$\mathbb{Z}_5$ Symmetries in F-theory, Homological Projective Duality and Modular Forms

PASCOS 2021

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Topological strings on torus fibered Calabi-Yau 3-folds

- Genus-one fibrations w/ N-sections for $N \leq 4$
  $\rightarrow Z_{top.}$ *can be expanded in* $\Gamma_1(N)$ *weak Jacobi forms*
  [Huang,Katz,Klemm'15],**[DelZotto,Gu,Huang,Kashani-Poor,Klemm,Lockhart'17]**,**[Lee,Lerche,Weigand'18]**,**[Cota,Klemm,T.S.'19]**
- Result of monodromies in stringy Kähler moduli space
  [T.S.'19],**[Cota,Klemm,T.S.'19]**

Torus fibered Calabi-Yau are abundant

[Huang,Taylor'19]: 99.99994% of 4d reflexive polytopes have structure indicating T^2 fibration

(see also e.g. **[Anderson,Gao,Gray,Lee'16]**)

Moreover, they encode F-theory vacua and play pivotal role in network of string dualities!

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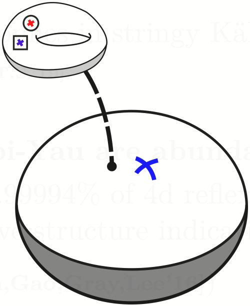
Not all torus fibrations are elliptic!

Some only have N-sections



Intersections experience monodromy along base!

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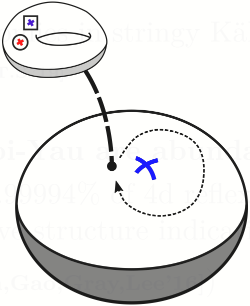

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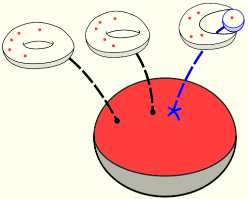
This talk: $N = 5 \rightarrow$ Rich in novel phenomena

published soon [Johanna Knapp, Emanuel Scheidegger, T.S.'21]



Genus-one fibered CY
with 5-sections

Higgs transitions
in M-/F-theory

Algebraic Geometry

Extremal transitions
toric $\rightarrow$ non-toric CY

Arithmetic Geometry

$\Gamma_1(5)$ Jacobi forms

Derived Equivalences,
Homological Projective Duality

Conjecture: For genus-one fibered Calabi-Yau with N-sections

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$$Z_{\text{top}}(\underline{t},\lambda) = Z_0(\tau,\lambda)\left(1+\sum_{\beta\in H_2(B,\mathbb{Z})} Z_\beta(\tau,\underline{m},\lambda)\,Q^\beta\right)$$

$Z_\beta(\tau,\underline{m},\lambda)$ are $\Gamma_1(N)$ lattice Jacobi forms

$\rightarrow$ *All-genus results for compact CY 3-folds!*

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$$Q^{\beta} \;=\; \exp\left(2\pi i \beta_j t^j\right)$$

t^j: (shifted) volumes of base curves

$$Z_{\text{top}}(\underline{t}, \lambda) = Z_0(\tau, \lambda)\left(1 + \sum_{\beta \in H_2(B,\mathbb{Z})} Z_{\beta}(\tau, \underline{m}, \lambda)\, Q^{\beta}\right)$$

τ: $\frac{1}{N}\times$ fiber volume

$\underline{m}$: Volumes of fiber components

$Z_{\beta}(\tau, \underline{m}, \lambda)$ ... obi forms

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Modularity from brane monodromies

Interpret cplx. volumes $\tau, \underline{m}, \underline{t}$ as 2-brane charges

→ *For $N \leq 4$ auto-equivalences of brane category generate $\Gamma_1(N)$-action*

- B-field shifts act as $\tau \mapsto \tau + 1$, $m \to m + 1$, ...
- Moreover, the **fiberwise conifold transformation** acts as

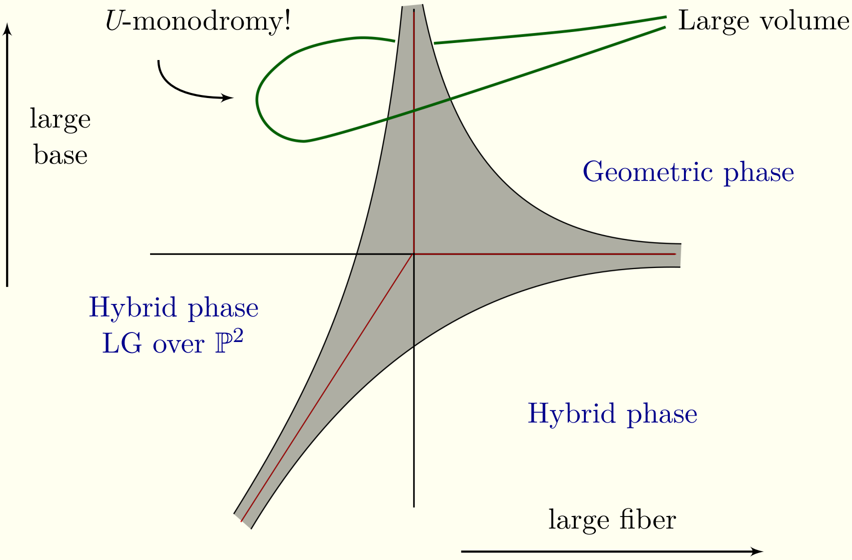
$$U: \begin{cases} \tau & \mapsto & \tau/(1+N\tau) \\ m_i & \mapsto & m_i/(1+N\tau)\,, \quad i = 1, ..., \mathrm{rk}(G) \\ Q_i & \mapsto & (-1)^{a_i} \exp\left(-\frac{N}{1+N\tau} \cdot \frac{1}{2} m^a m^b C^i_{ab} + \mathcal{O}(Q_i)\right) Q_i \end{cases}$$

[T.S.'19], **[Cota,Klemm,T.S.'19]**

Together leads to modular properties of $Z_{top.}$!

Moduli space of genus-1 fibration with $N \leq 4$-sections

(Example: genus one fibration w/ 2-sections over $\mathbb{P}^2$)



Puzzle 1:

For $N > 4$ the group $\Gamma_1(N)$ has at least 3 generators!

$\rightarrow$ Let's consider $N = 5$ to see how it works.

But how do we obtain genus-one fibrations with 5-sections?

Normal forms for genus-one curves

- **N=1:** Weierstraß form in $\mathbb{P}_{123}$

  $y^2 = x^3 + f \cdot xz^4 + g \cdot z^6$

  $$\vdots$$

- **N=4:** Intersection of two quadrics in $\mathbb{P}^3$

$$\begin{aligned} x^2 + c_1 xy + c_2 xz + c_3 xw + \cdots + c_9 w^2 &= 0 \\ c_{10} x^2 + c_{11} xy + c_{12} xz + c_{13} xw + \cdots + c_{19} w^2 &= 0 \end{aligned}$$

Note: All are complete intersections in toric varieties

What about $N = 5$?

Normal forms for genus-one curves w/ 5-sections

Can be mapped into **Pfaffian curve** in $\mathbb{P}^4$

- Consider anti-symmetric 5×5 matrix A with entries linear in $[x_1 : \cdots : x_5] \in \mathbb{P}^4$
- $\{\mathrm{rk}(A) \leq 2\} \subset \mathbb{P}^4$ is genus-one curve with "5-point"

  $\rightarrow$ *Vanishing locus of* 4×4 *Pfaffians*

Dual description as **complete intersection in Grassmanian!**

Plücker embedding $Gr(2,5) \rightarrow \mathbb{P}^9$ *also defined by* 4×4 *Pfaffians*

$\rightarrow$ Can also represent curve as codim 5 CI in $\mathrm{Gr}(2,5)$

see e.g. **[Fisher'06]**

Example of homological projective duality! **[Kuznetsov'05]**

1-fold analogon of Rødland Calabi-Yau **[Rødland'98]**

GLSM for Pfaffian curve constructed in **[Hori,Knapp'14]**

Gauge group $G = U(2)$, fields $p_{1,\ldots,5}$ in $\det^{-1}$, $x_{1,\ldots,5}$ in $\square$

Periods via localization using

[Benini,Cremonesi'12],**[Doroud,Gomis,Le Floch,Lee'12]**

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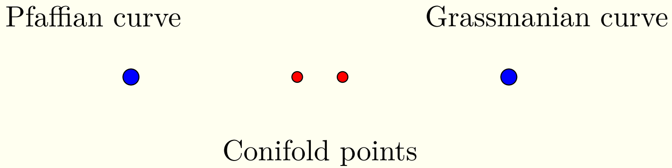
Moduli space:



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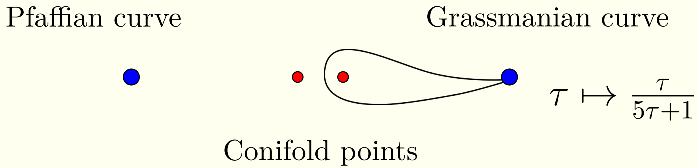

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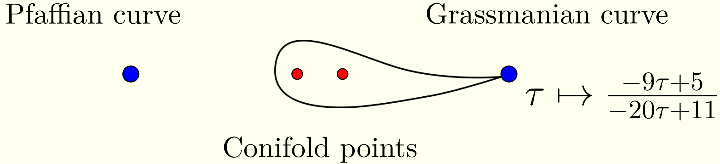

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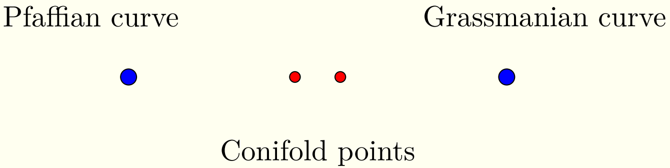

Together monodromies generate $\Gamma_1(5)$*!*

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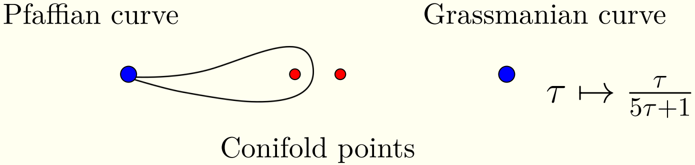

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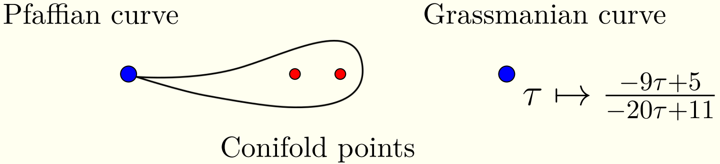


$$\tau \mapsto \frac{-9\tau+5}{-20\tau+11}$$

Together monodromies generate $\Gamma_1(5)$*!*

Let's move on to CY 3-folds!

The monodromies of fiber embed into moduli space of fibration.
→ *Can try to apply the modular bootstrap!*

But how do we get the such 3-folds?

Strategy: Promote coefficients to sections of bundles on base and construct associated **non-Abelian GLSM**, use **localization** to obtain quantum periods *(loc. w/ techniques from* **[Gerhardus,Jockers'15]***)*

But which bundles lead to smooth fibrations?

\+ **Puzzle 2:**

What do the two geometric phases mean in context of fibrations?

30 second introduction to F-theory

F-theory on CY 3-fold $\leftrightarrow$ 6d supergravity

Fibral divisors

$k-1$ independent N-sections

$$G = \frac{G_{\text{non-Abelian}}}{\mathbb{Z}_n} \times U(1)^k \times \mathbb{Z}_N$$

n-torsional sections

No section but only N-sections *or torsion 3-forms*

How do Higgs transitions work in M- and F-theory?

$U(1) \to \mathbb{Z}_N$ relates elliptic and genus-one fibrations!

Extremal transition from elliptic to genus-one fibration

see e.g. **[Morrison'99]**, **[Morrison,Braun'14]**, **[Morrison,Taylor'14]**
[Anderson,Garcia-Extebarria,Grimm,Keitel'14],
[Klevers,Mayorga Pena,Oehlmann,Piragua,Reuter'14],
[Mayrhofer,Palti,Till,Weigand'14],
[Cvetic,Donagi,Klevers,Piragua,Poretschkin'15],
[Oehlmann,T.S.'19]

We engineer elliptic fibers as CICY in Toric varieties

- Use GV-Spectroscopy to obtain generic F-theory spectrum **[Oehlmann,T.S.19]**

  → *Base-independent multiplicities of hypermultiplets*

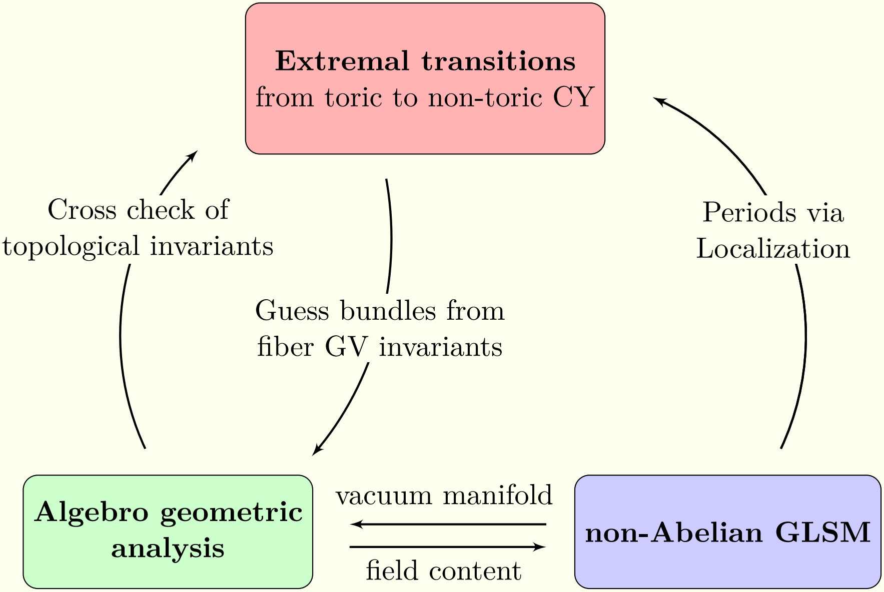
  → *F-theory allows Higgs transition to* $\mathbb{Z}_5$

- Can construct all smooth fibrations over e.g. $\mathbb{P}^2$

  → **Get Picard-Fuchs systems for 5-sect. fibrations**

  (all topological invariants can be deduced from GV-inv.)

Bonus: First F-theory vacua w/ charge 5 matter


Extremal transitions
from toric to non-toric CY
Cross check of
topological invariants
Guess bundles from
fiber GV invariants
Periods via
Localization
Algebro geometric
analysis
vacuum manifold
field content
non-Abelian GLSM

Let’s come back to

The Puzzle of the 2 Geometric Phases

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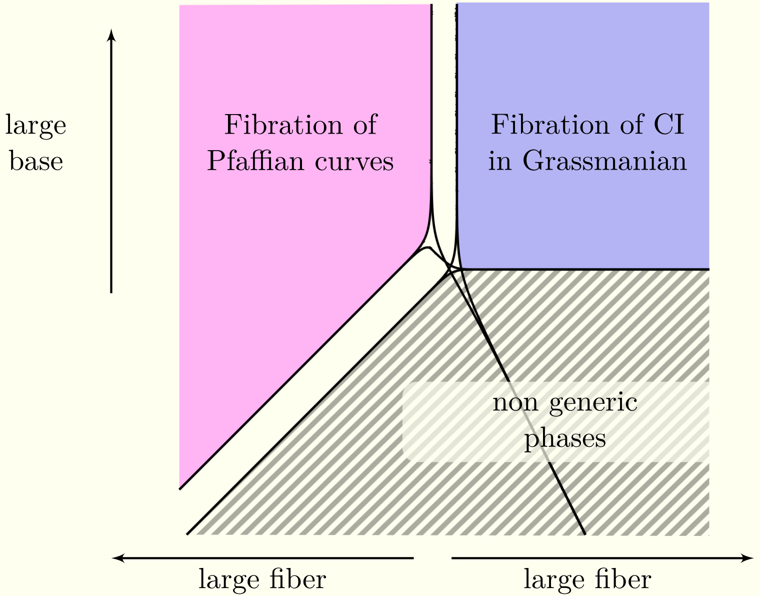
(Example: genus one fibration w/ 5-sections over $\mathbb{P}^2$)



Interlude:

A genus one fibration X has associated elliptic fibration
the **Jacobian fibration** $J(X)$.

F-theory only depends on $J(X)$, M-theory differs on X and $J(X)$!

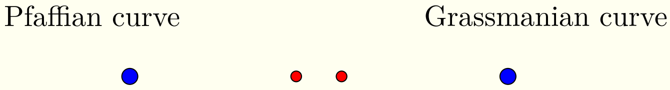
Genus one fibrations with same Jacobian fibration form
Tate-Shafarevich group Ш

Elements are Geometry X + Action of Jacobian fibration on X
((X, a) and (X, a^{-1}) in general different in Ш(X))

Pfaffian/Grassmanian phase

→ different elements of TS group!

Moreover, $Z_{\text{top.}}$ are related by $\Gamma_0(5)$ transformation



Conjecture: $Z_{\text{top.}}$ on *all* elements of TS group *for all* N are related by modular transformations. *(actually a bit more subtle)*

(soon published [**Knapp,Scheidegger,T.S.'21**], [**T.S.'21**])

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Pfaffian curve Grassmanian curve

$$\tau \mapsto \frac{2\tau-1}{5\tau-2}$$

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Summary

- Detailed study of T^2 fibered CY with 5-sections
  23 examples from extr. transitions + non-Abelian GLSMs

  published soon [Knapp,Scheidegger,T.S.'21]

- Pfaffian and Grassmanian phases
  derived equivalent and share Jacobian
  $\rightarrow$ *get all 4 non-elliptic geometries in $\mathbb{Z}_5$ TS-group*

  Related by relative homological projective duality!

- $Z_{\text{top.}}$ expressible in $\Gamma_1(5)$ Jacobi forms
  \+ related via $\Gamma_0(5)$ action!

  $\rightarrow$ *consequence of Higgs transition in M-/F-theory*

More general story ***published soon*** **[T.S.'21]**

Thank you for your attention!