

Analysis of a Real Time Approach to Quantum Tunneling

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July 21, 2021

Mark P. Hertzberg, Fabrizio Rompineve, and Neil Shah. “Quantitative Analysis of the Stochastic Approach to Quantum Tunneling”. In: *Phys. Rev. D* 102.7 (2020), p. 076003. DOI: 10.1103/PhysRevD.102.076003. arXiv: 2009.00017 [hep-th]

Overview

① Instanton

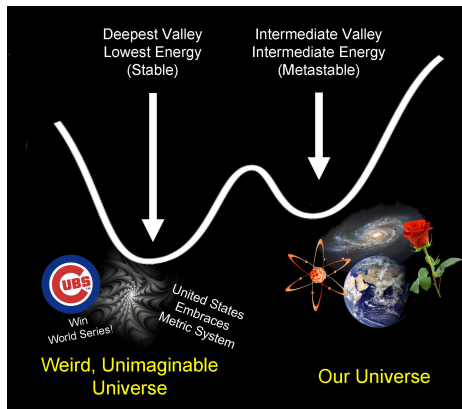
② Stochastic Method

③ Results

④ Conclusions

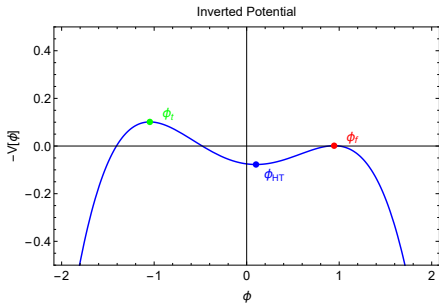
Tunneling

- Quantum tunneling central to physics
 - Higgs meta-stable, turnover at $E \sim \mathcal{O}(10^{11})$ GeV
 - String theory, exponentially many meta-stable vacua
 - Diodes, nuclear fusion
- In single particle QM tunneling from exact solution from Schrodinger Eq.
- In QFT, exact solution is difficult \Rightarrow approximation techniques



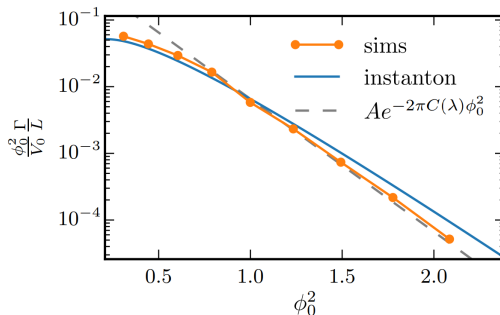
Instanton

- Standard approximation technique: Coleman Instanton [2]
- Involves classical solution in Euclidean spacetime ($t \rightarrow i\tau$)
- Instanton requires an $O(4)$ symmetry, broken in key regimes e.g. inflation
- If we can find real time tunneling method, can examine tunneling in time-dependent backgrounds!



Stochastic Method

- Past work by Linde [4] has shown parametric agreement between stochastic method and instanton
- Recent work by Braden, Johnson, Peiris, Pontzen, and Weinfurtner [1] claims excellent agreement between stochastic method and instanton



Stochastic Method

- Consider a potential $V(\phi)$ with at least two minima ϕ_f, ϕ_t .
- Initialize $\phi = \phi_f + \delta\phi, \pi = 0 + \delta\dot{\phi}$
- Draw $\delta\phi$ and $\delta\dot{\phi}$ from the free theory:

$$\Psi_{\text{free}}(\delta\phi) \propto \exp\left[-\frac{1}{2} \int \frac{d\mathbf{k}}{2\pi} \omega_{\mathbf{k}} |\delta\phi_{\mathbf{k}}|^2\right] \quad (1)$$

where $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + V''(\phi_f) = \mathbf{k}^2 + m_f^2$

- The 2-point correlation functions for $\delta\phi_{\mathbf{k}}$ and $\delta\dot{\phi}_{\mathbf{k}}$ are:

$$\langle \delta\phi_{\mathbf{k}}^* \delta\phi_{\mathbf{k}'} \rangle = \frac{1}{2\omega_{\mathbf{k}}} (2\pi) \delta(\mathbf{k} - \mathbf{k}') \quad \langle \delta\dot{\phi}_{\mathbf{k}}^* \delta\dot{\phi}_{\mathbf{k}'} \rangle = \frac{\omega_{\mathbf{k}}}{2} (2\pi) \delta(\mathbf{k} - \mathbf{k}') \quad (2)$$

- Use the Wigner distribution as a joint probability distribution to sample $\delta\phi_{\mathbf{k}}$ and $\delta\dot{\phi}_{\mathbf{k}}$ *simultaneously* to set initial conditions.
- Place ϕ in box of size L with periodic boundary conditions
 \Rightarrow discrete k -modes $k_n = 2\pi n/L$ with cutoff n_{cut} .

Stochastic Method

- Initial conditions:

$$\delta\phi(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^{n_{\text{cut}}} e^{ik_n x} \phi_{k_n} + c.c. \quad \Delta\phi_{k_n} = \sqrt{\langle |\phi_{k_n}|^2 \rangle} = \epsilon_\phi \frac{1}{\sqrt{2\omega_{k_n}}} \quad (3)$$

$$\delta\dot{\phi}(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^{n_{\text{cut}}} e^{ik_n x} \dot{\phi}_{k_n} + c.c. \quad \Delta\dot{\phi}_{k_n} = \sqrt{\langle |\dot{\phi}_{k_n}|^2 \rangle} = \epsilon_\pi \sqrt{\frac{\omega_{k_n}}{2}} \quad (4)$$

where ϵ_ϕ , ϵ_π are “fudge factors” to control the amplitude of fluctuations

Braden Method

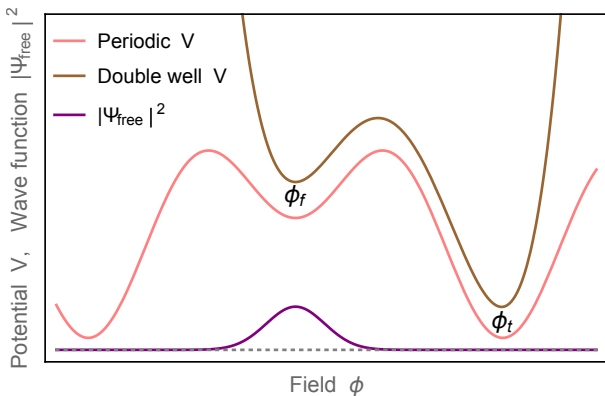
- Prepare an ensemble of classical fields $\{\phi_i\}$ with quantum initial conditions and evolve under the classical equations of motion:

$$\ddot{\phi}_i - \nabla^2 \phi_i + V'(\phi_i) = 0 \quad (5)$$

- Determine tunneling rate by examining timescale over which classical fields “tunnel”:
 - Let's define the following volume average for a field ϕ_i : $c_i(t)$
 - Choose some threshold \mathcal{T} , such that a field ϕ_i has “tunneled” at time t when $c_i(t) > \mathcal{T}$
 - Define the survival rate $F_{\text{survive}}(t) \equiv$ No. of fields that have not tunneled at time t .
 - Then using $F_{\text{survive}}(t) = e^{-\Gamma t}$, extract Γ as the tunneling rate.

Stochastic Method

- We will consider the follow potentials:
- For the periodic potential, $c_i(t) = \frac{1}{L} \int dx \cos(\phi_i(t, x)/\phi_0)$
- For the DW potential, $c_i(t) = \frac{1}{L} \int dx \phi_i(t, x)/\phi_0$



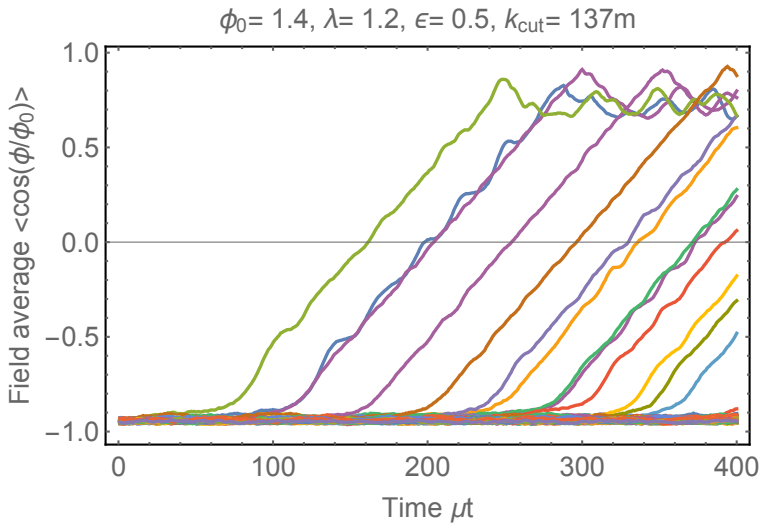
Cosine Potential

- The periodic potential has the form:

$$V(\phi) = V_0 \left(\cos \left(\frac{\phi}{\phi_0} \right) + \frac{\lambda^2}{2} \sin^2 \left(\frac{\phi}{\phi_0} \right) \right) \quad (6)$$

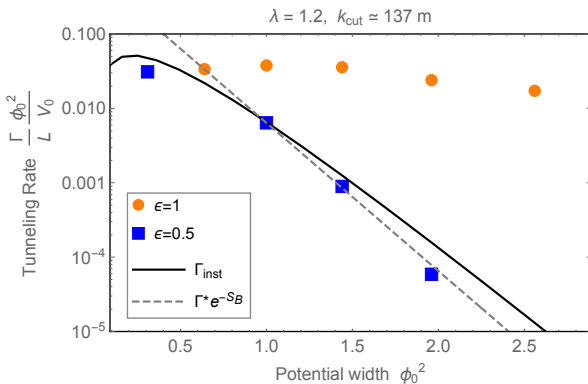
- This potential has infinite true and false vacua at $\phi_f = 2\pi m\phi_0$ and $\phi_\pi = m\pi\phi_0$ where $m \in \mathbb{N}$. Focus on two vacua $\phi_f = 0$ and $\phi_t = \pi\phi_0$
- $\cos(\phi/\phi_0)$ tracks tunneling: $\cos(\phi_f/\phi_0) = -1$, $\cos(\phi_t/\phi_0) = 1$
- ϕ_0 controls potential width, λ controls potential height and mass, V_0 is normalized to $V_0 = 0.008\phi_0^2$
- Set $\epsilon_\phi = \epsilon_\pi = \epsilon$

Cosine Potential



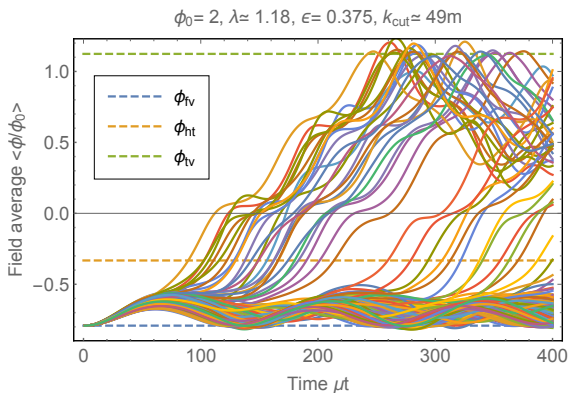
Cosine Potential

- Solid line is $\Gamma_{\text{inst}} = \Gamma_0 \left(\frac{S_B}{2\pi} \right)^2 e^{-S_B}$. Without renormalization, use $\Gamma_0 = Nm^2L$
- Dashed line is $\Gamma^* e^{-S_B}$ where $\Gamma^* = \Gamma_{\text{stoch}}(\epsilon = 0.5, \phi_0 = 1.0)$



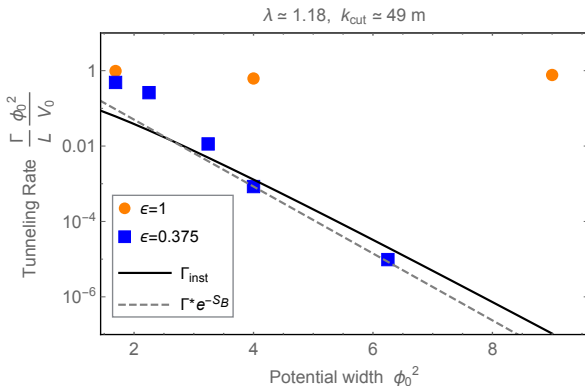
Double Well Potential

- Consider now: $V(\phi) = V_0 \left(\left(1 - \frac{\phi^2}{\phi_0^2} \right)^2 + \lambda \left(1 - \frac{\phi}{\phi_0} \right) \right)$
- New tunneling threshold $\frac{1}{L} \int dx \phi(t, x) / \phi_0 \geq V(\phi_{HT})$



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Renormalization

- The one-loop correction to the mass in 1+1 is:

$$m_R^2 = m_B^2 + \frac{g}{8\pi} \log \left(\frac{k_{\text{cut}} + m_B^2}{m_B^2} \right) \quad (7)$$

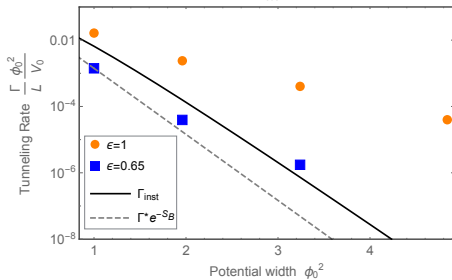
where $g_{\text{cos}} = V_0/\phi_0^4(1 - 4\lambda^2)$ and $g_{\text{DW}} = 24V_0/\phi_0^4$

- Requiring $|m_R^2 - m_B^2| < |m_B^2|$ gives us an upper bound on k_{cut} . Choosing new cutoffs, we get:

Renormalization

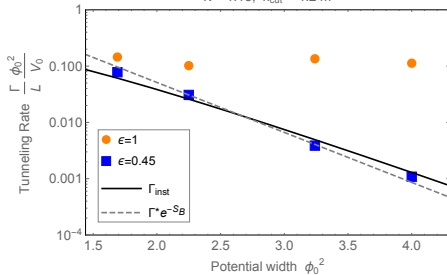
Cosine Potential

$\lambda = 1.2, k_{\text{cut}} \approx 2.7 \text{ m}$



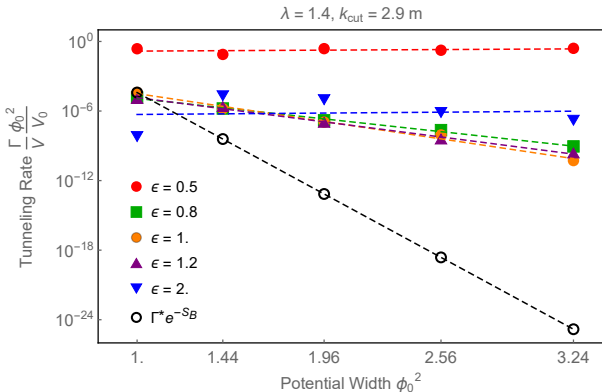
Double Well Potential

$\lambda \approx 1.18, k_{\text{cut}} \approx 4.2 \text{ m}$



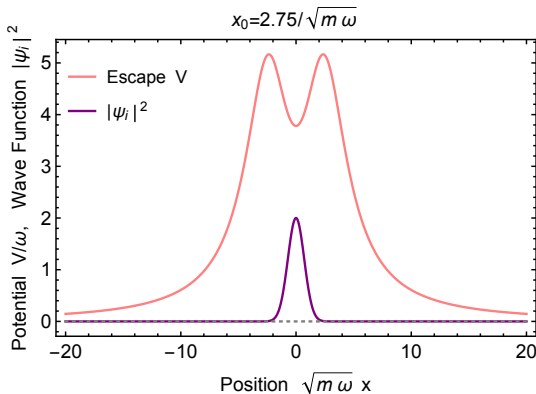
Other Physical States

- $\Delta\phi_k \Delta\dot{\phi}_k = \frac{\epsilon_\phi \epsilon_\pi}{2}$. So clearly $\epsilon_\phi = \epsilon_\pi = \epsilon < 1$ violates the uncertainty principle.
- Can modify fluctuation amplitudes while saturating uncertainty as follows:
 $\epsilon_\phi = 1/\epsilon_\pi = \epsilon$



Particle Escape

- The wavefunction starts in well, then spreads out. This is analogous to a particle escaping



Particle Escape

- Track variance in position over time:

$$\langle x^2 \rangle_Q(t) = \int_{-\infty}^{\infty} dx |\psi(x, t)|^2 x^2 \quad (8)$$

- Create an ensemble of 10^4 initial conditions for $\{x_i, p_i\}$ from gaussian distributions with variances:

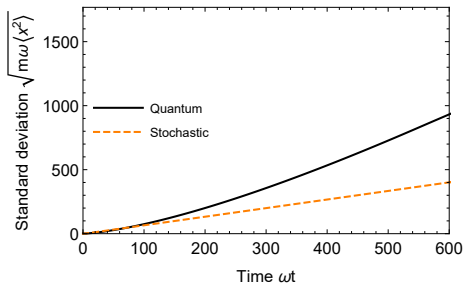
$$\sigma_{x,i}^2 = \frac{1}{2m\omega}, \quad \sigma_{p,i}^2 = \frac{m\omega}{2} \quad (9)$$

- Then evolve each x_i classically over time and ensemble average to obtain $\langle x^2 \rangle_S(t)$

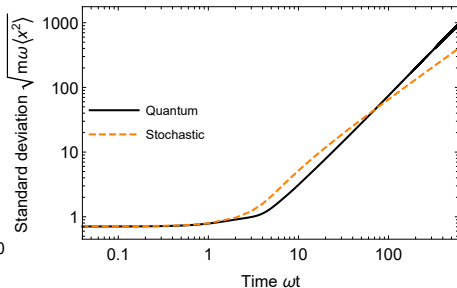
Particle Escape

- Choose large box to minimize edge effects: $x_0 = 2.75/\sqrt{m\omega}$, $L = 3536/\sqrt{m\omega}$

$$x_0 = 2.75/\sqrt{m\omega}$$



$$x_0 = 2.75/\sqrt{m\omega}$$



Conclusions

- The instanton is an imaginary-time approximation of tunneling rates that fails for certain time-dependent backgrounds
- Recent work introduced a real-time formalism that claimed excellent agreement to the instanton
- This isn't quite true, the stochastic method over-predicts tunneling rates unless fluctuations are artificially suppressed
- Various curing methods were applied, and the stochastic method continued to show only parametric agreement
- Future work: Develop a prescription for obtaining ideal “fudge factors”

Thank you!

References



Jonathan Braden et al. “New semiclassical picture of vacuum decay”. In: *Physical review letters* 123.3 (2019), p. 031601.



Sidney Coleman. “Fate of the false vacuum: Semiclassical theory”. In: *Physical Review D* 15.10 (1977), p. 2929.



Mark P. Hertzberg, Fabrizio Rompineve, and Neil Shah. “Quantitative Analysis of the Stochastic Approach to Quantum Tunneling”. In: *Phys. Rev. D* 102.7 (2020), p. 076003. DOI: 10.1103/PhysRevD.102.076003. arXiv: 2009.00017 [hep-th].



Andrei Linde. “Hard art of the universe creation”. In: *arXiv preprint hep-th/9110037* (1991).

A “classical” perspective

- The thresholds are chosen as following:
- For the Braden periodic potential:
 - Recall the volume average for a field ϕ_i : $c_i(t) = \frac{1}{L} \int dx \cos(\phi_i(t, x)/\phi_0)$
 - Then, define the ensemble values: $\bar{c}_T/\Delta c_T \equiv \text{Ensemble average/std. dev. of } \{c_i(0)\}$
 - Define the threshold $\mathcal{T}_{\text{Braden}} = \bar{c}_T + n_\sigma \Delta c_T$ where $5 \leq n_\sigma \leq 25$
- For the DW potential: $\mathcal{T}_{\text{DW}} = V(\phi_{\text{HT}})$

A “classical” perspective

- Let's define the Weyl transform of some operator \hat{A} :

$$\tilde{A}(q_{\pm k}, \pi_{\pm k}) = \int dx dy e^{-i\pi_k x - i\pi_{-k} y} \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \left| \hat{A} \right| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle \quad (10)$$

- Define the *Wigner function* $W \equiv (2\pi)^{-2} \tilde{\rho}$. If $W \geq 0 \Rightarrow$ phase space distribution
- By correspondence, define corresponding function of \hat{A} as:

$$\hat{q}_{\pm k} \rightarrow q_{\pm k}, \hat{\pi}_{\pm k} \rightarrow \pi_{\pm k} \Rightarrow \hat{A}(\hat{q}_{\pm k}, \hat{\pi}_{\pm k}) = A_C(q_{\pm k}, \pi_{\pm k}) \quad (11)$$

A “classical” perspective

- Define stochastic average of \hat{A} as:

$$\langle A \rangle_{\text{stoch}} = \int A_C(\mathbf{s}) W(\mathbf{s}) d^4 \mathbf{s} \quad (12)$$

where $\mathbf{s} \equiv (q_{\pm \mathbf{k}}, \pi_{\pm \mathbf{k}})$ is a 4-vector in phase space and $W(\mathbf{s})$ is the Wigner function.

- The Weyl transform has a key property we can use:

$$\langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A}) = \frac{1}{(2\pi)^2} \int \tilde{A}(\mathbf{s}) \tilde{\rho}(\mathbf{s}) d^4 \mathbf{s} = \int \tilde{A}(\mathbf{s}) W(\mathbf{s}) d^4 \mathbf{s} \quad (13)$$

- Clearly $\langle A \rangle_{\text{stoch}} = \langle \hat{A} \rangle$ if $A_C(\mathbf{s}) = \tilde{A}(\mathbf{s})$

Single Particle QM

- In QFT, we draw ϕ from Wigner function as joint distribution, then evolve classically.
- What if we move to SPQM and draw directly from a Gaussian wavefunction?
- Consider the following wavefunction:

$$\psi(x) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}m\omega x^2\right] \quad (14)$$

in the following potential:

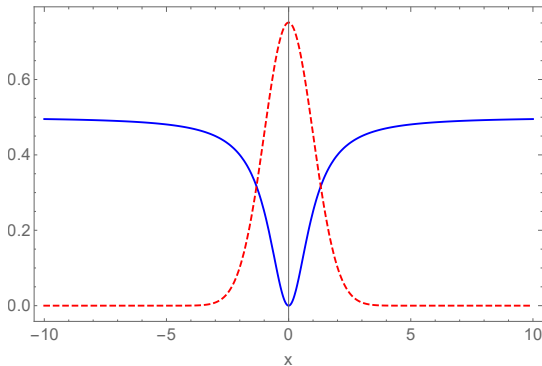
$$V(x) = \frac{1}{2}m\omega^2 x^2 \frac{1 - \frac{1}{2}\left(\frac{x}{x_0}\right)^2}{1 + \frac{1}{2}\left(\frac{x}{x_0}\right)^4} \quad (15)$$

Quantum Escape

- Define the potential:

$$V(x) = \frac{1}{2} \frac{x^2}{1 + x^2/\lambda^2} \quad (16)$$

- Initialize a Gaussian in the well, track its escape and compare to stochastic



Quantum Escape

