

7th Oct. 2020 @ CTPU seminar in JBS

Manifestly Covariant Theory of Stochastic Inflation



Yuichiro TADA Nagoya U.
w/ L. Pinol, S. Renaux-Petel 2008.07497

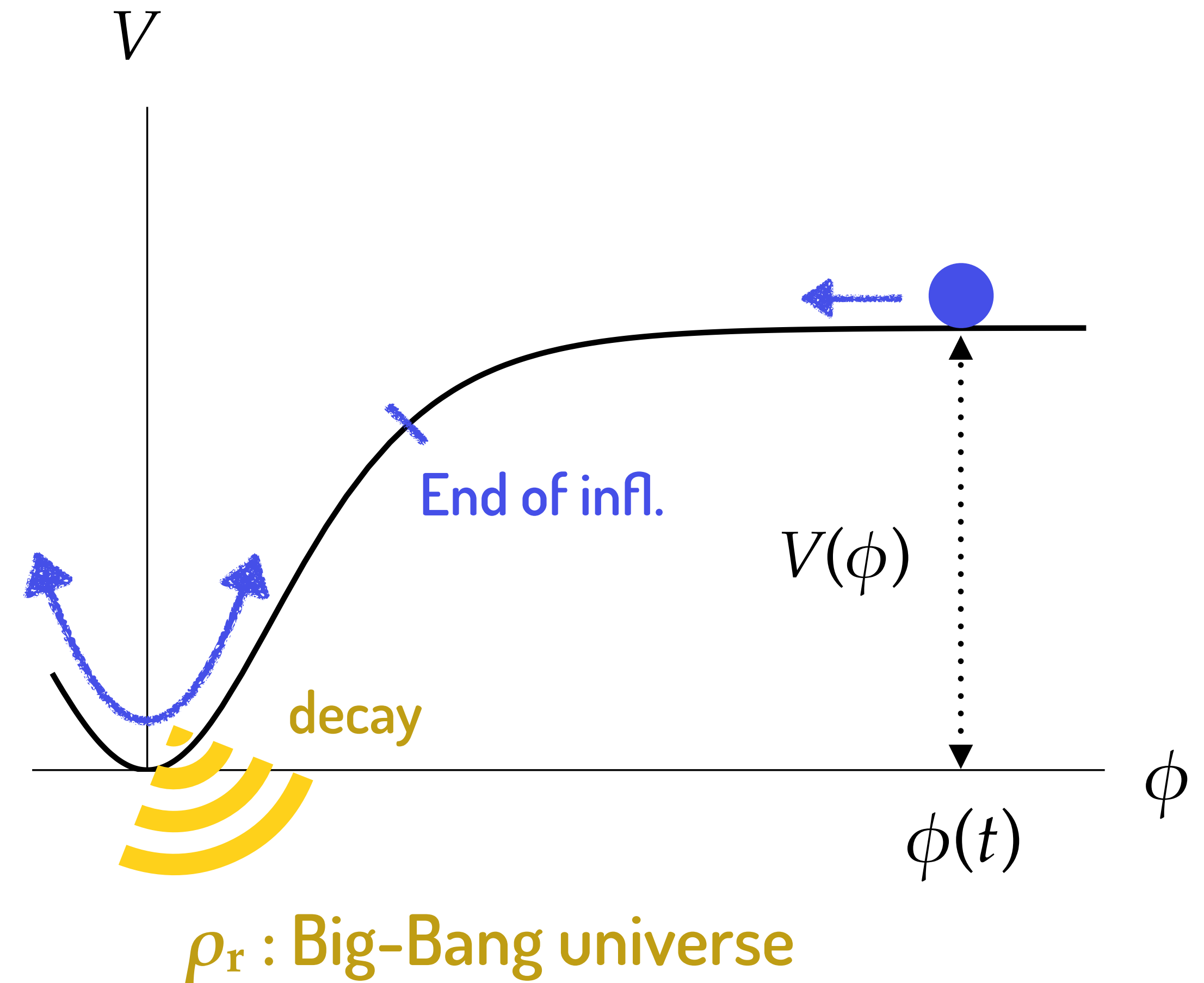
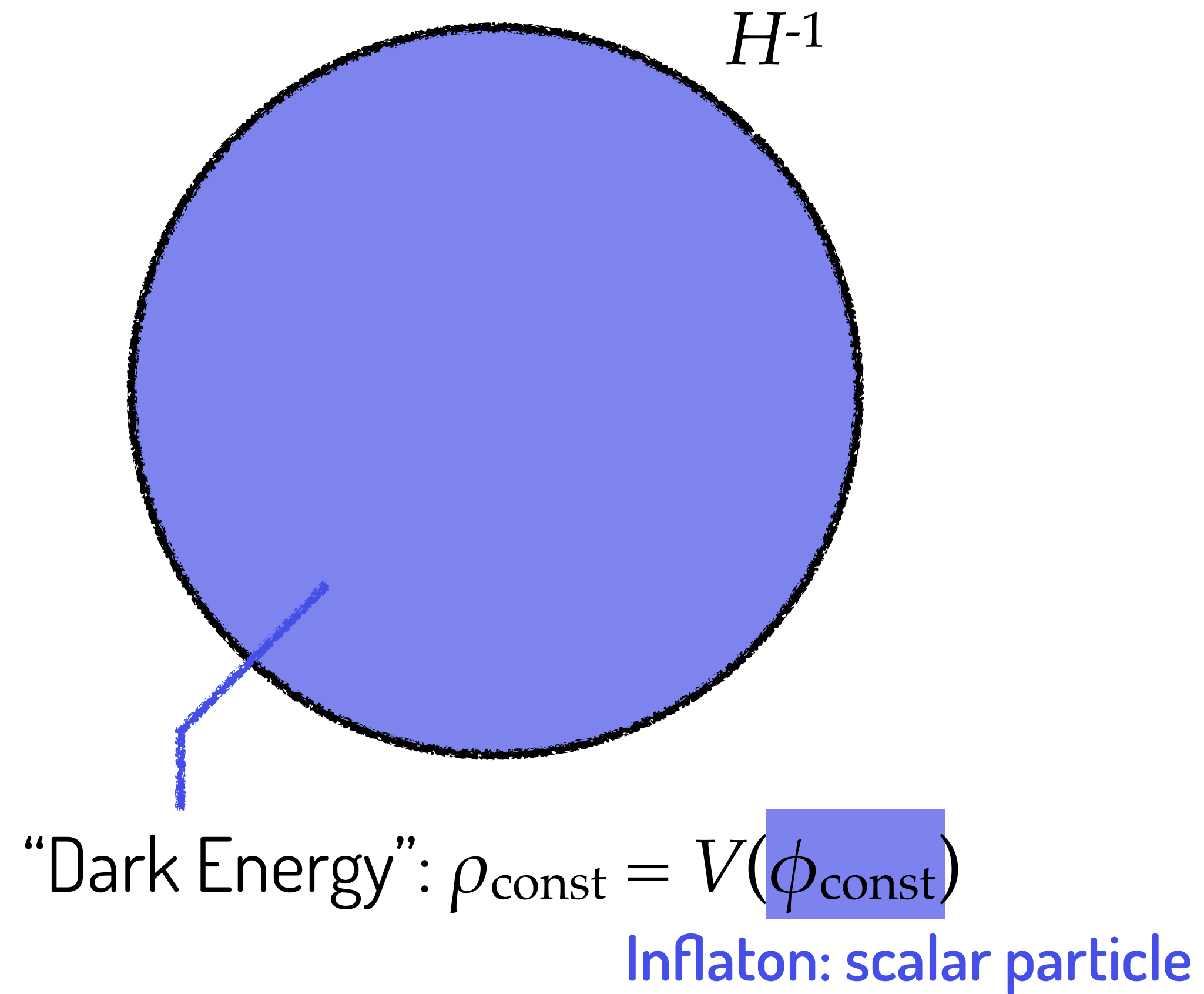
ref. Pinol, Renaux-Petel, Tada CQG **36** (2019) 07LT01
Renaux-Petel, Tada, Vennin in prep.

Contents

- ❖ Stochastic approach to inflation: *What? & How useful?*
- ❖ Ambiguity existed in formulation: *Stochastic Anomaly*
- ❖ Mathematically well-defined *covariant* formulation

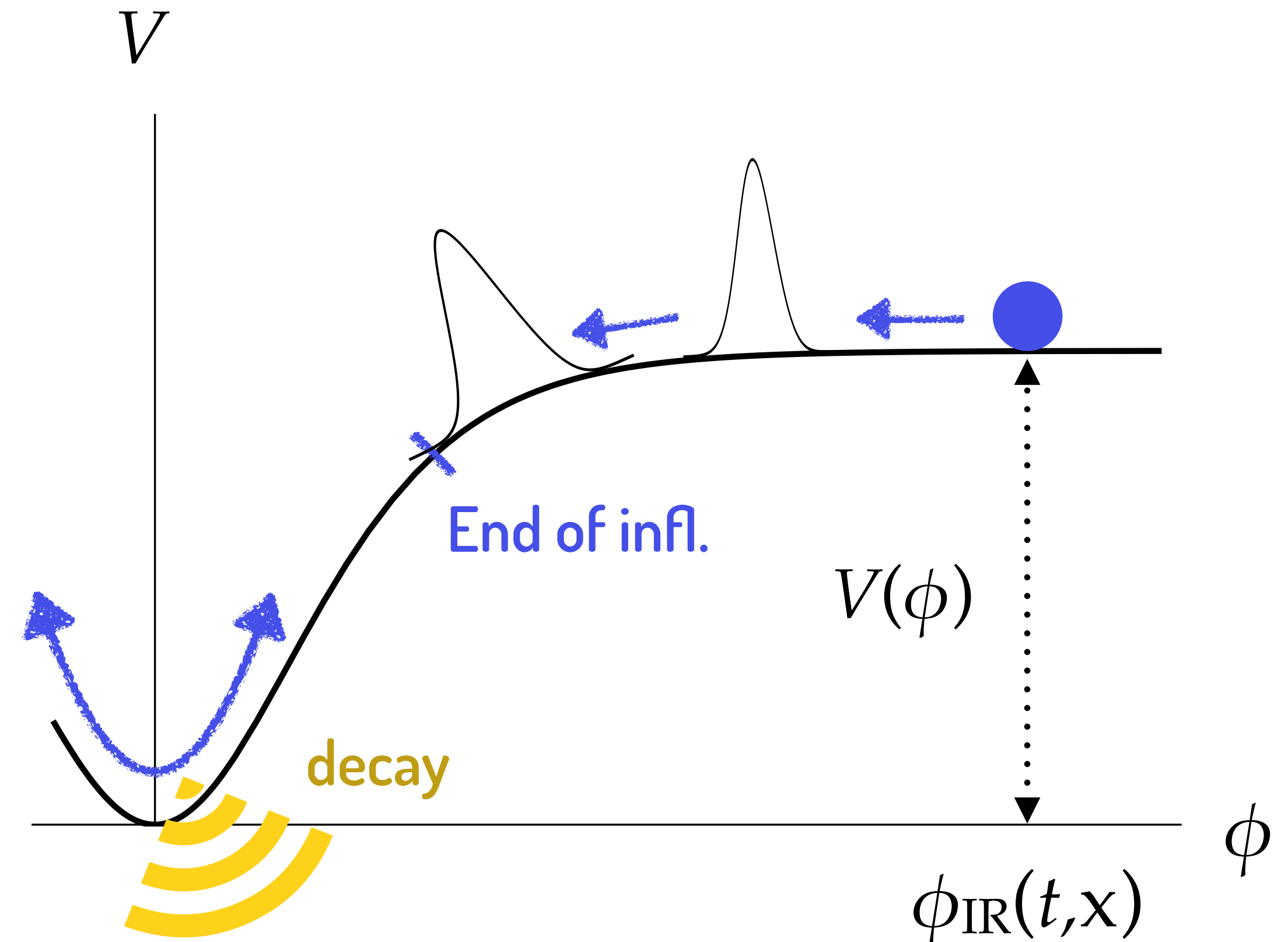
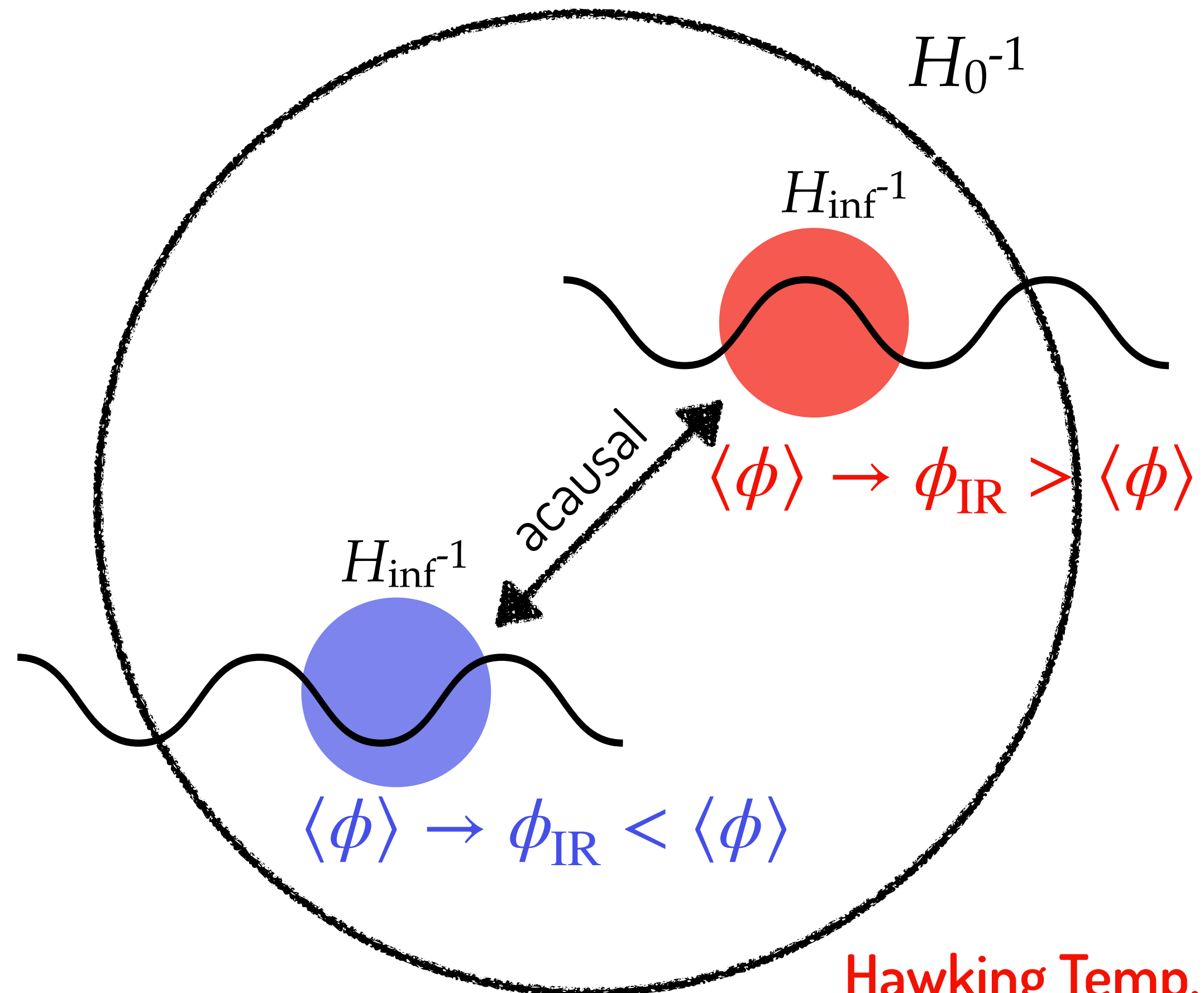
Inflation

pre-Big-Bang accelerated expansion



Stochastic Form.

Starobinsky '86



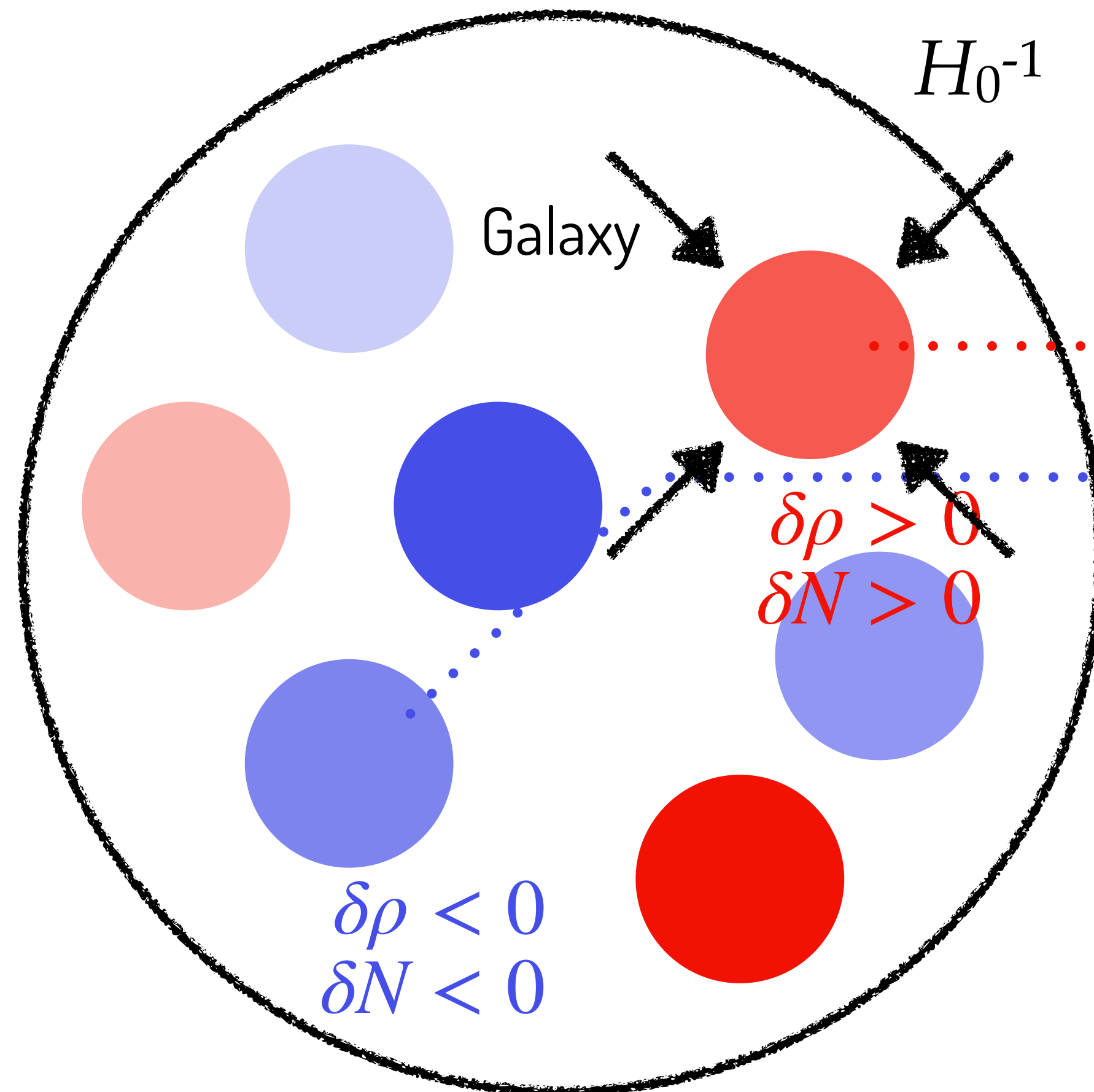
ρ_r : Big-Bang universe

❖ Stochastic EoM:
$$\frac{d\phi_{IR}}{dN} = -\frac{V'}{3H^2} + \frac{H}{2\pi} \xi$$

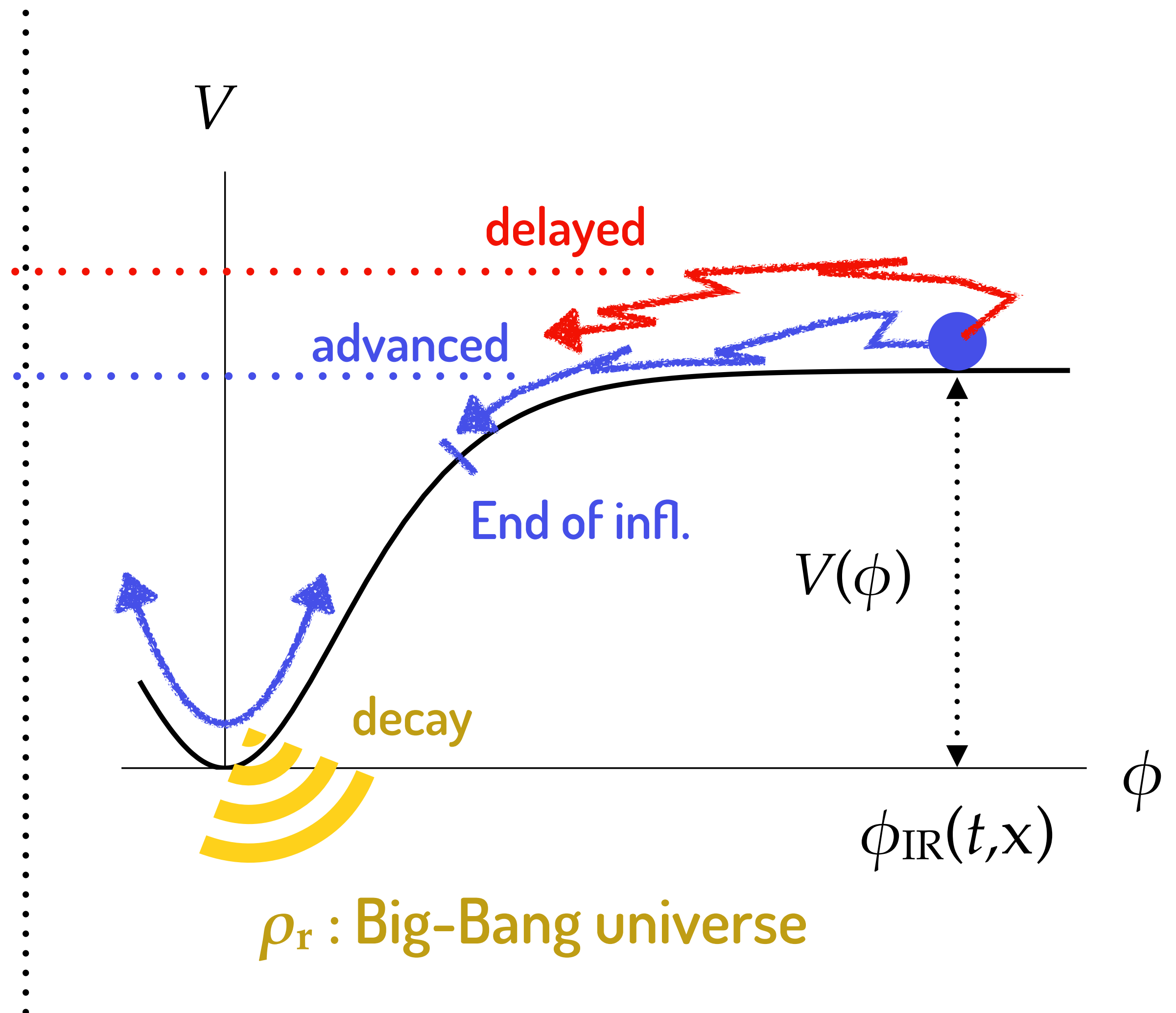
Hawking Temp. Gaussian Rand.

(conserved) δN Form.

Starobinsky '85



- ❖ Stochastic- δN formalism
Fujita, Kawasaki, YT, Takesato '13
Vennin & Starobinsky '15



PDE Approach

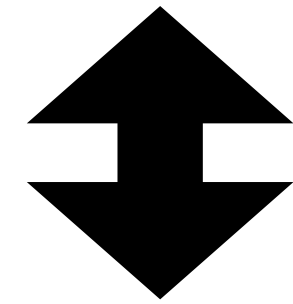
Vennin & Starobinsky '15

- ❖ Fokker-Planck eq. (diffusion)

$$\partial_N P(\phi^I; N) = -\partial_I [h^I P(\phi^I; N)] + \frac{1}{2} \partial_I \partial_J [A^{IJ} P(\phi^I; N)]$$

PDF of ϕ^I @ N

e.g. $h^I = -\frac{V^I}{3H^2}, \quad A^{IJ} = \left(\frac{H}{2\pi}\right)^2 \delta^{IJ}$



- ❖ adjoint FP eq.

$$\partial_{\mathcal{N}} \bar{P}(\mathcal{N}; \phi^I) = h^I \partial_I \bar{P}(\mathcal{N}; \phi^I) + \frac{1}{2} A^{IJ} \partial_I \partial_J \bar{P}(\mathcal{N}; \phi^I)$$

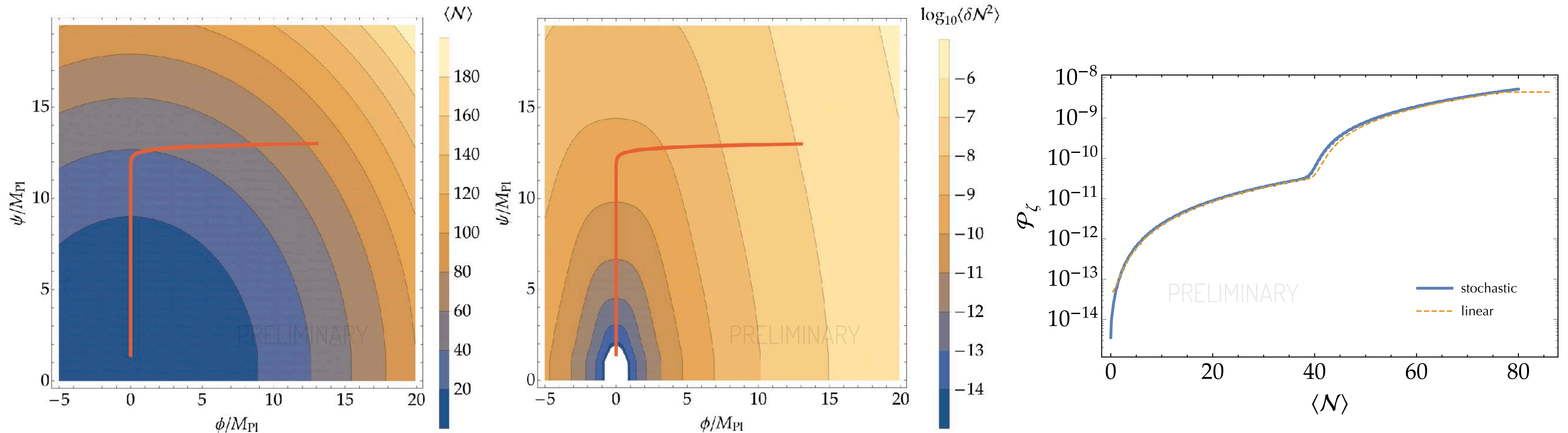
PDF of 1st. passage time $\mathcal{N}$ from ϕ^I

StocDeltaN.cpp

Renaux-Petel, YT, Vennin in prep.

❖ Double Mass-term

$$V = \frac{1}{2}M^2\phi^2 + \frac{1}{2}m^2\psi^2, \quad M = 9m = 10^{-5}M_{\text{Pl}}$$

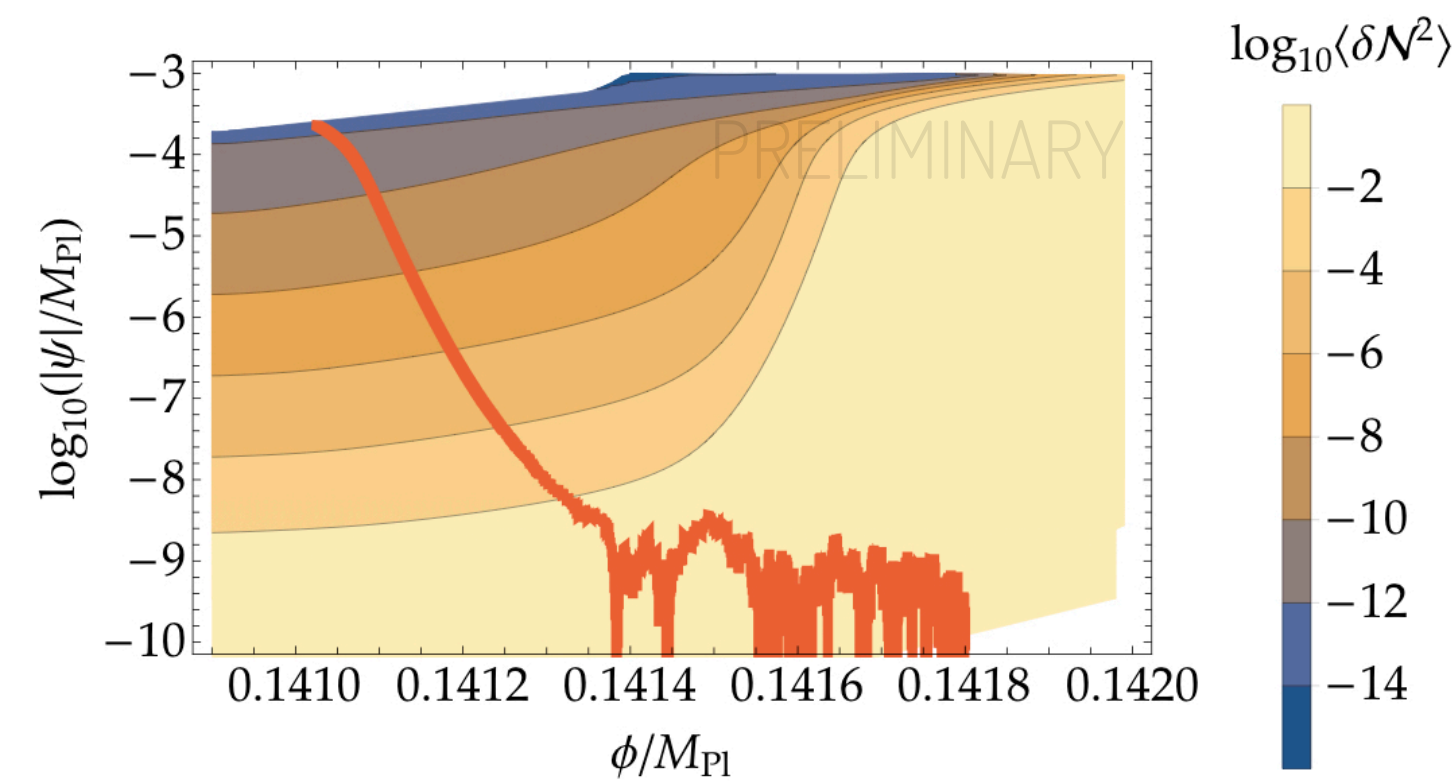
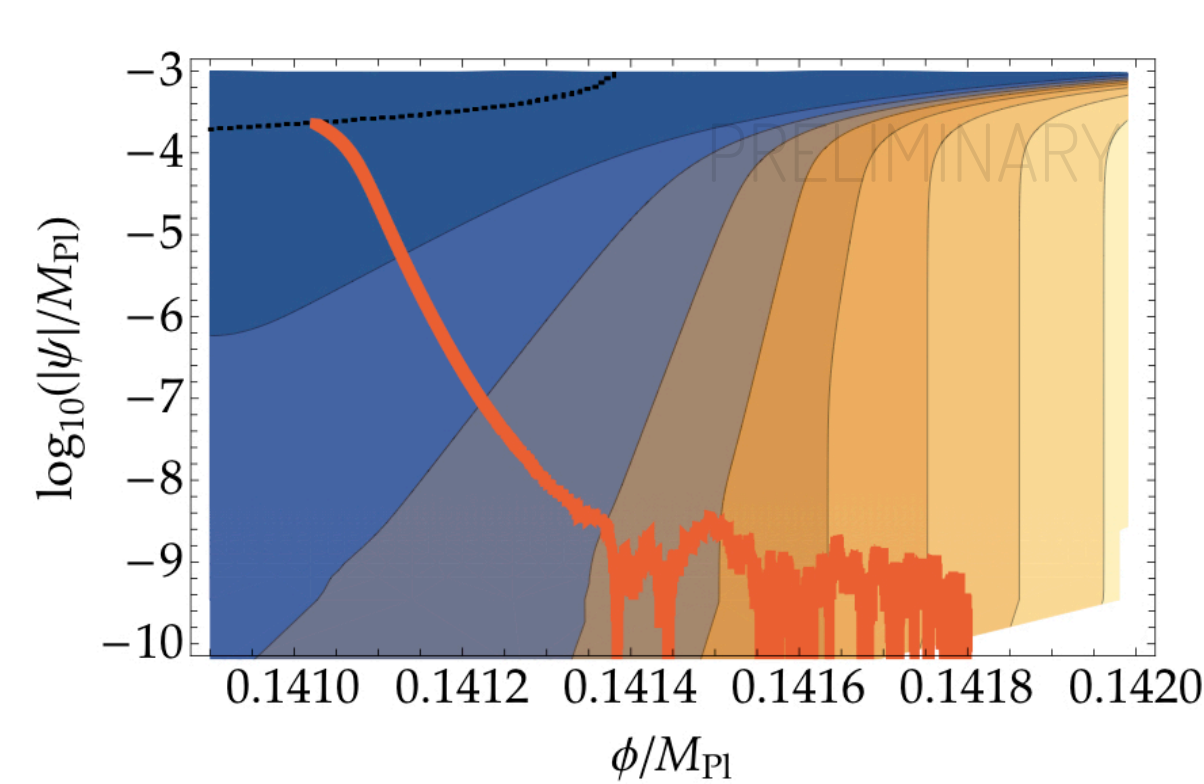
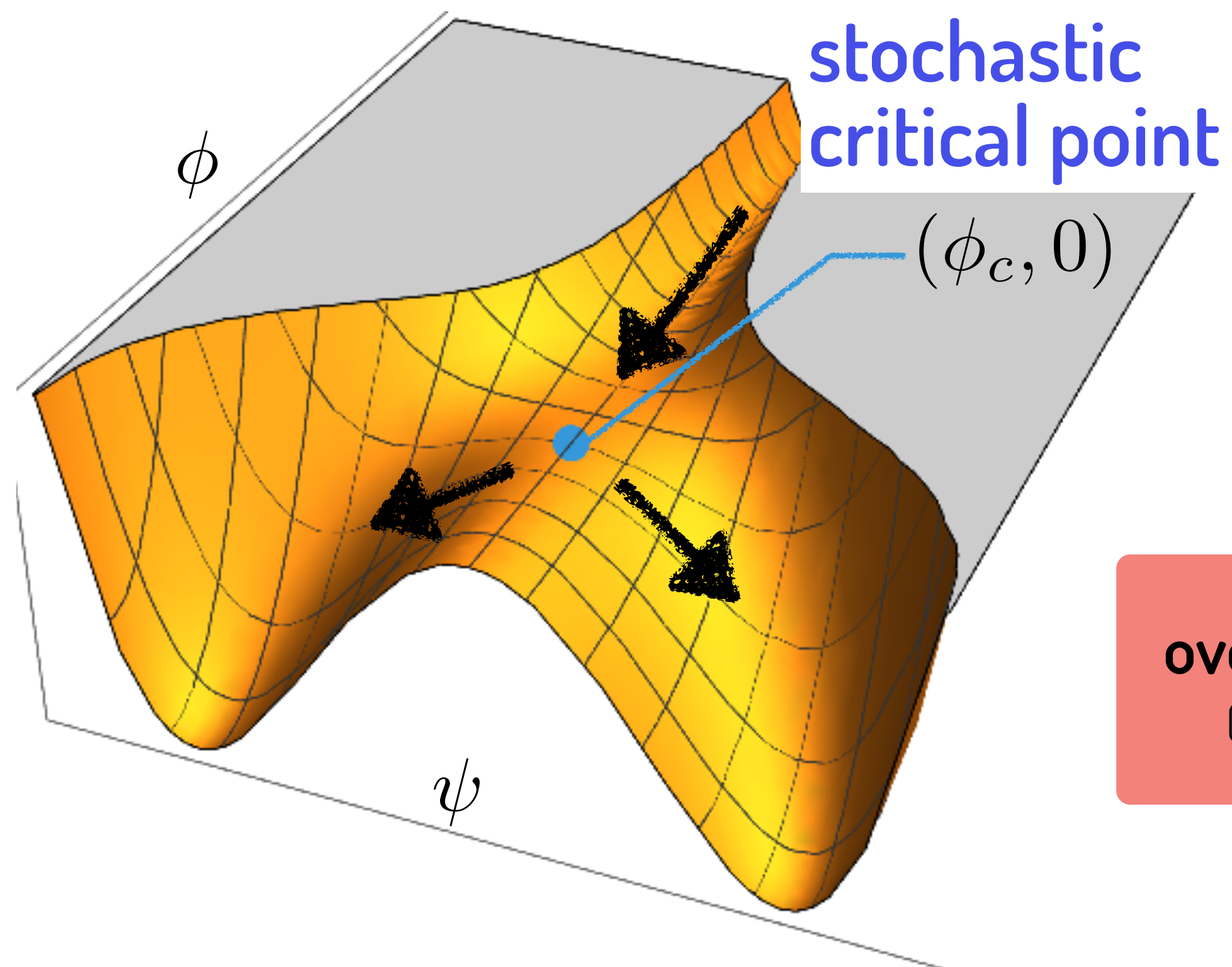


StocDeltaN.cpp

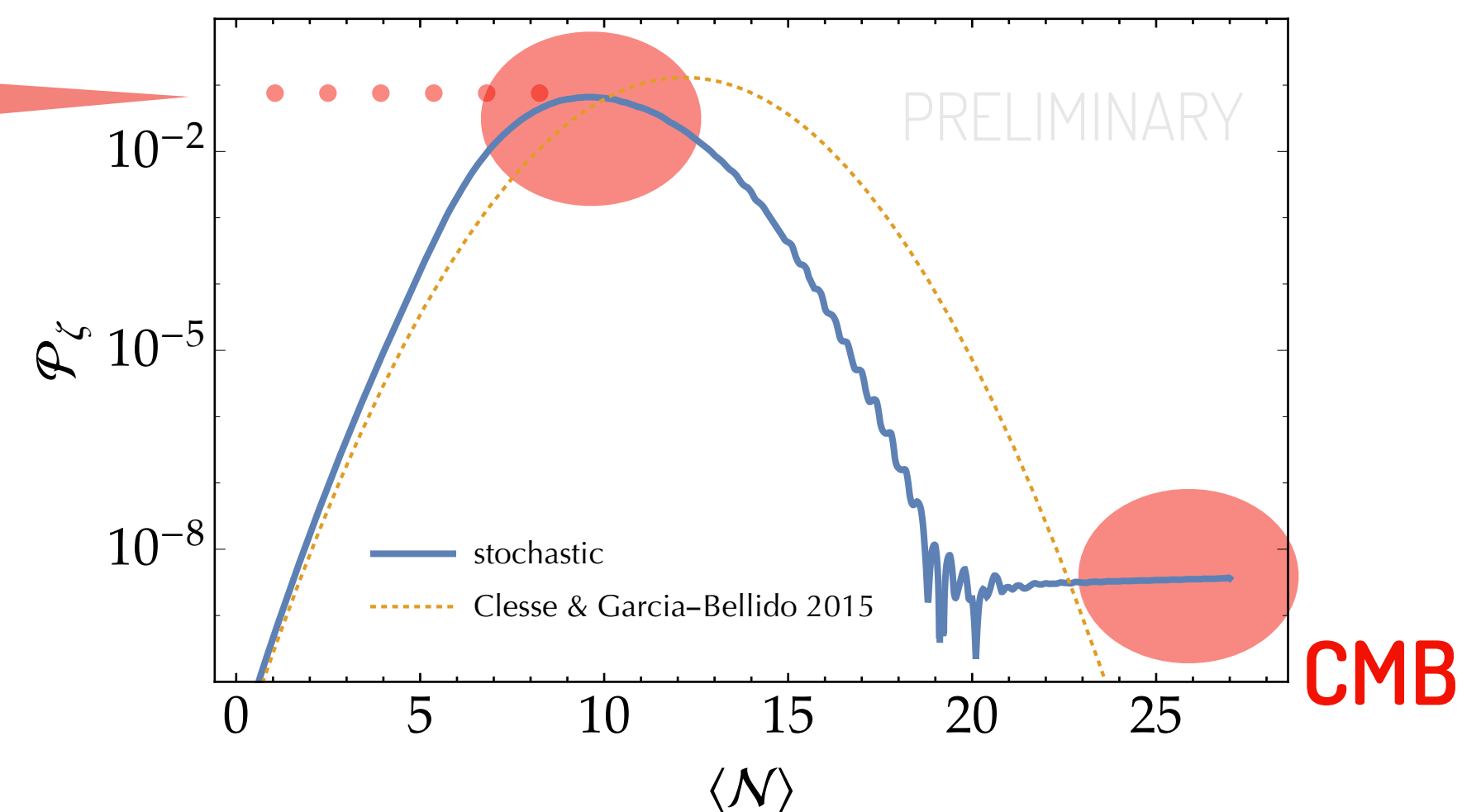
Renaux-Petel, YT, Vennin in prep.

❖ Hybrid Inflation

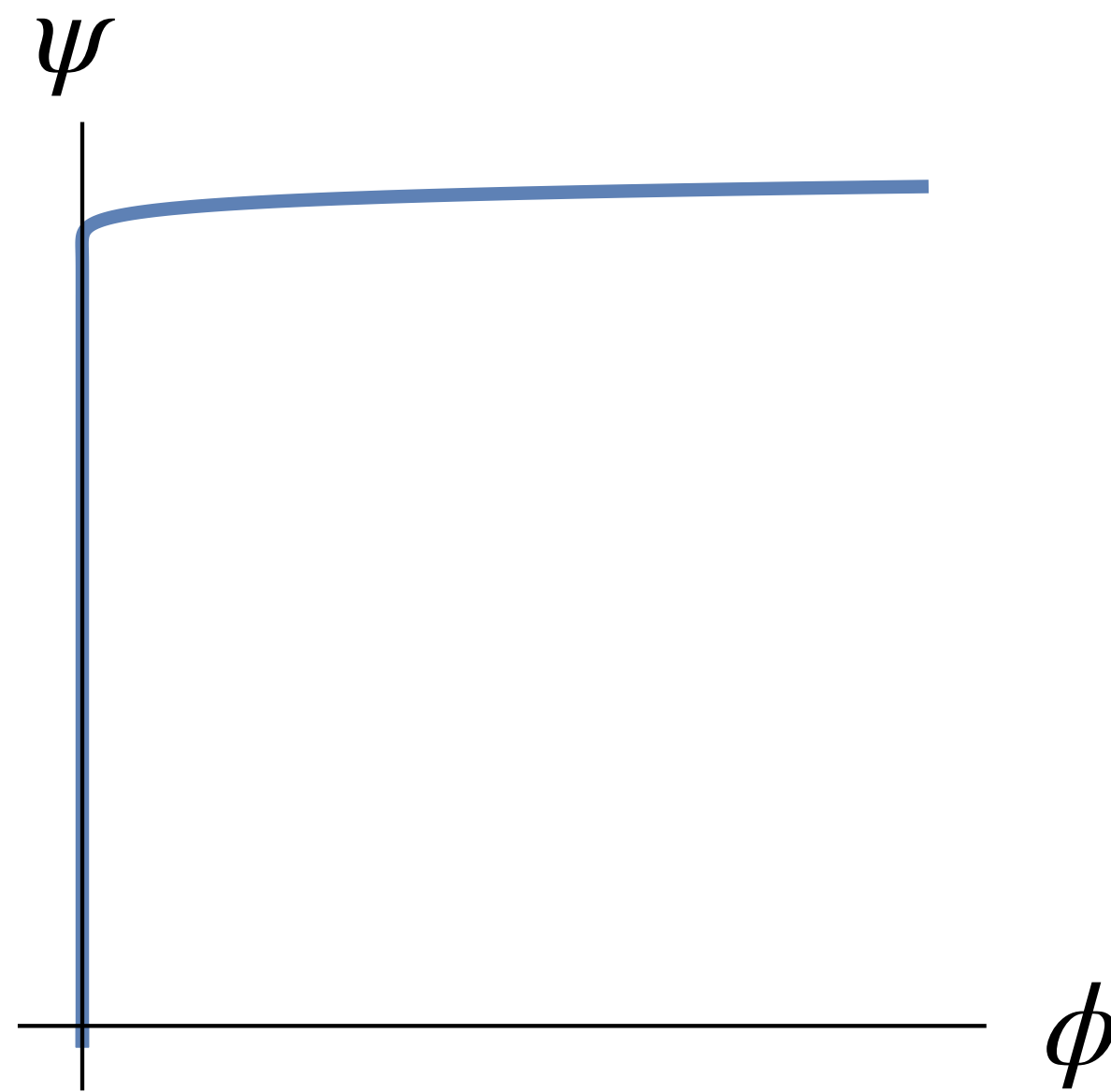
$$V = \Lambda^4 \left[\left(1 - \frac{\psi^2}{M^2} \right)^2 + 2 \frac{\phi^2 \psi^2}{\phi_c^2 M^2} + \frac{\phi - \phi_c}{\mu_1} - \frac{(\phi - \phi_c)^2}{\mu_2^2} \right]$$



overproduce PBHs
(Kawasaki, YT 2015)

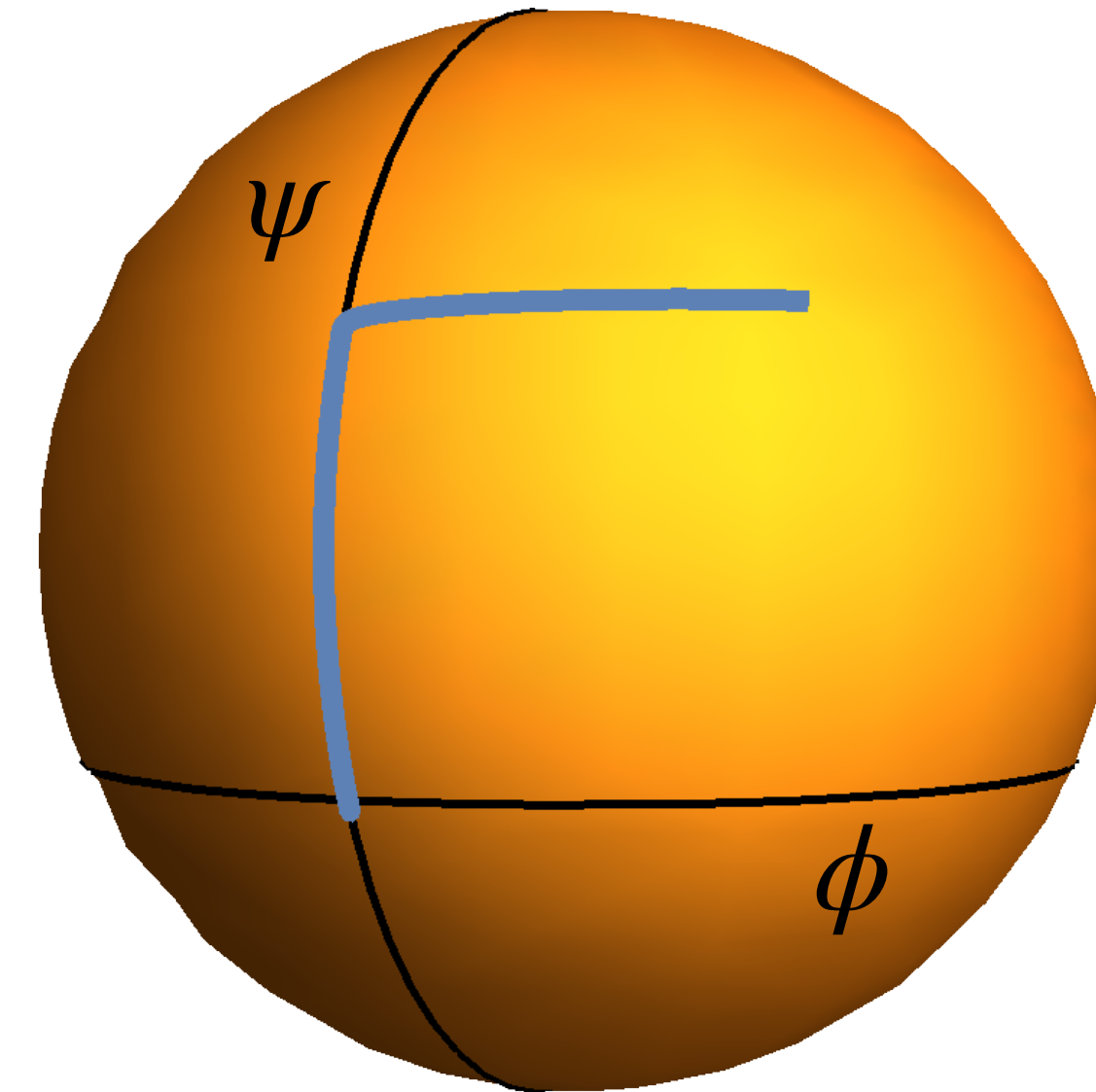


❖ Flat Fields



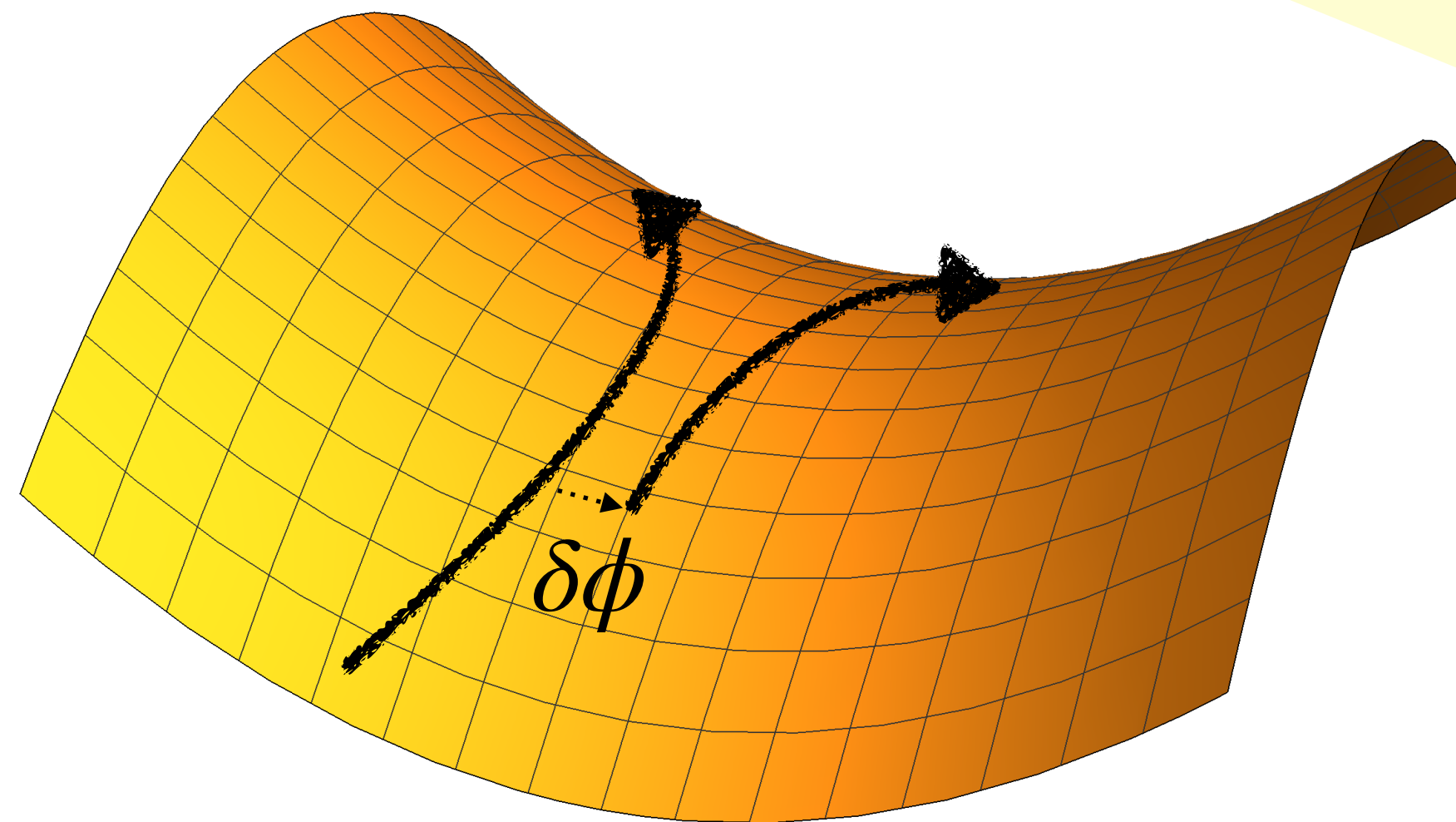
$$\begin{aligned}\mathcal{L}_{\text{kin}} &= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\psi)^2 \\ &= -\frac{1}{2}\delta_{IJ}\partial_\mu\phi^I\partial^\mu\phi^J\end{aligned}$$

❖ Curved Fields



$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}G_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J$$

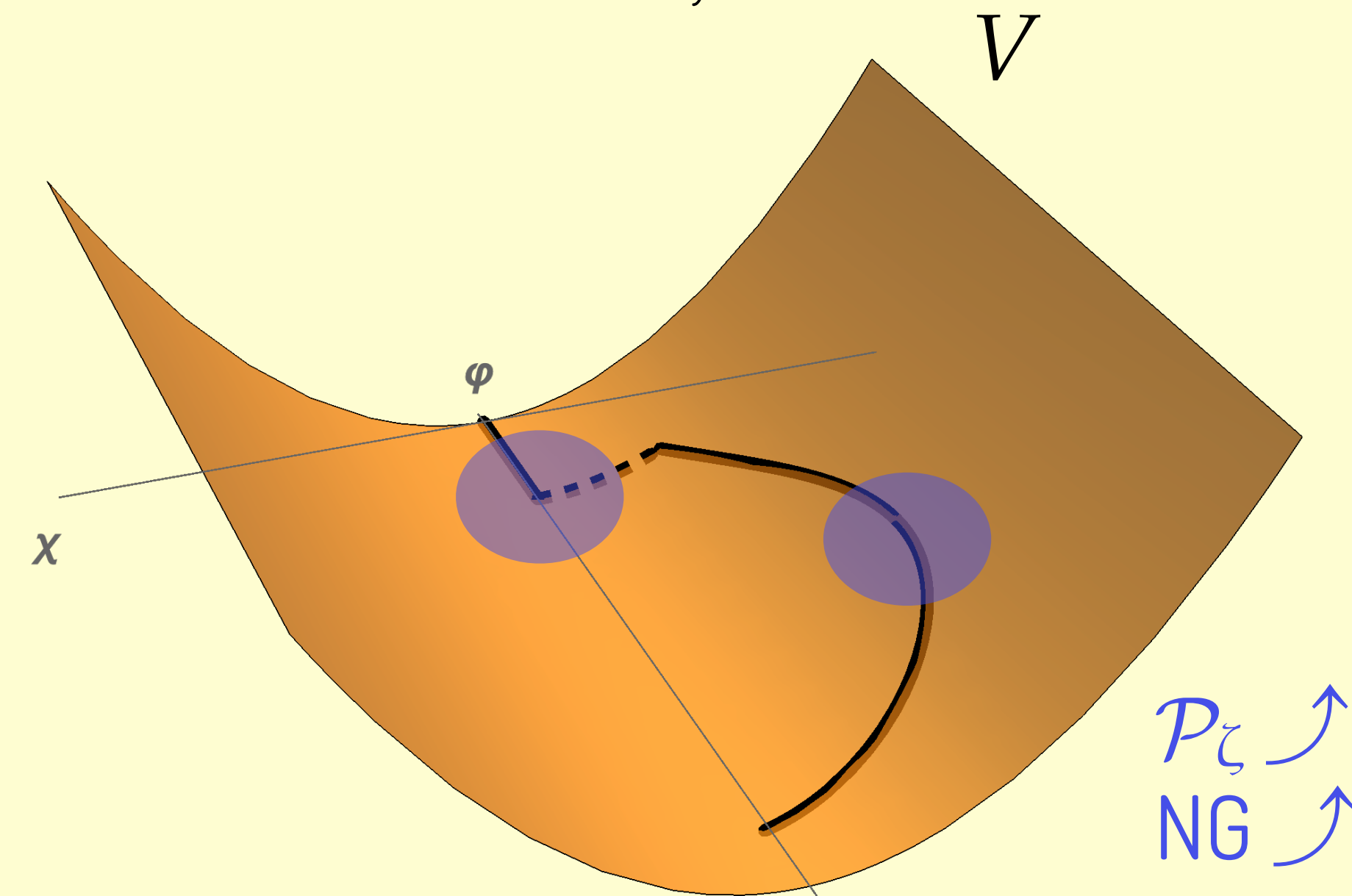
❖ Hyperbolic $R < 0$



unstable

Sidetracked Inflation

Gracia-Saenz, Renaux-Petel, Ronayne '18



Generalization?

Pinol, Renaux-Petel, YT '18

❖ General Multi-scalar $\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\overset{\text{Inflaton-space metric}}{G_{IJ}(\phi)}\partial_\mu\phi^I\partial_\nu\phi^J - V(\phi)$

Stochastic EoM ??

~~$$\frac{d\phi^I}{dN} = -\frac{G^{IJ}\partial_J V}{3H^2} + \xi^I, \quad \text{with } \langle \xi^I(N)\xi^J(N') \rangle = \left(\frac{H}{2\pi}\right)^2 G^{IJ}\delta(N-N')$$~~

- Covariance under $\phi^I \rightarrow \bar{\phi}^{\bar{I}} = \bar{\phi}^{\bar{I}}(\phi)$

and/or
$$\frac{d\phi^I}{dN} = -\frac{G^{IJ}\partial_J V}{3H^2} + \xi^I \quad \rightarrow \quad \frac{d\bar{\phi}^{\bar{I}}}{dN} \neq -\frac{G^{\bar{I}\bar{J}}\partial_{\bar{J}} V}{3H^2} + \bar{\xi}^{\bar{I}}$$

- Spurious Frame Dependence

$$\frac{d\phi^I}{dN} = -\frac{G^{IJ}\partial_J V}{3H^2} + \xi^I \neq -\frac{G^{IJ}\partial_J V}{3H^2} + \overset{\text{Rotation/Diagonalization}}{R^I_{\tilde{A}}}\tilde{\xi}^{\tilde{A}}$$



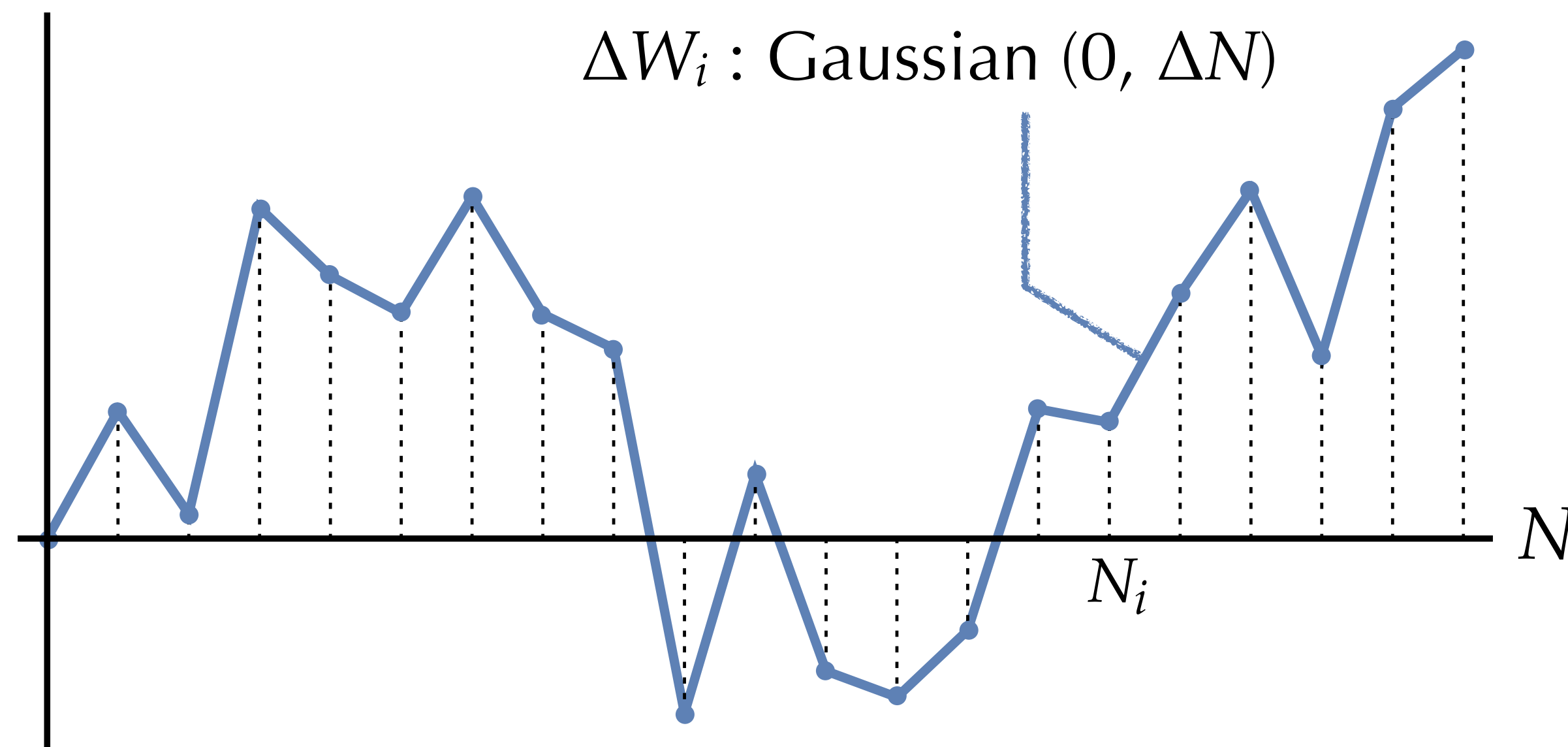
Manifestly Covariant Stochastic Inflation

Stoc. Calculus

❖ Stochastic Differential eq.

$$\frac{dX(N)}{dN} = A(X(N)) \xi(N), \quad \langle \xi(N) \xi(N') \rangle = \delta(N - N')$$

Brownian $W(N)$



$$\Delta X_i = A(X_{i+\alpha}) \Delta W_i$$

$$X_{i+\alpha} = (1 - \alpha)X_i + \alpha X_{i+1}$$
$$0 \leq \alpha < 1$$

Itô: $X_{i+0} = X_i$
Stratonovich: $X_{i+1/2} = (X_i + X_{i+1})/2$

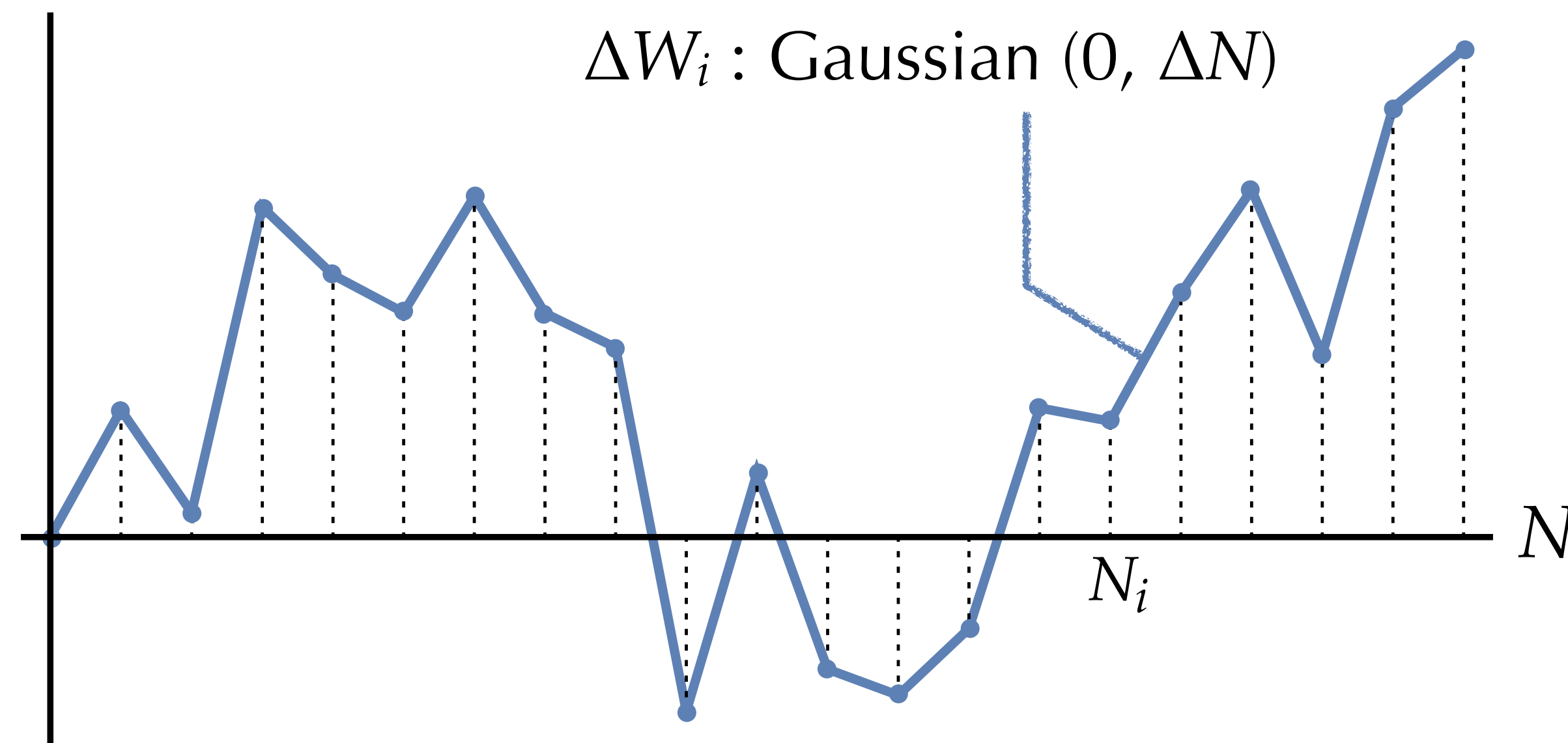
Stoc. Calculus

❖ Stochastic Differential eq.

$$\frac{dX(N)}{dN} = A(X(N)) \xi(N), \quad \langle \xi(N) \xi(N') \rangle = \delta(N - N')$$

X 's property (PDF) depends on the choice of α !

Brownian $W(N)$



$$\Delta X_i = A(X_{i+\alpha}) \Delta W_i$$

$$X_{i+\alpha} = (1 - \alpha)X_i + \alpha X_{i+1}$$

$$0 \leq \alpha < 1$$

Itô: $X_{i+0} = X_i$

Stratonovich: $X_{i+1/2} = (X_i + X_{i+1})/2$

Exercise

$$dZ = WdW$$

❖ Itô's scheme

$$\Delta Z_i = W_i \Delta W_i \quad \rightarrow \quad \langle \Delta Z_i \rangle = \langle W_i \Delta W_i \rangle = 0$$

No Correlation btw.
current position & noise

$$Z = \int dZ = \int WdW = \frac{1}{2}W^2 - \frac{1}{2}N$$

$\langle Z \rangle = 0$ & $\langle W^2 \rangle = N$

- Itô's lemma: $df(N, X) = f_N dN + f_X dX + \frac{1}{2} f_{XX} dX dX = A^2 dN$

$$d(W^2) = 2WdW + \frac{1}{2} \times 2dWdW = 2WdW + dN$$

$$\rightarrow dZ = \frac{1}{2}d(W^2) - \frac{1}{2}dN$$

Exercise

$$d\bar{Z} = W \circ dW$$

❖ Stratonovich's scheme

$$\Delta\bar{Z}_i = W_{i+1/2}\Delta W_i = \left(W_i + \frac{1}{2}\Delta W_i\right)\Delta W_i \quad \rightarrow \quad \langle\Delta\bar{Z}_i\rangle = \frac{1}{2}\langle(\Delta W_i)^2\rangle = \frac{1}{2}\Delta N$$

$$\bar{Z} = \int d\bar{Z} = \int W \circ dW = \frac{1}{2}W^2 \quad \langle\bar{Z}\rangle = \frac{1}{2}\langle W^2\rangle = \frac{1}{2}N$$

Std. Calculus !!

– Strato $\Leftrightarrow$ Itô

$$\Delta Y_i = h(N_{i+1/2}, Y_{i+1/2})\Delta N + g(N_{i+1/2}, Y_{i+1/2})\Delta W$$

$$= h(N_i, Y_i)\Delta N + g(N_i, Y_i)\Delta W + \frac{1}{2}g_Y g \Delta N$$

Noise-induced Drift

Stoc. Anomaly

Pinol, Renaux-Petel, YT '18

$$d\phi^I \stackrel{?}{=} -\frac{V^I}{3H^2}dN + \frac{H}{2\pi}e_a^I \circ_? dW^a \quad \begin{cases} e_a^I e_a^J = G^{IJ}, \\ dW^a dW^b = \delta^{ab}dN \end{cases}$$

❖ Covariance: $\phi^I \rightarrow \bar{\phi}^{\bar{I}} = \bar{\phi}^{\bar{I}}(\phi)$



Itô's lemma

$$d\bar{\phi}^{\bar{I}} = \frac{\partial \bar{\phi}^{\bar{I}}}{\partial \phi^J} d\phi^J + \frac{1}{2} \frac{\partial^2 \bar{\phi}^{\bar{I}}}{\partial \phi^J \partial \phi^K} d\phi^J d\phi^K$$

Break Covariance



Stratonovich

❖ Spurious Frame Dependence

$$d\phi^I = -\frac{V^I}{3H^2}dN + \frac{H}{2\pi}e_a^I \circ dW^a$$

$$= -\frac{V^I}{3H^2}dN + \frac{H}{2\pi}e_a^I dW^a + \frac{1}{2} \times \frac{H}{2\pi} e_{a,J}^I e_a^J dN$$



Spurious frame dependence in Strato

$$e_a^I \rightarrow \bar{e}_{\bar{a}}^{\bar{I}} = R_{\bar{a}}^{\bar{I}b}(\phi) e_b^I$$

❖ Stoc. vs. QFT Tokuda & Tanaka '17 & '18

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\lambda\phi^4$$



$$\mathcal{O}(\lambda^1) : \langle \phi^2 \rangle_{\text{Itô}} = \langle \phi^2 \rangle_{\text{QFT}}$$

🌀 non-Markovian origin of noise

$$\delta\hat{\phi}_{\mathbf{k}}^I = e_A^I Q_a^A \hat{a}_{\mathbf{k}}^a + e_A^I Q_a^{A*} \hat{a}_{-\mathbf{k}}^{a\dagger}$$

$$\xrightarrow{k \ll aH} e_A^I Q_a^A (\hat{a}_{\mathbf{k}}^a + \hat{a}_{-\mathbf{k}}^{a\dagger})$$

($\text{Im} Q_a^A \rightarrow 0$ up to const. phase)

CURRENT trs. (local frame $\rightarrow$ global coord.) : Strato

Quantum origin of noise $\rightarrow dW^a$

- Local frame mode function
- HISTORY of subhorizon dynamics : Itô

Preferred Frame

Pinol, Renaux-Petel, YT '20

$$\left\{ \begin{array}{l} d\phi^I = -\frac{V^I}{3H^2}dN + e_A^I \overset{\text{Strato}}{\underset{\text{Noise Amp. in Local Frame}}{Q_a^A}} dW^a \\ + \\ \mathcal{D}e_A^I = de_A^I + \Gamma_{JK}^I e_A^J \circ d\phi^K = 0 \quad (\text{or} = \Omega_A^B e_B^I dN) \end{array} \right.$$

Q_a^A is "Itô"-like

e_A^I is in itself a "Stoc. Variable" along the trajectory

$$d\phi^I = -\frac{V^I}{3H^2}dN + e_A^I Q_a^A dW^a + \frac{1}{2} Q_a^A \overset{\triangle}{de_A^I} dW^a = -\frac{1}{2} \Gamma_{JK}^I A^{JK} dN$$

$$A^{IJ} = e_A^I Q_a^A e_B^J Q_a^B \sim \left(\frac{H}{2\pi} \right)^2 G^{IJ}$$

$$\mathfrak{D}_N \phi^I := \frac{d\phi^I}{dN} + \frac{1}{2} \Gamma_{JK}^I A^{JK} = -\frac{V^I}{3H^2} + \xi^I$$

Itô Covariant Derivative Graham '85

No e_A^I -dependence

Cov. Stoc. EoM

Pinol, Renaux-Petel, YT '20

take-home message

Manifestly Covariant Stochastic EoM

$$\begin{cases} \mathfrak{D}_N \phi^I = \frac{\pi^I}{H} + \xi^{QI} \\ \mathfrak{D}_N \pi_I = -3\pi_I - \frac{V_I}{H} + \xi^{PI} \end{cases}$$

w/ Itô Cov. Derivatives $\mathfrak{D}\bar{\phi}^{\bar{I}} = \frac{\partial \bar{\phi}^{\bar{I}}}{\partial \phi^J} \mathfrak{D}\phi^J$, $\frac{\partial \bar{\phi}^{\bar{I}}}{\partial \phi^J} \mathfrak{D}\bar{\pi}_{\bar{I}} = \mathfrak{D}\pi_J$

$$\begin{cases} \mathfrak{D}\phi^I := d\phi^I + \frac{1}{2} \Gamma_{JK}^I A^{QQJK} dN \\ \mathfrak{D}\pi_I := \mathcal{D}\pi_I - \frac{1}{2} (\Gamma_{IJ,K}^S + \Gamma_{IJ}^M \Gamma_{KM}^S) \pi_S A^{QQJK} dN - \Gamma_{IJ}^K A^{QPJ}_K dN \end{cases}$$

Conclusions

- ❖ Stochastic form. is powerfull
- ❖ Stochastic Anomaly: Itô or Strato?
 - ~~Covariance~~ or Spurious frame dependence
- ❖ Manifestly covariant SDE in Itô