Higher-form symmetries and 3-group in axion electrodynamics

Ryo Yokokura (KEK)

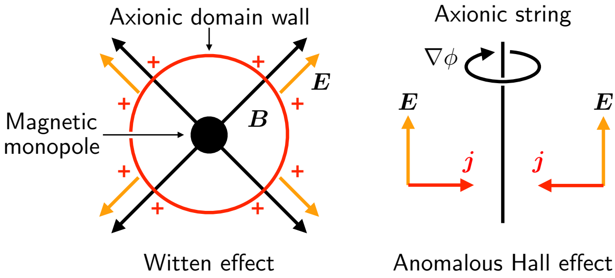
2020. 9. 24

Seminar @ IBS, online

based on

Y. Hidaka, M. Nitta, RY, Phys. Lett. B **808** (2020) 135672 [2006.12532]

What we did



- Axion electrodynamics in 4D possesses a 3-group structure.
- The Witten effect and the anomalous Hall effect can be understood as 3-group transformations.

Technically, we have shown

- higher-form symmetries in axion electrodynamics
- the 3-group structure by correlation functions of symmetry generators of the higher-form symmetries

Contents

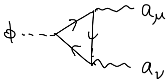
Axion electrodynamics: axion ϕ + photon a_μ + topological coupling

[Wilczek '87]

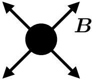
$$\frac{N}{4\pi^2}\phi \boldsymbol{E}\cdot\boldsymbol{B} = \frac{N}{32\pi^2}\phi\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma} = \frac{N}{8\pi^2}\phi da\wedge da$$

Features

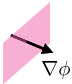
1. Topological coupling: determined by chiral anomaly (stable against higher-order corrections)



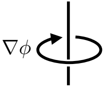
2. extended objects (temporally or spatially)



Magnetic monopole



Axionic domain wall



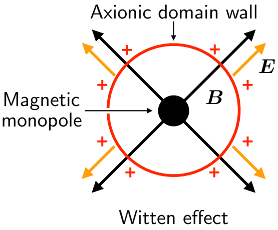
Axionic string

3. Ubiquitous in modern physics

   QCD axion, π^0 meson, axion insulator,...

$\frac{N}{4\pi^2}\phi\boldsymbol{E}\cdot\boldsymbol{B}+$ extended objects $\rightarrow$ rich and peculiar phenomena

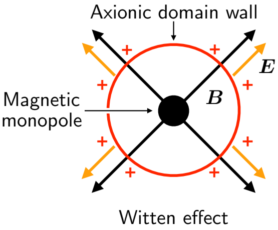
Witten effect for axionic domain wall



- Magnetic monopole + axionic domain wall → electric charge

[Witten '79; Sikivie '84; Kogan '92]

Witten effect for axionic domain wall



- Magnetic monopole + axionic domain wall → electric charge

  [Witten '79; Sikivie '84; Kogan '92]

- Due to modification of elec. Gauss law

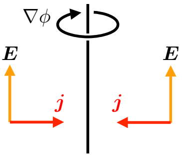
$$\nabla \cdot \boldsymbol{E} = \frac{N}{4\pi^2} \nabla\phi \cdot \boldsymbol{B}$$

(Original Witten effect: monopole becomes dyon in the presence of $\theta \boldsymbol{E} \cdot \boldsymbol{B}$)

Anomalous Hall effect for axionic string

Anomalous Hall effect = Hall effect without external magnetic field

Axionic string



Anomalous Hall effect

- Axionic string + electric field → electric current

[Sikivie '84; Wilczek '87; Qi, et al. '08; Teo & Kane '10]

Anomalous Hall effect for axionic string

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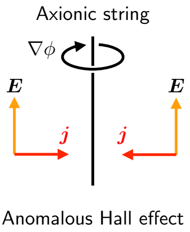


- Axionic string + electric field → electric current

  [Sikivie '84; Wilczek '87; Qi, et al. '08; Teo & Kane '10]

- Due to modification of Maxwell-Ampère law

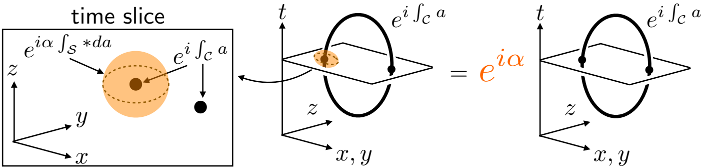
$$\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \frac{N}{4\pi^2} \nabla\phi \times \boldsymbol{E}$$

Q: Mathematical structure behind the phenomena for extended objects?

Candidate: Higher-form global symmetries [Gaiotto et al. '14]

Symmetries under transf. of p-dimensional extended objects

(Ordinary symmetries are 0-form symmetries: acting on local 0-dim. objects)



E.g., $U(1)$ 1-form symmetry in pure electromagnetism

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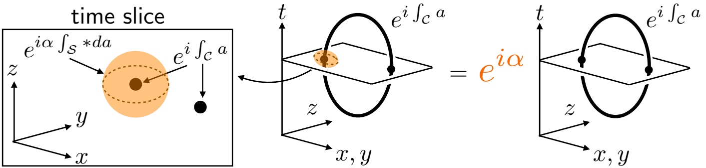

E.g., $U(1)$ 1-form symmetry in pure electromagnetism = conservation of electric flux

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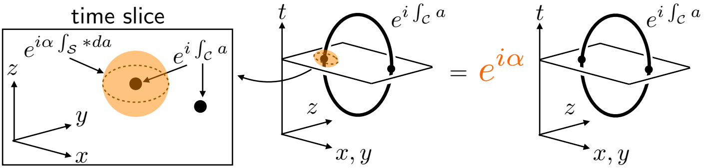

E.g., $U(1)$ 1-form symmetry in pure electromagnetism = conservation of electric flux

- Extended object: Wilson loop (1-dim.) $e^{i\int_C a_\mu dx^\mu}$ (Worldline of test particle)

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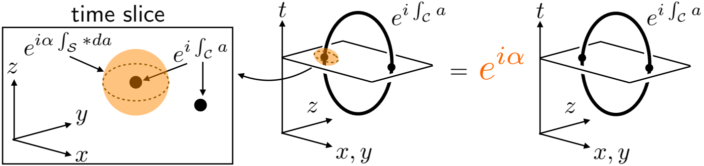

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- Extended object: Wilson loop (1-dim.) $e^{i\int_{\mathcal{C}} a_\mu dx^\mu}$ (Worldline of test particle)
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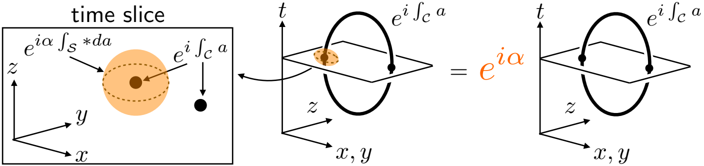

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= Nambu-Goldstone phase

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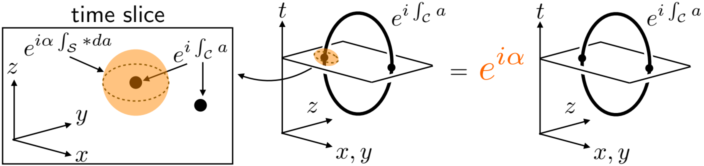

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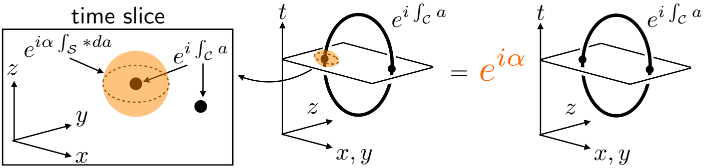

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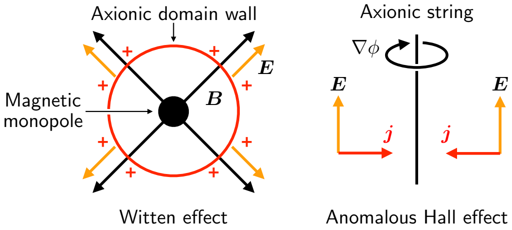
- Broken symmetry = 1-form symmetry: $\langle e^{i\int_C a_\mu dx^\mu} \rangle \neq 0$
- NG boson = photon

Q: Higher-form symmetries in **axion** electrodynamics?

Purpose of this talk



- We show higher-form symmetries and their group structure in axion electrodynamics.
- We consider the Witten effect and anomalous Hall effect from higher-form symmetries.

Higher-form symmetries in axion electrodynamics

Y. Hidaka, M. Nitta, RY, Phys. Lett. B **808** (2020) 135672

Message

There are 4 kinds of higher-form symmetries in axon electrodynamics.

	conservation law	Group	Charged object
0-form	EOM of ϕ	$\mathbb{Z}_N$	axion
Elec. 1-form	EOM of a_μ	$\mathbb{Z}_N$	Wilson loop
Mag. 1-form	Bianchi id. of a_μ	$U(1)$	't Hooft loop
2-form	Bianchi id. of ϕ	$U(1)$	worldsurface of axionic string

Set up [Wilczek '87]

Action

$$S = -\int \left(\frac{v^2}{2}|d\phi|^2 + \frac{1}{2e^2}|da|^2 - \frac{N}{8\pi^2}\phi da \wedge da\right)$$

v: decay constant, e: coupling constant

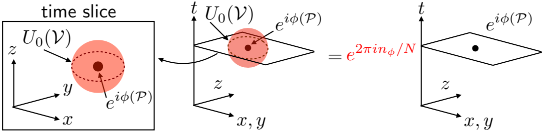
- Axion ϕ: periodic pseudo-scalar $\phi + 2\pi \sim \phi$ (NG boson $e^{i\phi}$)
- Photon $a = a_\mu dx^\mu$: $U(1)$ gauge field
- N: integer (number of massive Dirac fermions in UV)

Higher-form symmetries can be found by EOM and Bianchi identities

$$S = -\int d^4x \left(\frac{v^2}{2}|\partial_\mu \phi|^2 + \frac{1}{2e^2}|f_{\mu\nu}|^2 - \frac{N}{32\pi^2}\phi \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}\right), \; f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

$$S_{\mathrm{UV}} = -\int \left(\frac{1}{2e^2}|da|^2 + \frac{v^2}{2}|d\phi|^2 + i\bar\psi_i(\partial\!\!\!/ - i a\!\!\!/)\psi_i + yv\bar\psi_i e^{i\gamma_5 \phi}\psi_i\right), \; \psi_i\text{: Dirac fermions } i = 1, \ldots, N$$

$\mathbb{Z}_N$ 0-form symmetry: shift symmetry of axion $e^{i\phi(\mathcal{P})}$



$$\langle U_0(\mathcal{V}) e^{i\phi(\mathcal{P})} \rangle = e^{2\pi i n_\phi/N} \langle e^{i\phi(\mathcal{P})} \rangle$$

[derivation]

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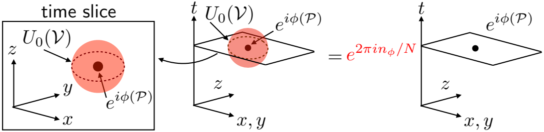

$$\langle U_0(\mathcal{V}) e^{i\phi(\mathcal{P})}\rangle = e^{2\pi i n_\phi/N}\langle e^{i\phi(\mathcal{P})}\rangle$$ [derivation]

Conservation law: EOM of axion

$$d(v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da) = 0 \quad (v^2 \Box\phi + \frac{N}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B} = 0)$$

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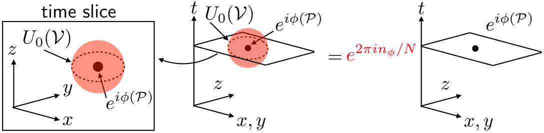

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1. Symmetry generator (3D):
   $U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) = \exp\left(\frac{2\pi i n_\phi}{N}\int_{\mathcal{V}}(-v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da)\right)$ ($\mathcal{V}$: 3D worldvolume)

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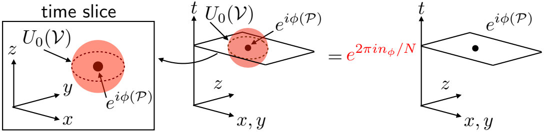

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2. Charged object (0D): $e^{i\phi(\mathcal{P})}$ (invariant under $\phi + 2\pi \sim \phi$)

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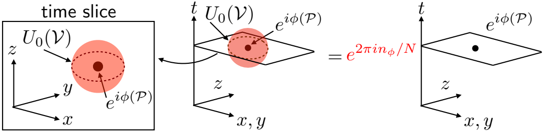

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2. Charged object (0D): $e^{i\phi(\mathcal{P})}$ (invariant under $\phi + 2\pi \sim \phi$)
3. Symmetry group: $\mathbb{Z}_N$ (Not $U(1)$ due to chiral anomaly) [detail]

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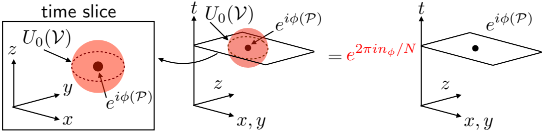

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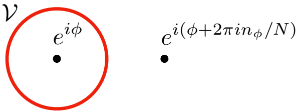
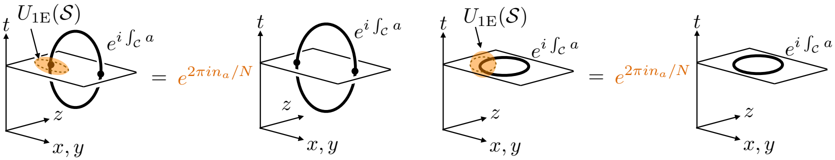
3. Symmetry group: $\mathbb{Z}_N$ (Not $U(1)$ due to chiral anomaly) [detail]

$U_0 =$ (external) axionic domain wall ($\phi(\mathcal{P})$ inside $\mathcal{V}$ differs from $\phi(\mathcal{P})$ outside $\mathcal{V}$ by $2\pi n_\phi/N$)



Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop



$$\langle U_{1\mathrm{E}}(\mathcal{S}) e^{i\int_{\mathcal{C}} a}\rangle = e^{2\pi i n_a/N}\langle e^{i\int_{\mathcal{C}} a}\rangle$$

[derivation]

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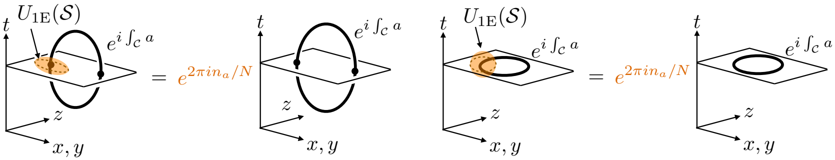

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Cons. law: elec. Gauss law & Maxwell-Ampère law

$$d(\tfrac{1}{e^2} * da - \tfrac{N}{4\pi^2}\phi da) = 0 \quad (\nabla\cdot\boldsymbol{D} = 0 \quad \& \quad \nabla\times\boldsymbol{H} - \tfrac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{0})$$

1. Symmetry generator: surface integral of elec. flux or line integral of mag. field

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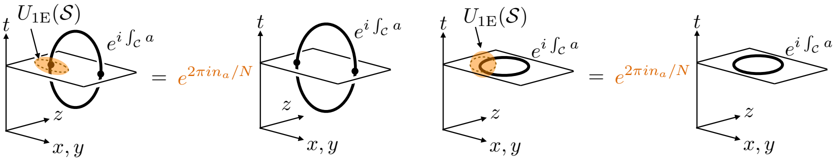

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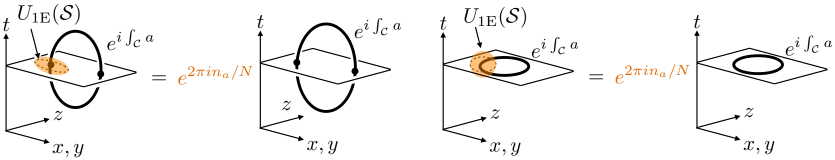

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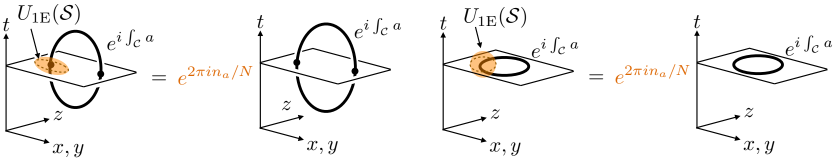
2. Charged object: Wilson loop $e^{i\int_{\mathcal{C}} a}$ (worldline of test particle or instantaneous current)
3. Symmetry group: $\mathbb{Z}_N$ (due to anomaly)

Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop



$$\langle U_{1\mathrm{E}}(\mathcal{S}) e^{i\int_\mathcal{C} a}\rangle = e^{2\pi i n_a/N}\langle e^{i\int_\mathcal{C} a}\rangle$$ [derivation]

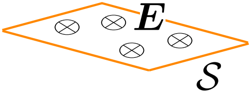
Cons. law: elec. Gauss law & Maxwell-Ampère law

$$d\left(\frac{1}{e^2} * da - \frac{N}{4\pi^2}\phi da\right) = 0 \quad \left(\nabla\cdot\boldsymbol{D} = 0 \quad \& \quad \nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t} = \boldsymbol{0}\right)$$

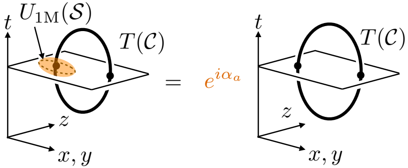
1. Symmetry generator: surface integral of elec. flux or line integral of mag. field

   $U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) = \exp\left(\frac{2\pi i n_a}{N}\int_\mathcal{S}\left(\frac{1}{e^2} * da - \frac{N}{4\pi^2}\phi da\right)\right)$
2. Charged object: Wilson loop $e^{i\int_\mathcal{C} a}$ (worldline of test particle or instantaneous current)
3. Symmetry group: $\mathbb{Z}_N$ (due to anomaly)

$U_{1\mathrm{E}}$: (external) elec. field



Magnetic $U(1)$ 1-form symmetry: phase rotation of 't Hooft loop



$$\langle U_{1\mathrm{M}}(\mathcal{S}) T(\mathcal{C}) \rangle = e^{i\alpha_a} \langle T(\mathcal{C}) \rangle$$

Cons. law: mag. Gauss law & Faraday law

$$dda = 0 \quad (\nabla \cdot \boldsymbol{B} = 0 \quad \& \quad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0})$$

1. Symmetry generator: surface integral of mag. flux $U_{1\mathrm{M}}(e^{i\alpha_a}, \mathcal{S}) = \exp\left(\frac{i\alpha_a}{2\pi} \int_{\mathcal{S}} da\right)$

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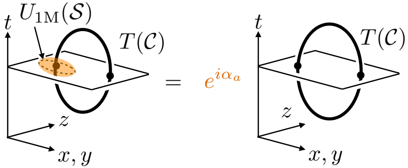

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   $\int_{\mathcal{S}} da = 2\pi$ in the presence of monopole

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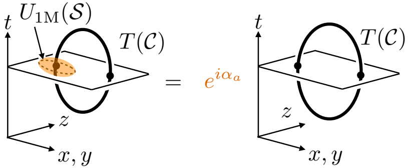

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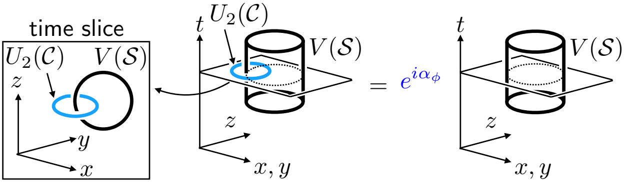
3. Symmetry group: $U(1)$ (magnetic flux is quantized)

$U(1)$ 2-form symmetry: phase rotation of axionic string



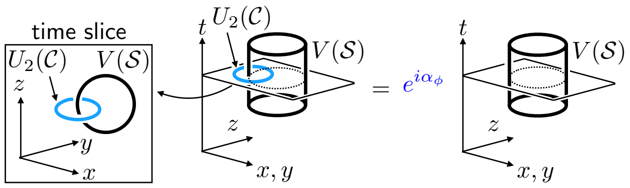
$$\langle U_2(\mathcal{C}) V(\mathcal{S}) \rangle = e^{i\alpha_\phi} \langle V(\mathcal{S}) \rangle$$

Cons. law: Bianchi identity of axion

$$dd\phi = 0 \quad (\nabla \times \nabla\phi = \mathbf{0})$$

1. Symmetry generator: winding number of axion $U_2(e^{i\alpha_\phi}, \mathcal{C}) = \exp\left(\frac{i\alpha_\phi}{2\pi} \int_{\mathcal{C}} d\phi\right)$

$U(1)$ 2-form symmetry: phase rotation of axionic string



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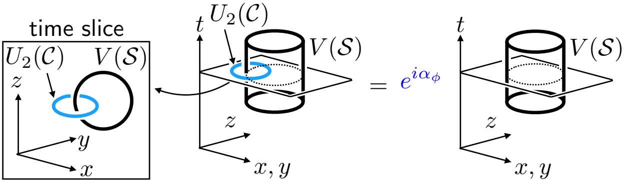
$U(1)$ 2-form symmetry: phase rotation of axionic string



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   $\int_{\mathcal{C}} d\phi = 2\pi$ in the presence of axionic string
3. Symmetry group: $U(1)$ (winding number is quantized)

Summary of higher-form symmetries

There are 4 kinds of higher-form symmetries

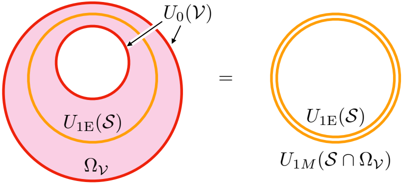
	group	transf. law	
0-form	$\mathbb{Z}_N$	shift of axion	$e^{i\phi(\mathcal{P})} \to e^{i(\phi(\mathcal{P})+2\pi/N)}$
Elec. 1-form	$\mathbb{Z}_N$	phase rot. of Wilson loop	$e^{i\int_{\mathcal{C}} a} \to e^{2\pi i/N} e^{i\int_{\mathcal{C}} a}$
Mag. 1-form	$U(1)$	phase rot. of 't Hooft loop	$T(\mathcal{C}) \to e^{i\alpha_1} T(\mathcal{C})$
2-form	$U(1)$	phase rot. of axionic string	$V(\mathcal{S}) \to e^{i\alpha_2} V(\mathcal{S})$

Witten effect and anomalous Hall effect from higher-form symmetries

Q: What is mathematical structure behind the phenomena?

Method: evaluating correlation functions of symmetry generators (extension of current algebra)

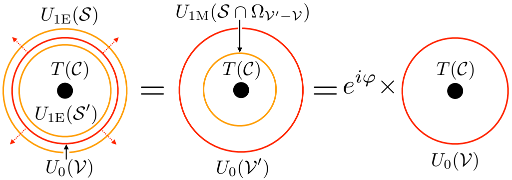
Witten effect: 0-form $\times$ elec. 1-form = mag. 1-form



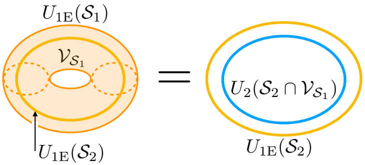
$$\langle U_0(e^{2\pi i n_\phi/N},\mathcal{V})U_{1\mathrm{E}}(e^{2\pi i n_a/N},\mathcal{S})\rangle = \langle U_{1\mathrm{M}}(e^{-2\pi i n_\phi n_a/N},\Omega_{\mathcal{V}}\cap\mathcal{S})U_{1\mathrm{E}}(e^{2\pi i n_a/N},\mathcal{S})\rangle$$

(Shifting ϕ of $U_{1\mathrm{E}} \sim \exp(i\int_{\mathcal{S}}\phi da)$ by U_0) [detail]

We can describe the Witten effect (Counting electric flux induced from axionic domain wall)



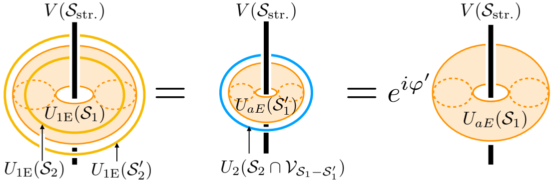
Anomalous Hall effect: elec. 1-form $\times$ elec. 1-form $=$ 2-form

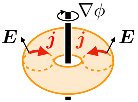


$$\langle U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}')\rangle = \langle U_2(e^{-2\pi i n_a n'_a/N}, \mathcal{V}_{\mathcal{S}} \cap \mathcal{S}') U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}')\rangle$$

(Shifting a of $U_{1\mathrm{E}} \sim \exp(\int_{\mathcal{S}} \phi da)$ by $U_{1\mathrm{E}}$) [detail]

We can describe the anomalous Hall effect (counting magnetic flux by Ampère law)

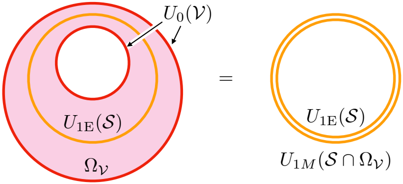




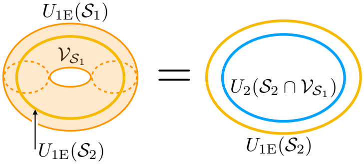
Note: Direction of Hall current $=$ parallel to torus

Q: What is mathematical structure behind the phenomena?

- Witten effect: 0-form $\times$ 1-form $=$ 1-form



- Anomalous Hall effect: 1-form $\times$ 1-form $=$ 2-form



- Cf. Ordinary current algebra based on ordinary group theory: 0-form $\times$ 0-form $=$ 0-form

We need a group structure beyond ordinary group theory

3-group in axion electrodynamics

3-group? Mathematically... [Conduché '84; Matrins & Picken '09]

It is called semistrict 3-group or 2-crossed module

Axioms (which we will use)

- 3 kinds of groups G, H, L
- Action $\triangleright$ of G on G, H, L: (extension of adjoint representation)

  $g \triangleright g' = g g' g^{-1}$, $g \triangleright h$, $g \triangleright l$
- Peiffer lifting: map from $h, h' \in H$ to L

  $\{h, h'\} \in L$
- ...

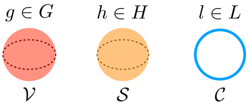
But the axioms seems abstract...

3-group? Physically...

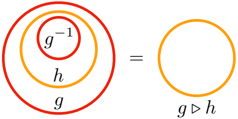
We can describe the 3-group in terms of higher-form symmetries

- Elements of G, H, L: Symmetry generators for 0-, 1-, 2-form symmetries, resp.

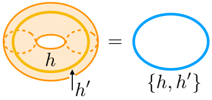
(they are the unitary representations, precisely)



- Action of $G \triangleright$: enclosing of elements by 0-form symmetry generators



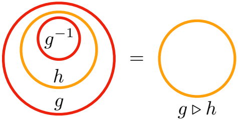
- Peiffer lifting $\{h, h'\}$: "link" of 1-form sym. generators (surface link) [Carter, et al '01]



The Witten effect & the anomalous Hall effect can be described!

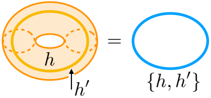
3-group in axion electrodynamics

- 3 kinds of groups: $G = \mathbb{Z}_N$ (0-form), $H = \mathbb{Z}_N \times U(1)$ (1-form), $L = U(1)$ (2-form)
- Witten effect = action of G:



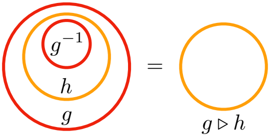
$$e^{2\pi i n_\phi/N} \triangleright (e^{2\pi i n_a/N}, e^{i\alpha_a}) = (e^{2\pi i n_a}, e^{-2\pi i n_\phi n_a/N} e^{i\alpha_a})$$

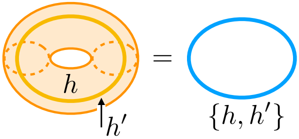
- Anomalous Hall effect = Peiffer lifting:



$$\{(e^{2\pi i n_a/N}, e^{i\alpha_a}), (e^{2\pi i n'_a/N}, e^{i\alpha'_a})\} = e^{-2\pi i n_a n'_a/N}$$

Summary





- Axion electrodynamics in 4D possesses a 3-group structure.
- The Witten effect and the anomalous Hall effect can be understood as 3-group transformations.
- Future work: relation to anomaly inflow, (including DOF on defects), higher-form symmetries for massive axions, etc.

Appendix

Group of 0-form symmetry is $\mathbb{Z}_N$ (1/4)

Large gauge invariance of $\int_{\mathcal{V}} a \wedge da$ is crucial

1. Symmetry group seems continuous, i.e. $U(1)$ because conserved current exists

$$j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da, \quad dj_{\phi 3} = 0$$

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4. However, the integral is not large gauge invariant.

   Further, the shift symmetry of axion should be broken to $\mathbb{Z}_N$ by chiral anomaly from UV viewpoint.

Group of 0-form symmetry is $\mathbb{Z}_N$ (2/4)

In the following, we consider the topological unitary operator

$$U_0(e^{i\alpha_0}, \mathcal{V}) = e^{-i\alpha_0 \int_{\mathcal{V}} (v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da)},$$

and show that the large gauge invariance of $U_0(e^{i\alpha_0}, \mathcal{V})$ requires $e^{i\alpha_0} \in \mathbb{Z}_N$.

Note: It is essentially the same as quantization of the level of Chern-Simons term in $(2+1)$ dim. [Henneaux & Teitelboim '86]

1. Since $d\phi$ is gauge invariant, we focus on the term $e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\mathcal{V}} a \wedge da}$.

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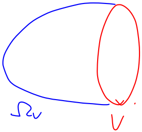
2. In order to make the integrand be manifestly gauge invariant, we define

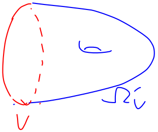
$$e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\mathcal{V}} a \wedge da} := e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\Omega_{\mathcal{V}}} da \wedge da}$$

on a 4d space $\Omega_{\mathcal{V}}$ satisfying $\partial \Omega_{\mathcal{V}} = \mathcal{V}$.



Group of 0-form symmetry is $\mathbb{Z}_N$ (3/4)

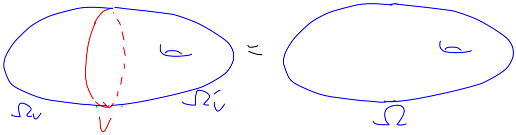
3. However, we have chosen a redundant space $\Omega_{\mathcal{V}}$, which may be replaced by another 4d space $\Omega'_{\mathcal{V}}$



4. The redundancy of the choice is absent if

$$e^{-i\alpha_0 \frac{N}{8\pi^2}\int_{\Omega_{\mathcal{V}}} da\wedge da} = e^{-i\alpha_0 \frac{N}{8\pi^2}\int_{\Omega'_{\mathcal{V}}} da\wedge da}, \quad \text{i.e.} \quad e^{-i\alpha_0 \frac{N}{8\pi^2}\int_{\Omega} da\wedge da} = 1$$

where $\Omega = \Omega_{\mathcal{V}} - \Omega'_{\mathcal{V}}$ is a closed 4d space. (Ω satisfies $\partial\Omega = \mathcal{V} - \mathcal{V} = 0$)



Group of 0-form symmetry is $\mathbb{Z}_N$ (4/4)

5. Since Ω is closed subspace $\partial\Omega = 0$, the Dirac quantization condition requires $\int_\Omega da \wedge da \in 2 \cdot (2\pi)^2 \mathbb{Z}$
6. Therefore, we have the condition $\alpha_0 \in \frac{2\pi}{N}\mathbb{Z}$, which means

$$e^{i\alpha_0} \in \mathbb{Z}_N.$$

[back]

Derivation of 0-form transformation 1/4

We evaluate the correlation function

$$\langle U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) e^{i\phi(\mathcal{P})}\rangle = \int \mathcal{D}[\phi, a] e^{iS + \frac{2\pi i n_\phi}{N}\int_{\mathcal{V}} j_{\phi 3} + i\phi(\mathcal{P})}$$

Let us integrate out $j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da$

1. local operators $\to$ spacetime integral

- $e^{i\phi(\mathcal{P})} = e^{i\int \phi \delta_4(\mathcal{P})}$, where $\delta_4(\mathcal{P}) = \delta^4(x - \mathcal{P}) dx^0 \wedge \cdots \wedge dx^3$
- $\int_{\mathcal{V}} j_{\phi 3} = \int_{\partial \Omega_{\mathcal{V}}} j_{\phi 3} = \int_{\Omega_{\mathcal{V}}} dj_{\phi 3} = \int dj_{\phi 3} \delta_0(\Omega_{\mathcal{V}})$,

  where $\Omega_{\mathcal{V}}$ is a 4d subspace satisfying $\partial \Omega_{\mathcal{V}} = \mathcal{V}$,

  $\delta_0(\Omega_{\mathcal{V}}) = \int_{\Omega_{\mathcal{V}}} \delta^4(x - y) dy^0 \wedge \cdots \wedge dy^3$.

Derivation of 0-form transformation 2/4

2. Action + symmetry generator can be rewritten by

Completing the square

$$S[\phi, a] + \frac{2\pi n_\phi}{N} \int dj_{\phi 3} \delta_0(\Omega_{\mathcal{V}})$$
$$= S[\phi - \frac{2\pi n_\phi}{N}\delta_0(\Omega_{\mathcal{V}}), a] + \frac{v^2}{2}\left(\frac{2\pi n_\phi}{N}\right)^2 \int \delta_1(\mathcal{V}) \wedge *\delta_1(\mathcal{V}).$$

Here, we have defined/used

- $\delta_1(\mathcal{V}) = \frac{\epsilon_{\mu\nu\rho\sigma}}{3!} dx^\sigma \int_{\mathcal{V}} \delta^4(x-y) dy^\nu \wedge dy^\rho \wedge dy^\sigma$
- $d\delta_0(\Omega_{\mathcal{V}}) = \delta_1(\partial\Omega_{\mathcal{V}}) = \delta_1(\mathcal{V})$
- $\int \omega_3 \wedge d\delta_0(\Omega_{\mathcal{V}}) = \int d\omega_3 \delta_0(\Omega_{\mathcal{V}}) = \int_{\Omega_{\mathcal{V}}} d\omega_3 = \int_{\partial\Omega_{\mathcal{V}}} \omega_3 = \int_{\mathcal{V}} \omega_3 = \int \omega_3 \wedge \delta_1(\mathcal{V})$

$S[\phi, a] = -\int \left(\frac{v^2}{2} d\phi \wedge *d\phi + \frac{1}{2e^2} da \wedge *da - \frac{N}{8\pi^2}\phi da \wedge da\right)$, $\quad j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da$

Derivation of 0-form transformation 3/4

3. By the redefinition $\phi - \frac{2\pi n_\phi}{N}\delta_0(\Omega_\mathcal{V}) \to \phi$, we have

Integrating out U_0

$$\langle U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) e^{i\phi(\mathcal{P})}\rangle = \int \mathcal{D}[\phi, a] e^{iS[\phi - \frac{2\pi n_\phi}{N}\delta_0(\Omega_\mathcal{V}), a] + i\int \phi\delta_4(\mathcal{P})}$$
$$= e^{\frac{2\pi i n_\phi}{N}\int \delta_0(\Omega_\mathcal{V})\delta_4(\mathcal{P})}\langle e^{i\phi(\mathcal{P})}\rangle,$$

($\int \delta_1(\mathcal{V}) \wedge *\delta_1(\mathcal{V})$ has been regularized by local counter term.)

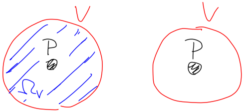
What is the phase factor $\int \delta_0(\Omega_\mathcal{V})\delta_4(\mathcal{P})$?

Derivation of 0-form transformation 4/4

Linking number

$$\int \delta_0(\Omega_{\mathcal{V}})\delta_4(\mathcal{P}) = \int_{\Omega_{\mathcal{V}}} \delta_4(\mathcal{P}) = \mathrm{Link}\,(\mathcal{V}, \mathcal{P}) \in \mathbb{Z}$$

Intersection of $\Omega_{\mathcal{V}}$ and $\mathcal{P}$ = link of $\mathcal{V}$ and $\mathcal{P}$



Therefore, we obtain

$\mathbb{Z}_{\mathbb{N}}$ 0-form transformation

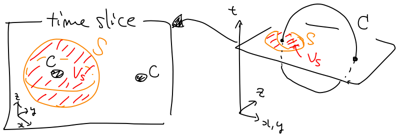
$$\langle U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) e^{i\phi(\mathcal{P})}\rangle = e^{\frac{2\pi i n_\phi}{N}\,\mathrm{Link}\,(\mathcal{V},\mathcal{P})}\langle e^{i\phi(\mathcal{P})}\rangle$$

[back]

Derivation of 1-form transformation 1/4

We evaluate the correlation function

$$\langle U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) e^{i\int_\mathcal{C} a}\rangle = \int \mathcal{D}[\phi, a] e^{iS + \frac{2\pi i n_a}{N}\int_\mathcal{S} j_{a2} + i\int_\mathcal{C} a}$$



Let us integrate out $j_{a2} = \frac{1}{e^2} * da - \frac{N}{4\pi^2}\phi da$

1. local operators $\to$ spacetime integral:

- $e^{i\int_\mathcal{C} a} = e^{i\int a\wedge\delta_3(\mathcal{C})}$, where $\delta_3(\mathcal{C}) = \frac{\epsilon_{\mu\nu\rho\sigma}}{3!} dx^\nu \wedge \cdots \wedge dx^\sigma \int_\mathcal{C} \delta^4(x-y) dy^\mu$
- $\int_\mathcal{S} j_{a2} = \int_{\partial\mathcal{V}_\mathcal{S}} j_{a2} = \int_{\mathcal{V}_\mathcal{S}} dj_{a2} = \int dj_{a2}\wedge\delta_1(\mathcal{V}_\mathcal{S})$,

  where $\mathcal{V}_\mathcal{S}$ is a 3d subspace satisfying $\partial\mathcal{V}_\mathcal{S} = \mathcal{S}$.

Derivation of 1-form transformation 2/4

2. Action + symmetry generator can be rewritten by

Completing the square

$$S[\phi,a]+\frac{2\pi n_\phi}{N}\int dj_{a2}\wedge\delta_1(\mathcal{V}_{\mathcal{S}})$$
$$=S[\phi,a+\frac{2\pi n_\phi}{N}\delta_1(\mathcal{V}_{\mathcal{S}})]+\frac{1}{2e^2}\left(\frac{2\pi n_a}{N}\right)^2\int\delta_2(\mathcal{S})\wedge *\delta_2(\mathcal{S}).$$

Here, we have defined/used

- $\delta_2(\mathcal{S})=\frac{\epsilon_{\mu\nu\rho\sigma}}{2!\cdot 2!}dx^\rho\wedge dx^\sigma\int_{\mathcal{S}}\delta^4(x-y)dy^\mu\wedge dy^\nu$
- $d\delta_1(\mathcal{V}_{\mathcal{S}})=-\delta_2(\partial\mathcal{V}_{\mathcal{S}})=-\delta_2(\mathcal{S})$
- $-\int\omega_2\wedge d\delta_1(\mathcal{V}_{\mathcal{S}})=\int d\omega_2\wedge\delta_1(\mathcal{V}_{\mathcal{S}})=\int_{\mathcal{V}_{\mathcal{S}}}d\omega_2=\int_{\partial\mathcal{V}_{\mathcal{S}}}\omega_2=\int_{\mathcal{S}}\omega_2=\int\omega_2\wedge\delta_2(\mathcal{S})$

$S[\phi,a]=-\int\left(\frac{v^2}{2}d\phi\wedge *d\phi+\frac{1}{2e^2}da\wedge *da-\frac{N}{8\pi^2}\phi da\wedge da\right),\quad j_{a2}=\frac{1}{e^2}*da-\frac{N}{4\pi^2}\phi da$

Derivation of 1-form transformation 3/4

3. By the redefinition $a + \frac{2\pi n_\phi}{N}\delta_1(\mathcal{V}_\mathcal{S}) \to a$, we have

Integrating out $U_{1\mathrm{E}}$

$$\begin{aligned}\langle U_{1E}(e^{2\pi i n_\phi/N}, \mathcal{V}) e^{i\phi(\mathcal{P})}\rangle &= \int \mathcal{D}[\phi, a] e^{iS[\phi, a + \frac{2\pi n_\phi}{N}\delta_1(\mathcal{V}_\mathcal{S})] + i\int a\wedge\delta_3(\mathcal{C})} \\ &= e^{\frac{2\pi i n_a}{N}\int \delta_3(\mathcal{C})\wedge\delta_1(\mathcal{V}_\mathcal{S})}\langle e^{i\int_\mathcal{C} a}\rangle,\end{aligned}$$

($\int \delta_2(\mathcal{S}) \wedge *\delta_2(\mathcal{S})$ has been regularized by local counter term.)

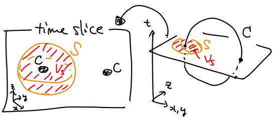
What is the phase factor $\int \delta_3(\mathcal{C}) \wedge \delta_1(\mathcal{V}_\mathcal{S})$?

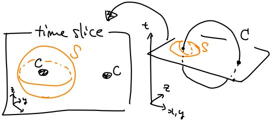
Derivation of 1-form transformation 4/4

Linking number

$$\int \delta_3(\mathcal{P}) \wedge \delta_1(\mathcal{V}_{\mathcal{S}}) = \int_{\mathcal{V}_{\mathcal{S}}} \delta_3(\mathcal{C}) = \text{Link}\,(\mathcal{S}, \mathcal{C}) \in \mathbb{Z}$$

Intersection of $\mathcal{V}_{\mathcal{S}}$ and $\mathcal{C}$ = link of $\mathcal{S}$ and $\mathcal{C}$





Therefore, we obtain

$\mathbb{Z}_N$ 1-form transformation

$$\langle U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) e^{i\int_{\mathcal{C}} a} \rangle = e^{\frac{2\pi i n_a}{N}\,\text{Link}\,(\mathcal{S},\mathcal{C})} \langle e^{i\int_{\mathcal{C}} a} \rangle$$

Correlation function $\langle U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S})\rangle$

We evaluate

$$\langle U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S})\rangle = \int \mathcal{D}[\phi, a] e^{iS[\phi,a] + \frac{2\pi i n_\phi}{N}\int_{\mathcal{V}} j_{\phi 3} + \frac{2\pi i n_a}{N}\int_{\mathcal{S}} j_{a2}}$$

- Integrating out $j_{\phi 3}$ can be done by the shift $\phi \to \phi + \frac{2\pi n_\phi}{N}\delta_0(\Omega_{\mathcal{V}})$
- $U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S})$ is shifted as

$$\begin{aligned} U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) \to{}& U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) e^{-\frac{2\pi i n_\phi n_a}{N}\cdot \frac{1}{2\pi}\int_{\mathcal{S}} \delta_0(\Omega_{\mathcal{V}}) da} \\ ={}& U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) U_{1\mathrm{M}}(e^{-2\pi i n_\phi n_a/N}, \Omega_{\mathcal{V}} \cap \mathcal{S}) \end{aligned}$$

[back]

$U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) = e^{\frac{2\pi i n_a}{N}\int_{\mathcal{S}} j_{a2}}, \quad j_{a2} = \frac{1}{e^2} * da - \frac{N}{4\pi^2}\phi da$

Correlation function $\langle U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}')\rangle$

We evaluate

$$\langle U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S}) U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}')\rangle = \int \mathcal{D}[\phi, a] e^{iS[\phi,a] + \frac{2\pi i n_a}{N}\int_{\mathcal{S}} j_{a2} + \frac{2\pi i n'_a}{N}\int_{\mathcal{S}'} j_{a2}}$$

- Integrating out $U_{1\mathrm{E}}(e^{2\pi i n_a/N}, \mathcal{S})$ can be done by the shift $a \to a - \frac{2\pi n_a}{N}\delta_1(\mathcal{V}_{\mathcal{S}})$
- $U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}')$ is shifted as ($\int_{\mathcal{S}'} * \delta_2(\mathcal{S})$ has been regularized)

$$\begin{aligned} U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') &\to U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') e^{\frac{2\pi i n_a n'_a}{N}\cdot\frac{1}{2\pi}\int_{\mathcal{S}'} \phi d\delta_1(\mathcal{V}_{\mathcal{S}})} \\ &= U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') e^{\frac{-2\pi i n_a n'_a}{N}\cdot\frac{1}{2\pi}\int_{\mathcal{S}'} d\phi\wedge\delta_1(\mathcal{V}_{\mathcal{S}})} \\ &= U_{1\mathrm{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') U_2(e^{-2\pi i n_a n'_a/N}, \mathcal{V}_{\mathcal{S}} \cap \mathcal{S}') \end{aligned}$$

[back]

Axioms of 3-group 1/3

A semistrict 3-group (2-crossed module) is a set $(L \overset{\partial_2}{\to} H \overset{\partial_1}{\to} G, \triangleright, \{-,-\})$ satisfying the following axioms. Here, G, H, and L are groups.

1. The maps

$$\partial_1 : H \to G, \quad \partial_2 : L \to H$$

are group homomorphisms $\partial_1(h_1 h_2) = (\partial_1 h_1)(\partial_1 h_2)$ and $\partial_2(l_1 l_2) = (\partial_2 l_1)(\partial_2 l_2)$ for $h_{1,2} \in H$ and $l_{1,2} \in L$, respectively. They satisfy

$$\partial_1 \circ \partial_2(l) = 1$$

for $l \in L$

Axioms of 3-group 2/3

2. $\triangleright$ is an action of $g \in G$ on $g' \in G$, $h \in H$, and $l \in L$ by automorphisms, $g \triangleright g' \in G$, $g \triangleright h \in H$, and $g \triangleright l \in L$. The action $g \triangleright g'$ is defined by conjugation,

$$g \triangleright g' := g g' g^{-1}.$$

3. $\partial_{1,2}$ are G-equivalent, that is,

$$g \triangleright (\partial_1 h) = \partial_1 (g \triangleright h), \quad g \triangleright (\partial_2 l) = \partial_1 (g \triangleright l),$$

Axioms of 3-group 3/3

4. The Peiffer lifting $\{-,-\}$ is a morphism $H \times H \to L$. In terms of the elements $h_{1,2} \in H$,

$$\{h_1, h_2\} \in L$$

For $l_{1,2} \in L$, the Peiffer lifting satisfies

$$\partial_2\{h_1, h_2\} = h_1 h_2 h_1^{-1} (\partial_1 h_1) \triangleright h_2^{-1},$$

$$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\},$$

$$\{\partial_2 l_1, \partial_2 l_2\} = l_1 l_2 l_1^{-1} l_2^{-1},$$

$$\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\} (\partial_1 h_1) \triangleright \{h_2, h_3\},$$

$$\{h_1, h_2 h_3\} = \{h_1, h_2\}\{h_1, h_3\}\{\partial_2\{h_1, h_3\}^{-1}, (\partial_1 h_1) \triangleright h_2\},$$

$$\{\partial_2 l_1, h_2\}\{h_2, \partial_2 l_1\} = l_1 (\partial_1 h) \triangleright l_1^{-1}.$$

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