Quantum curves for 6d $\mathcal{N} = (1,0)$ theories

Marcus Sperling

Yau Mathematical Sciences Center, Tsinghua University

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Tsinghua University

丘成桐数学科学中心
YAU MATHEMATICAL SCIENCES CENTER

Based on: `arXiv:2008.05155` with J. Chen, B. Haghighat, H-C. Kim

`arXiv:2103.16996` with J. Chen, B. Haghighat, H-C. Kim, X. Wang

`work in progress` with J. Chen, B. Haghighat, H-C. Kim, K. Lee, X. Wang

What is a quantum curve?

- Classical **algebraic curve** with variables $x, p \in \mathbb{K}$ for $\mathbb{K} = \mathbb{C}$ or $\mathbb{C}^\star$

$$\mathcal{C} = \left\{(x,p) \in \mathbb{K} \times \mathbb{K} \;\middle|\; H(p,x) = 0\right\} , \qquad \lambda = p \; \mathrm{d}x$$

- Quantise by defining the 2-form

$$\omega = \mathrm{d}\lambda = \mathrm{d}p \wedge \mathrm{d}x \qquad \Rightarrow \qquad [\hat{p}, \hat{x}] \sim \hbar$$

- Defining algebraic condition becomes the **Schrödinger type equation**

$$\widehat{H}(\hat{p}, \hat{x}; \hbar)\Psi = 0$$

Questions: Suppose H is a Seiberg-Witten curve with differential λ.

- How to naturally quantise H?
- Interpretation of wave function Ψ?

Example for 6d SW-curve

- **F-theory**: **non-compact elliptic CY3** M

$$\underset{2}{\mathfrak{su}(k)} \;\text{———}\; \underset{2}{\mathfrak{su}(k)} \;\text{———}\; \ldots \;\text{———}\; \underset{2}{\mathfrak{su}(k)}$$

  i.e. linear chain of -2 curves with fibre of Kodaira type I_k

- **M-theory**: N M5 branes (along $\mathbb{T}^2 \times \mathbb{R}^4$) transverse to $\mathbb{C}^2/\mathbb{Z}_k$ singularity
- **mirror CY3** W = cone over Riemann surface [Haghighat, Yan, Yau '17]

$$H_{N,k}(w,x) = t^N + v_1(x)t^{N-1} + \ldots + v_l(x)t^{N-l} + \ldots + v_N(x) = 0\,, \quad t = e^{2\pi i w}$$

  with $v_l(x)$ Jacobi forms in elliptic parameter x

$$\mathcal{C} = \left\{ (x,w) \in \mathbb{T}^2 \times \mathbb{C} \;\middle|\; H_{N,k}(w,x) = 0 \right\}$$

Quantisation and defect partition function

Quantisation in w and z amounts to

$$[\hat{w}, \hat{x}] \sim \hbar\,, \qquad Y = e^{-\hat{w}} \quad \text{a shift operator}$$

results in a **elliptic quantum curve**

$$\hat{H}_{N,k}(\hat{w}, \hat{x})\Psi = 0$$

Intuition

- 4d SW-curve = moduli space of vacua of **codim 2 defect**
- quantisation of phase space $\rightarrow$ Ψ is partition function of codim 2 defect
- 6d SCFT: codim 2 defect extended along $\mathbb{T}^2 \times \mathbb{R}^2$

Geometrically

- Insertion of special Lagrangian in mirror CY which intersects curve at a point

Outline

1. Class $\mathcal{S}_k$

2. E-string

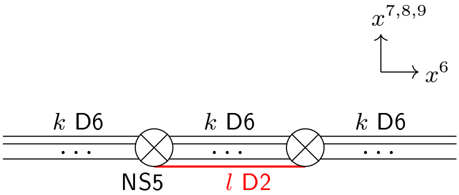
3. Summary and Outlook

Outline

Setup

- 2 M5 branes (along $\mathbb{T}^2 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$) transverse to $\mathbb{C}^2/\mathbb{Z}_k$
- partition function $Z^{6d} = Z_{\text{pert}} \cdot Z_{\text{str}}$ with $Z_{\text{str}} = \sum_{l=0}^{\infty} q_\phi^l \, Z_l$

Dual Type IIA realisation

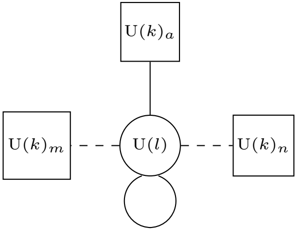


$\mathrm{SU}(k)$ gauge theory, $N_f = 2k$ flavours

+ 1 tensor multiplet

Elliptic genera Z_l via 2d $\mathcal{N} = (0, 4)$ theory

[Haghighat, Kozcaz, Lockhart, Vafa '13; Kim, Kim, Lee '15]

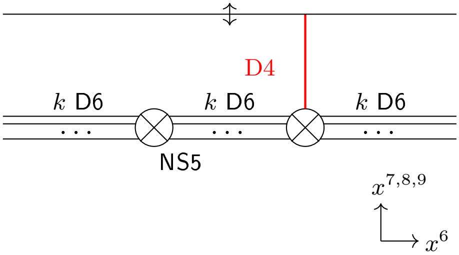


Codimension 2 defect

Higgsing to codim 2 defect

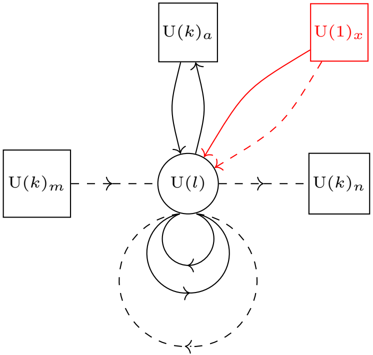
$\mathrm{SU}(k+1) \to \mathrm{SU}(k)$ + defect

[Gaiotto, Rastelli, Rasamat '12; Gaiotto, H.-C. Kim '14; ...]



additional D4 brane along $\mathbb{R}^2_{\epsilon_1} \subset \mathbb{R}^4_{\epsilon_1,\epsilon_2}$

Partition function is **modified by defect**



Elliptic genera Z_l via JK-residue prescription

[Benini, Eager, Hori, Tachikawa '13]

Quantisation

Expectation:

- codim 2 defect partition fct. satisfies **difference eq.** in Nekrasov Shatashvili limit

in NS-limit $$\widetilde{Z}^{\text{def}} := \lim_{\epsilon_2 \to 0} \frac{Z^{6d+\text{def}}}{Z^{6d}}$$ and $\epsilon_1 \sim \hbar$

Strategy:

- Quantisation: introduce shift operator Y via

$$\begin{cases} \text{defect parameter } X = e^{x} \\ \text{conjugate coord. } Y = e^{-w} \end{cases} \quad \text{with} \quad [\hat{w}, \hat{x}] \sim \hbar \quad \Rightarrow \quad Y \cdot X = p^{-1} X \cdot Y \; , p = e^{\hbar}$$

- Evaluate shift Y on defect partition function → **path integral** representation
- NS limit on defect partition function → saddle point eq. = difference eq.

Path integral representation

Derive **path-integral representation** for elliptic genera

$$Z_{\text{str}}^{\text{def}} = \sum_{l=0}^{\infty} \frac{1}{l!} q_\phi^l \oint \left(\prod_{p=1}^{l} \frac{\mathrm{d}u_p}{2\pi \mathrm{i}} \right) \left(\frac{2\pi\eta^3 \theta_1(\epsilon_1+\epsilon_2)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^l \prod_{\substack{p,q=1\\ p\neq q}}^{l} D(u_p - u_q) \prod_{p=1}^{l} Q(u_p) \prod_{p=1}^{l} V_{\text{def}}(u_p)$$

$$= \ldots$$

$$= \int \mathcal{D}\rho(u) \int \mathcal{D}\lambda'(u)\ \exp\left[\int \mathrm{d}u\, \mathrm{d}u' (\#)^2 \rho(u) \log(D(u-u'))\rho(u') \right.$$
$$\left. + \int \mathrm{d}u \left(\mathrm{i}\,\lambda'(u)\cdot\rho(u) + \# \left[\rho(u) \log\left(-q_\phi Q(u) V_{\text{def}}(u)\right) - \frac{1}{2\pi\, i} e^{-\mathrm{i}\,(\#)^{-1}\lambda'(u)} \right] \right) \right]$$

then evaluate NS-limit, c.f. [Ferrari, Piatek '12]

Saddle point analysis

- Evaluate **saddle point equation** in NS limit

$$1 + q_\phi M(u_*)\omega(u_*)\omega(u_* + \epsilon_1) = 0$$

- Analysis of pole properties, one equivalently finds

$$\frac{1 + q_\phi M(u-\epsilon_1)\omega(u)\omega(u-\epsilon_1)}{\omega(u)} = P(u) \quad \leftarrow \quad \text{to be determined}$$

- **Shift operator Y** on partition function

$$\begin{cases} Y\widetilde{Z}^{\text{def}}_{\text{pert}} = \frac{1}{P_0(x)}\widetilde{Z}^{\text{def}}_{\text{pert}} & \text{easy calculation} \\ Y\widetilde{Z}^{\text{def}}_{\text{str}} = \omega(x)P_0(x)\widetilde{Z}^{\text{def}}_{\text{str}} & \text{via path integral} \end{cases} \Rightarrow \quad Y\widetilde{Z}^{\text{def}} = \omega(x)\widetilde{Z}^{\text{def}}$$

- **Difference equation**

$$0 = \left[Y^{-1} + q_\phi M(x)\cdot Y - P(x+\epsilon_1)\right]\widetilde{Z}^{\text{def}}(x)$$

Wilson surface – codim 4 defect

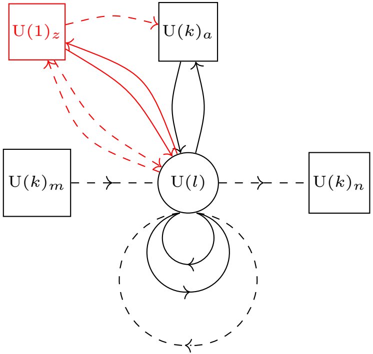
Guess: extra object P is a codim 4 defect

Wilson surface [Ganor '97; Chen, He, Wu, Zhang '07; Nekrasov '15, H-C. Kim '16; Bullimore, H.-C. Kim '14; Agrawal, Kim, Kim, Sciarappa '18]

introduce via another type of D4 brane:

- defect modifies D2 world-volume theory
- but retains $\mathcal{N} = (4,0)$ SUSY

Evaluate normalised codim 4 defect partition function in NS-limit.



Observation via explicit computation: new object $P(x) = \tilde{Z}_{\mathbf{Wilson}}$

Quantised SW-curve

Quantum curve [Chen, Haghighat, Kim, MS '20]

$$\left[Y^{-1} + q_\phi \prod_{l=1}^{2k} \theta(x-\mu_l) \cdot Y - \langle \mathcal{W} \rangle \right] \Psi = 0$$

- $\langle \mathcal{W} \rangle = \widetilde{Z}^{6d+\text{codim4}}$ = Wilson surface expectation value (codim 4 defect)
- $\Psi = \widetilde{Z}^{6d+\text{codim2}}$ = normalised codim 2 defect partition function

Compare to classical curve [Haghighat, Yan, Yau '18]

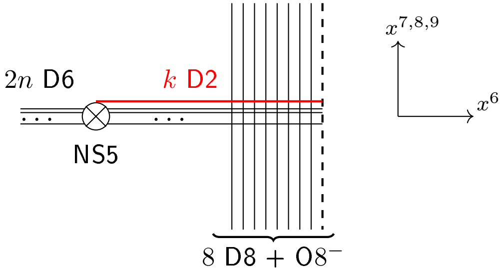
$$t + q_\phi \prod_{l=1}^{2k} \theta(x-\mu_l) t^{-1} - (1+q_\phi) \prod_{l=1}^{k} \theta(x-z_l) = 0 \qquad t = e^{2\pi i w}$$

Outline

Setup

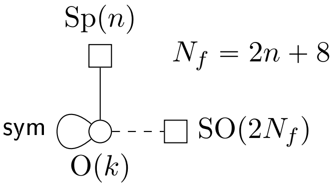
via **M5 near M9 on A-type singularity**

Higgsing $\mathrm{Sp}(1)$, $N_f = 10 \rightarrow$ E-string



Elliptic genera via D2 world-volume theory

[Kim, Kim, Lee, Park, Vafa '14; Kim, Kim, Lee '15]



- add codim 2 and codim 4 defects via additional branes or defect Higgsing
- evaluate defect partition functions in NS limit

Quantum curve

By the same logic as before (but more involved), one can derive the following **elliptic quantum curve** [Chen, Haghighat, Kim, MS, Wang '21]

$$\left[Y + q_\phi^2 \frac{\prod_{l=1}^{8} \theta_1(x \pm \mu_l + \hbar/2)}{\eta^{12}\theta_1(2x)\theta_1(2x+\hbar)^2\theta_1(2x+2\hbar)} Y^{-1}\right] \widetilde{Z}_{\rm str} = \mathcal{W}_\hbar^S(x)\widetilde{Z}_{\rm str}$$

- $\widetilde{Z}_{\rm str}$ = instanton part of normalised **codim 2 defect** partition function

  cf. from 5-brane webs [S.-S. Kim, Sugimoto, Yagi '20]

- $\mathcal{W}_\hbar^S(x)$ = Wilson surface expectation value (codim 4 defect)

Classical curve for $\mathrm{SO}(8) \times \mathrm{SO}(8) \subset E_8$ flavour symmetry [Haghighat, Kim, Yan, Yau '18]

$$\tilde{t} + \left(\frac{\prod_{l=1}^{4} \theta_1(x \pm \mu_l)}{\theta_1(2x)^2}\right)^2 \tilde{t}^{-1} = \left(2\frac{\prod_{l=1}^{4} \theta_1(x \pm \mu_l)}{\theta_1(2x)^2} + c\right)$$

Outline

Summary and Outlook

Defects play crucial role in **quantisation of SW-curve**

- Codim 2 "surface" defect = eigenfunction Ψ
- Codim 4 "Wilson" surface = eigenvalue $\langle \mathcal{W} \rangle$ of shift operator

Future directions

- Quantum curves of 6d SCFT on other $-n$ curves and chains of curves
- Relation to **integrable models**
  - Class S_k: elliptic Ruijsenaars-Schneider integrable system
  - E-string: van Diejen Model   see also [Nazzal, Razamat '18]
- World-volume theory living on the codim 2 defect