# Gravitational Positivity Bounds and the Standard Model

Toshifumi Noumi (Kobe U)


based on arXiv:2104.09682

w/Katsuki Aoki (YITP), Tran Quang Loc (Cambridge), Junsei Tokuda (Kobe U)

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Outline

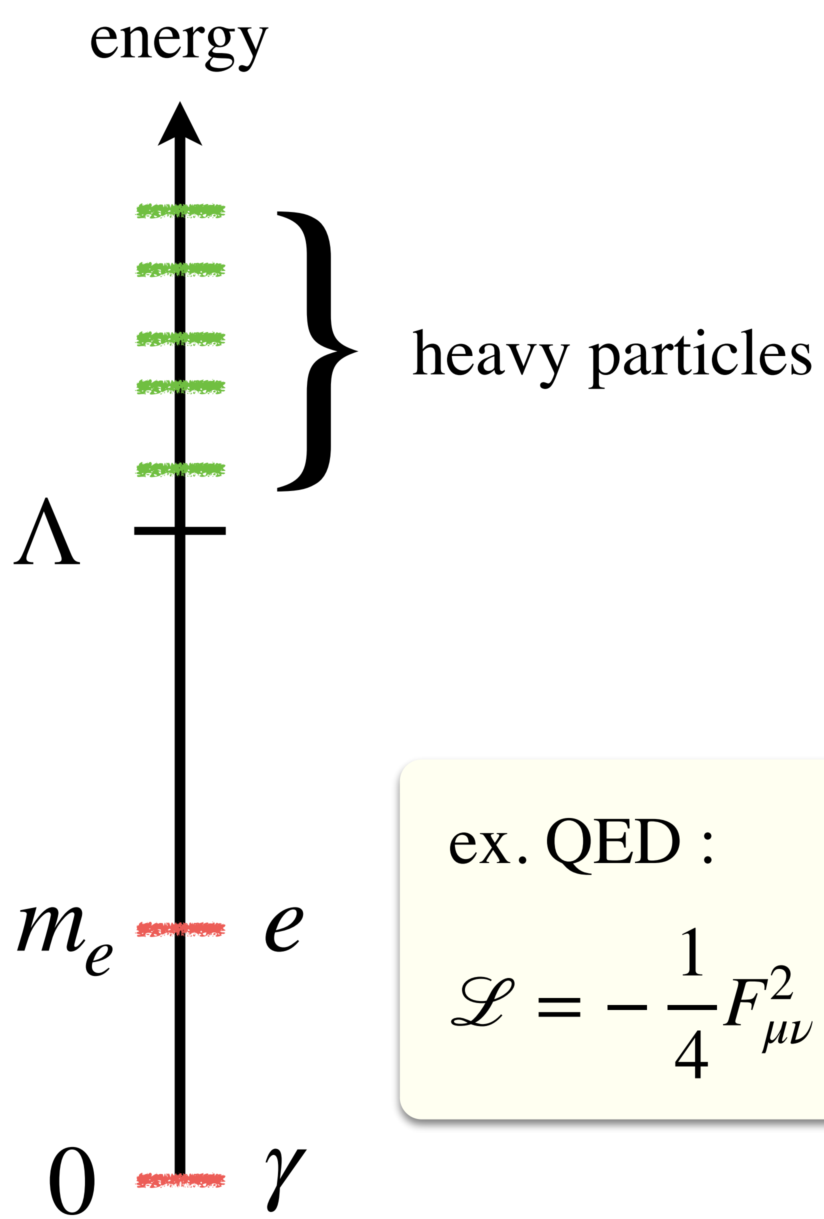
1. Positivity bounds on low-energy scattering amplitudes provide
   a criterion for a low-energy EFT to be UV completable in the standard manner
 - provides a Swampland condition when applied to gravitational EFTs

2. Puzzles on positivity in gravitational QED [Alberte-de Rham-Jaitly-Tolley '20]
 - implies a cutoff scale $\Lambda \sim 10^8$ GeV (too low to believe???)
 - implies that massless QED $m_e \to 0$ is in the Swampland (sounds strange???)

3. Positivity in gravitational Standard Model [Aoki-Loc-TN-Tokuda '21]
 - the cutoff scale is improved up to $\Lambda \sim 10^{16}$ GeV
 - massless limit $m_e \to 0$ is allowed if we take $m_W \to 0$ simultaneously

# 1. Positivity Bounds

# EFT and UV completion




ex. QED :

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 - \bar{\psi}(\not{D} + m_e)\psi + \text{higher dim. operators}$$

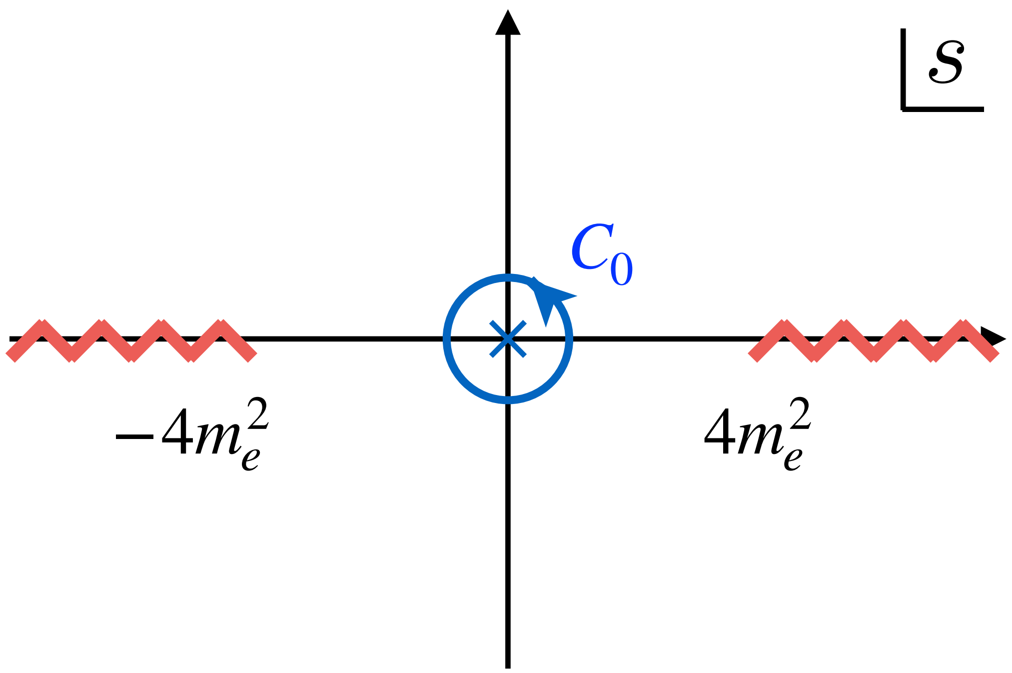
Q. What are the consistency conditions for an EFT to be UV completable?

Q. Where is the cutoff scale $\Lambda$ where the UV completion has to happen?

# Positivity Bounds (w/o gravity) [Adams et al ’06]

Consider an s-u crossing helicity sum of $\gamma\gamma \to \gamma\gamma$ scattering in the forward limit:

$$\mathcal{M} = \mathcal{M}_{++++} + \mathcal{M}_{----} + \mathcal{M}_{+-+-} + \mathcal{M}_{-+-+}$$




analytic structure of $\mathcal{M}(s, t = 0)$

IR behavior: $\mathcal{M}(s, t = 0) = \frac{c_2}{2} s^2 + \mathcal{O}\left(s^4\right)$

$$\frac{c_2}{2} = \oint_{C_0} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t = 0)}{s^3}$$

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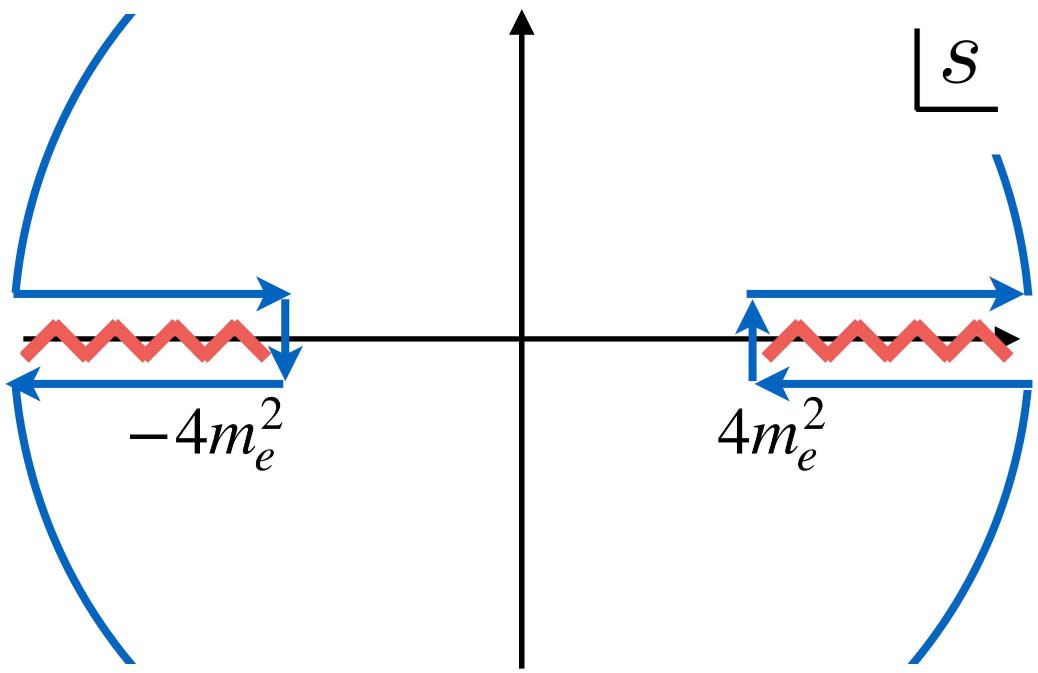


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$$\frac{c_2}{2} = \oint_{C_0} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t = 0)}{s^3}$$

Deform the integration contour to rewrite it in the UV language:

$$\frac{c_2}{2} = \frac{2}{\pi} \int_{4m_e^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}(s, t = 0)}{s^3} + \oint_{C_\infty} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t = 0)}{s^3}$$

※ used the s-u symmetry and Disc $\mathcal{M}(s, t = 0) = 2i\, \mathrm{Im}\, \mathcal{M}(s, t = 0)$

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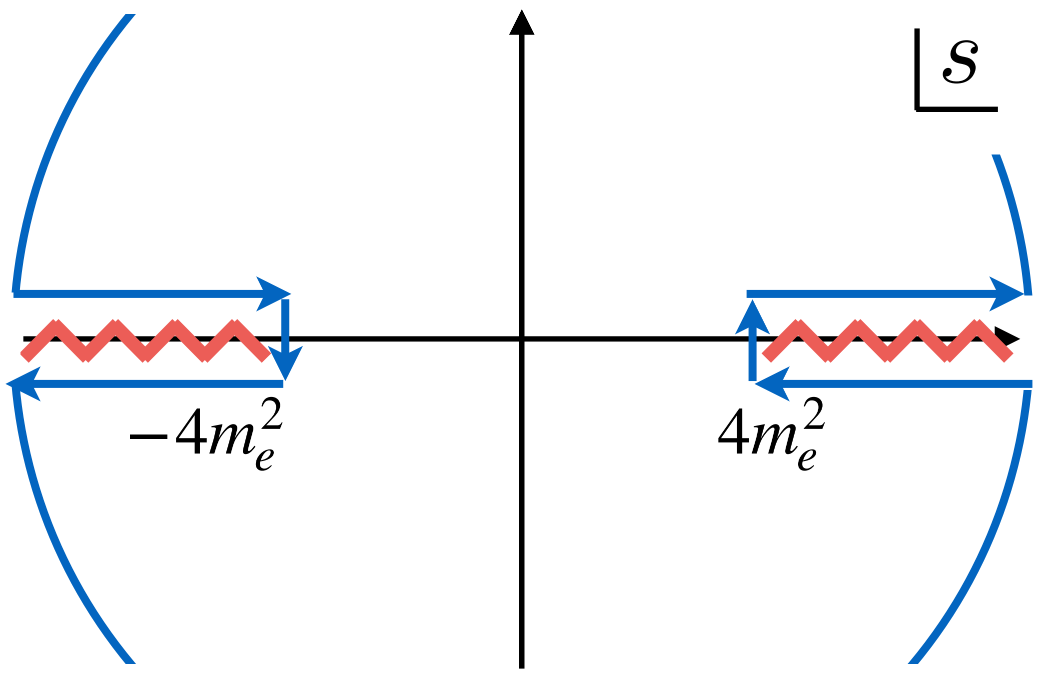


analytic s

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$$\frac{c_2}{2} = \oint_{C_0} \frac{ds}{2\pi i} \frac{\mathcal{M}(s, t=0)}{s^3}$$

Positive because of unitarity!

Deform the integration contour to rewrite it in the UV language:

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※ used the s-u symmetry and Disc $\mathcal{M}(s, t=0) = 2i\,\mathrm{Im}\,\mathcal{M}(s, t=0)$

※ assumed $|\mathcal{M}(s, t=0)| < |s|^2$ $(|s| \to \infty)$ (cf. Froissart bound)

# Positivity Bounds

# To summarize, unitarity and analyticity imply the positivity bound:

$$c_2 = \frac{4}{\pi}\int_{4m_e^2}^{\infty} ds \frac{\mathrm{Im}\mathscr{M}(s, t=0)}{s^3} > 0, \text{ where } \mathscr{M}(s, t=0) = \frac{c_2}{2}s^2 + \mathcal{O}\left(s^4\right)$$

# It is convenient to rewrite it as [Bellazzini '16, de Rham-Melville-Tolley-Zhou '17, …]

$$B(\Lambda) := c_2 - \frac{4}{\pi}\int_{4m_e^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathscr{M}(s, t=0)}{s^3} = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathscr{M}(s, t=0)}{s^3} > 0$$

- $B(\Lambda)$ is calculable within the EFT
- $B(\Lambda)$ monotonically decreases as $\Lambda$ increases

# Gravitational positivity bounds [Hamada-TN-Shiu '18, Tokuda-Aoki-Hirano '20, …]

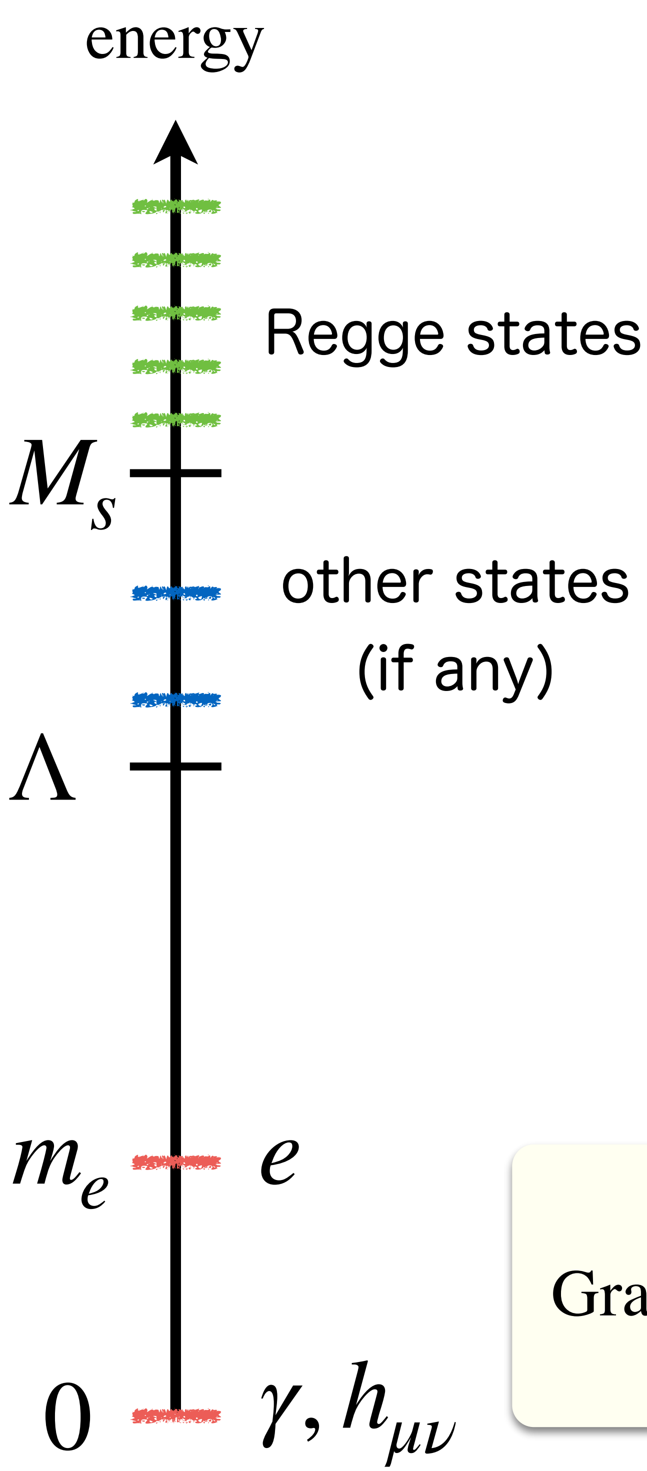
If we assume weakly coupled UV completion of gravity and a "single scaling" of Regge amplitudes at high energy, one can derive an approximate positivity

$$B(\Lambda) > -\mathcal{O}(1) \cdot \frac{1}{M_{\mathrm{Pl}}^2 M_s^2} \quad (M_s : \text{mass of higher spin Regge states})$$

# 2. Positivity in Gravitational QED

[Alberte-de Rham-Jaitly-Tolley ’20, see also Aoki-Loc-TN-Tokuda ’21]

# Gravitational QED as an EFT




UV completable?

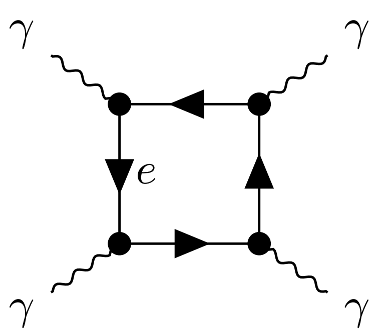
Where is the cutoff?

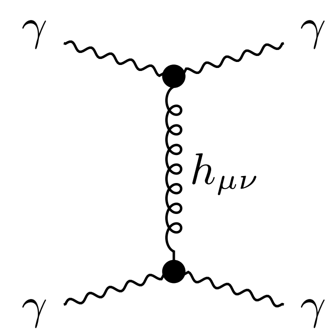
Gravitational QED: $\mathscr{L} = \frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}^2 - \bar{\psi}(\not{D} + m_e)\psi + \cdots$

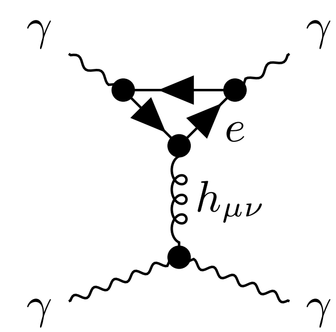
# Decomposition of scattering amplitudes

gravitational positivity: $B(\Lambda) := c_2 - \frac{4}{\pi}\int_{4m_e^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

- Decompose the $\gamma\gamma \to \gamma\gamma$ amplitude at IR as $\mathcal{M} = \mathcal{M}_{\mathrm{QED}} + \mathcal{M}_{\mathrm{GR}} + \mathcal{M}_{\mathrm{UV}}$

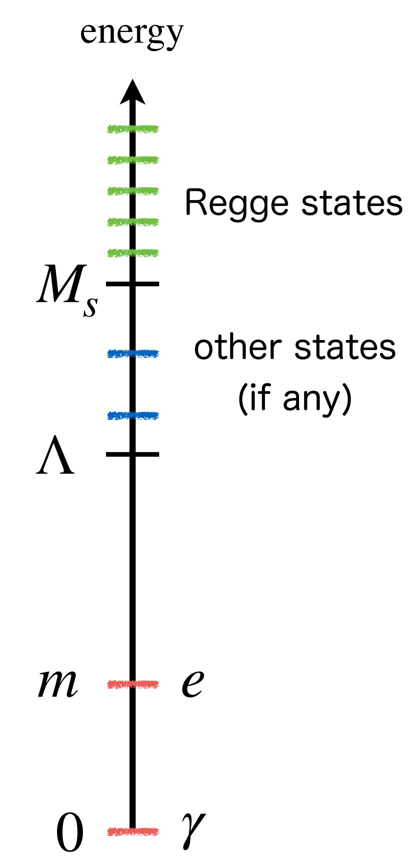
$\mathcal{M}_{\mathrm{QED}} =$ 

, $\mathcal{M}_{\mathrm{GR}} =$ 

$+$ 


$\mathcal{M}_{\mathrm{UV}}$ : effects of UV dof $\lesssim \frac{1}{\Lambda^4}$

- We perform similar decompositions, e.g., as $B = B_{\mathrm{QED}} + B_{\mathrm{GR}} + B_{\mathrm{UV}}$



# Evaluation of $B$'s

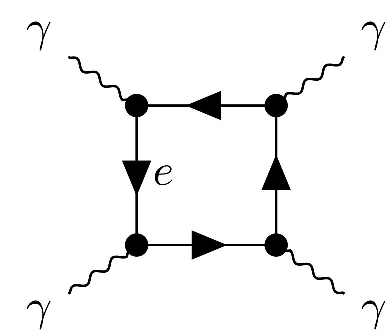
gravitational positivity: $B(\Lambda) := c_2 - \frac{4}{\pi}\int_{4m_e^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

- evaluation of $B_{\mathrm{QED}}$

Technically, it is convenient to remind $|\mathcal{M}_{\mathrm{QED}}(s,0)| < s^2$,

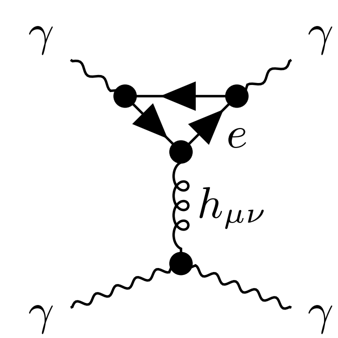
so that $c_{2,\mathrm{QED}} = \frac{4}{\pi}\int_{4m^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}_{\mathrm{QED}}(s,0)}{s^3}$ (cf. positivity in non-gravitational QED)

This implies $B_{\mathrm{QED}}(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}_{\mathrm{QED}}(s,0)}{s^3} = \frac{4e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m_e} - \frac{1}{4}\right)$.




※ Notice in particular that $\lim_{\Lambda\to\infty} B_{\mathrm{QED}}(\Lambda) = 0$.

- A straightforward computation shows $B_{\mathrm{GR}}(\Lambda) = -\frac{11e^2}{90\pi^2 m_e^2 M_{\mathrm{Pl}}^2}$




This gives a negative contribution that survives even in the limit $\Lambda \to \infty$.

# Cutoff scale of gravitational QED

gravitational positivity: $B(\Lambda) := c_2 - \frac{4}{\pi}\int_{4m_e^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathscr{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

Now the gravitational positivity bound reads

$$\frac{4e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m_e}-\frac{1}{4}\right)-\frac{11e^2}{90\pi^2 m_e^2 M_{\mathrm{Pl}}^2}+\frac{\alpha_{\mathrm{UV}}}{\Lambda^4} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2} \qquad (|\alpha_{\mathrm{UV}}| \lesssim 1)$$

Since $m_e \ll \Lambda \lesssim M_s$, we find $\frac{64\alpha^2}{\Lambda^4}\left(\ln\frac{\Lambda}{m_e}-\frac{1}{4}\right)+\frac{\alpha_{\mathrm{UV}}}{\Lambda^4} > \frac{22\alpha}{45\pi m_e^2 M_{\mathrm{Pl}}^2}$,

which gives an upper bound on the cutoff scale:

$$\Lambda \lesssim \min\left[\sqrt{e m_e M_{\mathrm{Pl}}},\, |\alpha_{\mathrm{UV}}|^{-1/4}\sqrt{m_e M_{\mathrm{Pl}}/e}\right] \sim 10^8\ \mathrm{GeV}.$$

↑

for QED parameters in our real world

Summary so far

- standard assumptions of positivity + the single scaling assumption implies

an approximate positivity bound $B(\Lambda) := c_2 - \frac{4}{\pi}\int_{4m_e^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1) \cdot \frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$.

- when applied to gravitational QED, this implies a cutoff $\Lambda \lesssim \sqrt{m_e M_{\mathrm{Pl}}/e} \sim 10^8$ GeV.

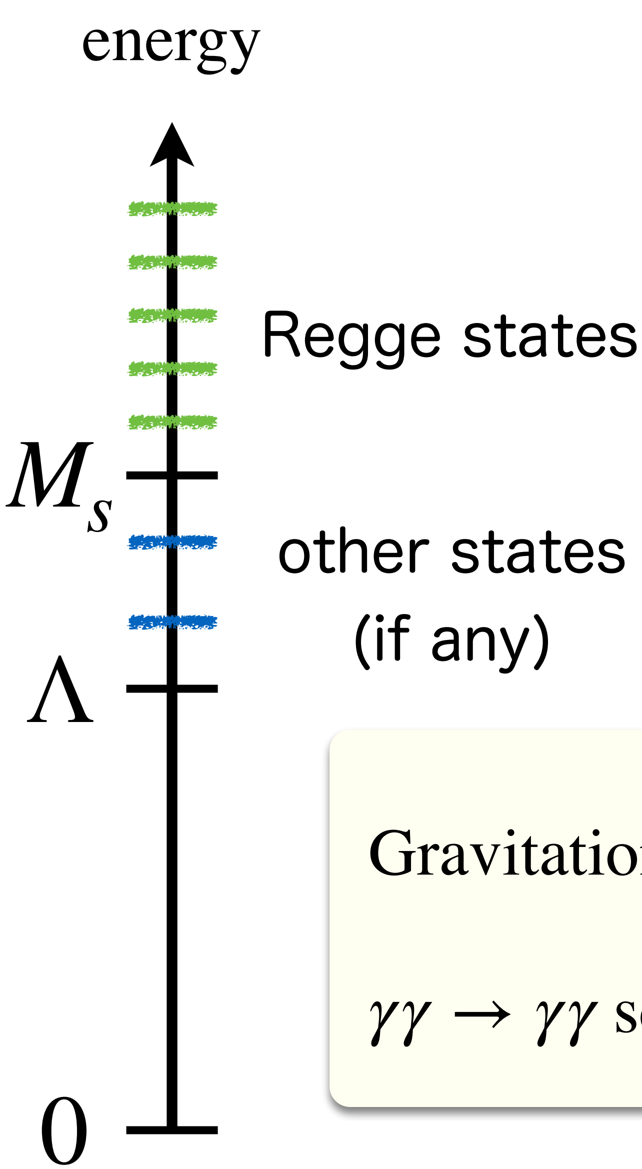
too small to believe the bound??? massless limit is not allowed???

→ we extended the analysis to the Standard Model

# 3. Positivity in Gravitational Standard Model

[Aoki-Loc-TN-Tokuda ’21]

# Gravitational Standard Model




Gravitational Standard Model: $\mathscr{L} = \mathscr{L}_{\rm SM} + \frac{M_{\rm Pl}^2}{2} R + \cdots$

$\gamma\gamma \to \gamma\gamma$ scattering: $\mathscr{M} = \mathscr{M}_{\rm QED} + \mathscr{M}_{\rm weak} + \mathscr{M}_{\rm QCD} + \mathscr{M}_{\rm GR} + \mathscr{M}_{\rm UV}$

What to do is the same as the QED case except for

(A) there exist charged spin 1 particles (W bosons)

(B) hadrons may contribute if some of $s, t, u$ is below the QCD scale

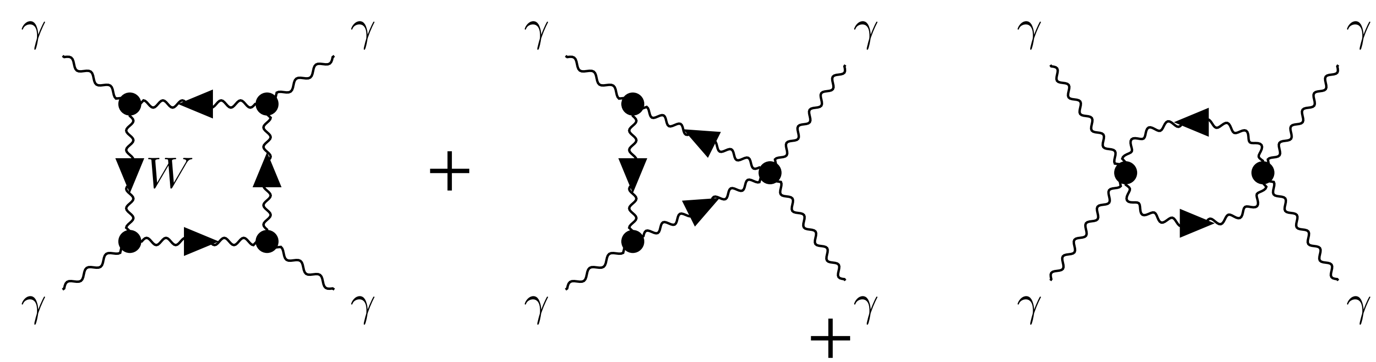
# Weak sector analysis

gravitational positivity: $B(\Lambda) := c_2 - \frac{4}{\pi}\int_{4m_e^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\rm Pl}^2 M_s^2}$

- just like the QED case, we have $B_{\rm weak}(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}_{\rm weak}(s,0)}{s^3}$.

- due to the spin 1 nature, W boson contributions grow faster than the QED case

$\mathcal{M}_{\rm weak} \simeq$




$$\simeq \frac{2e^4}{\pi^2 m_W^2} s \ln\frac{m_W^2}{-s} + (s \leftrightarrow -s) \qquad \text{cf. } \mathcal{M}_{\rm QED} \sim \ln^2 s$$

- we then find $B_{\rm weak}(\Lambda) = \frac{8e^4}{\pi^2 m_W^2 \Lambda^2} > B_{\rm QED}(\Lambda) = \frac{4e^4}{\pi^2\Lambda^4}\left(\ln\frac{\Lambda}{m} - \frac{1}{4}\right)$

- on the other hand, weak boson loops are sub-dominant in $B_{\rm GR}$

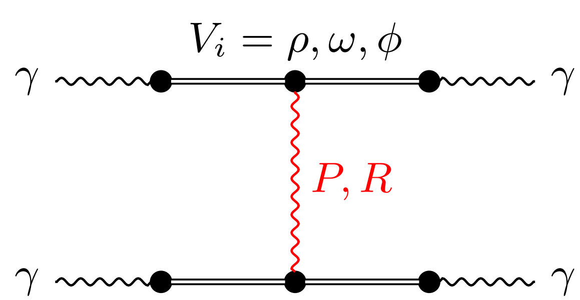
# QCD sector analysis

gravitational positivity: $B(\Lambda) := c_2 - \frac{4}{\pi}\int_{4m_e^2}^{\Lambda^2} ds \frac{\mathrm{Im}\mathcal{M}(s,0)}{s^3} > -\mathcal{O}(1)\cdot\frac{1}{M_{\mathrm{Pl}}^2 M_s^2}$

- again, we have $B_{\mathrm{QCD}}(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\mathcal{M}_{\mathrm{QCD}}(s,0)}{s^3}$.

- while the amplitude on the r.h.s. is high-energy, the momentum transfer is small

→ t-channel exchange of hadrons is relevant

$\mathrm{Im}\mathcal{M}_{\mathrm{QCD}} \simeq \mathrm{Im}$ 

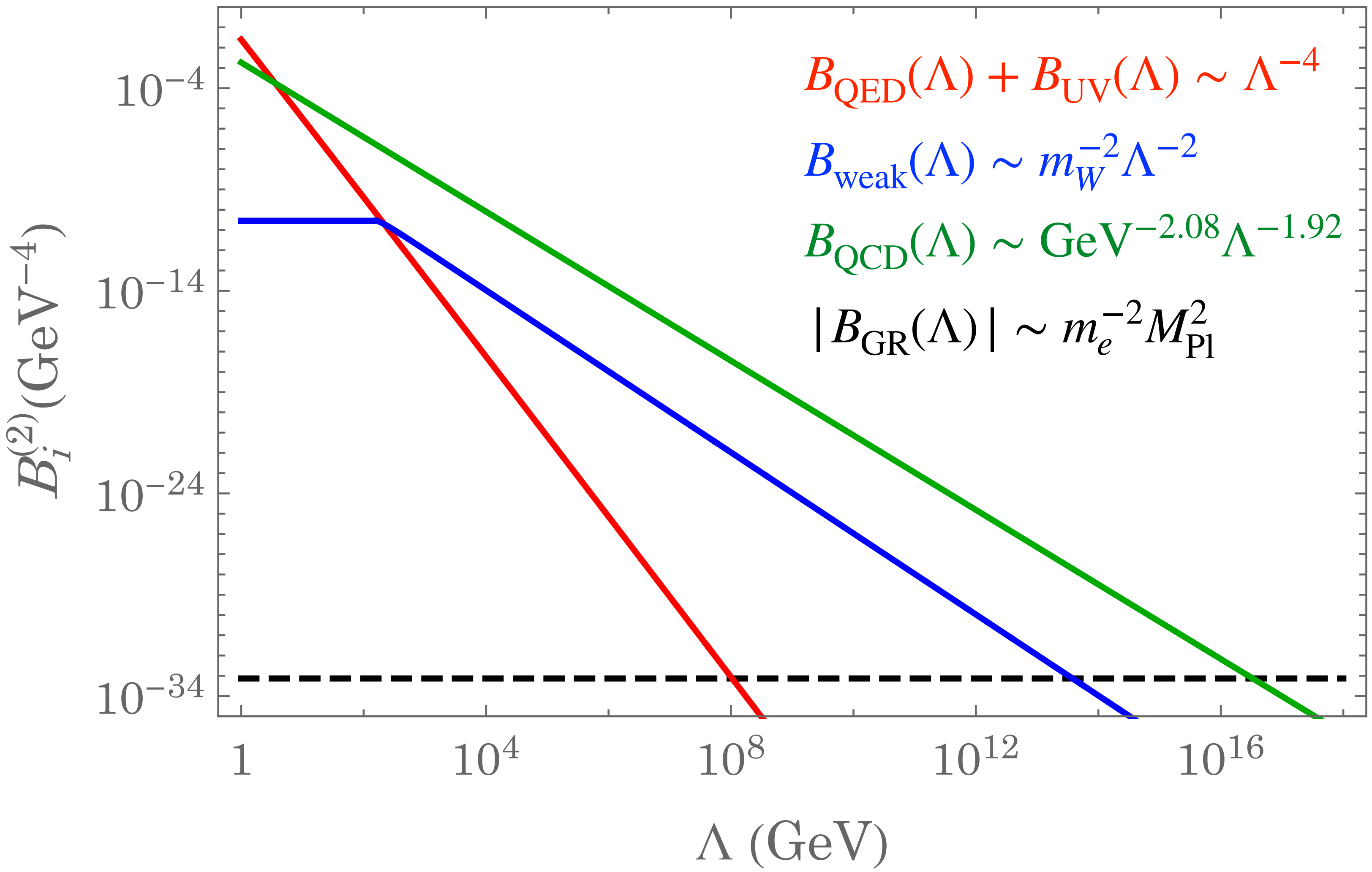
(P: Pomeroon, R: Reggeon)

- employing the Vector Meson Dominance (VDM) model,

$$\mathrm{Im}\mathcal{M}_{\mathrm{QCD}} \simeq \frac{25e^4}{16\pi^2}\left(\frac{s}{\mathrm{GeV}^2}\right)^{1.08}$$

(See our paper for model-(in)sensitivity)

# Cutoff scale of gravitational SM

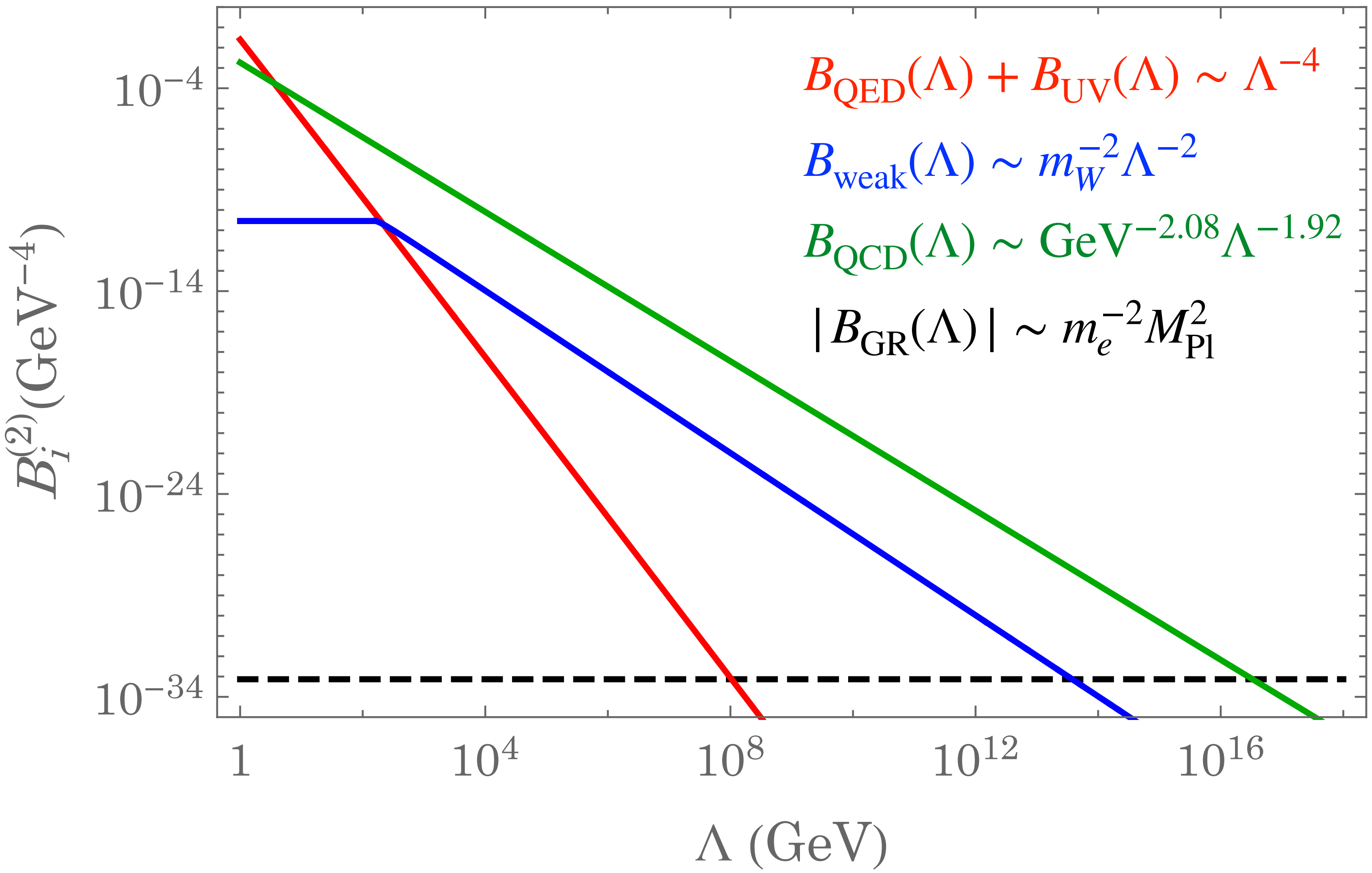



gravitational positivity:

$$B_{\rm QED}(\Lambda) + B_{\rm UV}(\Lambda) + B_{\rm weak}(\Lambda) + B_{\rm QCD}(\Lambda) > -B_{\rm GR}(\Lambda) - \mathcal{O}(1)\cdot\frac{1}{M_{\rm Pl}^2 M_s^2}$$

→ this defines the cutoff of the gravitational SM $\Lambda \simeq 3\times 10^{16}$ GeV.

# A remark on EW theory w/o QCD




gravitational positivity implies:

$$B_{\rm weak}(\Lambda) > -B_{\rm GR}(\Lambda) \quad \rightleftarrows \quad \frac{m_W}{M_{\rm Pl}} < \sqrt{\frac{720}{11}}\, e\, \frac{m_e}{\Lambda}$$

- Possible explanation for the hierarchy between the EW scale and the Planck scale??
- Massless limit $m_e \to 0$ is allowed if we take the limit $m_W \to 0$ simultaneously

# Summary and prospects

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   a criterion for a low-energy EFT to be UV completable in the standard manner

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## Future directions

- How generic the single scaling assumption is? → detailed study of string amplitudes
- connections to other principles such as energy conditions, entropy bounds?
- phenomenological applications

  e.g., bounds on scalar potentials [TN-Tokuda '21], dark matters, neutrinos, …
- possible implications for Higgs mechanism in string theory (brane recombination)?

Thank you!