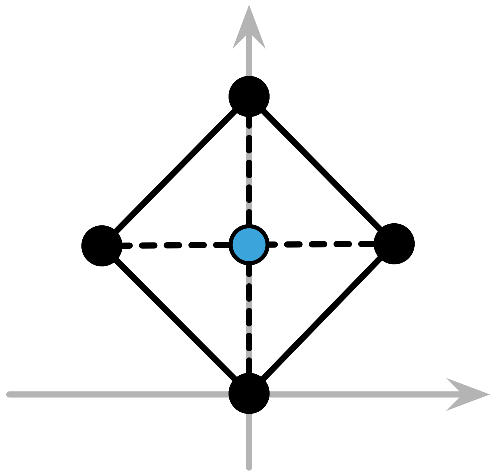
quiver gauge theory



D3 probing CY_3



brane tiling



toric diagram

NB BPS quiver of **pure KK** = single D0

⇔ single D3 = abelian quiver of **brane tiling**

e.g. [Franco, Hanany, Kennaway, Vegh, Wecht 96]

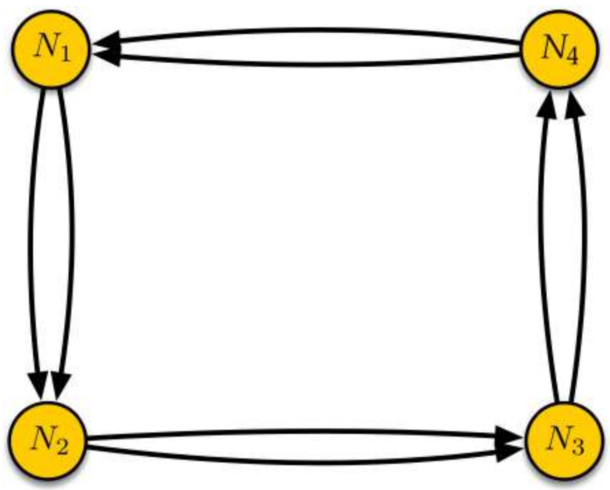
5D BPS quiver

Effective SUSY QM for BPS objects
in circle-compactified D=5 gauge theories

[1] (equal-ranked) quivers from brane tiling ➔ pure KK sector

[2] (unequal-ranked) extension ➔ non-trivial charged sector

e.g. BPS quiver of **5D SU(2)** gauge theory on circle

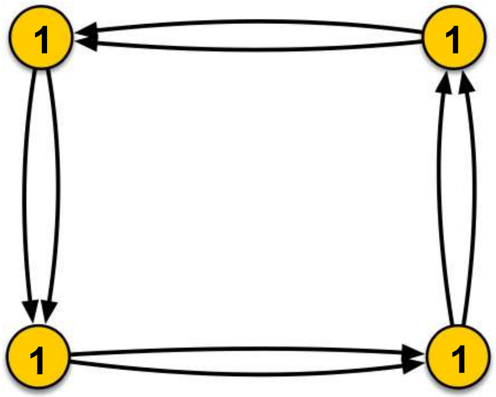


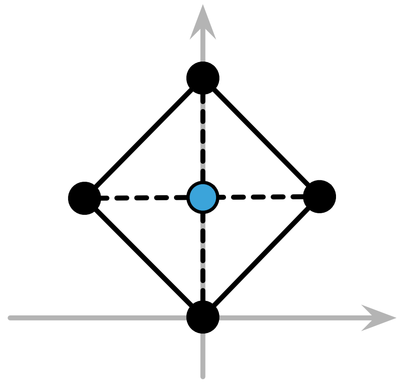
$$\Gamma = n_m \hat{m} + n_e \hat{e} + n_I \hat{I} + n_K \hat{K} = \sum_i N_i \gamma_i$$

[Banerjee, Longhi, Romo 18, 19]
[Closset, Del Zotto 19]

Remark

The **Higgs branch** of 4d N=1 **abelian** quiver corresponds to the probed **toric Calabi-Yau 3-fold**.





$$\mathcal{W} = X_{12}X_{23}Y_{34}Y_{41} + Y_{12}Y_{23}X_{34}X_{41} - X_{12}Y_{23}Y_{34}X_{41} - Y_{12}X_{23}X_{34}Y_{41}$$

$$\mathcal{M}^{(4d)}_{\text{Higgs}} = (\mathbb{C}[X\text{'s}]/\langle \partial \mathcal{W} = 0 \rangle)//U(1)^{N-1}$$

NB The quiver always involves **fine-tuned superpotential.**

NB **Coulomb** moduli pops up in 1d;

$$\mathcal{M}^{(1d)} = \mathbb{R}^3 \times (\mathbb{C}[X\text{'s}]/\langle \partial \mathcal{W} = 0 \rangle)//U(1)^{N-1}$$

e.g. [Franco, Hanany, Kennaway, Vegh, Wecht 96]

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[2] Witten Index of abelian BPS quivers (single KK)
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[Witten 82], [Alvarez-Gaume 83]

[3] Witten Index of non-abelian BPS quivers (higher KK, etc)
(Coulomb branch formula and weak-coupling chamber)

Witten Index

Definition

Twisted partition function with fermionic parity weight

[Witten 82]

$$\mathrm{Tr}\,(-1)^{\mathcal{F}} e^{-\beta\,\mathcal{H}} \quad \begin{cases} \mathcal{I} = \lim_{\beta \to +\infty} \mathrm{Tr}\,(-1)^{\mathcal{F}} e^{-\beta\,\mathcal{H}} \\ \Omega = \lim_{\beta \to 0^+} \mathrm{Tr}\,(-1)^{\mathcal{F}} e^{-\beta\,\mathcal{H}} \end{cases}$$

reduction to matrix integral or
localization leaving JK residue.

NB JK-residue formula ~~gives~~ *NOT ALWAYS* Witten index of N=4 SQM !

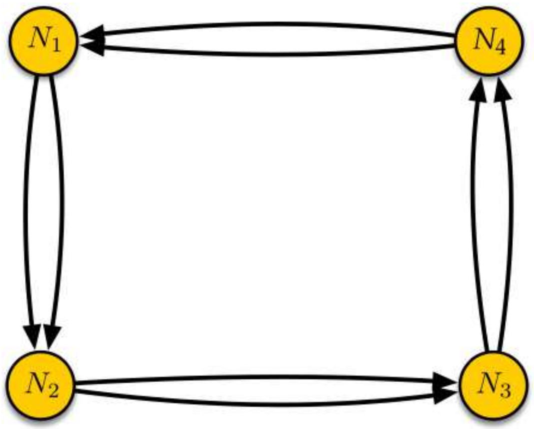
[Hori, Kim, Yi 14]

Failure of JK-residue formula

To be precise, Witten index counts L²-normalizable states. Contamination from continuous spectra must not be counted.

$$\mathrm{Tr}_{\mathcal{H}_{L^2}} (-1)^{\mathcal{F}} e^{-\beta \mathcal{H}}$$

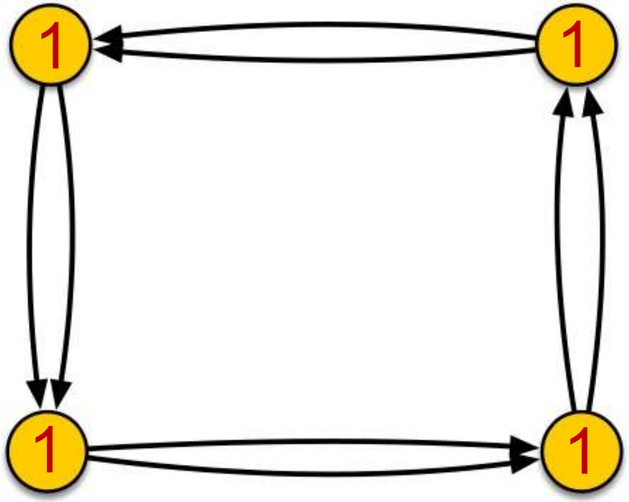
Furthermore, 5D BPS quivers entail **fine-tuned super-potentials** whereas localization formula assumes generic one.

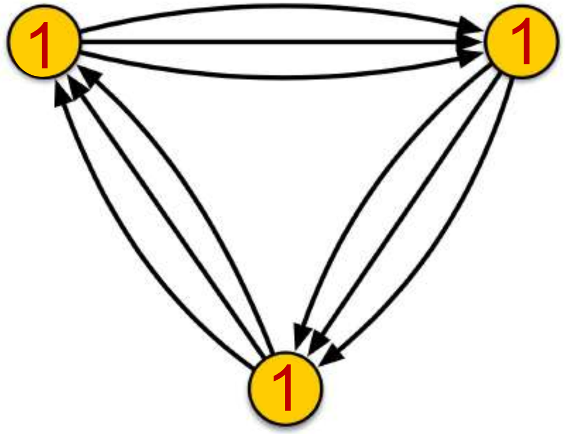


$$\begin{aligned}\mathcal{W} = &X_{12}X_{23}Y_{34}Y_{41} + Y_{12}Y_{23}X_{34}X_{41}\\ &-X_{12}Y_{23}Y_{34}X_{41} - Y_{12}X_{23}X_{34}Y_{41}\end{aligned}$$

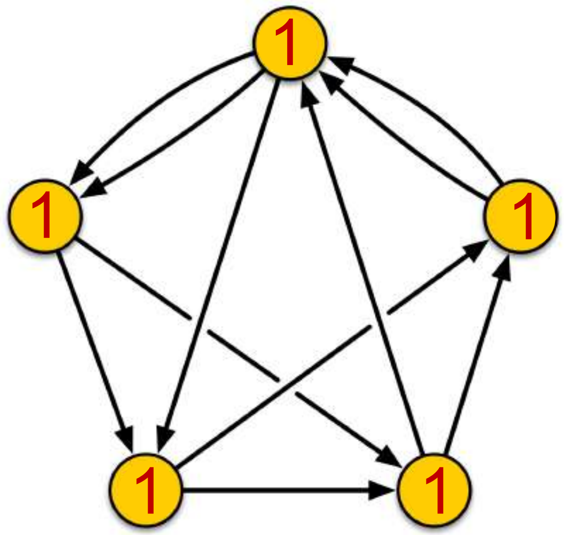
What to count ?

Abelian quivers





$$\Gamma = \sum_i \gamma_i = 1 \cdot \hat{K}$$

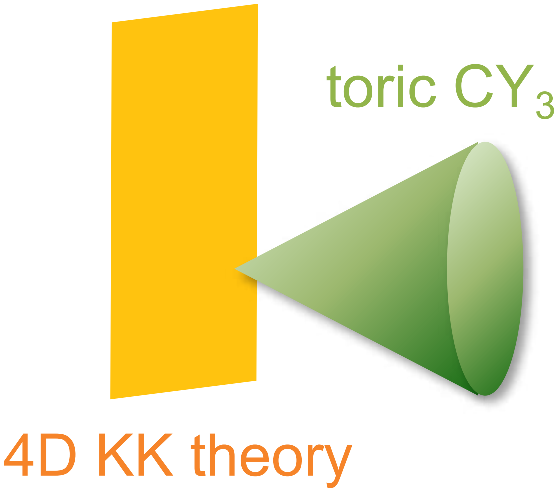


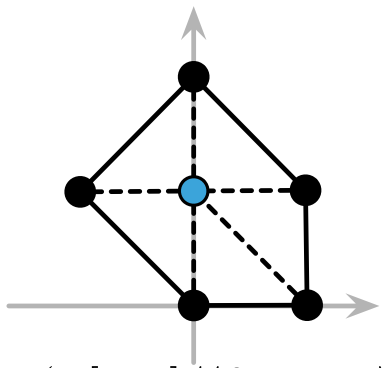
Witten index and L^2 cohomology

Witten index counts L^2-normalizable states.
Contamination from continuous spectra must not be counted.

$$\mathrm{Tr}_{\mathcal{H}_{L^2}} (-1)^{\mathcal{F}} e^{-\beta \mathcal{H}}$$

Wave function of corresponding particles correspond to entries of L^2 cohomology on target space.





~~$\mathcal{M}^{(4d)}_{\text{Higgs}} = (\mathbb{C}[X\text{'s}]/\langle \partial \mathcal{W} = 0 \rangle)//U(1)^{N-1}$~~

$$\mathcal{M}^{(1d)} = \mathbb{R}^3 \times (\mathbb{C}[X\text{'s}]/\langle \partial \mathcal{W} = 0 \rangle)//U(1)^{N-1}$$

Witten index and L^2 cohomology

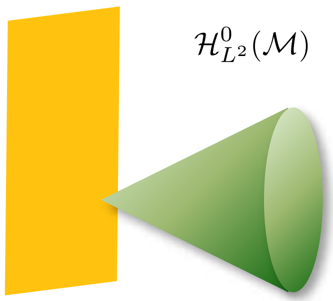
Relevant state of abelian (1,1,…,1) quiver of IIA frame

= **Scattering state along $\mathbb{R}^3$** $\otimes$ **L^2 normalizable state on CY_3**

$$\mathcal{M}^{(1d)} = \mathbb{R}^3 \times (\mathbb{C}[X\text{'s}]/\langle \partial\mathcal{W} = 0\rangle)//U(1)^{N-1}$$

$\mathcal{H}^0_{L^2}(\mathcal{M})$ $\mathcal{H}^1_{L^2}(\mathcal{M})$ $\mathcal{H}^2_{L^2}(\mathcal{M})$ $\mathcal{H}^3_{L^2}(\mathcal{M})$ $\mathcal{H}^4_{L^2}(\mathcal{M})$ $\mathcal{H}^5_{L^2}(\mathcal{M})$ $\mathcal{H}^6_{L^2}(\mathcal{M})$

4D KK theory



toric CY_3

NB There exists well-known machinery and mathematical theorem for counting L^2 cohomology of toric CY's.

L^2 cohomology counting

$\mathcal{H}^0_{L^2}(\mathcal{M})$ $\mathcal{H}^1_{L^2}(\mathcal{M})$ $\mathcal{H}^2_{L^2}(\mathcal{M})$ $\mathcal{H}^3_{L^2}(\mathcal{M})$ $\mathcal{H}^4_{L^2}(\mathcal{M})$ $\mathcal{H}^5_{L^2}(\mathcal{M})$ $\mathcal{H}^6_{L^2}(\mathcal{M})$

L^2 cohomology counting

[1] Pairing btw. L^2 cohomology $\int_{\mathcal{M}} \omega_{(k)} \wedge \omega_{(2d-k)}$

$\mathcal{H}^0_{L^2}(\mathcal{M})$ $\mathcal{H}^1_{L^2}(\mathcal{M})$ $\mathcal{H}^2_{L^2}(\mathcal{M})$ $\mathcal{H}^3_{L^2}(\mathcal{M})$ $\mathcal{H}^4_{L^2}(\mathcal{M})$ $\mathcal{H}^5_{L^2}(\mathcal{M})$ $\mathcal{H}^6_{L^2}(\mathcal{M})$

[2] Theorem for manifolds with conical metric
[Hausel, Hunsicker, Mazzeo 02]

$H^{p<3}(\mathcal{M}, \partial\mathcal{M})$ $\mathrm{Im}(H^3(\mathcal{M}, \partial\mathcal{M}) \to H^3(\mathcal{M}))$ $H^{3<p}(\mathcal{M})$

[3] Pairing btw. singular homology and de Rham cohomology $\int_{\Sigma(p)} \omega_{(p)}$

$H_{3<p}(\mathcal{M})$

L^2 cohomology counting

[1] Pairing btw. L^2 cohomology

$\mathcal{H}^0_{L^2}(\mathcal{M})$ $\mathcal{H}^1_{L^2}(\mathcal{M})$ $\mathcal{H}^2_{L^2}(\mathcal{M})$ $\mathcal{H}^3_{L^2}(\mathcal{M})$ $\mathcal{H}^4_{L^2}(\mathcal{M})$ $\mathcal{H}^5_{L^2}(\mathcal{M})$ $\mathcal{H}^6_{L^2}(\mathcal{M})$

[2] Theorem for manifolds with conical metric
[Hausel, Hunsicker, Mazzeo 02]

$H^{3<p}(\mathcal{M})$

[3] Pairing btw. singular homology
and de Rham cohomology

[4] Algebraicness & noncompactness

$H_4(\mathcal{M})$ $H_5(\mathcal{M})$ $H_6(\mathcal{M})$

Remark

L^2 Witten index of BPS quiver shares the following features:

[1] It counts the number of **4-cycles** of toric CY_3 of interest.

$$\mathcal{I} = -2 \cdot \dim H_4(\mathcal{M})$$

[2] symmetric Laurent polynomial in $SU(2)_R$ R-sym. fugacity **y**

$$\mathcal{I}(\mathbf{y}) = \dim H_4(\mathcal{M}) \cdot \left(-\mathbf{y} - \frac{1}{\mathbf{y}}\right)$$

NB grading by **y** : (p-d) for $\mathcal{H}^p_{L^2}$ cohomology and $d = \dim_{\mathbb{C}} \mathcal{M}$

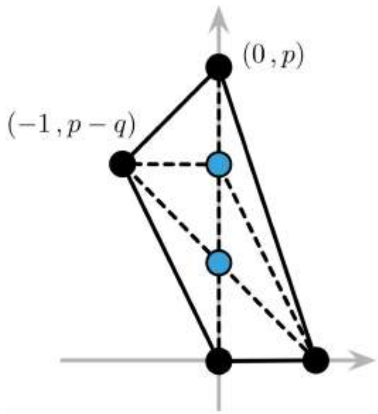
[3] SU(2) **spin** interpretation (Kähler 2-form J on CY_3)

$$L_+[\omega_{(n)}] = J \wedge \omega_{(n)} \;\;, \quad L_-[\omega_{(n)}] = i_J\, \omega_{(n)} \;\;, \quad L_3[\omega_{(n)}] = \frac{(n-3)}{2}\omega_{(n)}$$

Examples

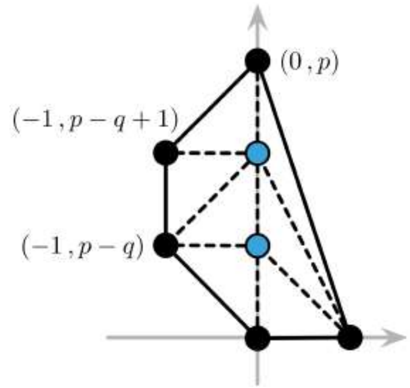
Two families of toric CY3, engineering 5D SU(p)$_q$ theory

$$\mathcal{I}(\mathbf{y}) = \dim H_4(\mathcal{M}) \cdot \left(-\mathbf{y} - \frac{1}{\mathbf{y}} \right)$$



$$\dim H_0 = 1\,, \quad \dim H_2 = p\,, \quad \dim H_4 = p-1\,, \quad \dim H_6 = 0\,.$$

$$\mathcal{I}_{Y^{p,q}} = (p-1) \times (-\mathbf{y} - \mathbf{y}^{-1})$$



$$\dim H_0 = 1\,, \quad \dim H_2 = p+1\,, \quad \dim H_4 = p-1\,, \quad \dim H_6 = 0\,,$$

$$\mathcal{I}_{X^{p,q}} = (p-1) \times (-\mathbf{y} - \mathbf{y}^{-1})$$

- $\dim H_4(\mathcal{M}_3)$ **= number of 4-cycles = rank of 5d SCFTs**

$$\mathcal{I}_{(1,1,\ldots,1)}(\mathbf{y}) = \text{rank} \times (-\mathbf{y} - \mathbf{y}^{-1})$$

NB **neutral KK modes = Cartan part of vector multiplets**

5D BPS quivers and KK towers

More information @ following QR links
60-min talks @ Fudan University and Imperial Quiver meeting

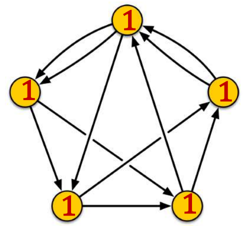
[1] Construction of D=5 BPS quiv
(facilitated by brane tiling of to

[2] Witten Index of abelian BPS q K)
(L^2 BPS states as L^2 cohomology of quiver Higgs moduli)

[3] Witten Index of non-abelian BPS quivers (higher KK, etc)
(Coulomb branch formula and weak-coupling chamber)
[Yi 97], [Sethi, Stern 97], [Manschot, Pioline, Sen 10-14]

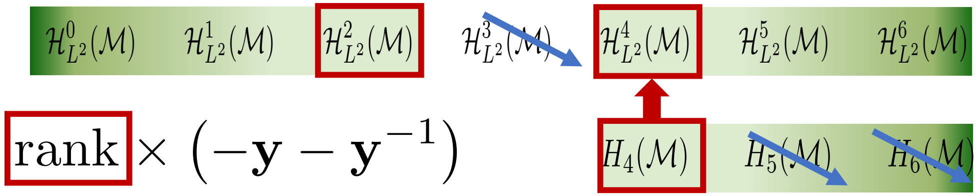
Summary

- Single KK mode



L^2 cohomology counting on Coulomb / Higgs branch of quiver

➔ **Scattering state along $\mathbb{R}^3$** $\otimes$ **L^2 normalizable state on CY_3**



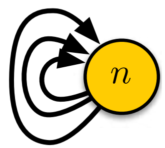
FI-independent! $\mathcal{I}_{(1,1,\ldots,1)}(\mathbf{y}) = \text{rank} \times \left(-\mathbf{y} - \mathbf{y}^{-1}\right)$

- Higher KK modes

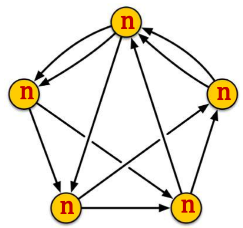
$$\mathcal{N}_n \equiv \left[(\mathcal{M}_3)^n \times (\mathbb{R}^3)^n\right]/S_n = \mathbb{R}^3 \times \left[(\mathcal{M}_3)^n \times (\mathbb{R}^3)^{n-1}\right]/S_n$$

$$\simeq \left[\mathbb{R}^3 \times \mathcal{M}_3\right] \times \left[(\mathbb{R}^6)^{n-1} \times (\mathbb{R}^3)^{n-1}\right]/S_n$$

$$\Psi \simeq \Psi_{\text{geometric}} \otimes \Psi_{SU(n)_*}$$

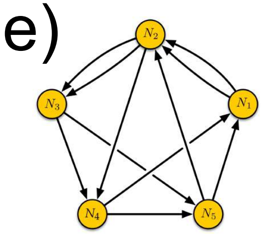


$\mathcal{I}_{SU(n)} = 1$



FI-independent! $\mathcal{I}_{(n,n,\ldots,n)}(\mathbf{y}) = \text{rank} \times \left(-\mathbf{y} - \mathbf{y}^{-1}\right)$

- Electric / flavor particles (NO magnetic/instanton charge)



MPS <u>wall-crossing</u> formula with **quiver invariant** hypothesis

FI-dependent! $\mathcal{I}_{\text{inv}}(\gamma_i) = 1$ for **single-node quiver**

Thank you