An introduction to higher-form symmetries

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Based on Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, JHEP **02** (2015) 172, [arXiv:1412.5148 [hep-th]], and so on.

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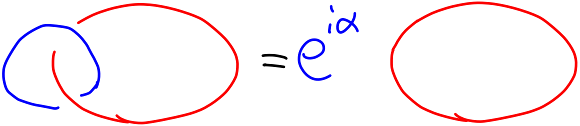
What are higher p-form symmetries?

Symmetries under transformations of p-dim. extended objects



e.g. a transformation of a Wilson loop (red loop)

- In this talk, I would like to explain 1-form global symmetries in 4D Maxwell theory.

Why we consider such symmetries?

Motivations

- Particle physics: based on symmetry for particles (local objects),

  e.g. $SU(3) \times SU(2) \times U(1)$ gauge symmetry

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  e.g. $SU(3) \times SU(2) \times U(1)$ gauge symmetry

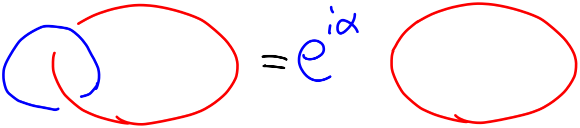
However, there can be extended objects in

- Cosmology: domain walls, cosmic strings,...
- String theory: fundamental strings, branes,...
- SUSY/SUGRA: BPS solitons,...
- Condensed matter: magnetic vortices, magnetic domain walls,...
- ...

Motivations

We can classify phases of matter based on extended objects

E.g. A global symmetry for a Wilson loop W: $W \to e^{i\alpha} W$



- Confinement phases $\langle W \rangle \to 0$: Symmetric phase
- Deconfinement phases $\langle W \rangle \to 1$: Symmetry broken phase

  Charged object W develops VEV.

  We can apply this classification to many systems admitting extended objects

Why global symmetries?

- Global symmetries are physical.

  They lead to degeneracy of energy states (as in QM).

- 't Hooft anomalies for global symmetries constrain phase structures,

  in particular forbidding non-degenerate gapped vacuum

- Gauge symmetries are introduced as local transformations of global ones.

- Quantum gravity may forbid global symmetries.

  Higher-form symmetries may give us some constraints on effective theories

  consistent with quantum gravity [Banks & Seiberg '10; Montero et al., '17].

Purpose of this talk

Introducing higher-form symmetries in terms of field theories.

- I focus on the way to generalize ordinary symmetries to higher-form symmetries.
- I show how to derive higher-form symmetry transformations.

For concreteness, I only consider $(3+1)$ D field theories.

0-form symmetries

as ordinary symmetries

Message

Rephrasing ordinary symmetries in terms of **topology**.

Symmetry transformation

$$\langle U(g,\mathcal{V})\Phi^i(y)\rangle = R(g)^i{}_j\langle\Phi^j(y)\rangle \quad \text{(if linked)}$$

1. Existence of symmetry $\to$ Existence of topological object $U(g,\mathcal{V})$
2. Symmetry generators are conserved/commute with Hamiltonian

   $\to U(g,\mathcal{V})$ is topological.
3. Symmetry transformation is generated by

   commutator $\to$ link of $U(g,\mathcal{V})$ and the charged object $\Phi^j(y)$

Noether's theorem

Continuous symmetry $\rightarrow$ conserved current

Assumption

- Φ^i: fields (scalars or fermions, for simplicity)
- Action $S[\Phi^i]$ is invariant $\delta S = 0$ under $\delta \Phi^i = \epsilon M^i{}_j \Phi^j$ (ϵ: infinitesimal parameter)

  $M^i{}_j$: generator of symmetry group G

Existence of conserved current $\partial_\mu j^\mu = 0$

- Position dependent transf. is given by
  $S[\Phi^i + \epsilon(x) M^i{}_j \Phi^j] - S[\Phi^i] = -\int \epsilon(x) \partial_\mu j^\mu$
- The deviation should vanish by EOM

In quantum theory, Noether theorem is generalized to

Ward-Takahashi identity

$$i\langle \partial_\mu j^\mu(x) \Phi^i(y) \rangle = \delta^4(x-y) M^i{}_j \langle \Phi^j(y) \rangle$$

$\langle \dots \rangle$: VEV in the path integral formalism

- Physical meaning: charged object $\Phi^i(y)$ is a source of j^μ.

Brief derivation: Use the reparameterization

$$\Phi^i(x) \to \Phi^i(x) + \epsilon(x) M^i{}_j \Phi^j(x)$$

in the path integral [detail]

By integrating the Ward-Takahashi identity,

we have

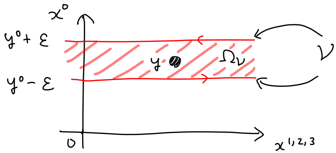
Symmetry transf. by conserved charge

$$i\langle[Q,\Phi^i(y)]\rangle_{\mathrm{cq.}} = M^i{}_j\langle\Phi^j(y)\rangle_{\mathrm{cq.}}.$$

$\langle\rangle_{\mathrm{cq}}$: VEV in the canonical quantization formalism

Derivation

- Integrating both hand sides of $i\langle\partial_\mu j^\mu(x)\Phi^i(y)\rangle = \delta^4(x-y)M^i{}_j\langle\Phi^j(y)\rangle$ over $\Omega_{\mathcal{V}} := [y^0+\epsilon, y^0-\epsilon]\times\mathbb{R}^3$

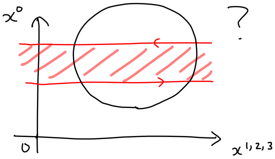


- Using Stokes theorem
  $\int_{\Omega_{\mathcal{V}}} d^4x\partial_\mu j^\mu = \int d^3\boldsymbol{x} j^0(y^0+\epsilon,\boldsymbol{x}) - \int d^3\boldsymbol{x} j^0(y^0-\epsilon,\boldsymbol{x}) = Q(y^0+\epsilon) - Q(y^0-\epsilon)$
- Explicitly writing the time-ordered product
  $\langle(Q(y^0+\epsilon)-Q(y^0-\epsilon))\Phi^i(y)\rangle = \langle\mathrm{T}\,(Q(y^0+\epsilon)-Q(y^0-\epsilon))\Phi^i(y)\rangle_{\mathrm{cq}} = \langle[Q(y^0),\Phi^i(y)]\rangle_{\mathrm{cq}}$

Some technical problems

If we consider transformations of **extended objects**,

it may be difficult to generalize the transf. directly.



It may be difficult (or not systematic) to formulate

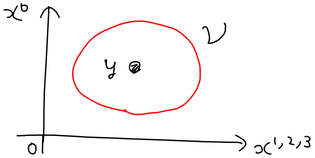
- Decomposition of time ordering for temporally extended objects
- Symmetry transformation in terms of commutation relation

We would like to rewrite the symmetry transf. so that they are also suitable for extended objects.

Rewriting ordinary symmetry transf.

Symmetry transf. based on link

$$i\langle Q(\mathcal{V})\Phi^i(y)\rangle = \mathrm{Link}\,(\mathcal{V},y)M^i{}_j\langle\Phi^j(y)\rangle$$



- Charge Q on a time slice $\to$ Charge $Q(\mathcal{V})$ on 3D closed subspace $\mathcal{V}$

$$Q(\mathcal{V}) := \int_{\mathcal{V}} \frac{\epsilon_{\mu\nu\rho\sigma}}{3!} j^\mu(x) dV^{\nu\rho\sigma}$$

($dV^{\mu\nu\rho}$ is a volume element)

- Commutation relation $[,] \to$ link of $Q(\mathcal{V})$ and $\Phi^i(y)$

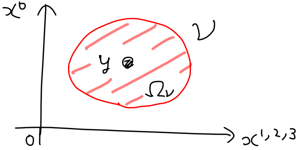
Q. How to derive this relation?

Integrating the Ward-Takahashi identity

Integral over 4D subspace $\Omega_{\mathcal{V}}$

$$i\langle Q(\mathcal{V})\Phi^i(y)\rangle = \left(\int_{\Omega_{\mathcal{V}}} d^4x\delta^4(x-y)\right) M^i{}_j\langle\Phi^j(y)\rangle$$

$\Omega_{\mathcal{V}}$: 4D subspace whose boundary is 3D subspace $\mathcal{V}$, $\partial\Omega_{\mathcal{V}} = \mathcal{V}$.



For the left-hand side, I have used/defined

- Stokes theorem

$$\int_{\Omega_{\mathcal{V}}} d^4x\partial_\mu j^\mu(x) = \int_{\mathcal{V}} \frac{\epsilon_{\mu\nu\rho\sigma}}{3!} j^\mu(x)dV^{\nu\rho\sigma}$$

- Charge on $\mathcal{V}$ ($dV^{\mu\nu\rho}$ is a volume element)

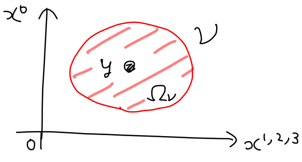
$$Q(\mathcal{V}) := \int_{\mathcal{V}} \frac{\epsilon_{\mu\nu\rho\sigma}}{3!} j^\mu(x)dV^{\nu\rho\sigma}$$

For the right-hand side, what is $\int_{\Omega_{\mathcal{V}}} d^4x\delta^4(x-y)$?

$\int_{\Omega_{\mathcal{V}}} d^4x\delta^4(x-y)$: linking number of $\mathcal{V}$ and y

Linking number

$$\int_{\Omega_{\mathcal{V}}} d^4x\delta^4(x-y) = \mathrm{Link}\,(\mathcal{V}, y)$$



- $\int_{\Omega_{\mathcal{V}}} d^4x\delta^4(x-y) =$ intersection number of $\Omega_{\mathcal{V}}$ and y.
- It is equal to the linking number of $\mathcal{V}$ and y.

Finally, we arrive at

Symmetry transf.

$$i\langle Q(\mathcal{V})\Phi^i(y)\rangle = \mathrm{Link}\,(\mathcal{V}, y) M^i{}_j \langle \Phi^j(y)\rangle$$

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We remark that linking number in RHS is a topological invariant.

How about LHS?

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$$i\langle Q(\mathcal{V})\Phi^i(y)\rangle = \mathrm{Link}\,(\mathcal{V}, y) M^i{}_j \langle \Phi^j(y)\rangle$$

We remark that linking number in RHS is a topological invariant.

How about LHS?

- Is $Q(\mathcal{V})$ in LHS topological? $\rightarrow$ Yes.
- What is the origin of the topological property? $\rightarrow$ Conservation law

Finally, we arrive at

Symmetry transf.

$$i\langle Q(\mathcal{V})\Phi^i(y)\rangle = \mathrm{Link}\,(\mathcal{V}, y) M^i{}_j \langle \Phi^j(y)\rangle$$

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How about LHS?

- Is $Q(\mathcal{V})$ in LHS topological? $\to$ Yes.
- What is the origin of the topological property? $\to$ Conservation law

This implies the following generalization:

$$Q \text{ is conserved} \to Q(\mathcal{V}) \text{ is topological}$$

Let us see it.

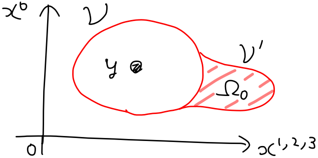
Conservation $\to$ Topology

Rephrasing "conservation law"

Q is conserved $\to$ $Q(\mathcal{V})$ is topological

Under a deformation, $\mathcal{V} \to \mathcal{V}' = \mathcal{V} + \partial\Omega_0$, $y \notin \Omega$

$i\langle Q(\mathcal{V})\Phi^i(y)\rangle = \text{Link}\,(\mathcal{V}, y) M^i{}_j \langle \Phi^j(y)\rangle$ is invariant.



- LHS: conservation law $Q(\mathcal{V}') = Q(\mathcal{V}) + \int_{\partial\Omega_0} \frac{\epsilon_{\mu\nu\rho\sigma}}{3!} j^\mu dV^{\nu\rho\sigma} = Q(\mathcal{V}) + \int_{\Omega_0} d^4x \partial_\mu j^\mu = Q(\mathcal{V})$.
- RHS: topological invariance $\text{Link}\,(\mathcal{V}', y) = \text{Link}\,(\mathcal{V}, y)$

We also have a finite symmetry transformation.

Finite symmetry transformation

By the exponent of $Q(\mathcal{V})$, we also have

Finite symmetry transformation

$$\langle U(g,\mathcal{V})\Phi^i(y)\rangle = R(g)^i{}_j\langle\Phi^j(y)\rangle \quad \text{(if linked)}$$

$U(g,\mathcal{V})$ is a topological unitary operator given by

- symmetry group $g \in G$
- 3D closed subspace $\mathcal{V}$

Comments

- U satisfies $\frac{d}{d\alpha}U(e^{i\alpha},\mathcal{V})|_{\alpha=0} = iQ(\mathcal{V})$

  $R(g)$: representation matrix of g

Summary (continuous symmetry)

Cont. symmetry under $G \to$ Conserved charge $Q \to$ Topological object $U(g, \mathcal{V})$

Finite symmetry transformation

$$\langle U(g, \mathcal{V}) \Phi^i(y) \rangle = R(g)^i{}_j \langle \Phi^j(y) \rangle \quad \text{(if linked)}$$

Some terminology

- $U(g, \mathcal{V})$: symmetry generator, $g \in G$
- $\Phi^i(y)$ charged object
- This symmetry is called a G 0-form symmetry: the charged object is 0-dim.

Q: How about discrete symmetries?

There may be no conserved currents.

A: Topological objects exist even for the discrete ones.

Review of discrete symmetry

We consider internal discrete symmetries given by

- $g \in G$: discrete group
- $U(g)$: unitary operator commuting with Hamiltonian & momentum,

  $[U(g), P^\mu] = 0$
- Unitary transformation (in can. quant. formalism)

$$\langle U(g)\Phi^i(y)U(g)^{-1}\rangle_{\rm cq} = R(g)^i{}_j \langle \Phi^j(y)\rangle_{\rm cq},$$

$R(g)^i{}_j$: representation matrix

I would like to show that

Existence of $U(g)$ $\to$ existence of topological operator $U(g, \mathcal{V})$

Existence of topological objects for discrete symmetries

Discrete symmetry transf.

$$\langle U(g,\mathcal{V})\Phi^i(y)\rangle = R(g)^i{}_j\langle\Phi^j(y)\rangle \quad \text{(if linked)}$$

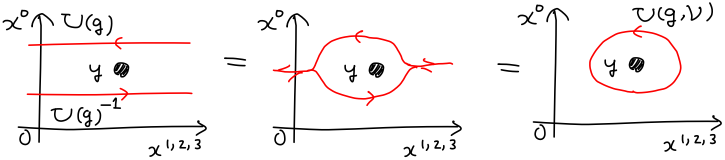
- $[U(g), P^\mu] = 0$ implies $U(g)$ can continuously move.

  $\rightarrow U(g)$ is topological.

- By the topological deformation of $\langle U(g)\Phi^i(y)U(g)^{-1}\rangle_{\text{cq}}$,

  we have a symmetry generator $U(g,\mathcal{V})$

$$\langle U(g)\Phi^i(y)U(g)^{-1}\rangle_{\text{cq}} = \langle U(g,\mathcal{V})\Phi^i(y)\rangle$$



Summary (ordinary symmetry)

Existence of symmetry under $g \in G$ $\rightarrow$ Existence of topological object $U(g, \mathcal{V})$

(rather than invariance of the action)

Symmetry transformation

$$\langle U(g, \mathcal{V}) \Phi^i(y) \rangle = R(g)^i{}_j \langle \Phi^j(y) \rangle \quad \text{if linked}$$

$R(g)^i{}_j$: representation matrix

- $U(g, \mathcal{V})$: Symmetry generator

  3D, topological object

- $\Phi^i(y)$ Charged object

  0D (not necessarily topological)

- This symmetry is called G 0-form symmetry.

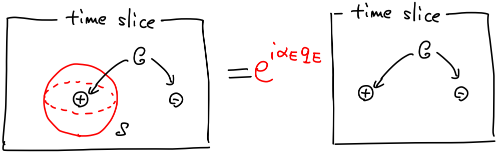
Finding symmetries $\rightarrow$ Finding topological objects

1-form symmetries in Maxwell theory

Just conservation laws of electric & magnetic fluxes

Message

There are $U(1)$ electric & magnetic 1-form symmetries.



- Symmetry generator: surface integrals of electric & magnetic fluxes

  2D topological objects

- Charged objects: Wilson loop & 't Hooft loop

  1D objects (not necessarily topological)

- Symmetry groups: $U(1)$ for electric & magnetic symmetries

  due to Dirac quantization

I begin with

Maxwell equations without matter

$$\frac{1}{e^2}\partial_\mu f^{\mu\nu} = 0, \quad \partial_\mu \tilde{f}^{\mu\nu} = 0.$$

$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$: field strength of $U(1)$ gauge field a_μ

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Maxwell equations without matter

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$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$: field strength of $U(1)$ gauge field a_μ

There are conserved quantities ($\mathcal{S}$: 2D closed surface e.g., a sphere S^2)

- Electric flux

$$Q_{\mathrm{E}}(\mathcal{S}) = \frac{1}{e^2}\int_{\mathcal{S}} \frac{\epsilon_{\mu\nu\rho\sigma}}{2!2!} f_{\mu\nu} dS^{\rho\sigma} \sim \int_{\mathcal{S}} \boldsymbol{E}\cdot d\boldsymbol{S},$$

- Magnetic flux

$$Q_{\mathrm{M}}(\mathcal{S}) = \frac{1}{2\pi}\int_{\mathcal{S}} \frac{1}{2!} f_{\mu\nu} dS^{\mu\nu} \sim \int_{\mathcal{S}} \boldsymbol{B}\cdot d\boldsymbol{S},$$

We know that $Q_{\mathrm{E}}(\mathcal{S})$ & $Q_{\mathrm{M}}(\mathcal{S})$ are topological under deformation of $\mathcal{S}$.

There are symmetries for electric/magnetic fluxes!

Symmetry for fluxes...?

Symmetry generators are

$$U_{\mathrm{E}}(\mathcal{S}) \sim \exp(iQ_{\mathrm{E}}(\mathcal{S})) \quad \text{and} \quad U_{\mathrm{M}}(\mathcal{S}) \sim \exp(iQ_{\mathrm{M}}(\mathcal{S}))$$

Symmetry for fluxes...?

Symmetry generators are

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Two questions

1. What are charged objects (= source) for the symmetries?

   For an ordinary symmetry, a charged object is $\Phi^i(y)$

   $i\langle \partial_\mu j^\mu(x) \Phi^i(y) \rangle = \delta^4(x-y) M^i{}_j \langle \Phi^j(y) \rangle$

2. What are groups parameterizing symmetries?

Symmetry for fluxes...?

Symmetry generators are

$$U_{\mathrm{E}}(\mathcal{S}) \sim \exp(iQ_{\mathrm{E}}(\mathcal{S})) \quad \text{and} \quad U_{\mathrm{M}}(\mathcal{S}) \sim \exp(iQ_{\mathrm{M}}(\mathcal{S}))$$

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2. What are groups parameterizing symmetries?

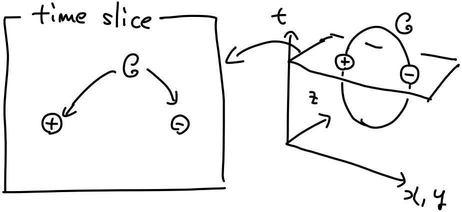
Answers

1. Charged objects are

   a Wilson loop for the electric flux,

   an 't Hooft loop for the magnetic flux.
2. Symmetry groups are $U(1)$ for electric/magnetic fluxes

Wilson loop $W(q_{\rm E}, \mathcal{C}) = e^{iq_{\rm E}\int_{\mathcal{C}} a_\mu dx^\mu}$



Worldline of probe electric particle

- Probe electric particle = Source of electric flux

  Electric Gauss law: $\frac{1}{e^2}\partial_\mu f^{\mu\nu}(x) = q_{\rm E}\int_{\mathcal{C}} \delta^4(x-y)dy^\nu$

- Worldline is closed: gauge invariance under $a_\mu \to a_\mu + \partial_\mu \lambda$

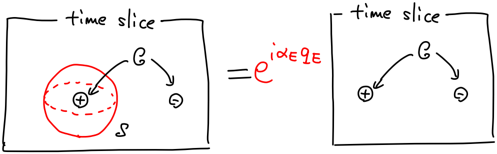
- $q_{\rm E} \in \mathbb{Z}$: charge of the probe particle

  The quantization is required if the gauge group is $U(1)$ [detail]

$U(1)$ symmetry for electric flux

Symmetry transformation of Wilson loop

$$\langle U_{\rm E}(e^{i\alpha_{\rm E}},\mathcal{S}) e^{iq_{\rm E}\int_{\mathcal{C}} a_\mu dx^\mu}\rangle = e^{i\alpha_{\rm E} q_{\rm E}\,{\rm Link}\,(\mathcal{S},\mathcal{C})}\langle e^{iq_{\rm E}\int_{\mathcal{C}} a_\mu dx^\mu}\rangle$$



- $U_{\rm E}(e^{i\alpha_{\rm E}},\mathcal{S}) = \exp\left(i\alpha_{\rm E} Q_{\rm E}(\mathcal{S})\right)$: unitary operator of electric flux,
- ${\rm Link}\,(\mathcal{S},\mathcal{C})$: linking number of $\mathcal{S}$ and $\mathcal{C}$
- Symmetry group is $U(1)$, $\alpha_{\rm E}+2\pi \sim \alpha_{\rm E}$ due to the quantization of $q_{\rm E}$.

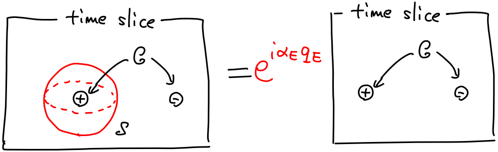
How to derive it?

- Schwinger-Dyson equation (quantum version of EOM) for $\alpha_{\rm E} \ll 1$
- Field redefinition (for finite $\alpha_{\rm E}$) [derivation]

Summary of $U(1)$ symmetry for electric flux

Symmetry transformation of Wilson loop

$$\langle U_{\mathrm{E}}(e^{i\alpha_{\mathrm{E}}},\mathcal{S})e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu}\rangle = e^{i\alpha_{\mathrm{E}} q_{\mathrm{E}}\,\mathrm{Link}\,(\mathcal{S},\mathcal{C})}\langle e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu}\rangle$$



- Symmetry generator: $U_{\mathrm{E}}(e^{i\alpha_{\mathrm{E}}},\mathcal{S}) = \exp\left(i\alpha_{\mathrm{E}} Q_{\mathrm{E}}(\mathcal{S})\right)$

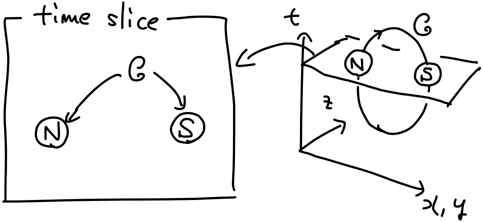
  2D topological object

- Charged object: $e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu}$
- Symmetry group: $e^{i\alpha_{\mathrm{E}}} \in U(1)$

This symmetry is called a electric $U(1)$ 1-form symmetry, since the charged object is 1D.

How about magnetic flux?

Charged object: 't Hooft loop $T(q_{\mathrm{M}}, \mathcal{C})$



Worldline of probe magnetic monopole

- Probe magnetic particle = Source of magnetic flux

  Magnetic Gauss law: $\int_{\mathcal{S}} \frac{1}{2!} f_{\mu\nu} dS^{\mu\nu} = 2\pi q_{\mathrm{M}}$

- Worldline is closed: gauge invariance of dual photon

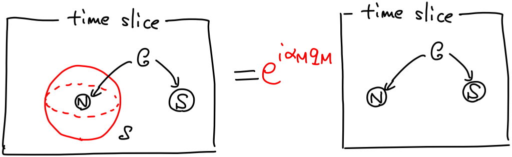
- $q_{\mathrm{M}} \in \mathbb{Z}$: charge of the monopole

  A monopole with quantized charge can exist if gauge group is $U(1)$ [detail]

$U(1)$ symmetry for conservation of magnetic flux

Symmetry transformation of 't Hooft loop

$$\langle U_{\mathrm{M}}(e^{i\alpha_{\mathrm{M}}}, \mathcal{S}) T(q_{\mathrm{M}}, \mathcal{C})\rangle = e^{i\alpha_{\mathrm{M}} q_{\mathrm{M}} \, \mathrm{Link}\,(\mathcal{S},\mathcal{C})} \langle T(q_{\mathrm{M}}, \mathcal{C})\rangle$$



- $U_{\mathrm{M}}(e^{i\alpha_{\mathrm{M}}}, \mathcal{S}) = \exp\left(i\alpha_{\mathrm{M}} Q_{\mathrm{M}}(\mathcal{S})\right)$: unitary operator of magnetic flux
- $e^{i\alpha_{\mathrm{M}}}$: $U(1)$ parameter, $\alpha_{\mathrm{M}} + 2\pi \sim \alpha_{\mathrm{M}}$ due to the quantization of q_{M}.
- $\mathrm{Link}\,(\mathcal{S},\mathcal{C})$: linking number of $\mathcal{S}$ and $\mathcal{C}$

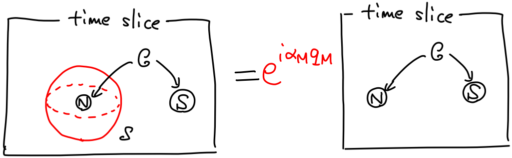
How to derive it?

- Using $Q_{\mathrm{M}}(\mathcal{S}) = q_{\mathrm{M}}$ in the presence of $T(q_{\mathrm{M}}, \mathcal{C})$ if $\mathcal{S}$ and $\mathcal{C}$ are linked.
- Dualizing the theory to the Maxwell theory with the dual photon.

Summary of $U(1)$ symmetry for magnetic flux

Symmetry transformation of 't Hooft loop

$$\langle U_{\mathrm{E}}(e^{i\alpha_{\mathrm{E}}}, \mathcal{S}) T(q_{\mathrm{M}}, \mathcal{C})\rangle = e^{i\alpha_{\mathrm{E}} q_{\mathrm{E}}\, \mathrm{Link}\,(\mathcal{S},\mathcal{C})} \langle T(q_{\mathrm{M}}, \mathcal{C})\rangle$$



- Symmetry generator: $U_{\mathrm{M}}(e^{i\alpha_{\mathrm{M}}}, \mathcal{S}) = \exp\left(i\alpha_{\mathrm{M}} Q_{\mathrm{M}}(\mathcal{S})\right)$

  2D topological object

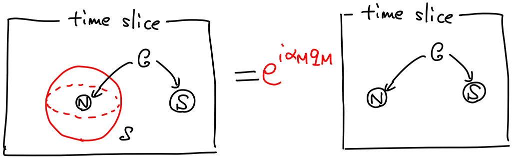
- Charged object: 't Hooft loop $T(q_{\mathrm{M}}, \mathcal{C})$

- Symmetry group: $e^{i\alpha_{\mathrm{M}}} \in U(1)$

This symmetry is also called a $U(1)$ 1-form symmetry, since the charged object is 1D.

Summary of higher-form symmetries in Maxwell theory without matter

There are $U(1)$ electric & magnetic 1-form symmetries.



- Symmetry generator: $U_{\mathrm{E}}(e^{i\alpha_{\mathrm{E}}}, \mathcal{S})$ & $U_{\mathrm{M}}(e^{i\alpha_{\mathrm{M}}}, \mathcal{S})$

  2D topological objects

- Charged objects: Wilson loop $e^{iq_{\mathrm{E}} \int_{\mathcal{C}} a_\mu dx^\mu}$ & 't Hooft loop $T(q_{\mathrm{M}}, \mathcal{C})$,

  1D objects (not necessarily topological)

- Symmetry groups: $e^{i\alpha_{\mathrm{E}}} \in U(1)$ & $e^{i\alpha_{\mathrm{M}}} \in U(1)$ due to Dirac quantization

Generalization

G p-form symmetry in D dimensions is given by

- Symmetry generator $U(g, \Sigma_{D-p-1})$: $(D-p-1)$-dim. topological object
- Charged object $W(q, \mathcal{C}_p)$: p-dim. object
- Symmetry transformation

$$\langle U(g, \Sigma_{D-p-1}) W(q, \mathcal{C}_p) \rangle = R(g)^q \langle W(q, \mathcal{C}_p) \rangle \quad \text{if linked}$$

Summary

Message

1. Existence of symmetry = Existence of topological objects
2. Symmetry transf. = link of symmetry generators & charged objects

I have reviewed

- Ordinary symmetry: 0-form symmetry
  - Symmetry generator $U(g, \mathcal{V})$: 3D topological object
  - Charged object $\Phi^i(y)$: 0D object
- Electric & magnetic $U(1)$ 1-form symmetries in Maxwell theory
  - Symmetry generators $U_{\mathrm{E}}(e^{i\alpha_{\mathrm{E}}}, \mathcal{S})$ & $U_{\mathrm{M}}(e^{i\alpha_{\mathrm{M}}}, \mathcal{S})$: 2D topological objects
  - Charged objects: Wilson & 't Hooft loops

Appendix

Derivation of 0-form symmetry transf. - I

I will show that

Ward-Takahashi identity

$$i\langle \partial_\mu j^\mu(x)\Phi^i(y)\rangle = \delta^4(x-y){M^i}_j\langle \Phi^j(y)\rangle$$

Derivation: in the path integral formalism, $\langle \partial_\mu j^\mu(x)\Phi^i(y)\rangle$ is given by

$$\langle \partial_\mu j^\mu(x)\Phi^i(y)\rangle = \mathcal{N}\int \mathcal{D}\Phi \partial_\mu j^\mu(x)\Phi^i(y)e^{iS[\Phi]}.$$

$\partial_\mu j^\mu(x)$ can be written as

$$\partial_\mu j^\mu(x) = -\frac{\delta}{\delta\epsilon(x)}S[\Phi^i + \epsilon(x){M^i}_j\Phi^j]\Big|_{\epsilon(x)=0}$$

since $j^\mu(x)$ is given by $S[\Phi^i + \epsilon(x){M^i}_j\Phi^j] - S[\Phi^i] = -\int d^4x\epsilon(x)\partial_\mu j^\mu(x)$.

Derivation of 0-form symmetry transf. - II

$\langle \partial_\mu j^\mu(x) \Phi^i(y) \rangle$ is then rewritten as

$$\begin{aligned}\langle \partial_\mu j^\mu(x) \Phi^i(y) \rangle =& -\mathcal{N} \int \mathcal{D}\Phi \ \tfrac{\delta}{\delta\epsilon(x)} S[\Phi^i + \epsilon(x) M^i{}_j \Phi^j] \Phi^i(y) e^{iS[\Phi]} \Big|_{\epsilon(x)=0} \\ =& -\ \frac{1}{i} \tfrac{\delta}{\delta\epsilon(x)} \mathcal{N} \int \mathcal{D}\Phi \Phi^i(y) e^{iS[\Phi^i + \epsilon(x) M^i{}_j \Phi^j]} \Bigg|_{\epsilon(x)=0} .\end{aligned}$$

By redefinition $\Phi^i(x) + \epsilon(x) M^i{}_j \Phi^j(x) \to \Phi^i(x)$, we have

$$\begin{aligned}\langle \partial_\mu j^\mu(x) \Phi^i(y) \rangle =& -\ \frac{1}{i} \tfrac{\delta}{\delta\epsilon(x)} \mathcal{N} \int \mathcal{D}\Phi (\Phi^i(y) - \epsilon(y) M^i{}_j \Phi^j(y)) e^{iS[\Phi^i]} \Bigg|_{\epsilon(x)=0} \\ =& \frac{1}{i} \delta^4(x-y) M^i{}_j \mathcal{N} \int \mathcal{D}\Phi \Phi^j(y) e^{iS[\Phi^i]} .\end{aligned}$$

(if there is an anomaly, redefinition of $\mathcal{D}\Phi$ should also be considered)

Derivation of 0-form symmetry transf. - III

Therefore, we obtain

$$i\langle \partial_\mu j^\mu(x)\Phi^i(y)\rangle = \delta^4(x-y)M^i{}_j\langle \Phi^j(y)\rangle$$

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Large gauge invariance of Wilson loop $e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu}$

$q_{\mathrm{E}} \in \mathbb{Z}$ is required for $U(1)$ gauge theory.

If the gauge group is $U(1)$,

- The gauge parameter λ in $e^{i\lambda} \in U(1)$ can be periodic, $\lambda + 2\pi \sim \lambda$.

  (The gauge parameter and gauge field are normalized by the periodicity.)

- λ can have a winding number: $\int_{\mathcal{C}} \partial_\mu \lambda dx^\mu \in 2\pi\mathbb{Z}$.
- Wilson loop should be gauge invariant even for winding λ:

$$e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu} \to e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu} e^{iq_{\mathrm{E}}\int_{\mathcal{C}} \partial_\mu \lambda dx^\mu}$$

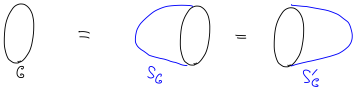
- Thus, we have $e^{iq_{\mathrm{E}}\int_{\mathcal{C}} \partial_\mu \lambda dx^\mu} = 1$ implying $q_{\mathrm{E}} \in \mathbb{Z}$.

Large gauge invariance $\to$ Dirac quantization

Dirac quantization

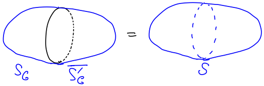
- By the Stokes theorem, we also have

$$e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu} = e^{iq_{\mathrm{E}}\int_{\mathcal{S}_{\mathcal{C}}} \frac{1}{2!} f_{\mu\nu} dS^{\mu\nu}} = e^{iq_{\mathrm{E}}\int_{\mathcal{S}'_{\mathcal{C}}} \frac{1}{2!} f_{\mu\nu} dS^{\mu\nu}},$$



which implies

$$e^{iq_{\mathrm{E}}\int_{\mathcal{S}_{\mathcal{C}} \cup \overline{\mathcal{S}'_{\mathcal{C}}}} \frac{1}{2!} f_{\mu\nu} dS^{\mu\nu}} = e^{iq_{\mathrm{E}}\int_{\mathcal{S}} \frac{1}{2!} f_{\mu\nu} dS^{\mu\nu}} = 1.$$



- For $q_{\mathrm{E}} = 1$, we have

$$\int_{\mathcal{S}} \frac{1}{2!} f_{\mu\nu} dS^{\mu\nu} \in 2\pi\mathbb{Z},$$

implying the existence of a magnetic monopole with a quantized charge inside $\mathcal{S}$.

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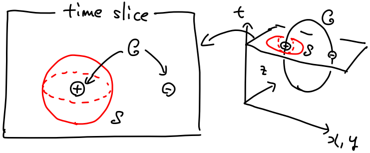
Derivation of 1-form transformation 0/4

We will evaluate

Correlation function

$$\langle U_{\mathrm{E}}(e^{i\alpha_{\mathrm{E}}},\mathcal{S})e^{iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu}\rangle = \int \mathcal{D}a e^{iS+i\alpha_{\mathrm{E}}Q_{\mathrm{E}}(\mathcal{S})+iq_{\mathrm{E}}\int_{\mathcal{C}} a_\mu dx^\mu}$$

where $S[a_\mu] = -\frac{1}{4e^2}\int d^4x f^{\mu\nu}f_{\mu\nu}$.



The correlator can be evaluated by eliminating $Q_{\mathrm{E}}(\mathcal{S}) = \frac{1}{e^2}\int_{\mathcal{S}} \frac{\epsilon_{\mu\nu\rho\sigma}}{2!2!} f^{\mu\nu} dS^{\rho\sigma}$

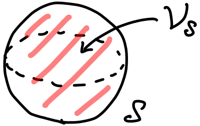
$Q_{\mathrm{E}}(\mathcal{S})$ can be absorbed to the action.

Derivation of 1-form transformation 1/4

1. Use of Stokes theorem

1st order derivative → 2nd order derivative

$$\int_{\mathcal{S}} \frac{\epsilon_{\mu\nu\rho\sigma}}{2!2!} f^{\mu\nu} dS^{\rho\sigma} = \int_{\partial\mathcal{V}_{\mathcal{S}}} \frac{\epsilon_{\mu\nu\rho\sigma}}{2!2!} f^{\mu\nu} dS^{\rho\sigma} = \int_{\mathcal{V}_{\mathcal{S}}} \frac{\epsilon_{\nu\rho\sigma\tau}}{3!} \partial_\mu f^{\mu\nu} dV^{\rho\sigma\tau}$$



- $\mathcal{V}_{\mathcal{S}}$ is a 3d subspace satisfying $\partial\mathcal{V}_{\mathcal{S}} = \mathcal{S}$.
- $dV^{\rho\sigma\tau}$ is a volume element.

Derivation of 1-form transformation 2/4

2. Use of delta function

Volume integral → spacetime integral

$$\int_{\mathcal{V}_\mathcal{S}} \frac{\epsilon_{\nu\rho\sigma\tau}}{3!}\partial_\mu f^{\mu\nu} dV^{\rho\sigma\tau} = \int d^4x \partial_\mu f^{\mu\nu} J_\nu(\mathcal{V}_\mathcal{S})$$

- $J_\nu(\mathcal{V}_\mathcal{S})$ is an abbreviation of delta function current

$$J_\nu(x;\mathcal{V}_\mathcal{S}) = \int_{\mathcal{V}_\mathcal{S}} \frac{\epsilon_{\nu\rho\sigma\tau}}{3!}\delta^4(x-y)dV^{\rho\sigma\tau}(y).$$

$J_\nu(x;\mathcal{V}_\mathcal{S})$ is non-zero on $\mathcal{V}_\mathcal{S}$.

Derivation

$$\begin{aligned}\int_{\mathcal{V}_\mathcal{S}} \frac{\epsilon_{\nu\rho\sigma\tau}}{3!}\partial_\mu f^{\mu\nu}(y) dV^{\rho\sigma\tau}(y) &= \int d^4x \int_{\mathcal{V}_\mathcal{S}} \frac{\epsilon_{\nu\rho\sigma\tau}}{3!}\partial_\mu f^{\mu\nu}(x)\delta^4(x-y) dV^{\rho\sigma\tau} \\ &= \int d^4x \partial_\mu f^{\mu\nu}(x) \int_{\mathcal{V}_\mathcal{S}} \frac{\epsilon_{\nu\rho\sigma\tau}}{3!}\delta^4(x-y) dV^{\rho\sigma\tau}\end{aligned}$$

Derivation of 1-form transformation 3/4

By the redefinition $a_\mu(x) - \alpha_{\mathrm{E}} J_\mu(x; \mathcal{V}_{\mathcal{S}}) \to a_\mu$, we can

Integrate out $Q_{\mathrm{E}}(\mathcal{S})$

$$\int \mathcal{D}a e^{iS + i\alpha_{\mathrm{E}} Q_{\mathrm{E}}(\mathcal{S}) + iq_{\mathrm{E}} \int_{\mathcal{C}} a_\mu dx^\mu} = e^{iq_{\mathrm{E}} \alpha_{\mathrm{E}} \int_{\mathcal{C}} J_\nu(\mathcal{V}_{\mathcal{S}}) dx^\mu} \langle e^{iq_{\mathrm{E}} \int_{\mathcal{C}} a_\mu dx^\mu} \rangle$$

Derivation

$$S[a_\mu - \alpha_{\mathrm{E}} J_\mu(\mathcal{V}_{\mathcal{S}})] = S[a_\mu] + \frac{\alpha_{\mathrm{E}}}{e^2} \int d^4x J_\nu(\mathcal{V}_{\mathcal{S}}) \partial_\mu f^{\mu\nu} + \frac{\alpha_{\mathrm{E}}^2}{4e^2} \int (\partial_\mu J_\nu(\mathcal{V}_{\mathcal{S}}) - \partial_\nu J_\mu(\mathcal{V}_{\mathcal{S}}))^2$$

$$= S[a_\mu] + \alpha_{\mathrm{E}} Q_{\mathrm{E}}(\mathcal{S})$$

The last term in the 1st line can be regularized by a local counter term.

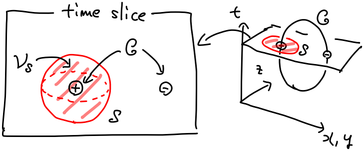
What is $\int_{\mathcal{C}} J_\nu(\mathcal{V}_{\mathcal{S}}) dx^\mu$?

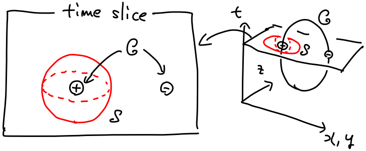
Derivation of 1-form transformation 4/4

Linking number

$$\int_{\mathcal{C}} J_\nu(\mathcal{V}_{\mathcal{S}}) dx^\mu = \mathrm{Link}\,(\mathcal{S}, \mathcal{C}) \in \mathbb{Z}$$

- $\int_{\mathcal{C}} J_\nu(\mathcal{V}_{\mathcal{S}}) dx^\mu =$ intersection number of $\mathcal{V}_{\mathcal{S}}$ and $\mathcal{C}$
- It is equal to the linking number of $\mathcal{S}$ and $\mathcal{C}$





Therefore, we obtain

Correlation function

$$\langle U_{\mathrm{E}}(e^{i\alpha_{\mathrm{E}}}, \mathcal{S}) e^{iq_{\mathrm{E}} \int_{\mathcal{C}} a_\mu dx^\mu} \rangle = e^{i\alpha_{\mathrm{E}} q_{\mathrm{E}} \,\mathrm{Link}\,(\mathcal{S},\mathcal{C})} \langle e^{iq_{\mathrm{E}} \int_{\mathcal{C}} a} \rangle$$

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