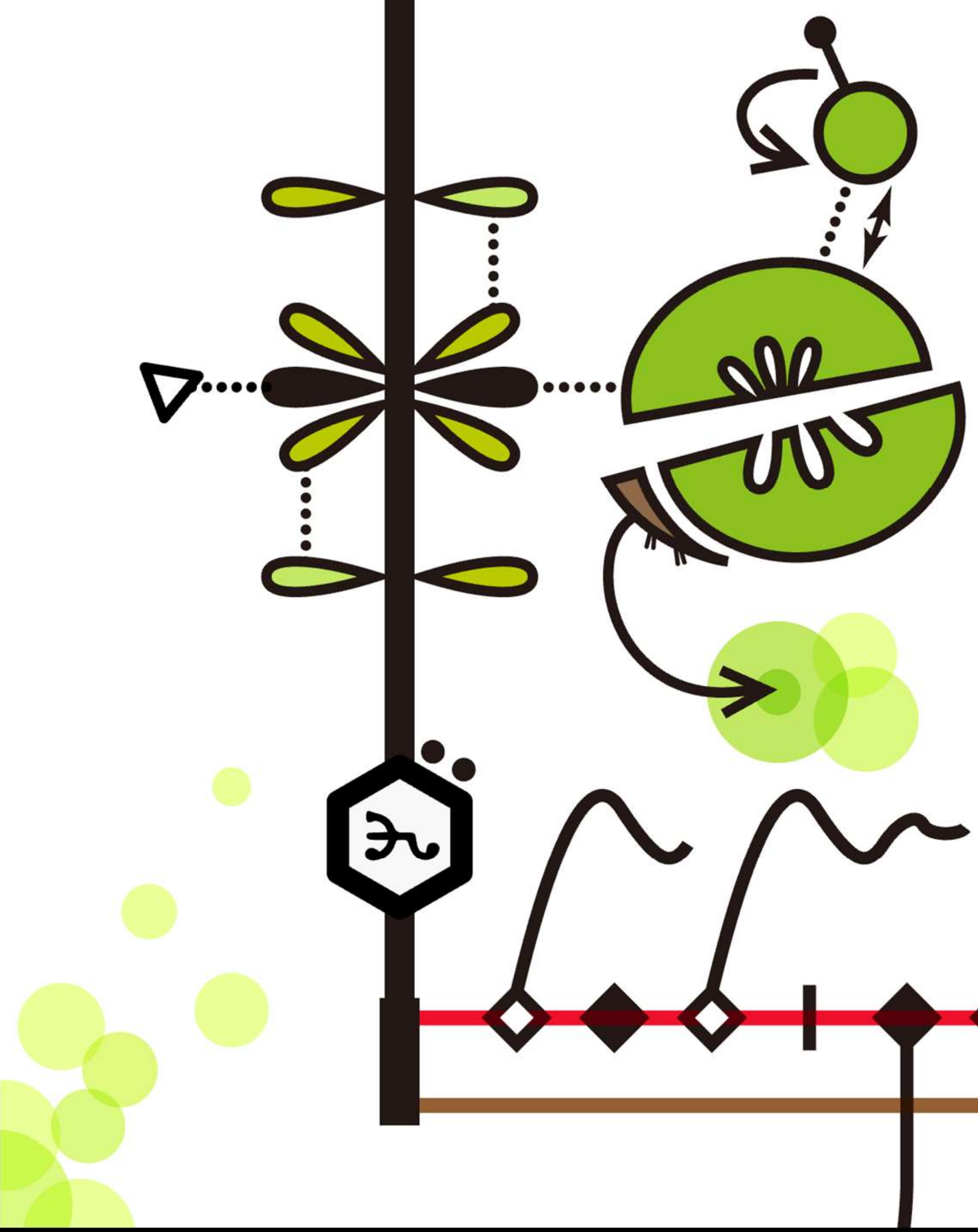
# Re:presentation



*Introducing Myself*

*Representation I – Tensor Graphs*

*Representation II – Thermodynamics*

*Re:presentation – Listen to Tensors*

JoonHwi Kim 2021

# Introducing Myself

## Joon-Hwi Kim 김준휘

HEP-Theory
Interested in Quantum Gravity, Scattering Amplitudes, Geometrical Ideas in Physics

2021.09.- Caltech, to pursue a Ph.D. degree

2017-2020 Seoul National University, B.S. in Physics

2014-2016 Seoul Science High School for the Gifted

**J.-H. Kim**, J.-W. Kim, and S. Lee (2021). The Relativistic Spherical Top as a Massive Twistor. [arXiv:2102.07063]

**J.-H. Kim** and J. Nam (2021). Thermodynamic Identities with Sunray Diagrams. *European Journal of Physics 43*(2). [arXiv:1912.11485]

**J.-H. Kim**, M. S. H. Oh, and K.-Y. Kim (2021). Boosting Vector Calculus with the Graphical Notation. *American Journal of Physics 89*(2). [arXiv:1911.00892]

# Introducing Myself

**[2102.07063]**

Prepared for submission to JHEP

## The Relativistic Spherical Top as a Massive Twistor

**Joon-Hwi Kim**[a] **Jung-Wook Kim**[b] **Sangmin Lee**[a,c,d]

[a] *Department of Physics and Astronomy, Seoul National University, Seoul 08826, Korea*
[b] *Centre for Research in String Theory, School of Physics and Astronomy, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom*
[c] *Center for Theoretical Physics, Seoul National University, Seoul 08826, Korea*
[d] *College of Liberal Studies, Seoul National University, Seoul 08826, Korea*

*E-mail:* joonhwi.kim@snu.ac.kr, jung-wook.kim@qmul.ac.uk, sangmin@snu.ac.kr

Abstract: We prove the equivalence between two traditional approaches to the classical mechanics of a massive spinning particle in special relativity. One is the spherical top model of Hanson and Regge, recast in a Hamiltonian formulation with improved treatment of covariant spin constraints. The other is the massive twistor model, slightly generalized to incorporate the Regge trajectory relating the mass to the total spin angular momentum. We establish the equivalence by computing the Dirac brackets of the physical phase space carrying three translation and three rotation degrees of freedom. Lorentz covariance and little group covariance uniquely determine the structure of the physical phase space. The Regge trajectory does not affect the phase space but enters the equations of motion. Upon quantization, the twistor model produces a spectrum that agrees perfectly with the massive spinor-helicity description proposed by Arkani-Hamed, Huang and Huang for scattering amplitudes for all masses and spins.

### Vectorial Description

$$\omega = d\left(p_\mu dx^\mu + \frac{1}{2} S_{\mu\nu} \Lambda^\mu_A d\Lambda^{\nu A}\right)$$

$$\phi_0 = \frac{1}{2}(p^2 + m^2)\,, \quad \phi_a = \frac{1}{2}(\hat{p}^\mu + \Lambda^\mu{}_0) S_{\mu\nu} \Lambda^\nu{}_a\,,$$

$$\chi^0 = \frac{1}{p^2} x^\mu p_\mu\,, \quad \chi^a = \hat{p}_\mu \Lambda^{\mu a}\,.$$

Equivalent!

$$p_{\alpha\dot{\alpha}} = -\lambda_\alpha^I \bar{\lambda}_{I\dot{\alpha}}\,, \quad \Lambda^a_{\alpha\dot{\alpha}} = \frac{1}{|\det(\lambda)|} \lambda_\alpha^I (\sigma^a)_I{}^J \bar{\lambda}_{J\dot{\alpha}}$$

$$\mu^{\dot{\alpha}I} + z^{\dot{\alpha}\alpha} \lambda_\alpha^I = 0\,, \quad z^\mu = x^\mu + i y^\mu$$

$$L_{\mu\nu} = (x \wedge p)_{\mu\nu}\,, \quad S_{\mu\nu} = *(y \wedge p)_{\mu\nu}$$

### Spinorial (Twistor) Description

$$\omega = i\, d\bar{Z}_I{}^{\mathrm{A}} \wedge dZ_{\mathrm{A}}{}^I$$

$$\phi = -\frac{1}{2}\left(\det(\lambda) - m(\tilde{S}^2)\right)\,, \quad \bar{\chi} = \frac{1}{\det(\lambda)} \langle \bar{\mu} \lambda \rangle\,,$$

$$\bar{\phi} = -\frac{1}{2}\left(\det(\bar{\lambda}) - m(\tilde{S}^2)\right)\,, \quad \chi = \frac{1}{\det(\bar{\lambda})} [\bar{\lambda} \mu]\,.$$

# Introducing Myself

❖ Today's Talk

**J.-H. Kim**, M. S. H. Oh, and K.-Y. Kim (2021). Boosting Vector Calculus with the Graphical Notation. *American Journal of Physics 89*(2). [arXiv:1911.00892]

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# Introducing Myself

❖ Today's Talk

✓ The papers were in fact written for (talented) undergrads, not for PhDs.

✓ Thus, not only just strictly following the two papers, I also want to discuss the big picture:

- The role of notations in physics,
  the "representation cycle" of physics-algebra-geometry
- What determines good or bad notations?
  Why are graphical notations powerful?
- My (unpolished) idea of "notationology": RG flow of notational systems

✓ After that, I will add a little twist: re-presentation!

# 1. Representation

# Notation Engineering

❖ The History of "Notation Engineering" in Physics

| | |
|---|---|
| • Invariant theory graphs | (Sylvester, Cayley, … 1870-1880s) |
| • **Vector calculus** | (Heaviside, 1884) |
| • **Bra-ket notation** | (Dirac, 1939) |
| • **Feynman diagrams** | (Feynman, 1948) |
| • $3n$-$j$ symbols | (Levinson, Yutsis, Vanagas, … 1956) |
| • Penrose graphical notation | (Penrose, 1957) |
| • Birdtracks | (Cvitanović, 1976) |
| • Category theory | (Joyal, Street, Selinger, … 1990s) |
| • Trace diagrams | (Peterson, 2006) |
| • ZX Calculus | (Coecke, Duncan, … 2011) |
| • On-shell diagrams | (Hodges, Arkani-Hamed, … 2010s) |
| • Sunray diagrams | (**JHK**, 2019) |

**[JHK 1911.00892]**

# Notation Engineering

❖ The History of "Notation Engineering" in Physics

- Invariant theory graphs (Sylvester, Cayley, … 1870-1880s)
- **Vector calculus** (Heaviside, 1884)
- **Bra-ket notation** (Dirac, 1939)
- **Feynman diagrams** (Feynman, 1948)
- $3n$-$j$ symbols (Levinson, Yutsis, Vanagas, … 1956)
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- Trace diagrams (Peterson, 2006)
- ZX Calculus (Coecke, Duncan, … 2011)
- On-shell diagrams (Hodges, Arkani-Hamed, … 2010s)
- Sunray diagrams (JHK, 2019)

**What is the role of notations in physics?**

[JHK 1911.00892]

# Notation Engineering

## 1. Vector Notation (Heaviside)

| | | |
|---|---|---|
| $\begin{cases} \mu\alpha = \partial H/\partial y - \partial G/\partial z \\ \mu\beta = \partial F/\partial y - \partial H/\partial z \\ \mu\gamma = \partial G/\partial y - \partial F/\partial z \end{cases}$ | $\vec{B} = \vec{\partial} \times \vec{A}$ | $F = dA$ |
| $\begin{cases} f = -\partial F/\partial t - \partial \Psi/\partial x \\ g = -\partial G/\partial t - \partial \Psi/\partial y \\ h = -\partial H/\partial t - \partial \Psi/\partial z \end{cases}$ | $\vec{E} = -\dot{\vec{A}} - \vec{\partial} V$ | |
| $\partial f/\partial x + \partial g/\partial y + \partial h/\partial z = e$ | $\vec{\partial} \cdot \vec{E} = \rho$ | $d * F = J$ |
| $\begin{cases} \partial\gamma/\partial y - \partial\beta/\partial z = p + \frac{\partial f}{\partial t} \\ \partial\alpha/\partial z - \partial\gamma/\partial x = q + \frac{\partial g}{\partial t} \\ \partial\beta/\partial x - \partial\alpha/\partial y = r + \frac{\partial h}{\partial t} \end{cases}$ | $\vec{\partial} \times \vec{B} = \vec{J} + \dot{\vec{E}}$ | |
| $\partial e/\partial t + \partial p/\partial x + \partial q/\partial y + \partial r/\partial z = 0$ | $\dot{\rho} + \vec{\partial} \cdot \vec{J} = 0$ | $dJ = 0$ |
| $\begin{cases} P = ef + \mu(q\gamma - r\beta) \\ Q = eg + \mu(r\alpha - p\gamma) \\ R = eh + \mu(p\beta - q\alpha) \end{cases}$ | $\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B}$ | $f = *((*F) \wedge (*J))$ |

# Notation Engineering

## 2. Bra-ket Notation (Dirac)

$\alpha$ → $\alpha$

$\psi$ → $\left|\psi\right\rangle$

$\phi$ → $\left\langle\phi\right|$

$\left\langle\phi, \psi\right\rangle$ → $\left\langle\phi\middle|\psi\right\rangle$

"Syntax Highlight"

```python
class Dirac:
    """ Dirac """
    def __init__(self, Dirac):
        # Dirac
        self.Dirac = Dirac
        for Dirac in range(0,3):
            print(Dirac, self.Dirac)
```

$$\left(\text{Buffalo}^{\lambda}{}_{\kappa}\ \text{buffalo}^{\kappa}\right)\left(\left(\text{Buffalo}^{\rho}{}_{\sigma}\ \text{buffalo}^{\sigma}\right)\left(\text{buffalo}_{\rho}\right)^{\mu}{}_{\lambda}\right)\text{buffalo}_{\nu\mu}\left(\text{Buffalo}^{\nu}{}_{\xi}\ \text{buffalo}^{\xi}\right)$$

Physical significance

# Manifest symmetry/covariance

| | |
|---|---|
| Euclidean vector calculus: | SO($n$) covariance |
| Maxwell electromagnetism: | SO(1,3) covariance |
| Bra-ket notation: | GL($n$) covariance |

Practical aspects

# Enhanced readability & Save papers

Physical significance

# Manifest symmetry/covariance

| | |
|---|---|
| Euclidean vector calculus: | SO($n$) covariance |
| Maxwell electromagnetism: | SO(1,3) covariance ··· *Hints SR!* |
| Bra-ket notation: | GL($n$) covariance |

(Discovery of SR = Unearthing the Lorentz symmetry inherent in Maxwell EM)

Practical aspects

# Enhanced readability & Save papers

# Notation Engineering

## 3. Feynman Diagrams

*The “Representation Cycle”*

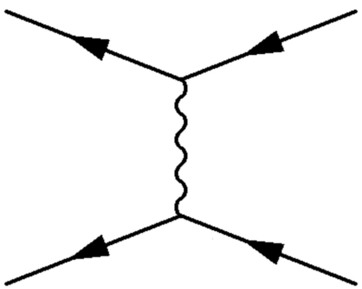
$$(ie)^2 \int d^4x\, d^4x'\, \underset{\leftarrow}{\psi}(x)\, \underline{\gamma^\mu}\, \underset{\rightarrow}{\psi}(x)\, \underset{\leftarrow}{\psi}(x')\, \underline{\gamma^\nu}\, \underset{\rightarrow}{\psi}(x')\, \boxed{A_\mu(x) A_\nu(x')}$$

*Algebra*

Represent

Interacting
Quantum
Particles

!



*Physics*

*Geometry*

Also consider:
- Non-commutative field theory and open strings
- Anyons and braiding of knots

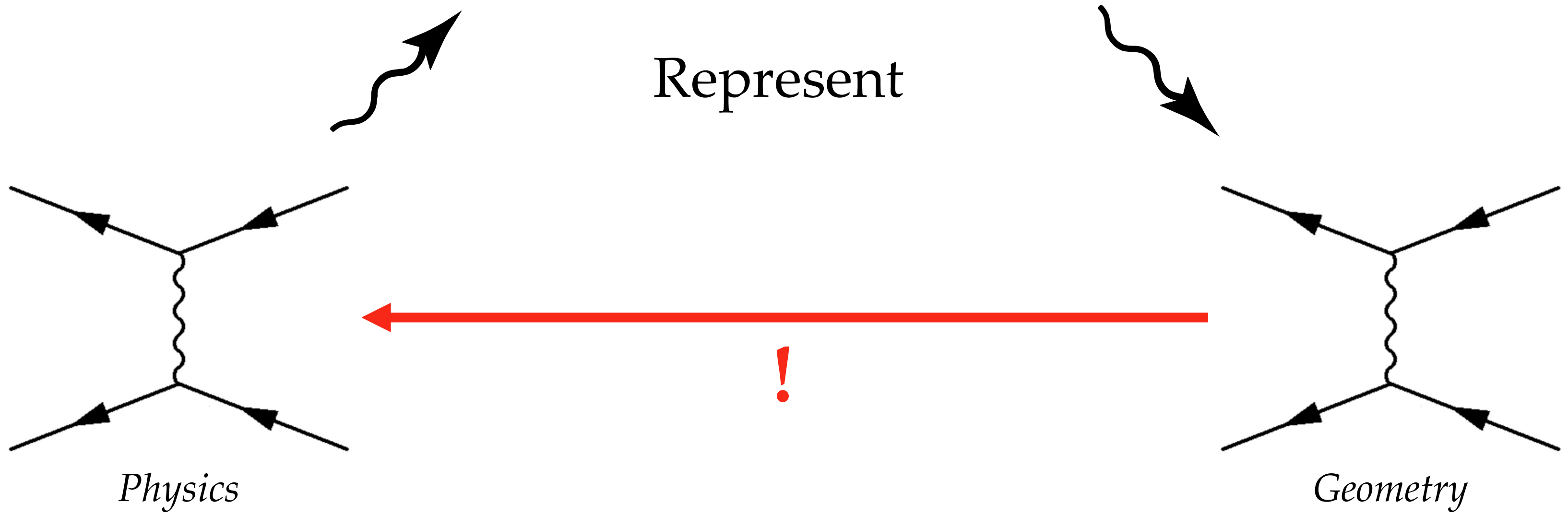
# Notation Engineering

## 3. Feynman Diagrams

*The "Representation Cycle"*

$$(ie)^2 \int d^4x\, d^4x'\, \underleftarrow{\psi}(x)\, \underline{\gamma^\mu}\, \underrightarrow{\psi}(x)\, \underleftarrow{\psi}(x')\, \underline{\gamma^\nu}\, \underrightarrow{\psi}(x')\, \boxed{A_\mu(x) A_\nu(x')}$$

*Algebra*




Also consider:
- Non-commutative field theory and open strings
- Anyons and braiding of knots

Physical significance

**Manifest symmetry/covariance**

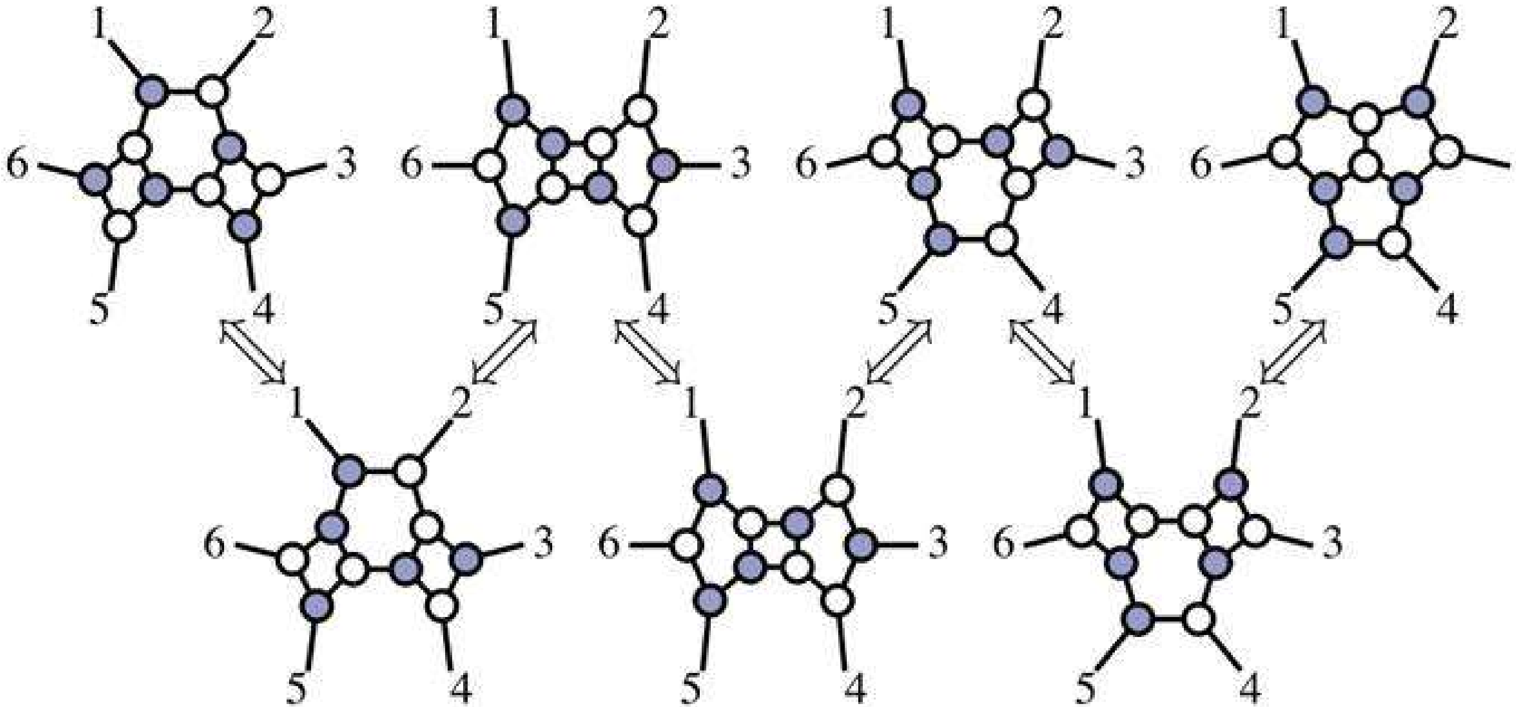
*May guide us to new physics!*

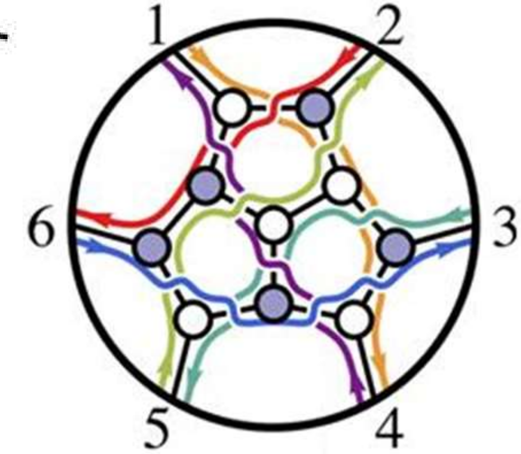
**Representation(s) of the physical reality**

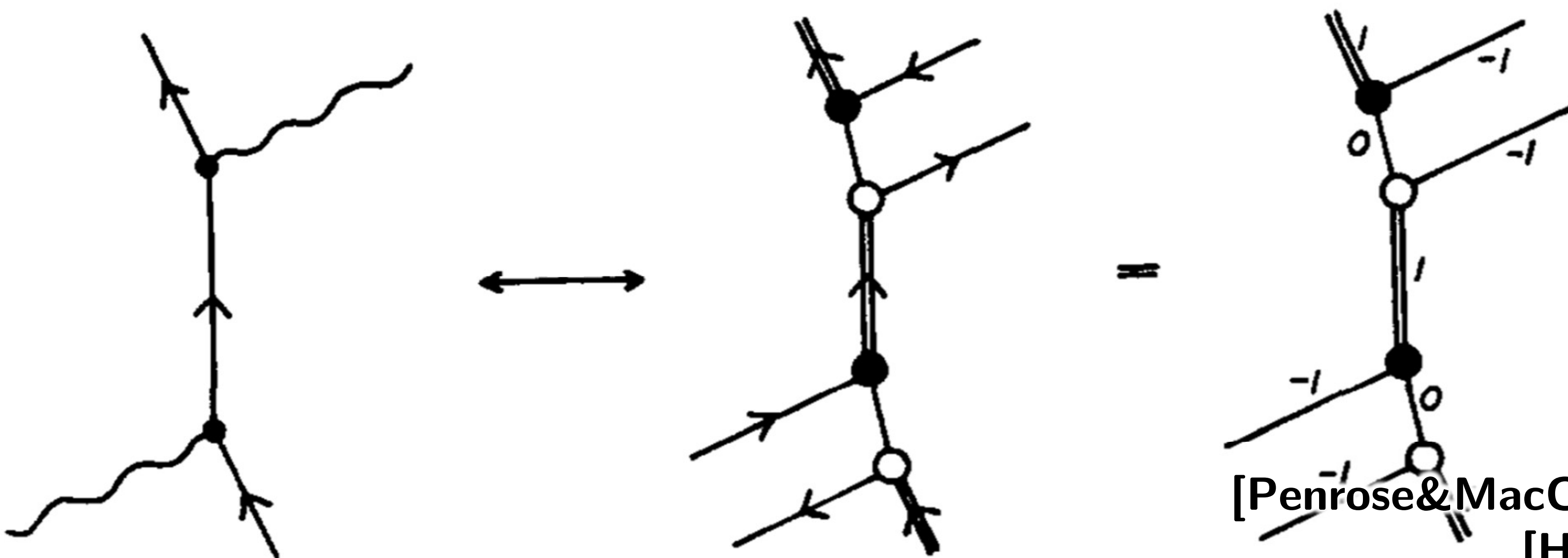
(Make fundamental constituents visible)

Practical aspects

**Enhanced readability & Save papers**




**[Penrose&MacCullum 1972]**
**[Hodges 1982]**
**[1604.03479]**

Physical significance

**Manifest symmetry/covariance**

*May guide us to new physics!*

**Representation(s) of the physical reality**
(Make fundamental constituents visible)

Practical aspects

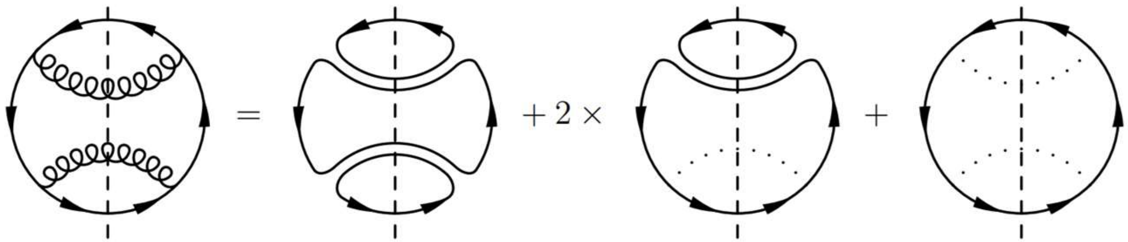
**Enhanced readability & Save papers**

**Intuitive manipulation in graphical syntax**

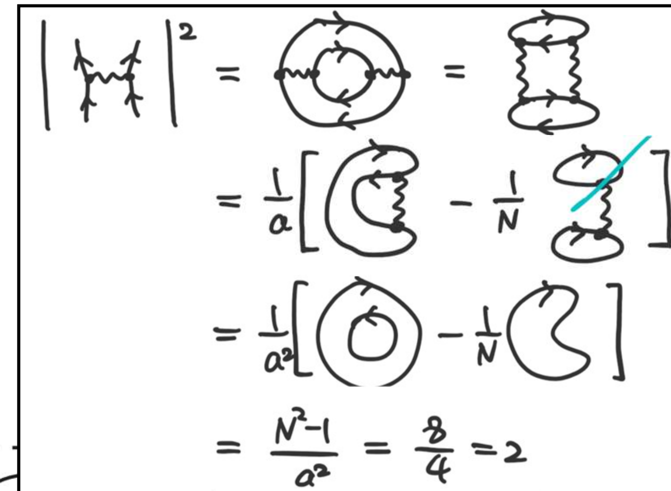
# Notation Engineering

## 3. Feynman Diagrams

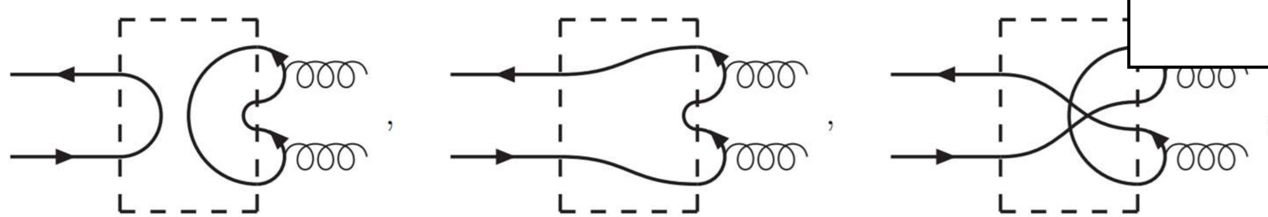
✓ QCD color factor calculation (Cvitanović's birdtracks)



$$= \frac{1}{4}\left(N^3 + 2N^2\left(\frac{-1}{N}\right) + N\left(\frac{-1}{N}\right)^2\right) = \frac{N}{4}\left(N - \frac{1}{N}\right)^2$$

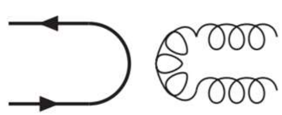
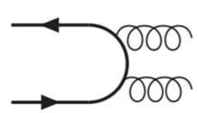
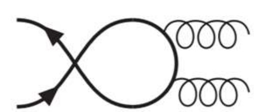



in all possible ways. The non-zero possibilities are

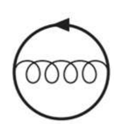


(82)

yielding

$c_1 =$  , $c_2 =$  , $c_3 =$  , (83)

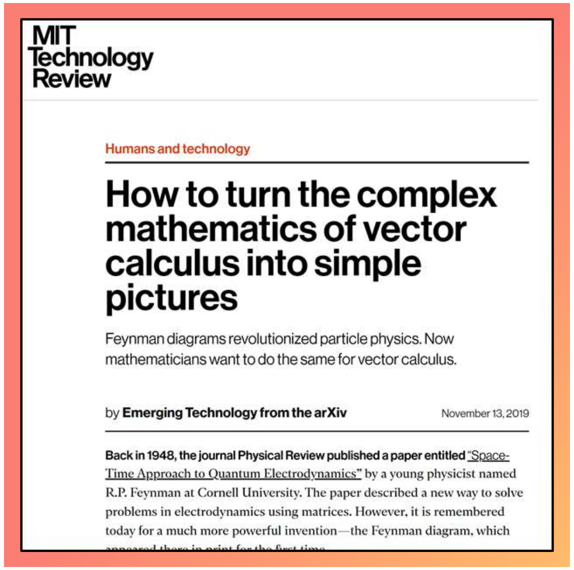
where we have omitted irrelevant prefactors. Note that in general the $c_j$ are not mutually orthogonal, e.g.

$\langle c_1, c_2\rangle = \mathrm{tr}(c_1^\dagger c_2) =$  $= T_R(N^2-1) = C_F N\,,$ (84)

[1206.3700]
[1707.07280]

# Notation Engineering

## ❖ My Contributions


MIT Technology Review

Humans and technology

How to turn the complex mathematics of vector calculus into simple pictures

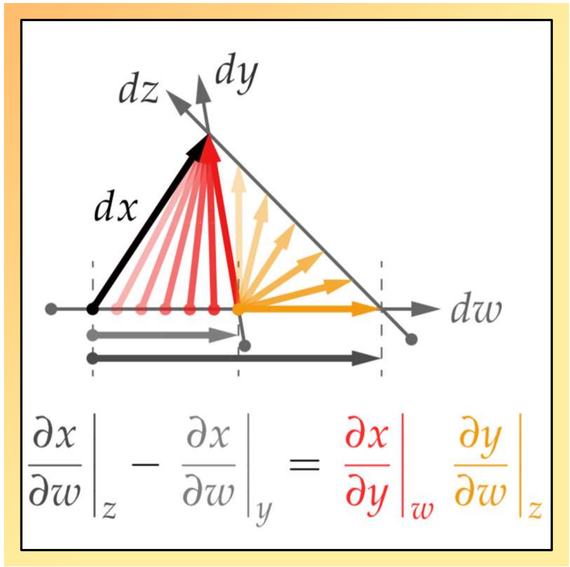
Feynman diagrams revolutionized particle physics. Now mathematicians want to do the same for vector calculus.

by Emerging Technology from the arXiv November 13, 2019

Back in 1948, the journal Physical Review published a paper entitled "Space-Time Approach to Quantum Electrodynamics" by a young physicist named R.P. Feynman at Cornell University. The paper described a new way to solve problems in electrodynamics using matrices. However, it is remembered today for a much more powerful invention—the Feynman diagram, which

**Joon-Hwi Kim**, M. S. H. Oh, and K.-Y. Kim (2021). Boosting Vector Calculus with the Graphical Notation. *American Journal of Physics 89*(2). [arXiv:1911.00892]

>> Application of the Penrose graphical notation to vector differential & integral calculus

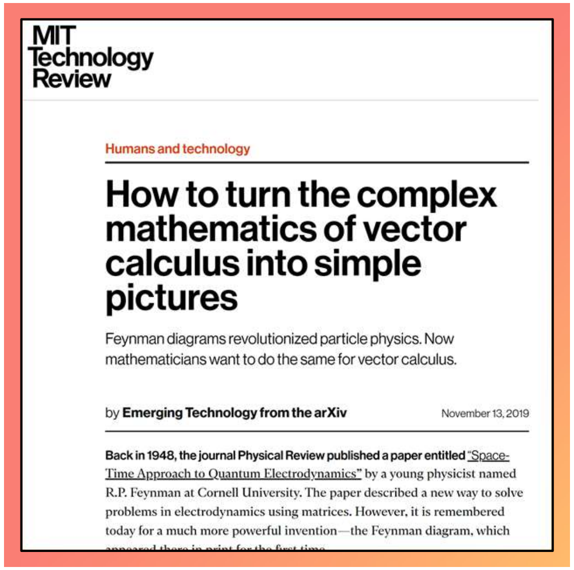



**Joon-Hwi Kim** and J. Nam (2020). Thermodynamic Identities with Sunray Diagrams. *European Journal of Physics 43*(2). [arXiv:1912.11485]

>> A novel graphical method for deriving partial derivative identities in thermodynamics

# Notation Engineering

## ❖ My Contributions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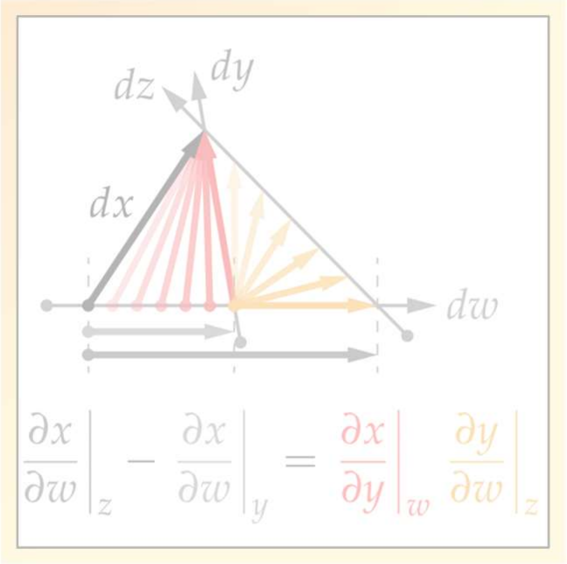
>> Application of the Penrose graphical notation to vector differential & integral calculus

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>> A novel graphical method for deriving partial derivative identities in thermodynamics

# 1. Penrose Graphical Notation

## ❖ First Look

✓ Bra-ket-like index-free notation for one-forms & vectors

Basis-free notion of a vector and a one-form

$$\overrightarrow{A} = \vec{\mathrm{e}}_i A^i \qquad \overleftarrow{B} = B_i \overleftarrow{\mathrm{e}}^i$$

Components

Cf. Wald's GR book

$$\begin{pmatrix} A^{\mathrm{A}} = \mathrm{e}_i^{\ \mathrm{A}} A^i \\ B_{\mathrm{A}} = B_i \mathrm{e}^i_{\ \mathrm{A}} \end{pmatrix}$$

$$\overleftarrow{B} \qquad \overrightarrow{A}$$

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$$\overleftrightarrow{BA}$$

cf. $\langle B | A \rangle$

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## ❖ First Look

✓ A transition to *Penrose graphical notation*

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Looks like "Lego blocks" $\overleftarrow{B}$ and $\overrightarrow{A}$ being attached?

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$$\overleftrightarrow{BA} = \overleftarrow{B}\;\overrightarrow{A} = \boxed{B}\!-\!\!-\!\!-\boxed{A}$$

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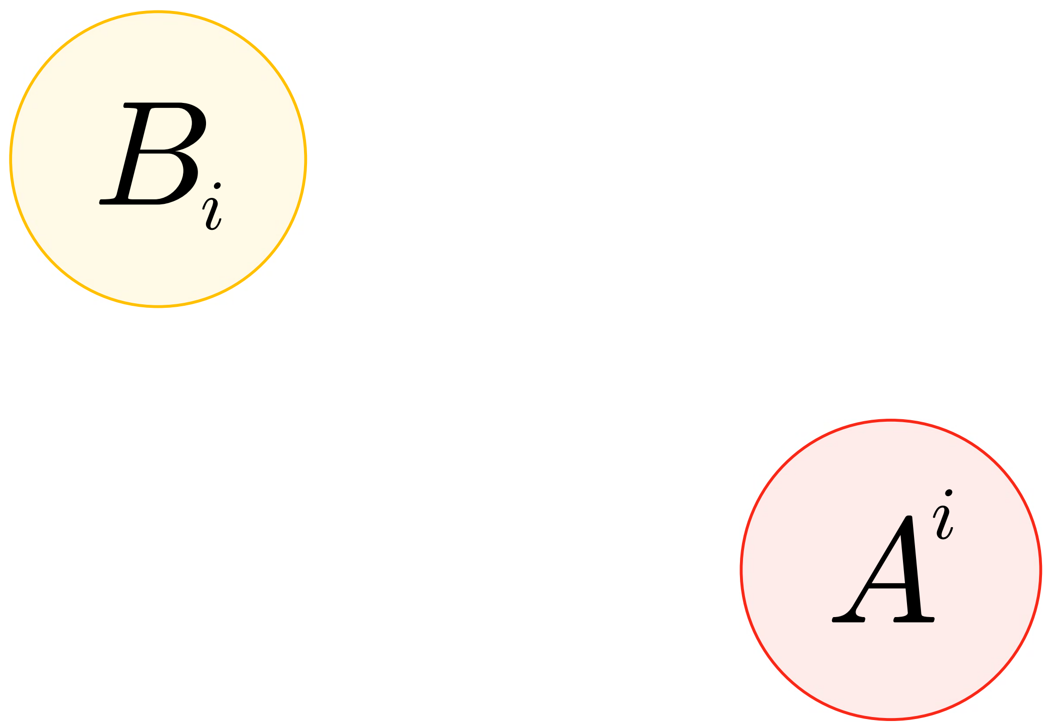
# 1. Penrose Graphical Notation

## ❖ First Look

✓ Recall chemistry

- Contracted indices ↔ Electron pairs (internal lines)
- Unpaired indices ↔ Unpaired electrons (external lines)
- Left-heading/Right-heading lines ↔ Contra-/Co-variant indices



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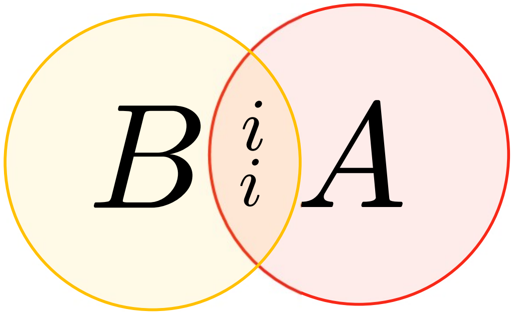

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## ❖ First Look

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$$B - A$$

# 1. Penrose Graphical Notation

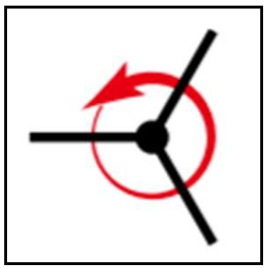
## ❖ First Look

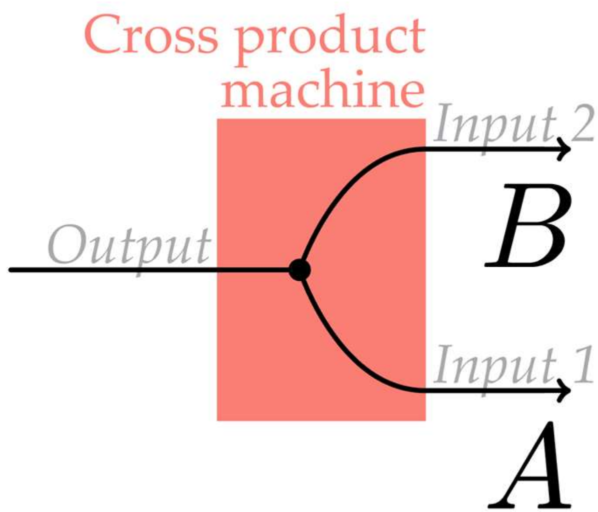
- ✓ Cross product: a machine that takes two vectors as input and gives one vector as output

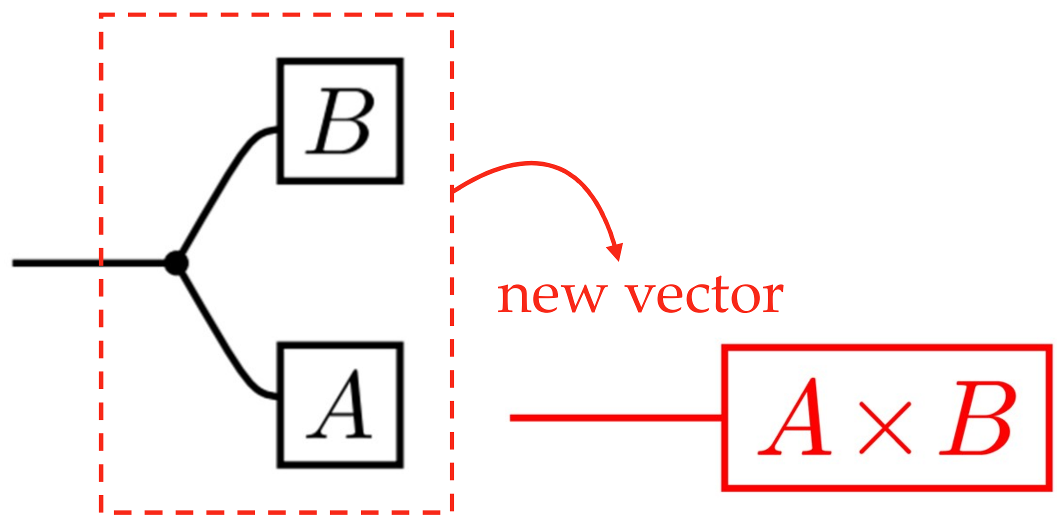
$$\times : \; \overrightarrow{A}, \overrightarrow{B} \;\mapsto\; \overrightarrow{A} \times \overrightarrow{B} = \vec{\mathrm{e}}_i \varepsilon^i{}_{jk} A^j B^k$$

- ✓ Graphical representation?

orientation





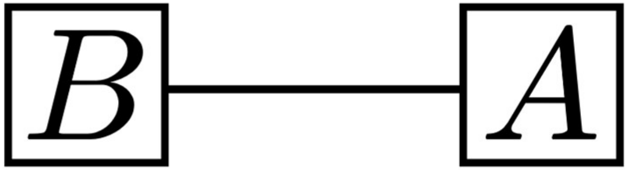



# 1. Penrose Graphical Notation

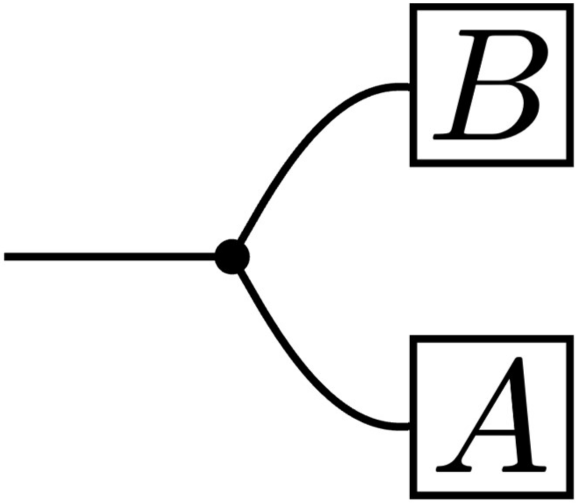
❖ First Look

✓ Component language: leave only the "bones"

$$\overleftrightarrow{BA} = B_i \delta^i_j A^j \qquad \delta^i_j =$$



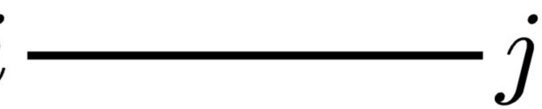

$$\vec{A} \times \vec{B} = \vec{\mathrm{e}}_i \varepsilon^i{}_{jk} A^j B^k \qquad \varepsilon^i{}_{jk} = i$$



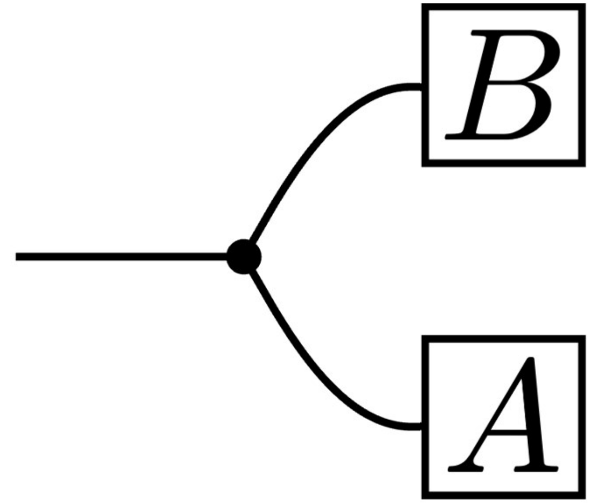
## ❖ First Look

✓ Component language: leave only the "bones"

$$\overleftrightarrow{B}\overrightarrow{A} = B_i \delta^i_j A^j$$

$\delta^i_j = i$ 


$$\vec{A} \times \vec{B} = \vec{\mathrm{e}}_i \varepsilon^i{}_{jk} A^j B^k$$

$\varepsilon^i{}_{jk} = i$ 

## ❖ First Look

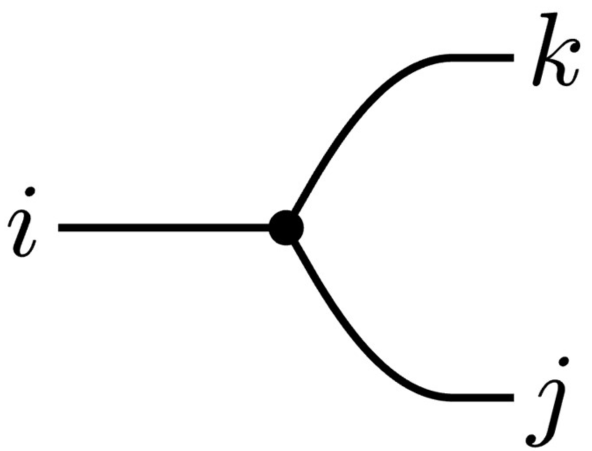
✓ Component language: leave only the "bones"

$$\overleftarrow{B}\overrightarrow{A} = B_i \delta^i_j A^j$$

$$\delta^i_j =$$


$$\vec{A} \times \vec{B} = \vec{\mathrm{e}}_i \varepsilon^i{}_{jk} A^j B^k$$

$$\varepsilon^i{}_{jk} =$$

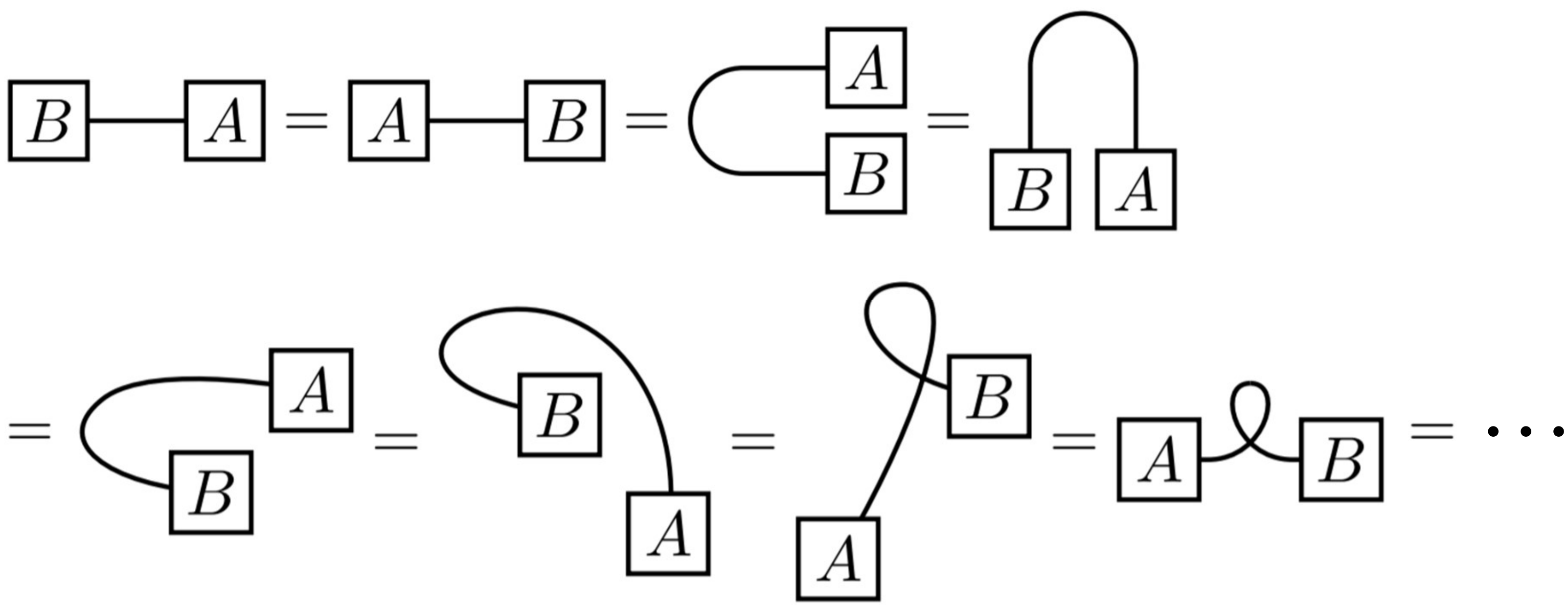


# 1. Penrose Graphical Notation

## ❖ First Look

✓ If we raise & lower indices by the Euclidean metric $\delta_{ij}$ …

- All indices can be lowered: no need for "left-right hierarchy" (distinguishing b/w co- and contra-variance)



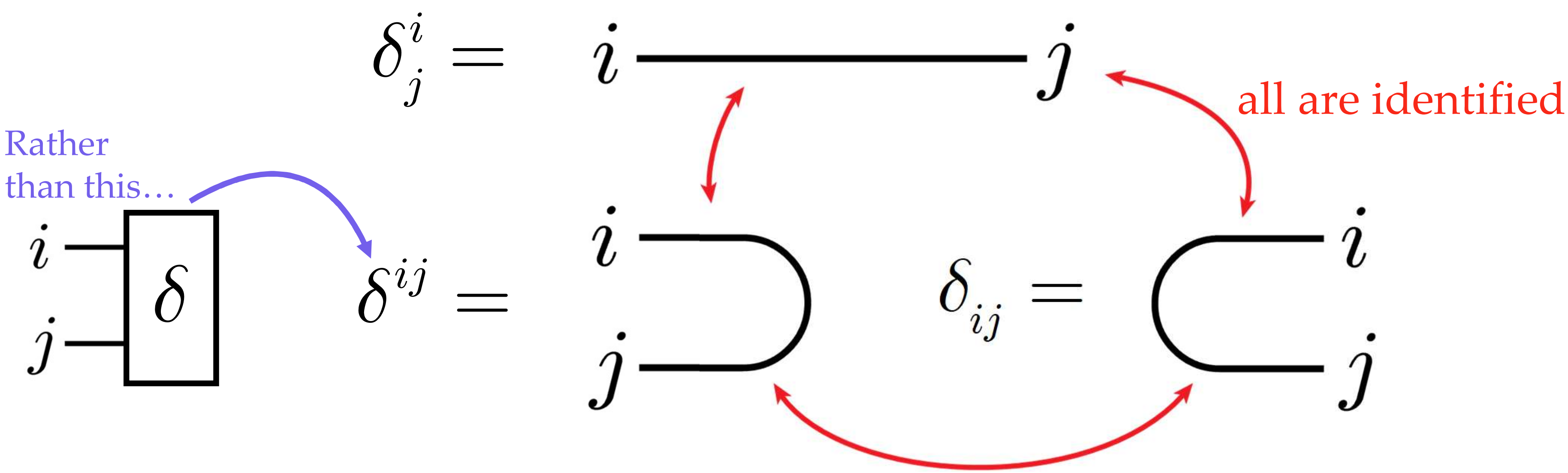

**Planar isotopy preserves the value of diagrams**

# 1. Penrose Graphical Notation

## ❖ First Look

✓ If we raise & lower indices by the Euclidean metric $\delta_{ij}$ …

- Following the spirit of **"topological computation,"** the Euclidean metric and its inverse are represented as…



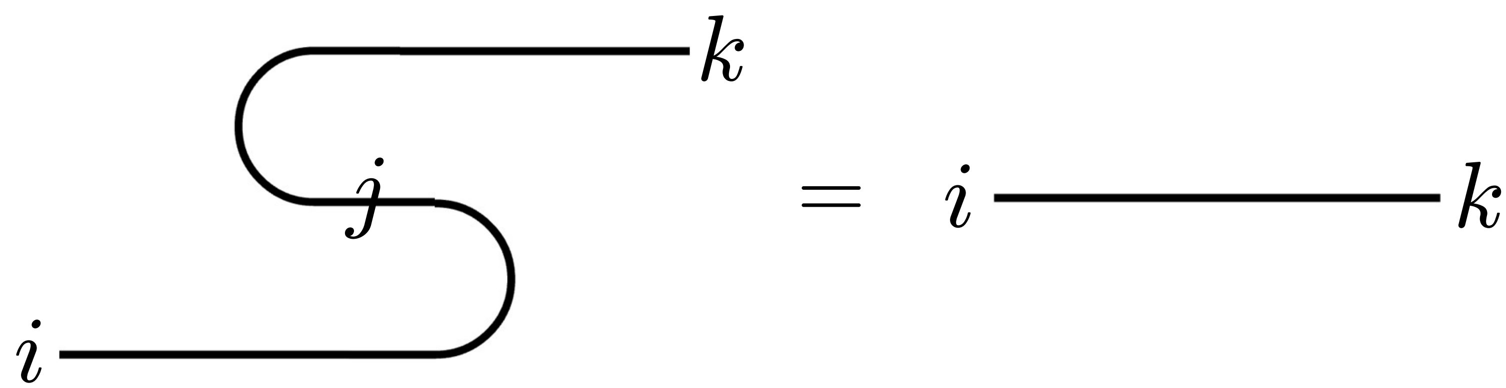

$$\delta^i_{\ j} = \quad \delta^{ij} = \quad \delta_{ij} =$$

**Planar isotopy preserves the value of diagrams**

## ❖ First Look

✓ If we raise & lower indices by the Euclidean metric $\delta_{ij}$ …

- Following the spirit of **"topological computation,"** the Euclidean metric and its inverse are represented as…




$\delta^{ij}\delta_{jk}$ $\qquad$ $\delta^{i}_{k}$

**Planar isotopy preserves the value of diagrams**

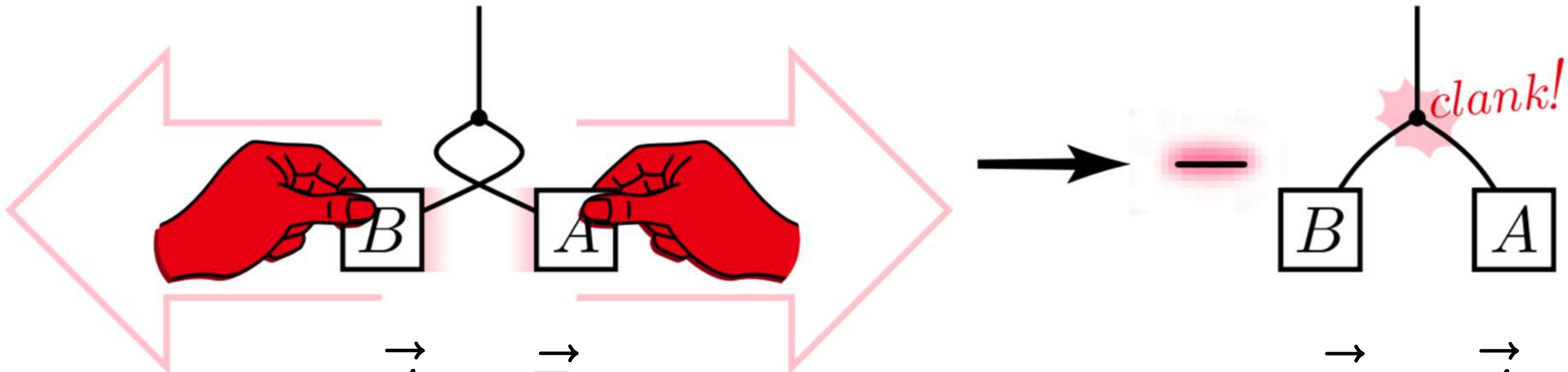
# 1. Penrose Graphical Notation

## ❖ First Look

✓ Antisymmetry of the volume form $\varepsilon_{ijk}$

- Swap-then-yanking a cross product machine:
  as a **"discontinuous process,"** the diagram gains a factor of $-1$

not an isotopy but
parity-violating




$\vec{A} \times \vec{B}$ $\qquad$ $-\vec{B} \times \vec{A}$

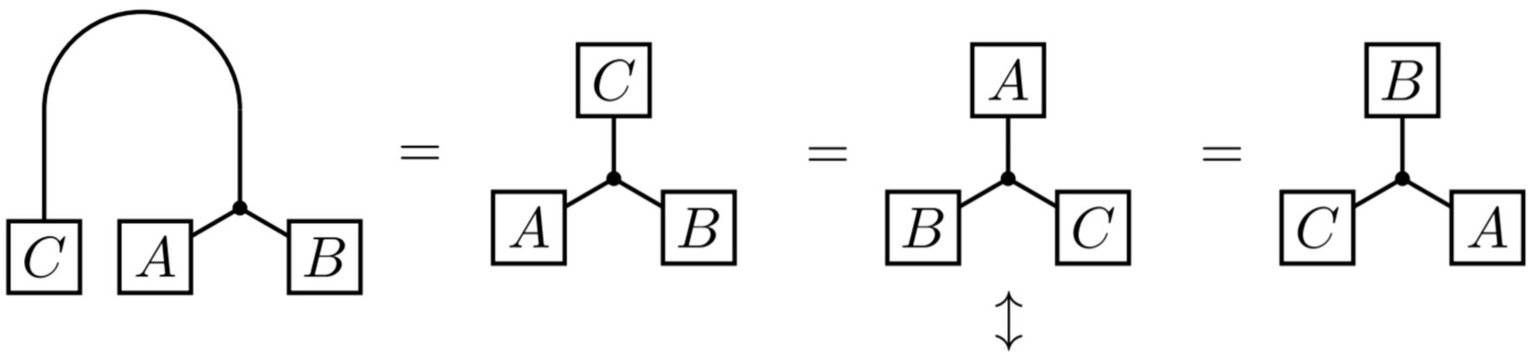
**Planar isotopy preserves the value of diagrams**

# 1. Penrose Graphical Notation

## ❖ First Look

✓ Cyclic symmetry of the volume form $\varepsilon_{ijk}$

- *Self-explanatory design*: already reflected in its graphical design!
- *The economy of graphical notations*: redundant expressions are brought to the same or at least evidently equivalent diagram

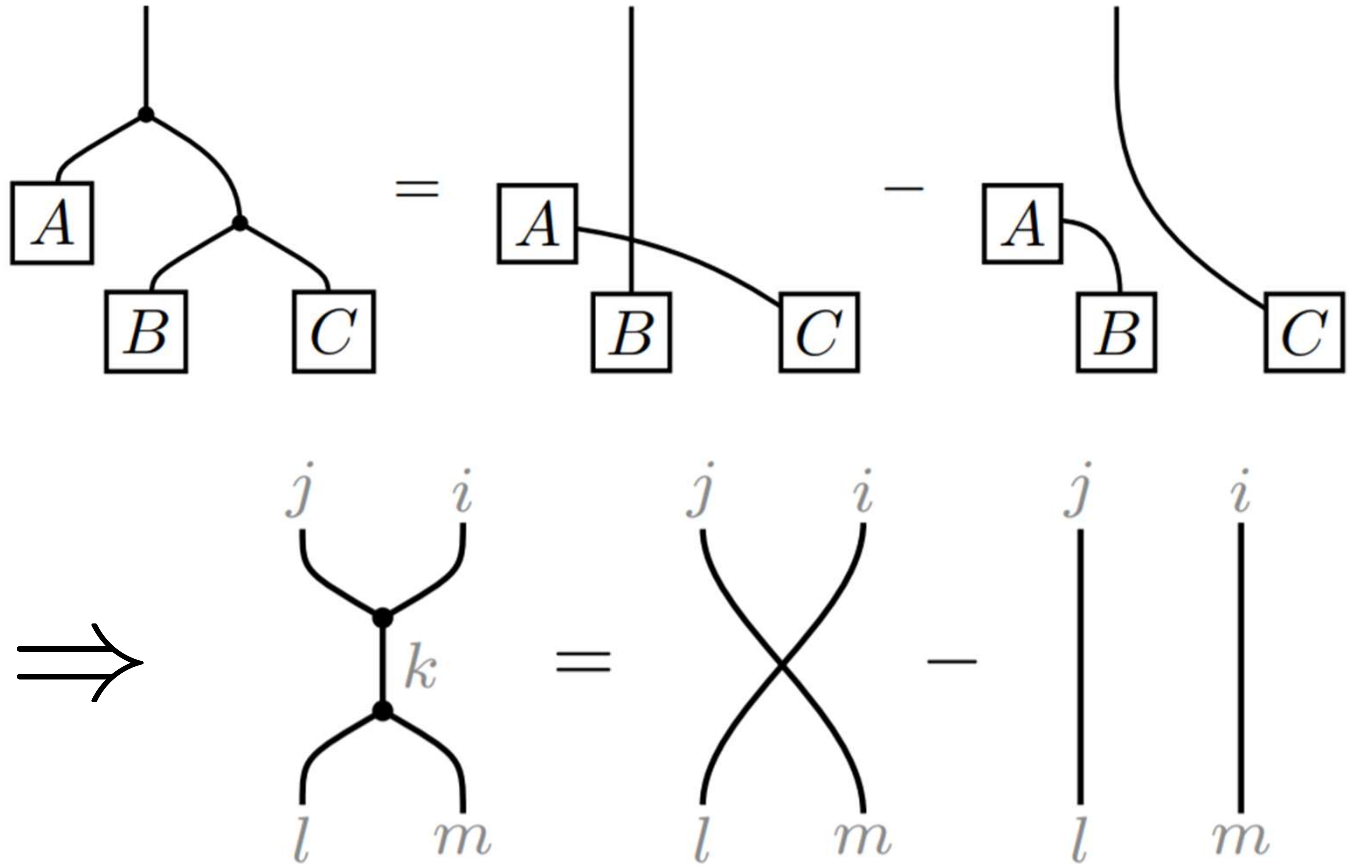



$$\vec{C}\cdot(\vec{A}\times\vec{B}) \;=\vec{A}\cdot(\vec{B}\times\vec{C}) \;=\vec{B}\cdot(\vec{C}\times\vec{A})$$

# 1. Penrose Graphical Notation

## ❖ First Look

✓ The BAC-CAB identity




$\varepsilon_{ijk}\varepsilon_{klm}$ $\qquad$ $\delta_{jm}\delta_{il} - \delta_{jl}\delta_{im}$

# 1. Penrose Graphical Notation

## ❖ Advantages of the Tensor Graphical Notation

✓ It is a "good notation": What is a "good notation" anyway?

- Automated calculations by topological computations
- *The economy of graphical notations*: reduced redundancy
- *The self-explanatory design*: reduced arbitrariness (cf. linguistics)
- Clarity: transparent view of index contractions, tensors syntax-highlighted

✓ Cognitive aspects

- Graphical intuition can be faster than "plaintext" algebraic manipulations

✓ Pedagogical benefits

- Easy to generate/classify various tensor expressions & identities (cf. linguistics… may involve Broca's area☺)
- It is fun as Lego blocks or magnetic building sticks! **[JHK 1911.00892]**

# 1. Penrose Graphical Notation

## ❖ A Little Formalism

✓ Penrose (1971) Applications of Negative Dim Tensors: **"Abstract Tensor System"**

**Applications of Negative Dimensional Tensors**

ROGER PENROSE

*Birkbeck College, University of London, England*

I wish to describe a theory of "abstract tensor systems" (abbreviated ATS) and indicate some applications. Unfortunately I shall only be able to give a very brief outline of the general theory here.†

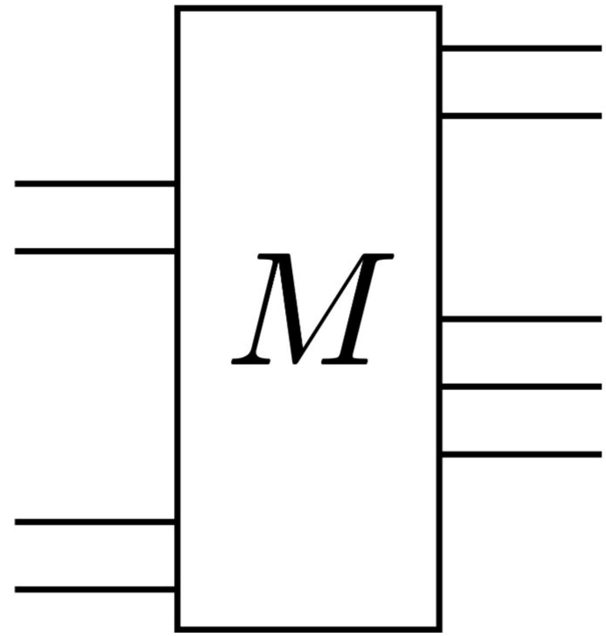
I take as my model, the conventional tensor index notation with Einstein's summation convention, which has become so familiar in physics and in what is now referred to as "old fashioned" differential geometry. The elements of an ATS may be denoted by kernel symbols with indices in a way formally identical with the tensor index notation, but now the meanings of the indices are to be quite different. This will enable more general types of object than ordinary tensors to be considered. Some of these (for example, "negative dimensional" tensors) will not be representable in terms of components in the ordinary way.

Each index is to be simply a *label* and does not stand for, say, 1, 2, ..., *n*. Thus an element $\xi^a$ (a "vector") of an ATS is not a set of components, but a single element of a vector space (or module) $\mathscr{T}^a$ over a field (or ring) $\mathscr{T}$. Since I wish to mirror the ordinary index notation and allow expressions such as

## ❖ A Little Formalism

✓ What does a line mean? (basis-independently) — **the vector (carrier) space $\mathcal{V}$**

$$\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_n \otimes \vec{e}_o \; M^{ij}{}_{klm}{}^{no}{}_{pq} \; \overleftarrow{e}^k \otimes \overleftarrow{e}^l \otimes \overleftarrow{e}^m \otimes \overleftarrow{e}^p \otimes \overleftarrow{e}^q =$$




$$\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V} \longleftarrow \mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}$$

a $\binom{4}{5}$-tensor is a map $\underbrace{\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}}_{\text{Four output lines}} \leftarrow \underbrace{\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}}_{\text{Five input lines}}$

# 1. Penrose Graphical Notation

## ❖ A Little Formalism

- ✓ Not only to GL($n$) tensors, Penrose graphical notation can also be applied to representations of Lie groups
- ✓ Adding invariant symbols ⇒ Lie algebras
- ✓ Cvitanović's birdtracks

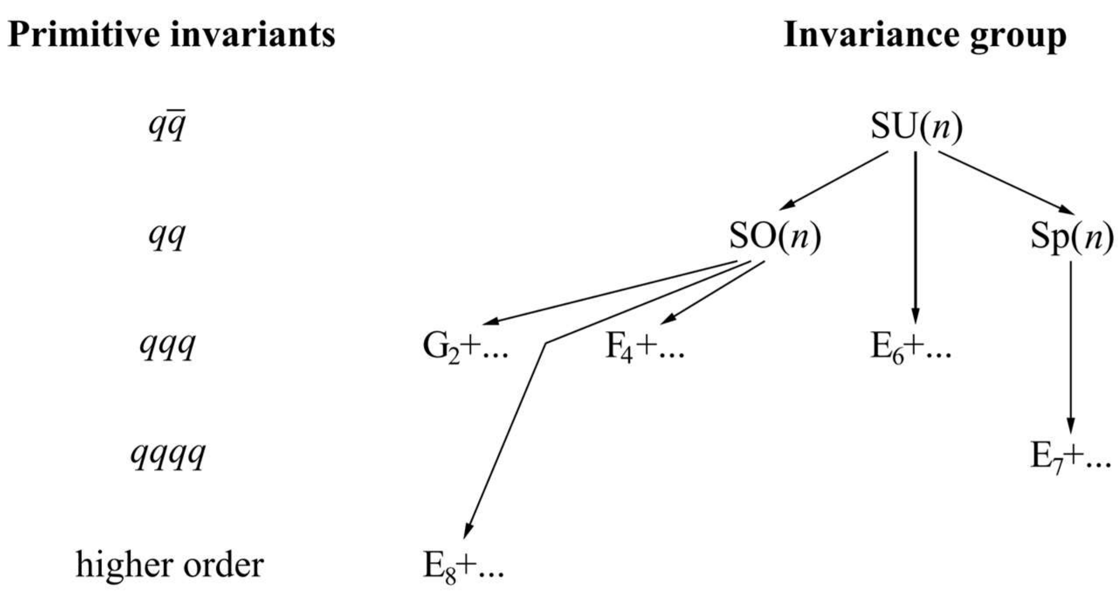
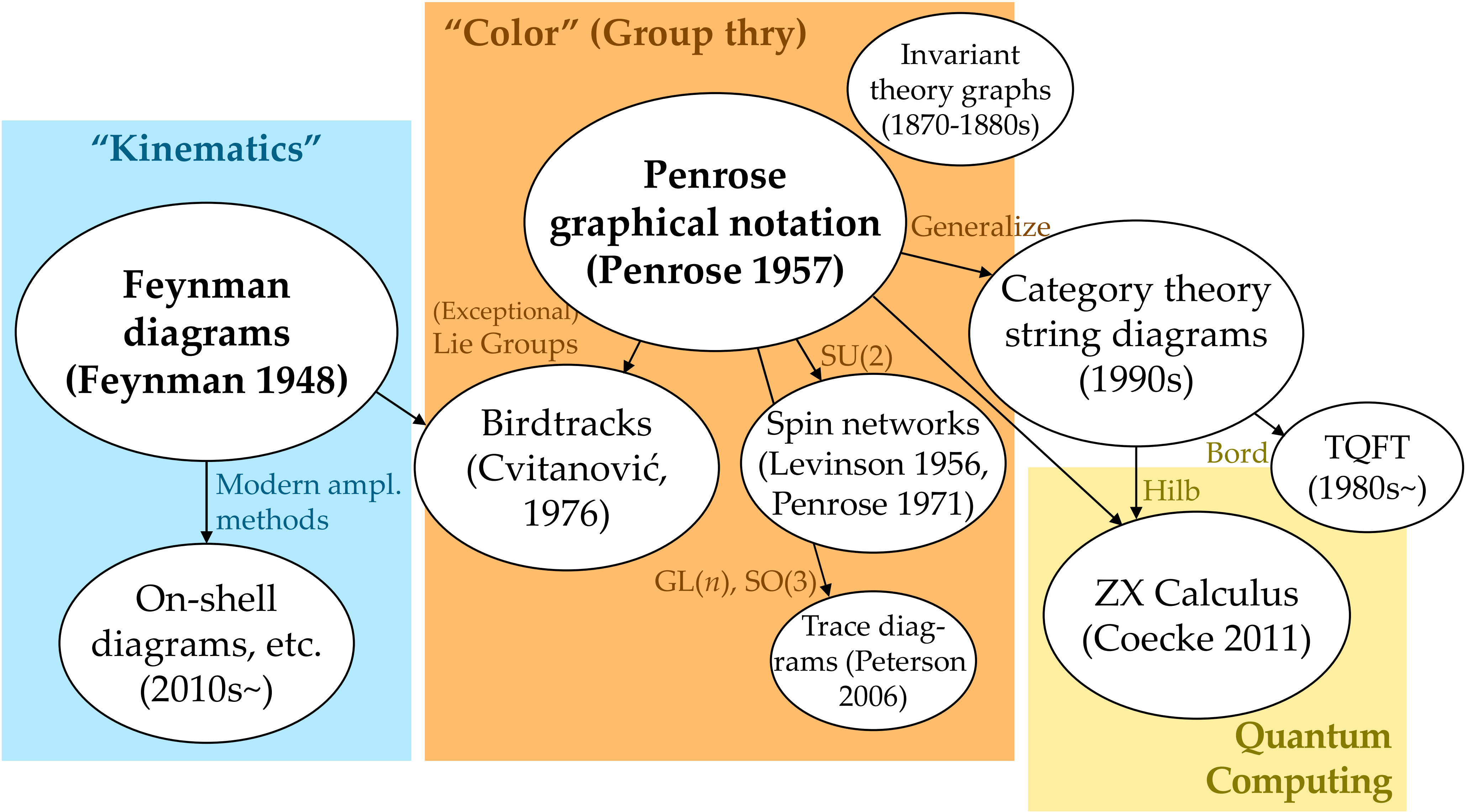



Figure 2.1 Additional primitive invariants induce chains of invariance subgroups.

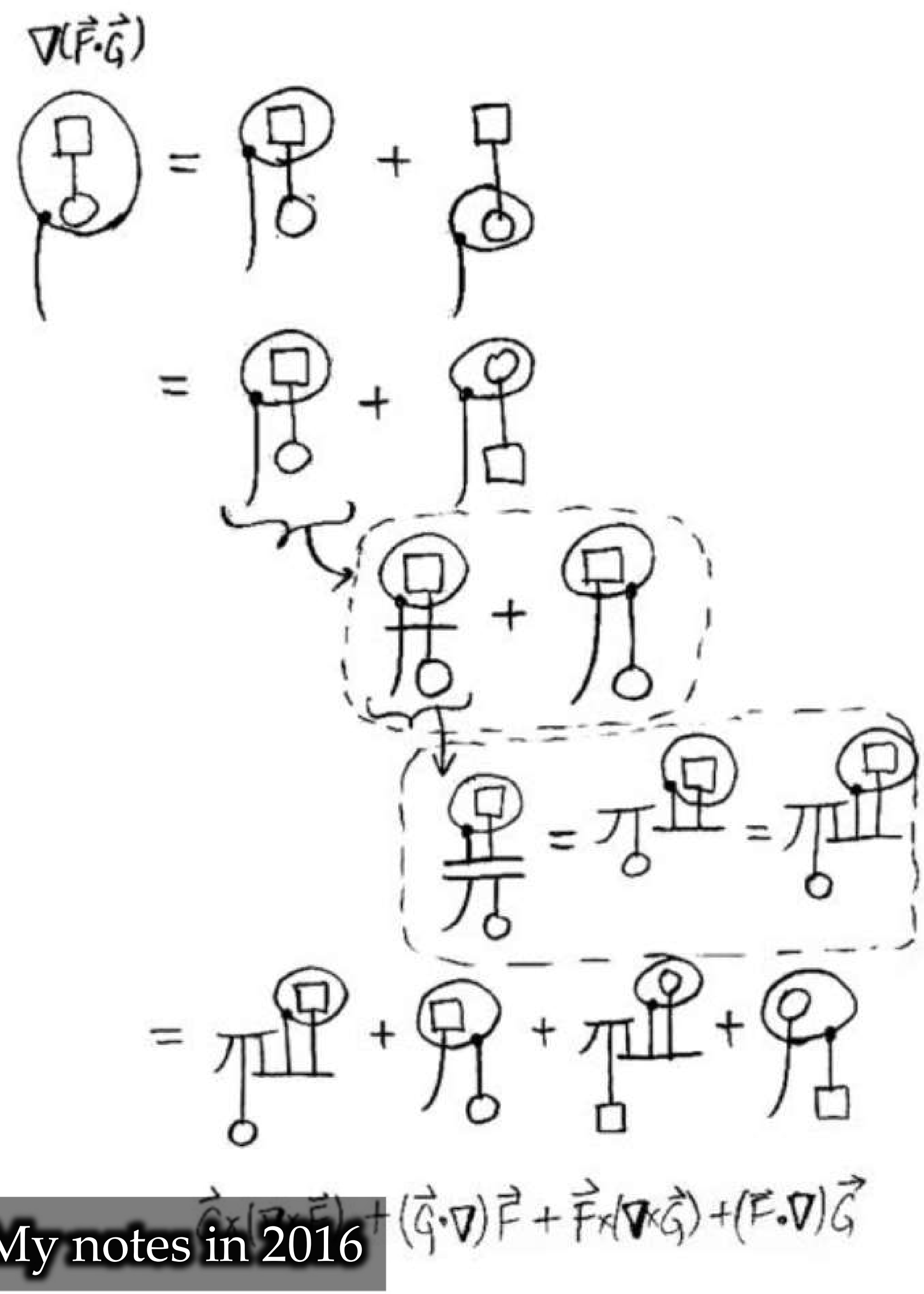
**[Cvitanovic 2009]**

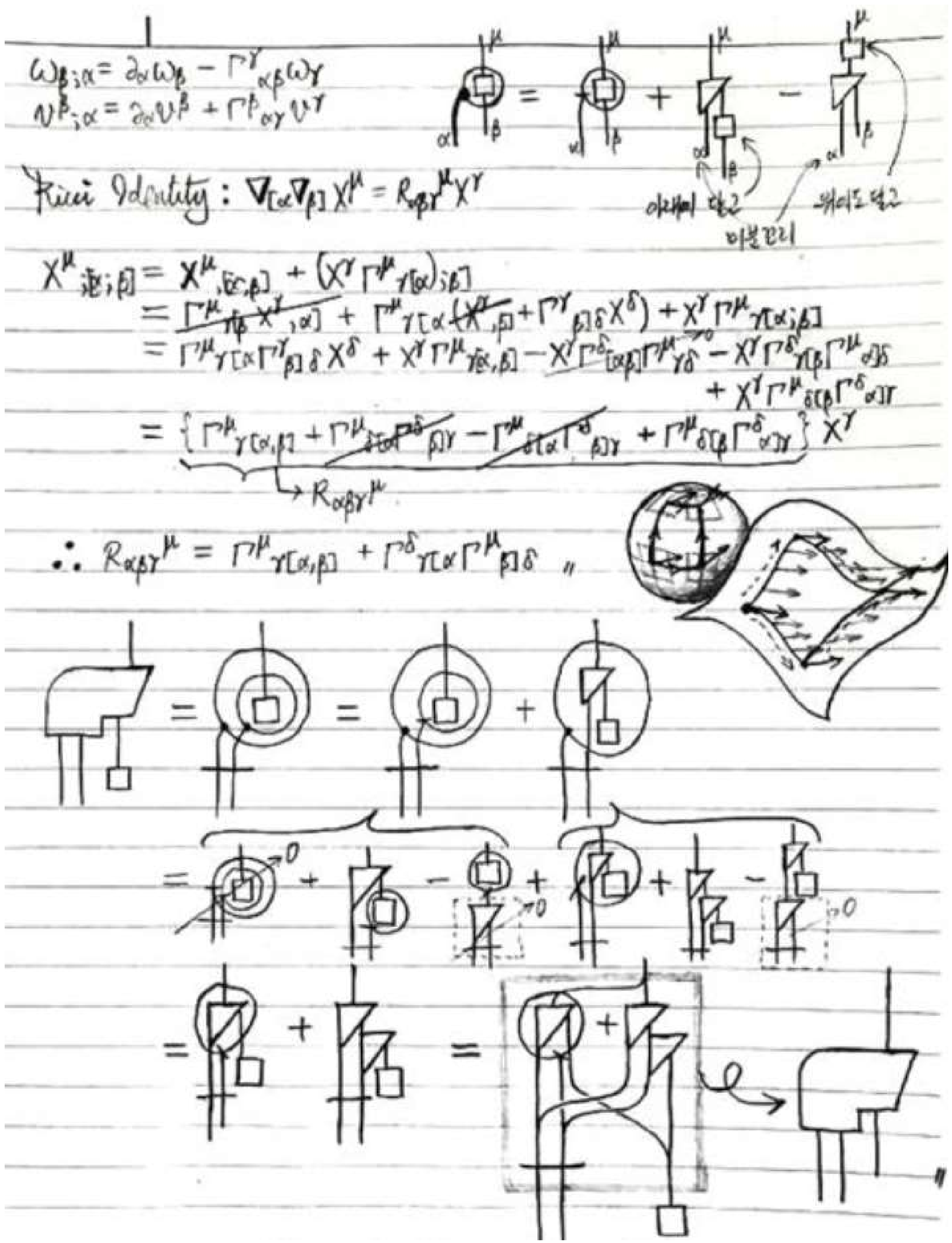
# 1. Penrose Graphical Notation




[Cvitanovic 2009] [JHK 1911.00892]

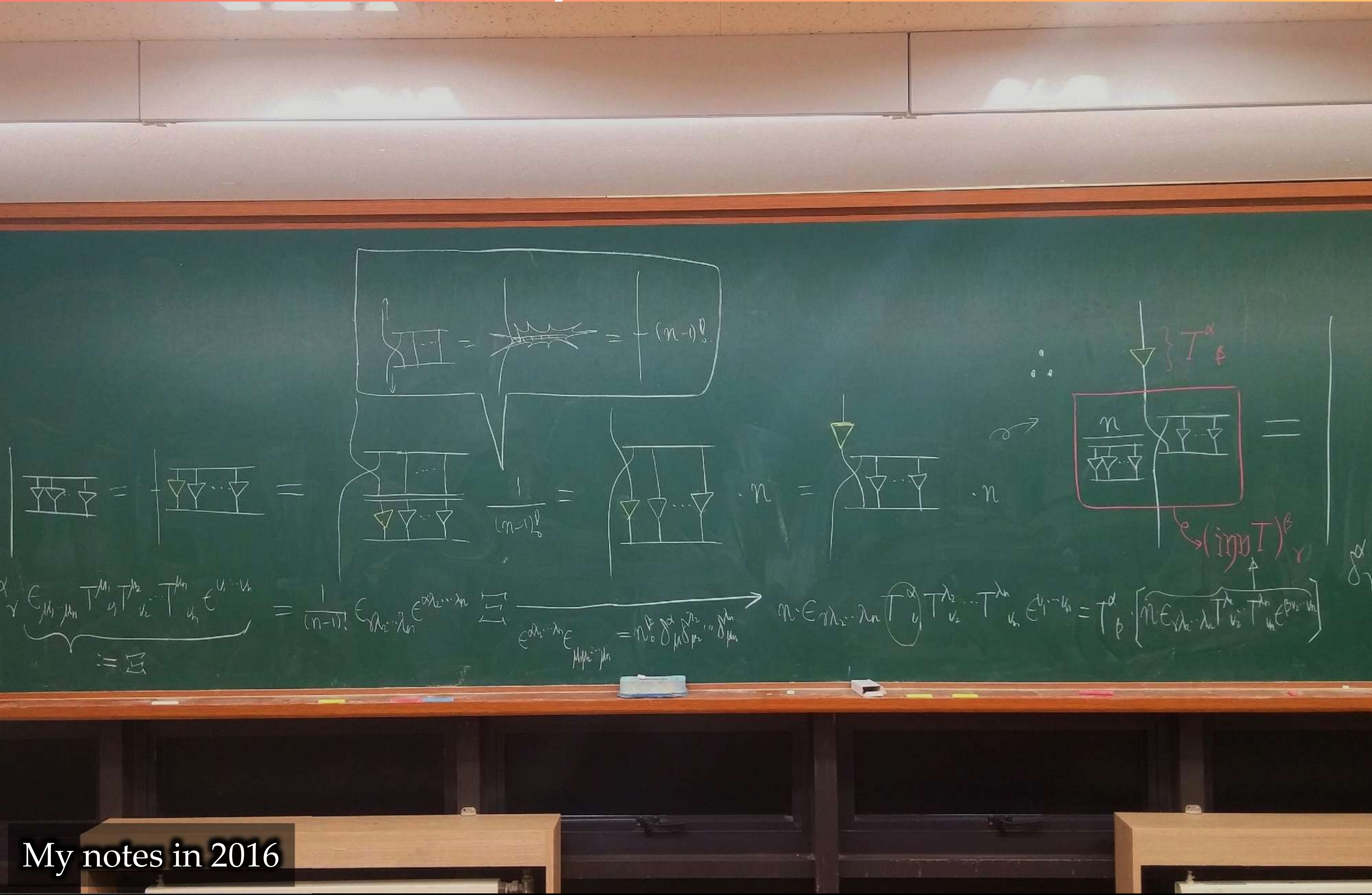
# 1. Penrose Graphical Notation





My notes in 2016

# 1. Penrose Graphical Notation

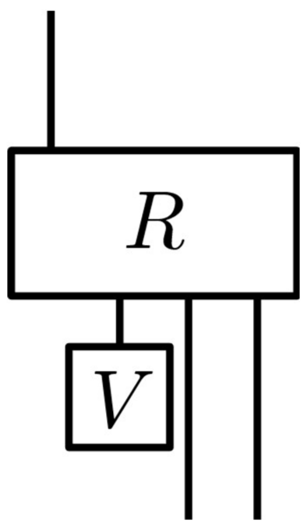


My notes in 2016

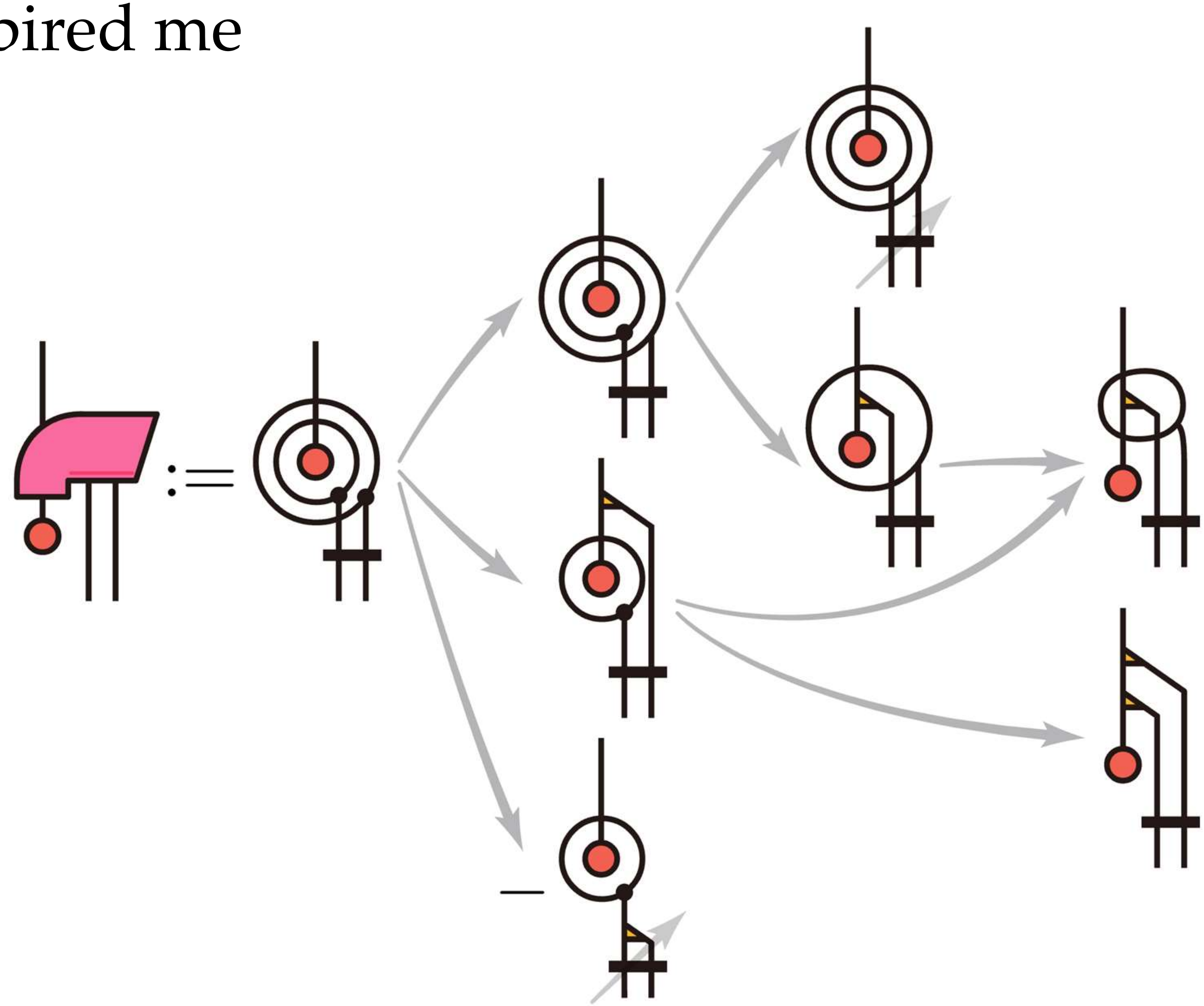
# 1. Penrose Graphical Notation

❖ Penrose Style: More Visual Abstraction

✓ Artistically inspired me



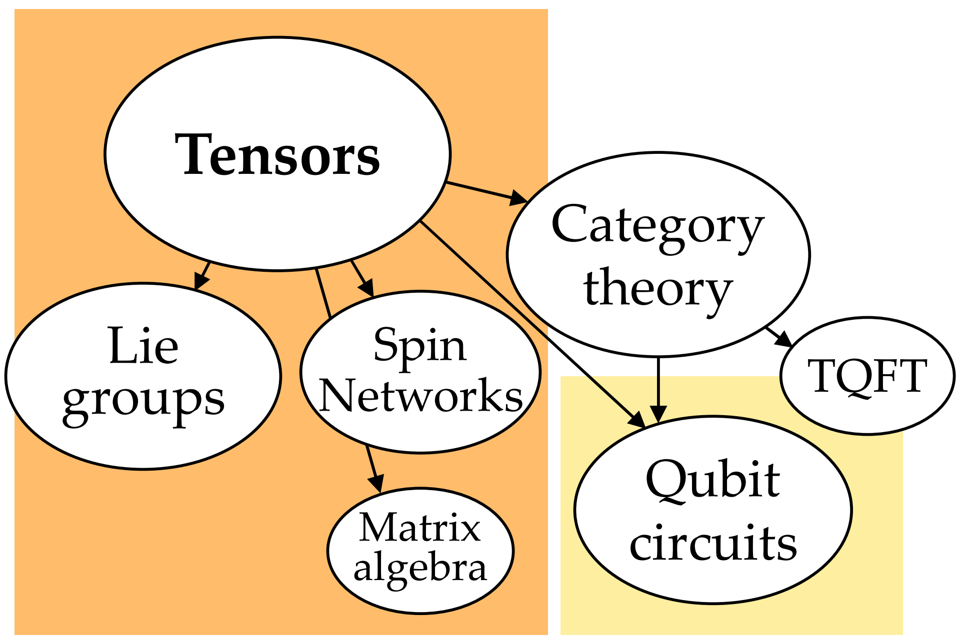

$$R^{\mu}{}_{\nu\rho\sigma}V^{\nu}$$



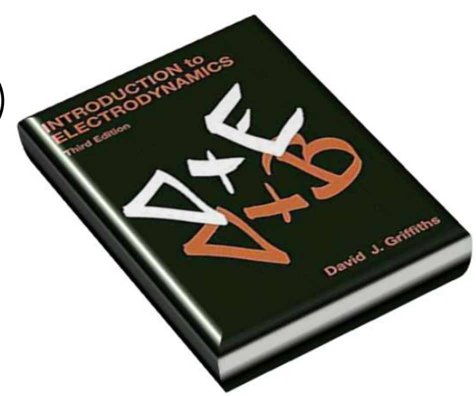
# 1. Penrose Graphical Notation

❖ Applications?

| | |
|---|---|
| ✓ SO(3), SU(2) tensors | Penrose (1971) |
| ✓ Lie groups, spinors, etc. | Cvitanović, Kennedy, … (1970-80s) |
| ✓ Matrix algebra & SO(3) | Peterson (2006) |
| ✓ Qubit circuits | Coecke, Duncan, … (2011) |
| ✓ Vector ***calculus*** | Questioned by MSH Oh in 2018 |




**Everyone forgot what it was like to take an EM course for the first time?**




# 1. Penrose Graphical Notation

❖ Collaboration

✓ Main article + 29-pages supplementary material

MIT Technology Review

Humans and technology

## How to turn the complex mathematics of vector calculus into simple pictures

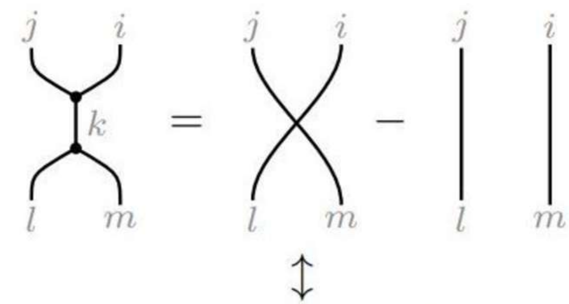
Feynman diagrams revolutionized particle physics. Now mathematicians want to do the same for vector calculus.

by **Emerging Technology from the arXiv** November 13, 2019

**Back in 1948, the journal Physical Review published a paper entitled** "Space-Time Approach to Quantum Electrodynamics" by a young physicist named R.P. Feynman at Cornell University. The paper described a new way to solve problems in electrodynamics using matrices. However, it is remembered today for a much more powerful invention—the Feynman diagram, which appeared there in print for the first time.

Of course, many other areas of physics rely on complex mathematics. And that raises the interesting question of whether graphics-based innovations could simplify these calculations and perhaps kick-start a new era of innovation, just as Feynman did.

Enter Joon-Hwi Kim at Seoul National University in South Korea and a couple of colleagues who have come up with a similar innovation for vector calculus—a graphics-based shorthand for one of the most common and powerful mathematical tools in science. "We anticipate that graphical vector calculus will lower the barriers in learning and practicing vector calculus, as Feynman diagrams did in quantum field theory," they say.




$$\epsilon_{ijk}\epsilon_{klm} = \delta_{jm}\delta_{il} - \delta_{jl}\delta_{im}$$

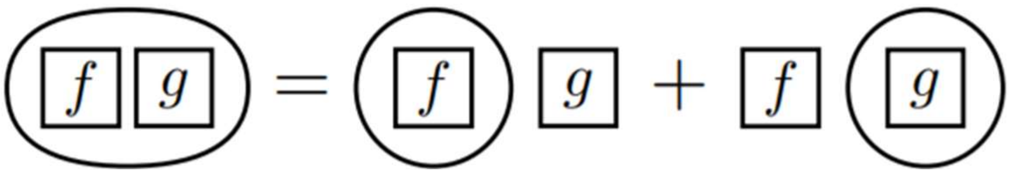
First some background. Vector calculus is the branch of mathematics that deals with the differentiation and integration of vector fields. The reason it is so important in physics is that more or less everything in the universe can be

**[JHK 1911.00892]**

# 1. Penrose Graphical Notation

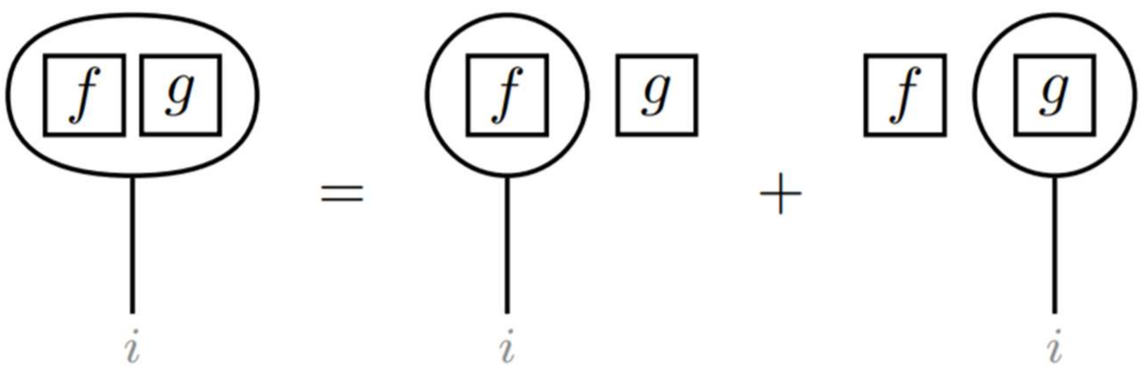
## ❖ Derivatives

✓ Leibniz rule




$\leftrightarrow (fg)' = f'g + fg'$

✓ Vector-ize




$\updownarrow$

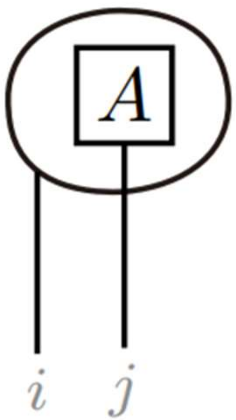
$$\partial_i(fg) = \partial_i(f)\, g + f\, \partial_i(g)$$

*reminds me of*
*CFT topological*
*surface operators…*

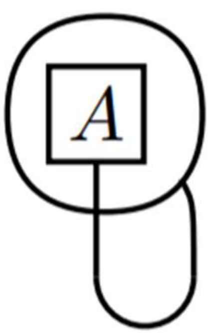
# 1. Penrose Graphical Notation

❖ **Derivatives**

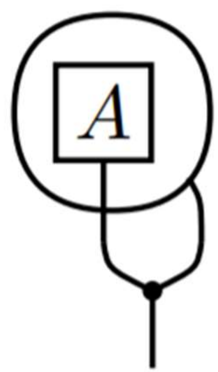
✓ Divergence and curl




$$= \partial_i A_j = (\text{“}\nabla \vec{A}\text{”})_{ij}$$



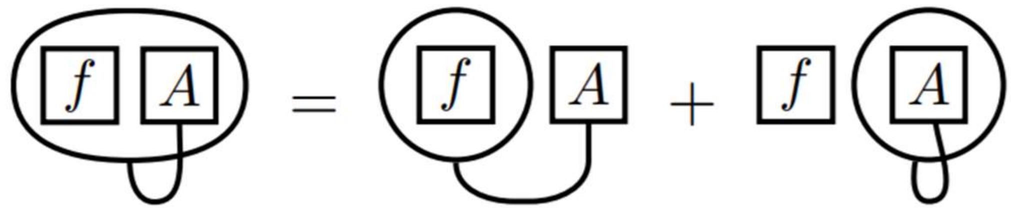

$$= \nabla \cdot \vec{A} \,,$$




$$= \nabla \times \vec{A}$$

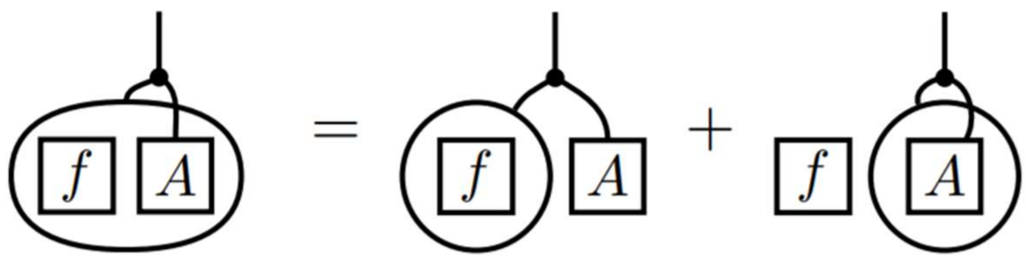
## (1) Vector Differential Calculus

✓ Deriving vector calculus identities in the graphical notation




$$\updownarrow$$

$$\nabla \cdot (f\vec{A}) = \nabla f \cdot \vec{A} + f\,\nabla \cdot \vec{A}$$




$$\updownarrow$$

$$\nabla \times (f\vec{A}) = \nabla f \times \vec{A} + f\,\nabla \times \vec{A}$$

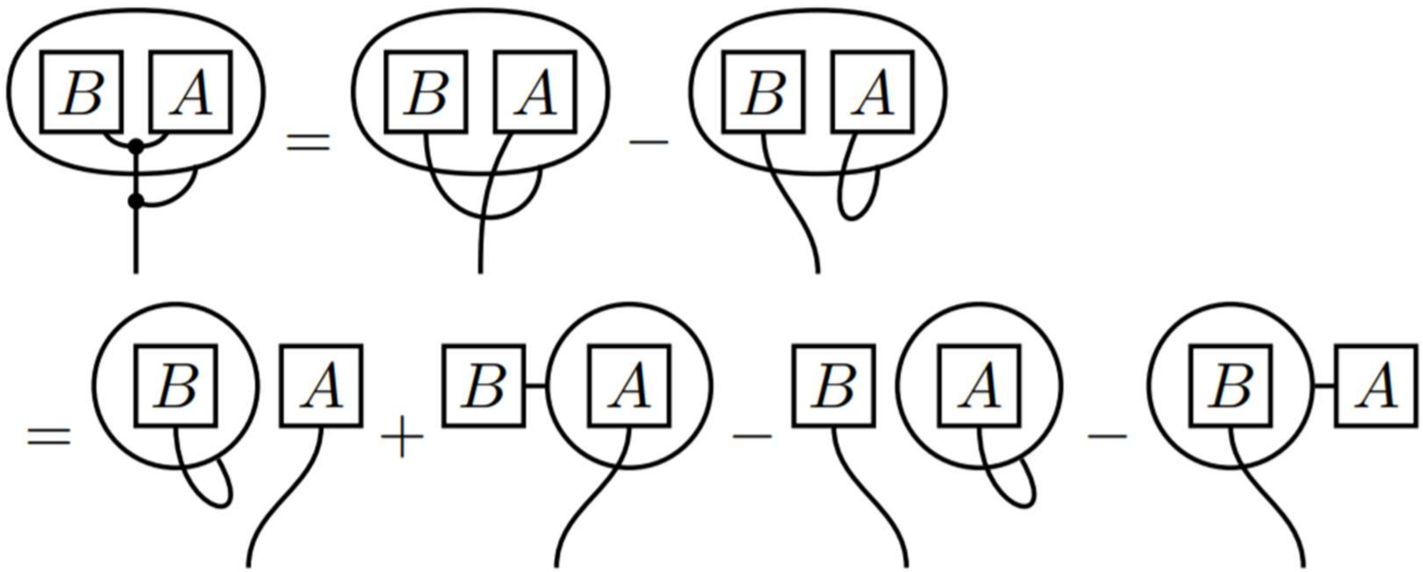
[JHK 1911.00892]

# 1. Penrose Graphical Notation

## (1) Vector Differential Calculus

✓ Deriving vector calculus identities in the graphical notation

$$\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} + (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} - (\vec{A} \cdot \nabla)\vec{B}.$$



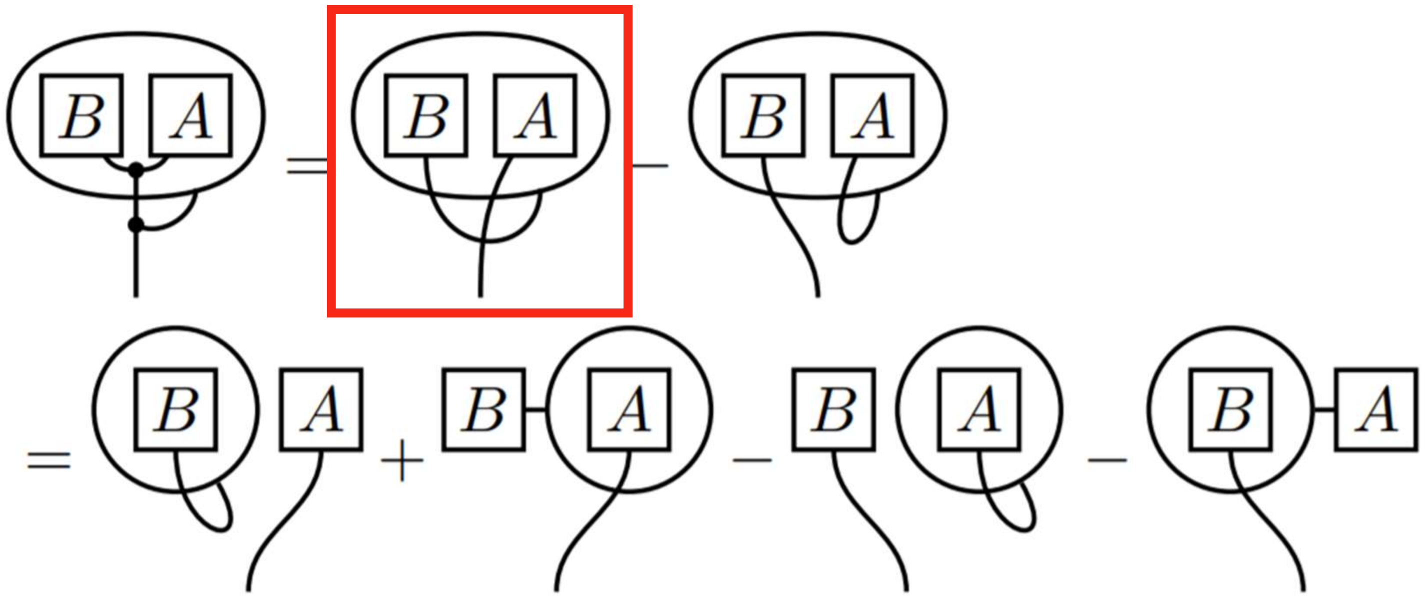

Just doodle with diagrams, and you will get the answer.

**[JHK 1911.00892]**

## (1) Vector Differential Calculus

✓ Deriving vector calculus identities in the graphical notation

$$\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} + (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} - (\vec{A} \cdot \nabla)\vec{B}$$



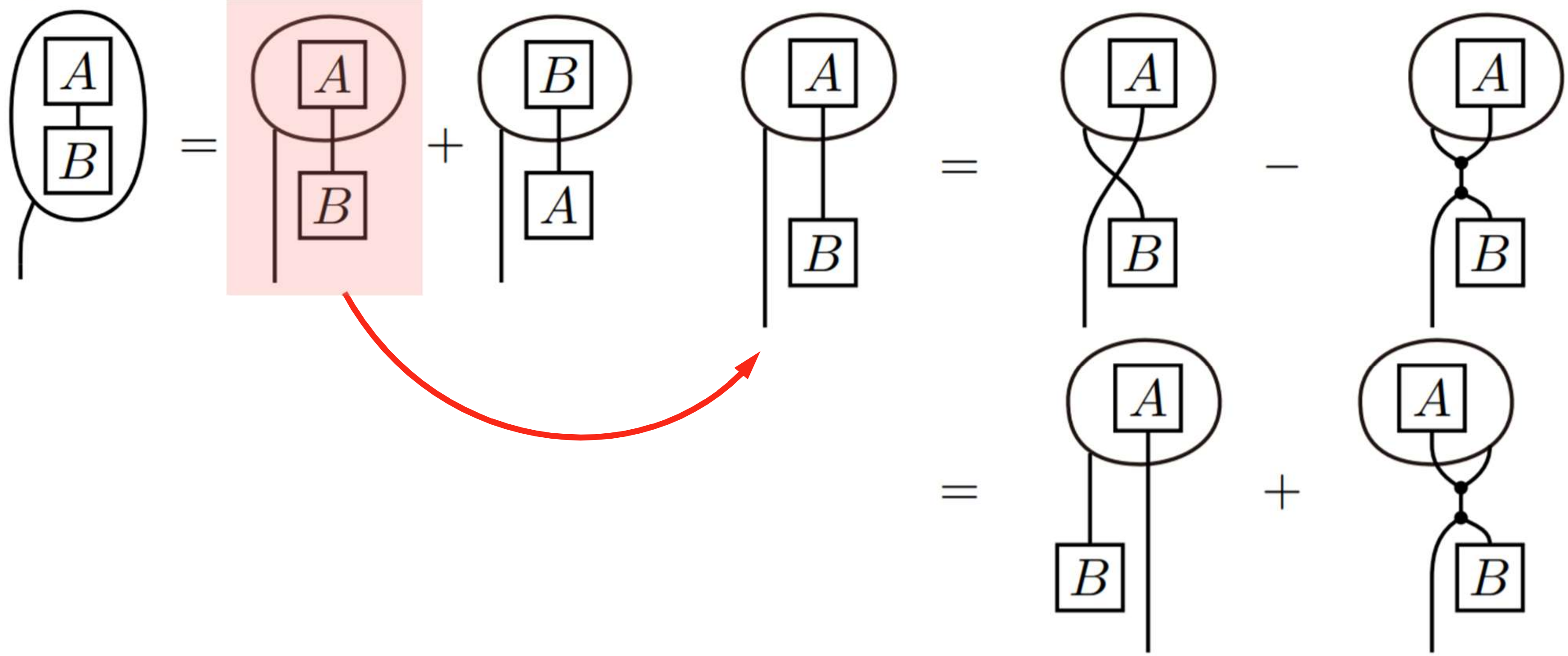

Just doodle with diagrams, and you will get the answer.

[JHK 1911.00892]

## (1) Vector Differential Calculus

✓ Deriving vector calculus identities in the graphical notation

$\nabla(\vec{A}\cdot\vec{B})$




$$= (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A})$$

[JHK 1911.00892]

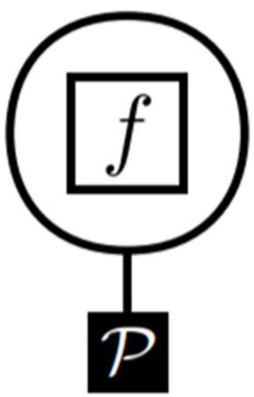
# 1. Penrose Graphical Notation

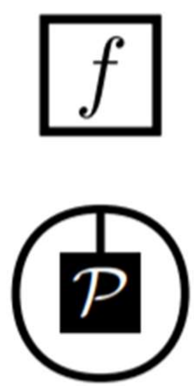
## (2) Vector Integral Calculus

✓ Graphical notation for integral calculus (Stokes' thm): inspired by the duality b/w chains and cochains

$\int_{\mathcal{P}} dl_i \leftrightarrow$ 

and $\int_{\partial\mathcal{P}} \leftrightarrow$ 

$=$ 

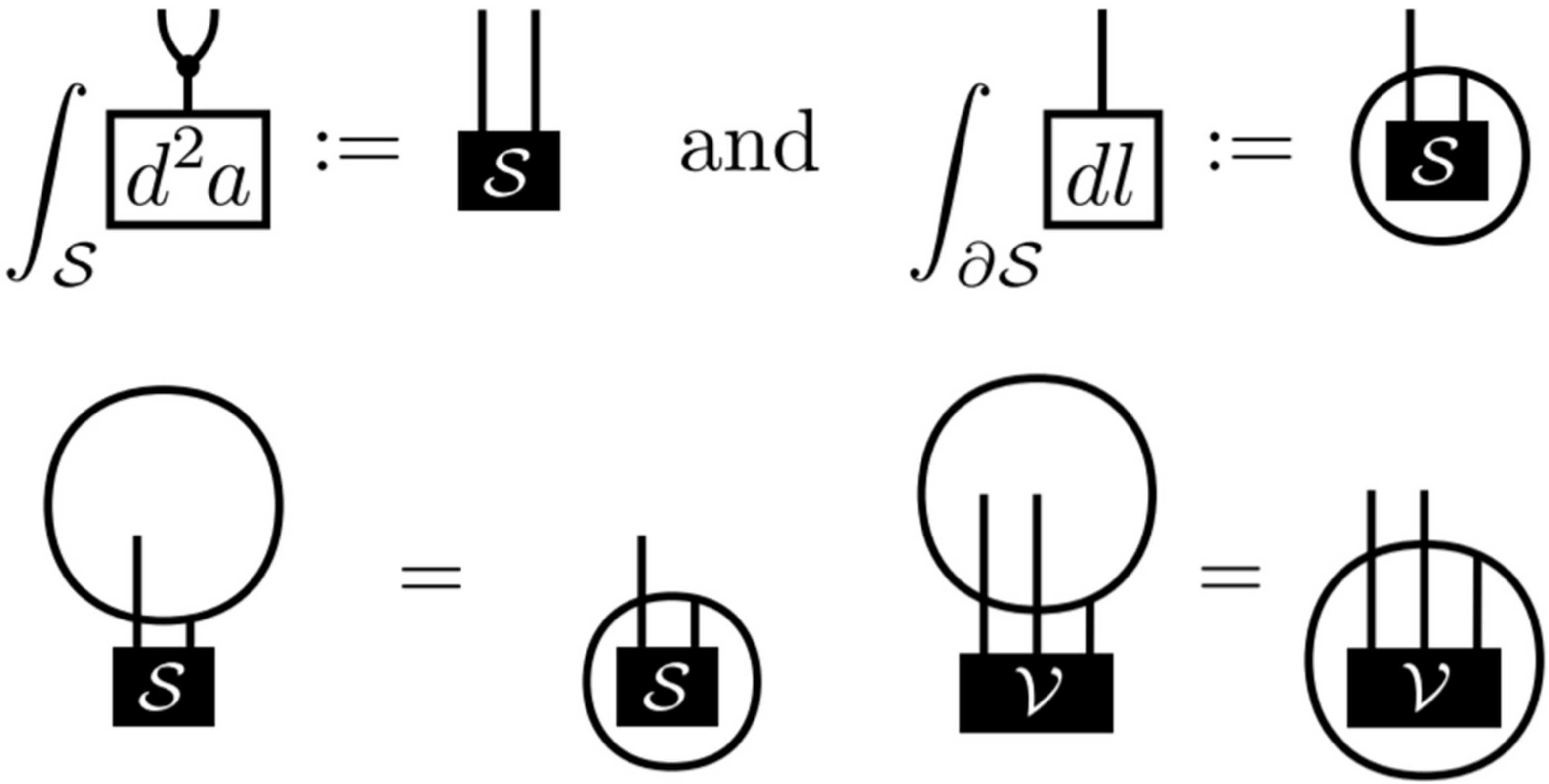

$$\int_{\mathcal{P}} d\vec{l} \cdot \nabla f = \int_{\partial\mathcal{P}} f$$

[JHK 1911.00892]

# 1. Penrose Graphical Notation

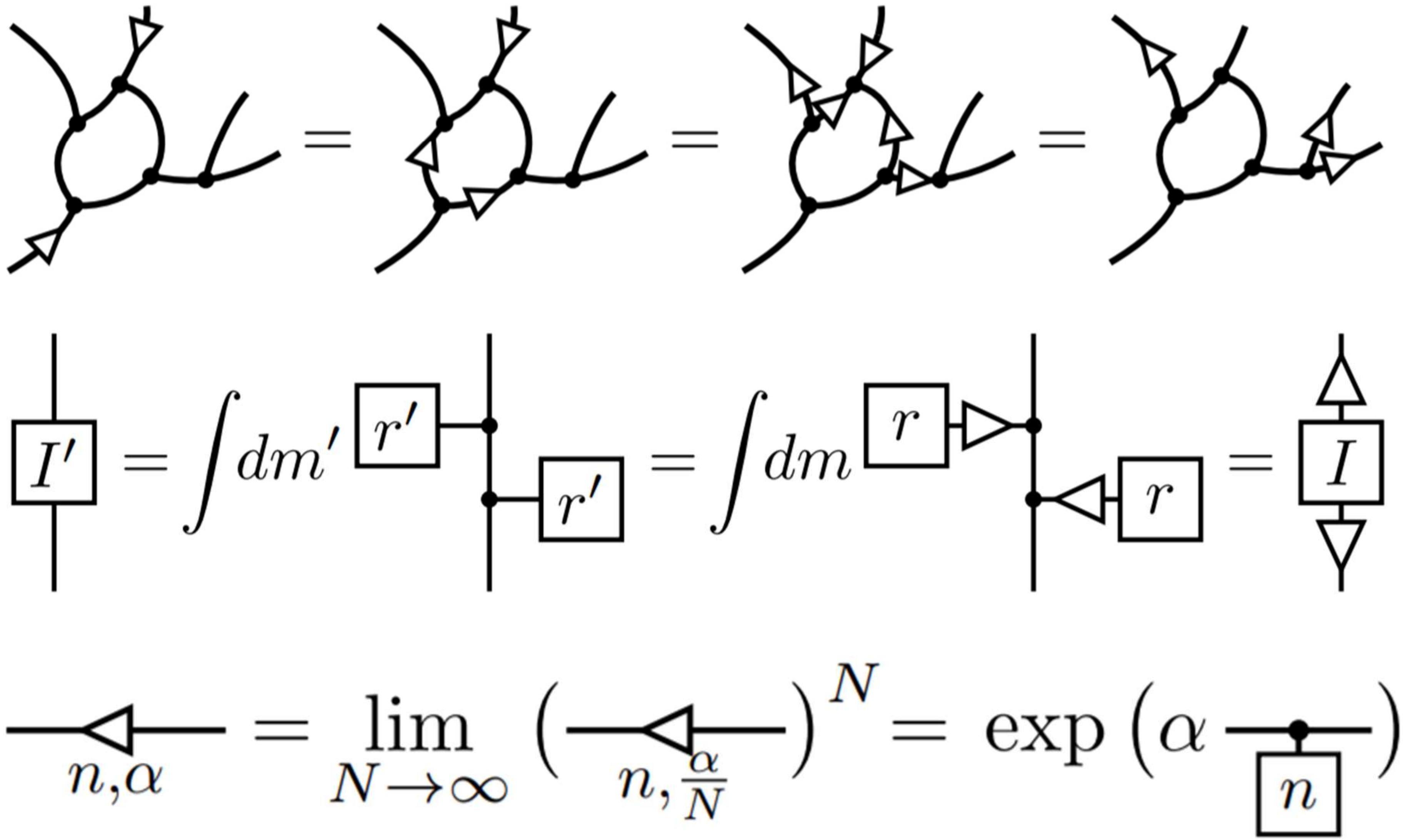
## (2) Vector Integral Calculus

✓ Graphical notation for integral calculus (Stokes' thm): inspired by the duality b/w chains and cochains



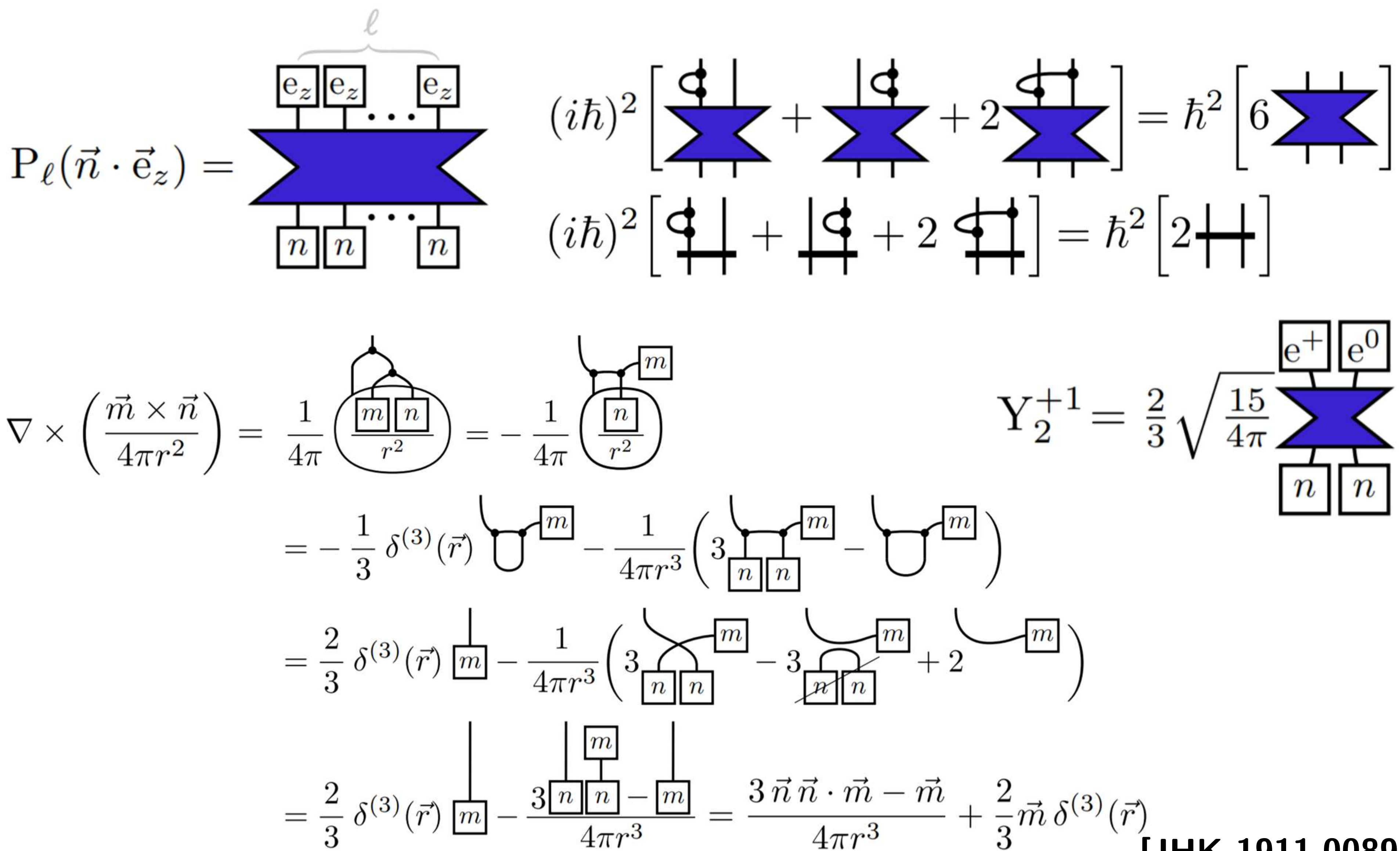

[JHK 1911.00892]

## (3) Rotation matrices, SO(3) covariance




[JHK 1911.00892]

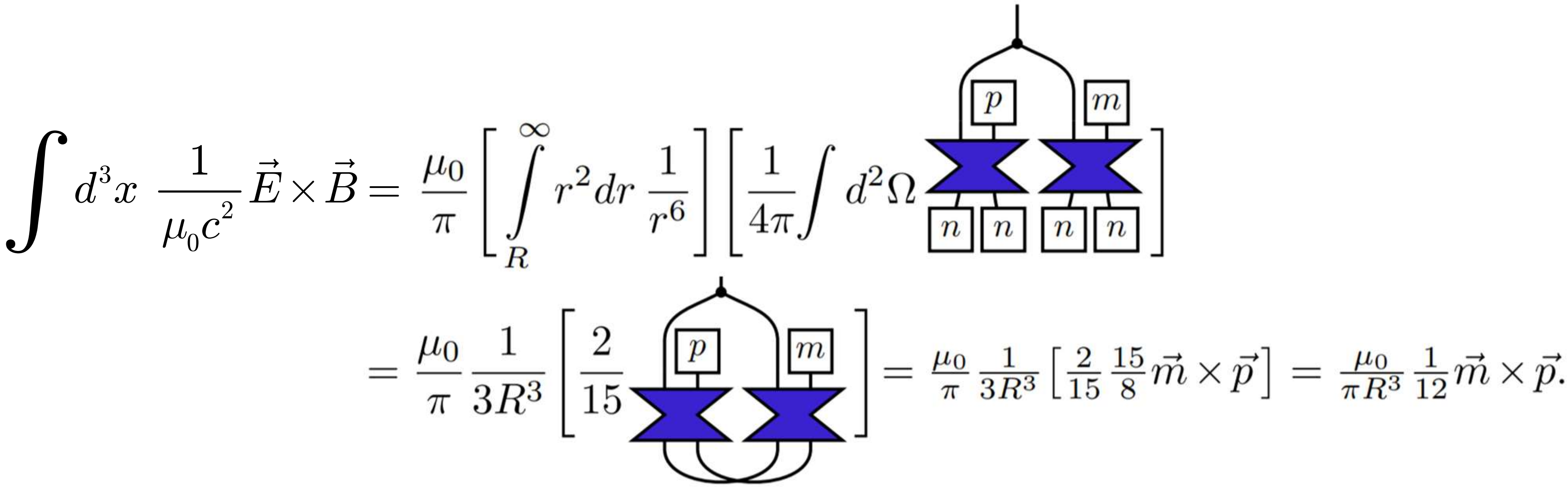
## (4) Rep thry of SO(3), harmonic functions, appl. to EM

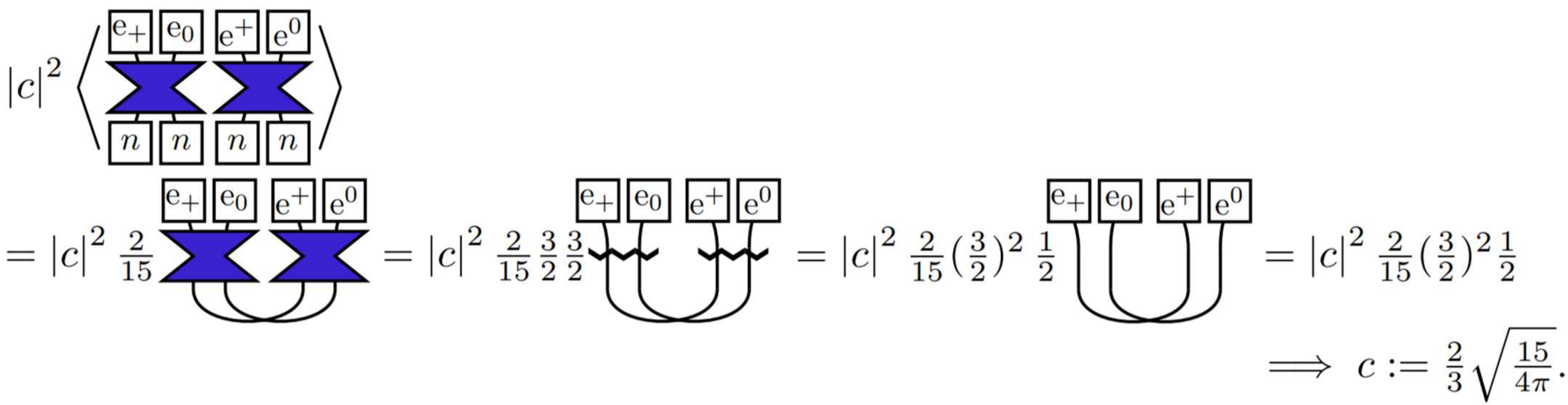



[JHK 1911.00892]

# 1. Penrose Graphical Notation

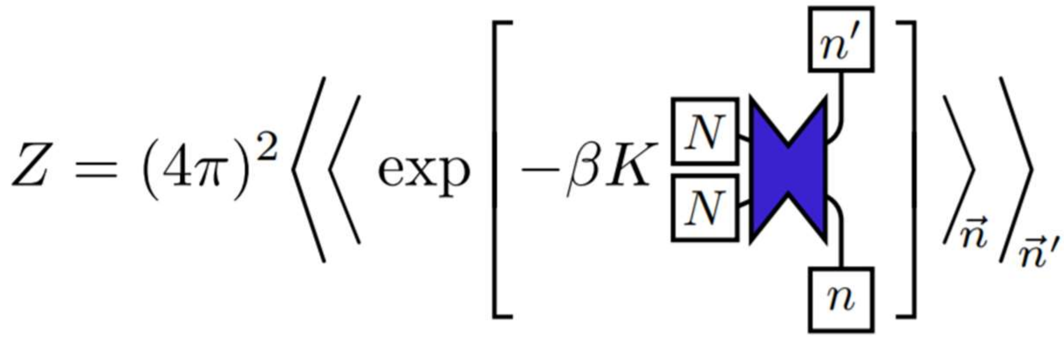
## (5) Algebro-combinatorial treatment of spherical integrals

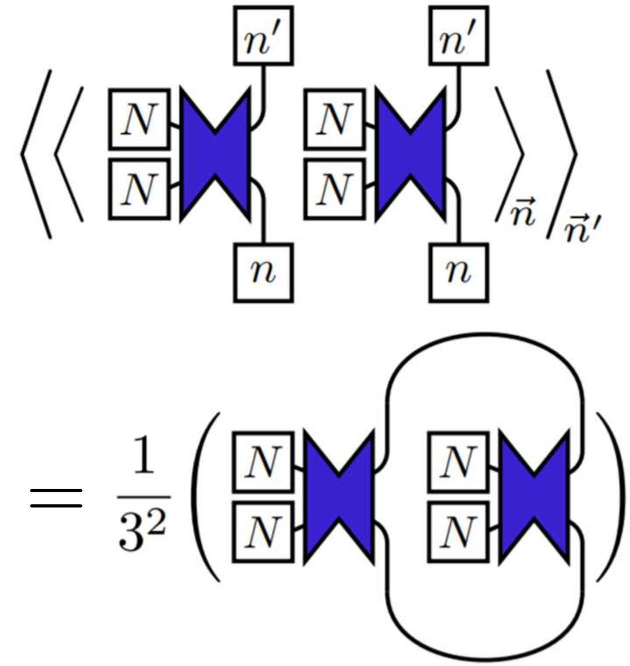


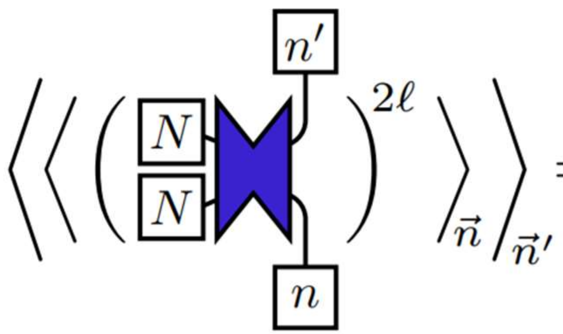




**[JHK 1911.00892]**

## (6) Diagrammatic perturbation: dipole-dipole interaction



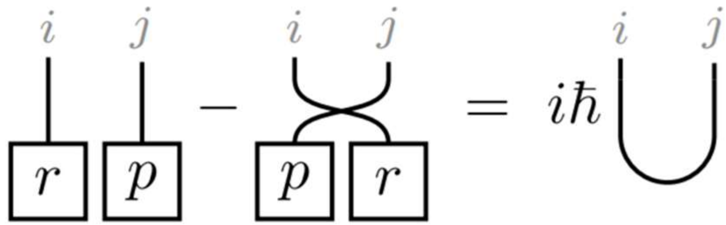



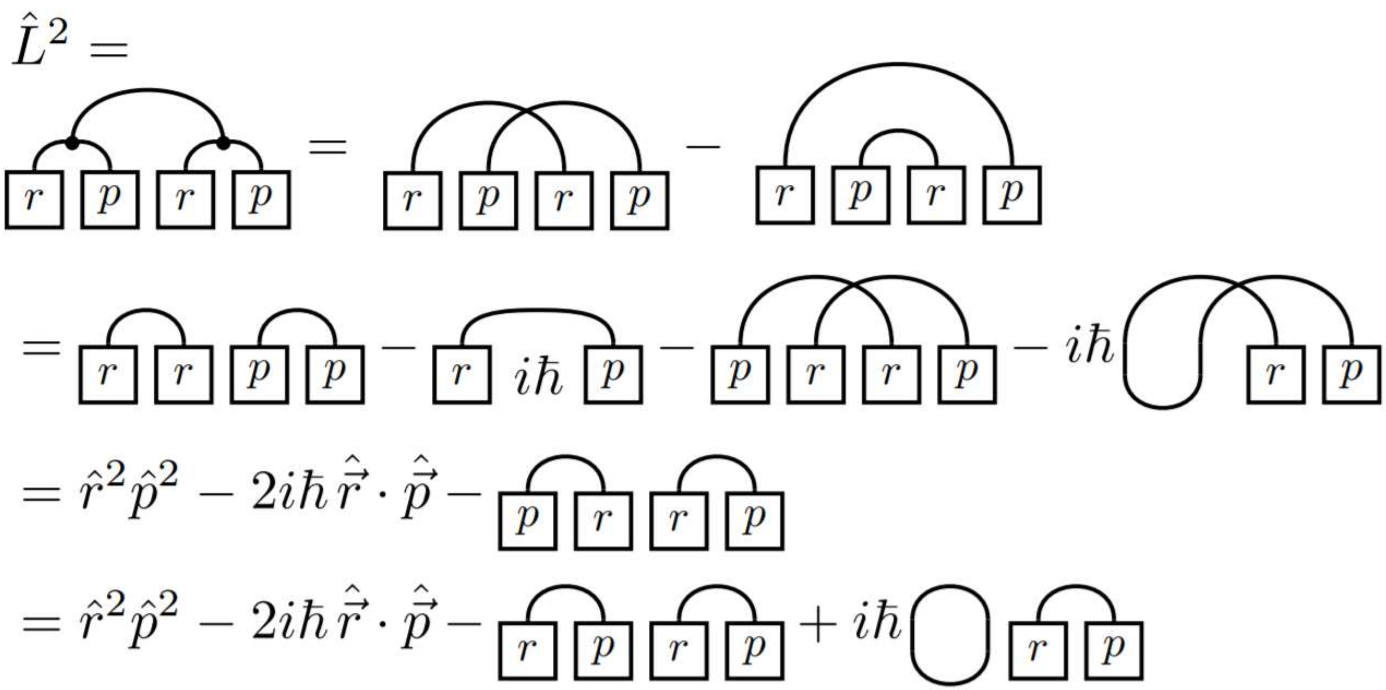

$$= \frac{1 + 2^{1-2\ell} p_\ell(3)}{(2\ell+1)^2},$$

$$p_\ell(d) := \begin{cases} 1 & (\ell = 1) \\ 1 + \sum_{j=1}^{\ell-1} \binom{\ell}{j} d^{2j-1}(d-2j) & (\ell > 1) \end{cases}$$

[JHK 1911.00892]

## (7) Quantum Mechanics




$$= \hat{r}^2\hat{p}^2 - 2i\hbar\,\hat{\vec{r}}\cdot\hat{\vec{p}} - (\hat{\vec{r}}\cdot\hat{\vec{p}})^2 + 3i\hbar\,\hat{\vec{r}}\cdot\hat{\vec{p}}$$

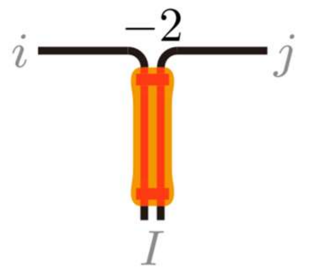
$$= \hat{r}^2\hat{p}^2 - (\hat{\vec{r}}\cdot\hat{\vec{p}})^2 + i\hbar\,\hat{\vec{r}}\cdot\hat{\vec{p}}$$

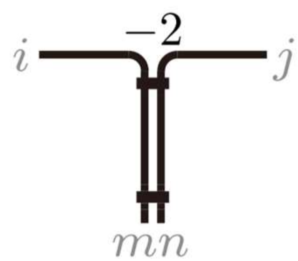
**[JHK 1911.00892]**

# 1. Penrose Graphical Notation

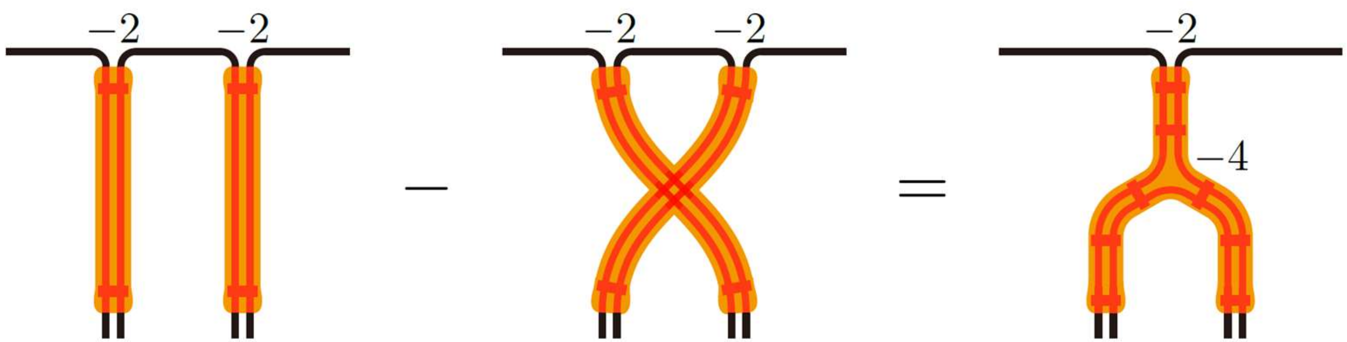
## ❖ Collective Indices

✓ "Self-similar" tensor systems in SO(3)

$(M_I)_{ij} =$ 


$(M_I)_{ij} \leftrightarrow$ 


$= -2\delta_{i[m}\delta_{n]j}$



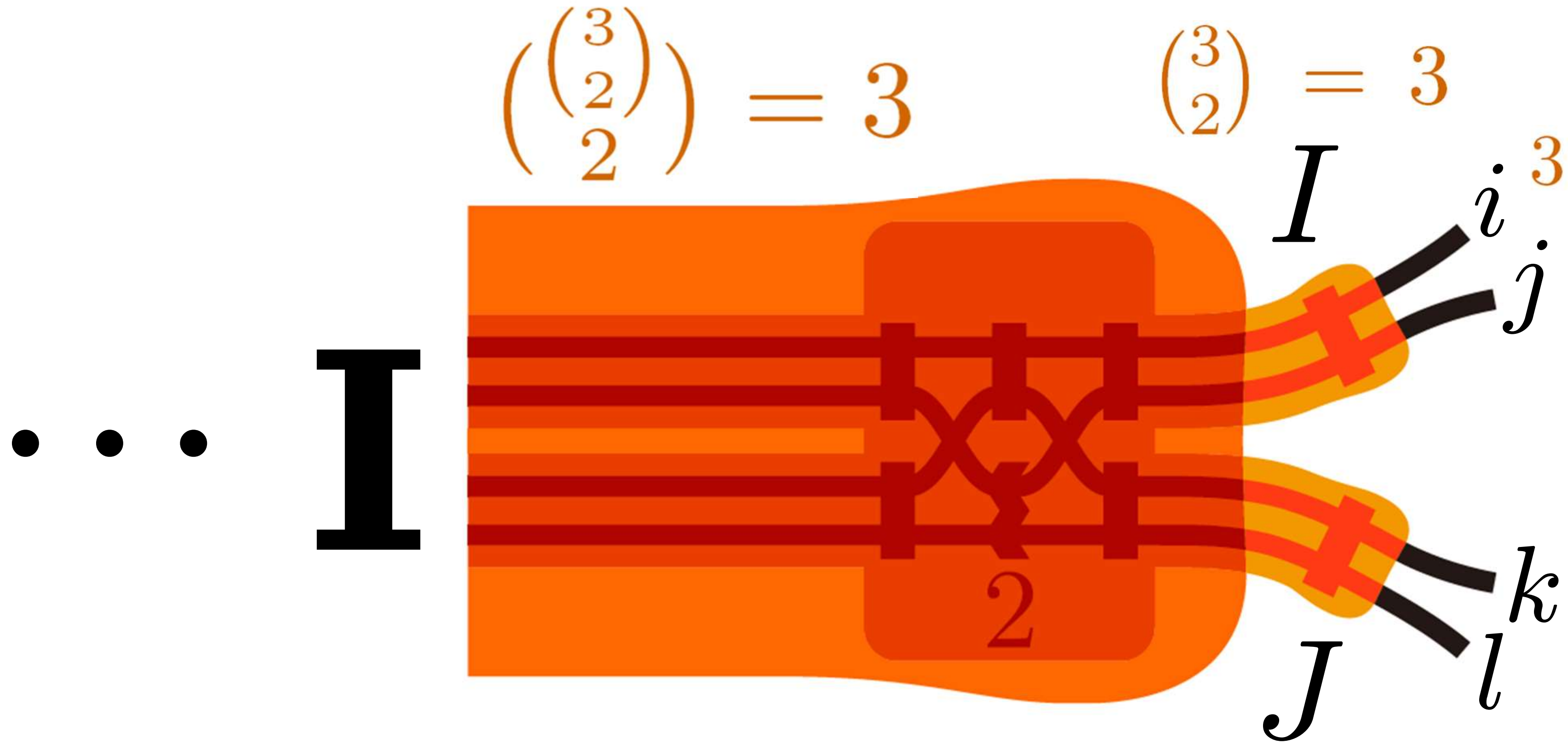

$$(M_I)_{ik}(M_J)_{kj} - (M_J)_{ik}(M_I)_{kj} = f_{KIJ}(M_K)_{ij}$$

"self-similar" if $\varepsilon_{KIJ} := \frac{\pm i}{\sqrt{2}} f_{KIJ}$

# 1. Penrose Graphical Notation

❖ Collective Indices

✓ Infinite tower of self-similarity

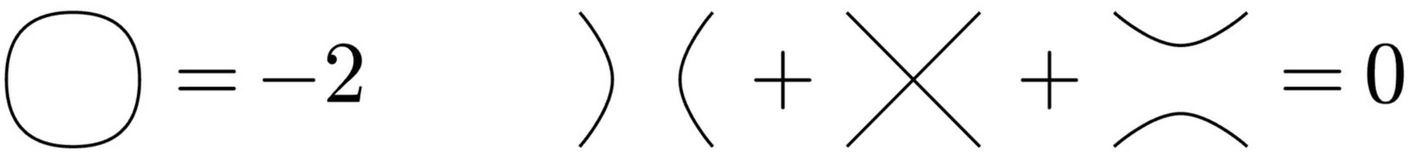



*"critical point"*

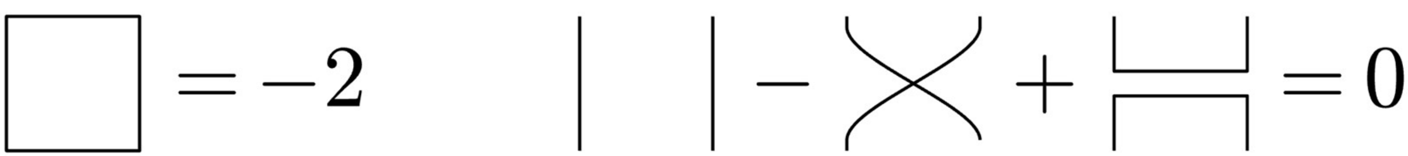
# 1. Penrose Graphical Notation

❖ Collective indices

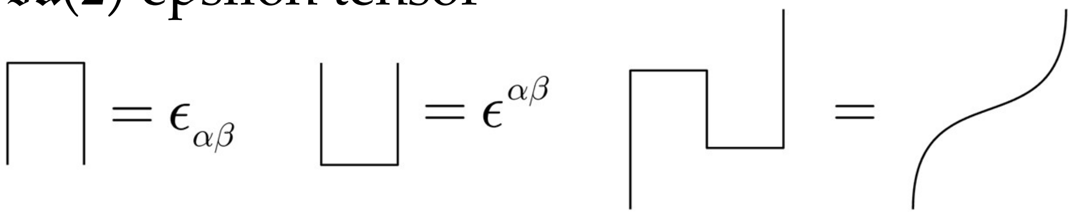
✓ Penrose's (−2)-dimensional tensors $\mathfrak{so}(-2) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$




✓ Interpretation (Sym→ASym)




$\mathfrak{su}(2)$ epsilon tensor




$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$$

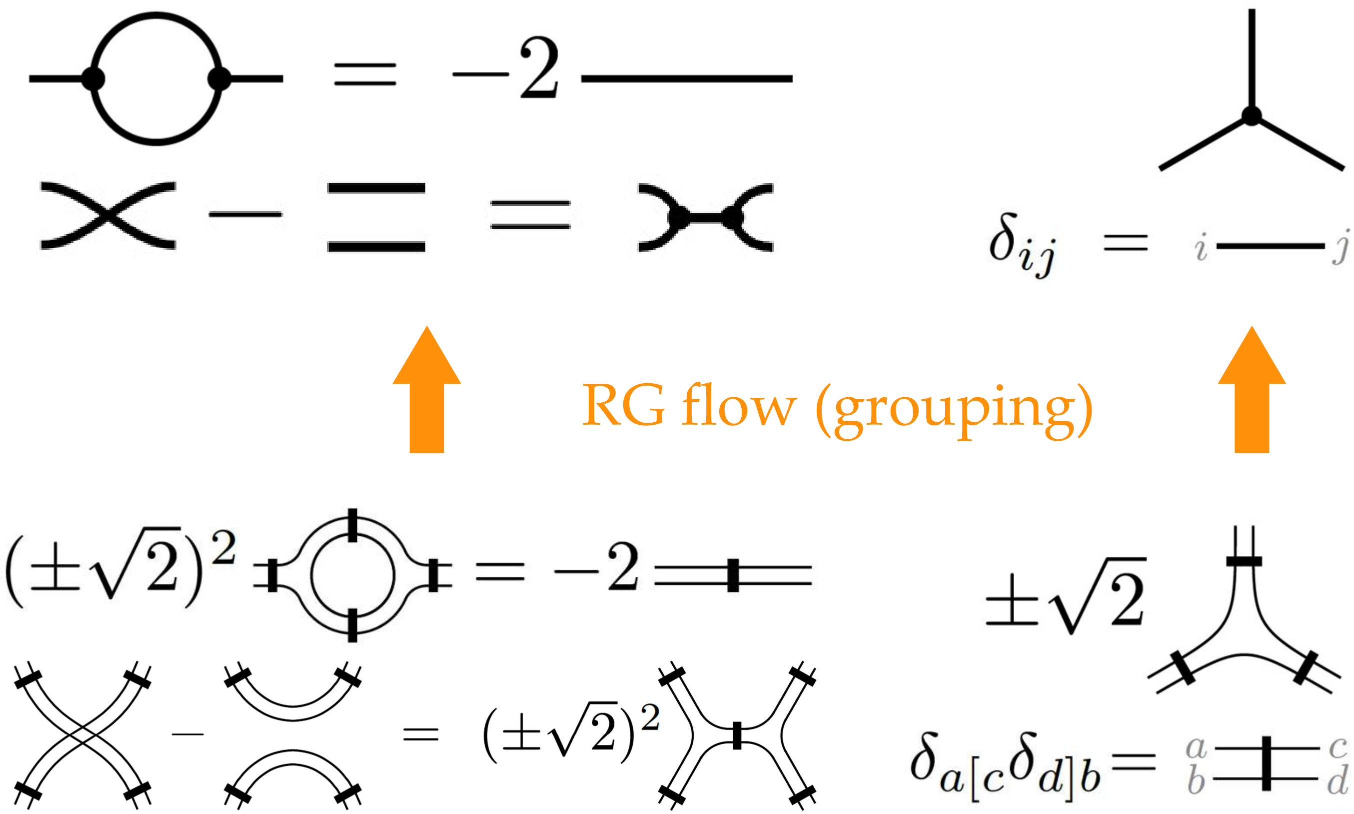
[Penrose 1971]
[Cvitanovic&Kennedy 1982]

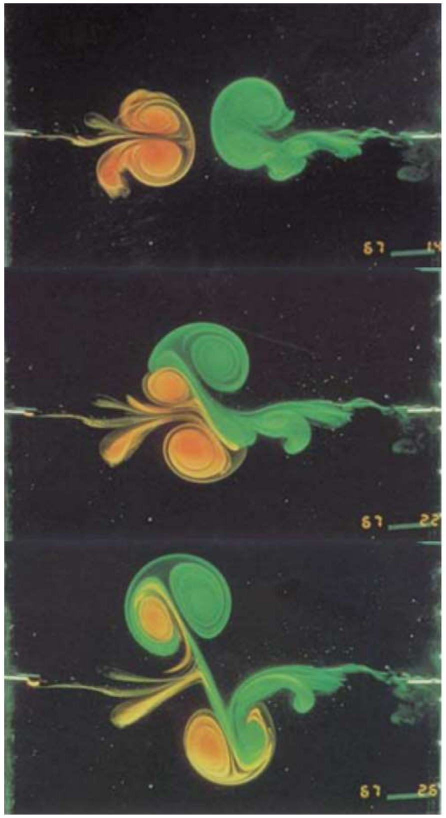
# 1. Penrose Graphical Notation

❖ Collective indices

✓ The "B1-floor" of the tower: $\binom{-2}{2} = 3$

Penrose's (−2)-dimensional tensors $\mathfrak{so}(-2) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$

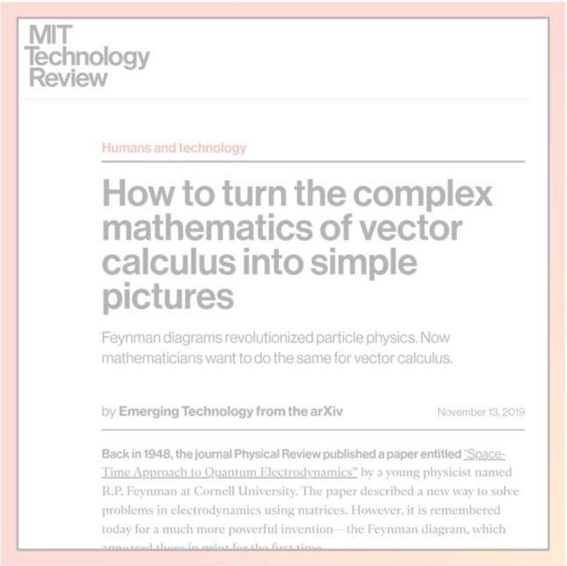


Notational systems in physics themselves may be investigated through physics

# Notation Engineering

## ❖ My Contributions


MIT Technology Review

Humans and technology

How to turn the complex mathematics of vector calculus into simple pictures

Feynman diagrams revolutionized particle physics. Now mathematicians want to do the same for vector calculus.

by Emerging Technology from the arXiv — November 13, 2019

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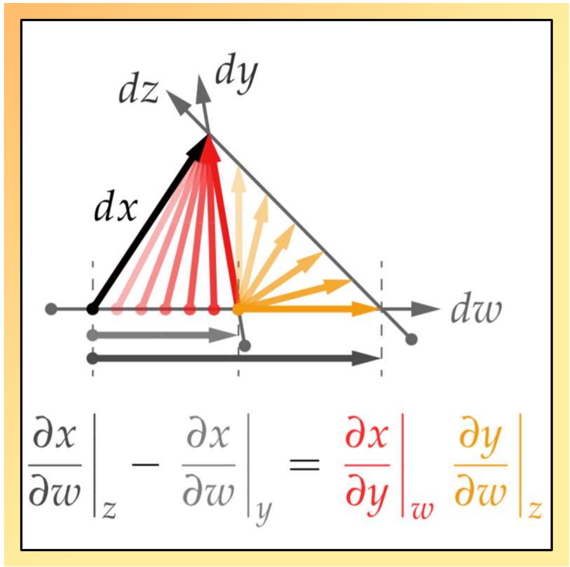
>> Application of the Penrose graphical notation to vector differential & integral calculus

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>> A novel graphical method for deriving partial derivative identities in thermodynamics

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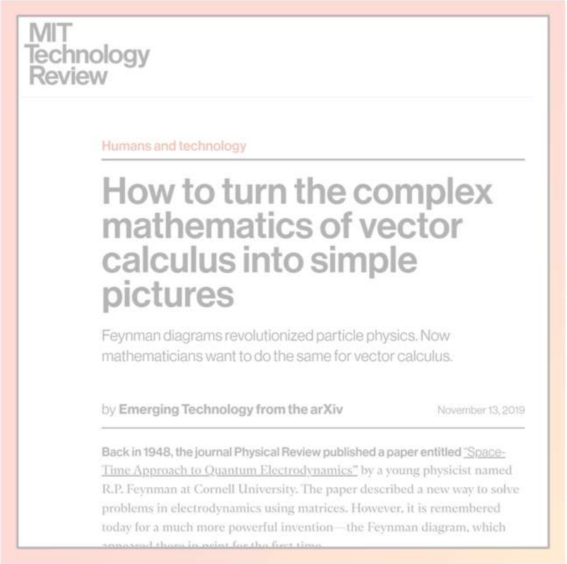
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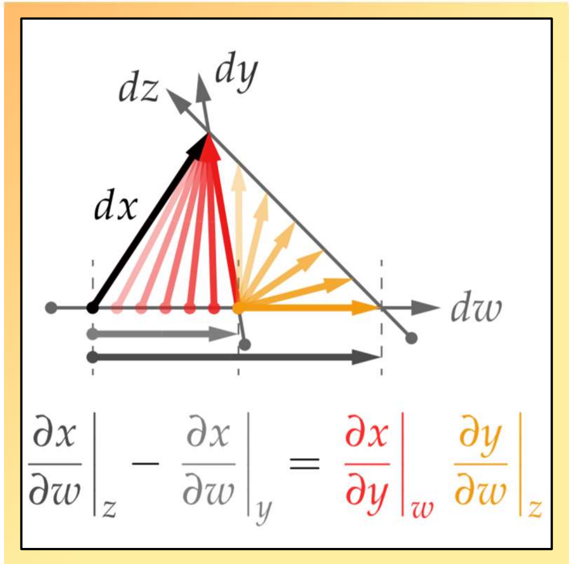
Juno Nam, SNU (undergrad)

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# 2. Sunray Diagrams

## ❖ Partial Derivative Identities in Thermodynamics

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial x}\right|_z = 1$$

Derivation complicated!

$$\left.\frac{\partial x}{\partial w}\right|_z - \left.\frac{\partial x}{\partial w}\right|_y = \left.\frac{\partial x}{\partial y}\right|_w \left.\frac{\partial y}{\partial w}\right|_z$$

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1$$

Why $-1$? Any intuitive explanations possible?

$$\cdots$$

$$\left.\frac{\partial x}{\partial y}\right|_w \left.\frac{\partial y}{\partial z}\right|_w = \left.\frac{\partial x}{\partial z}\right|_w$$

$$C_P - C_V = \frac{{\alpha_P}^2 TV}{\kappa_T}$$

Classification & systematic understanding of various identities?

# 2. Sunray Diagrams

❖ Partial Derivative Identities in Thermodynamics

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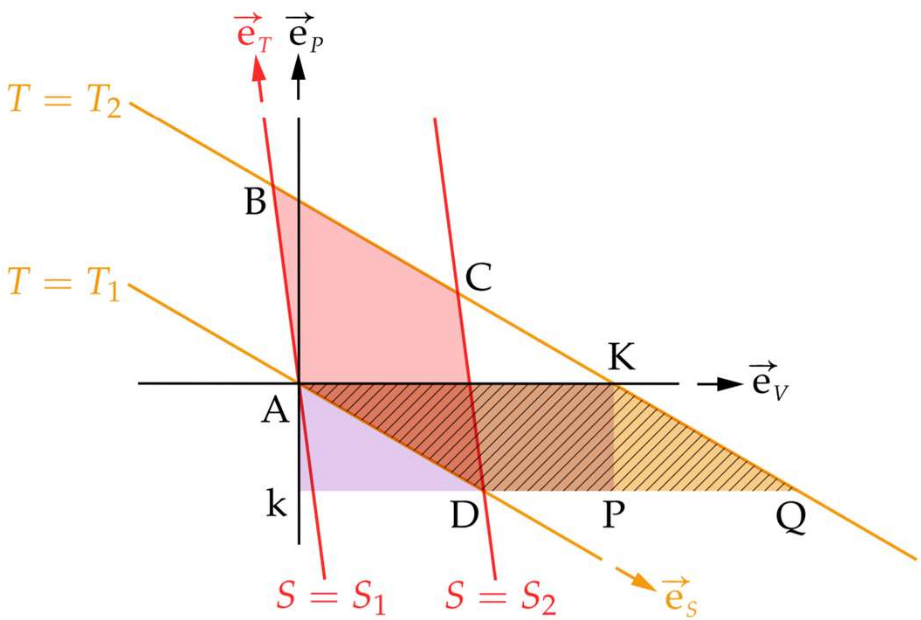
Classification & systematic understanding of various identities?

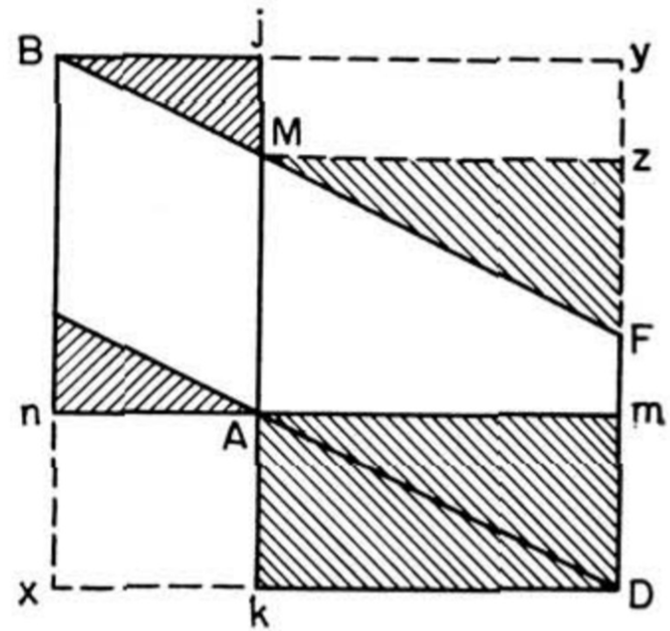
# 2. Sunray Diagrams

❖ Previous Works on the *Geometric Language* of Thermodynamics

✓ Maxwell (1871), reproduced in Nash (1964)




The equal-area sliding for deriving Maxwell relations

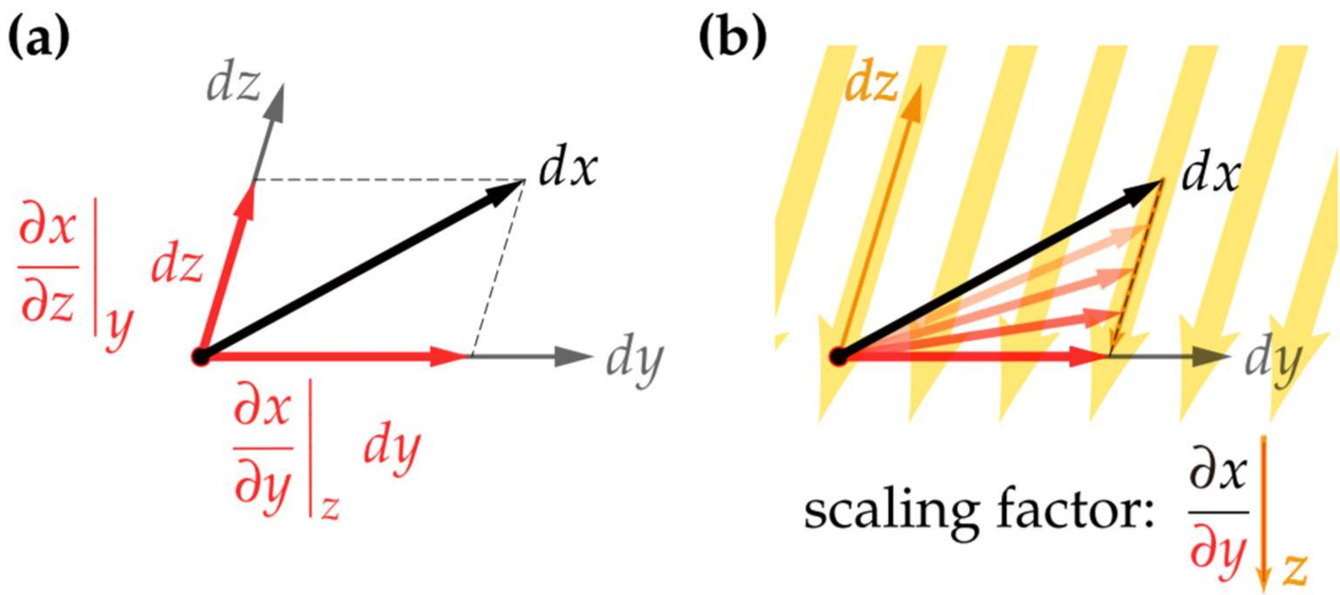



The equal-area sliding for deriving the $C_P - C_V$ identity

✓ Based on ***area manipulations*** ⇒ Inconvenient!

# 2. Sunray Diagrams

- Sunray Diagrams: Based on *Arrow Manipulations*
  - ✓ Devised by JHK in 2014
  - ✓ Started as a **black magic** (It works but why?)




Why does it work, and what's the meaning of "arrow *dx*"? "Constant-*z*" equals to "parallel to *dz*"?

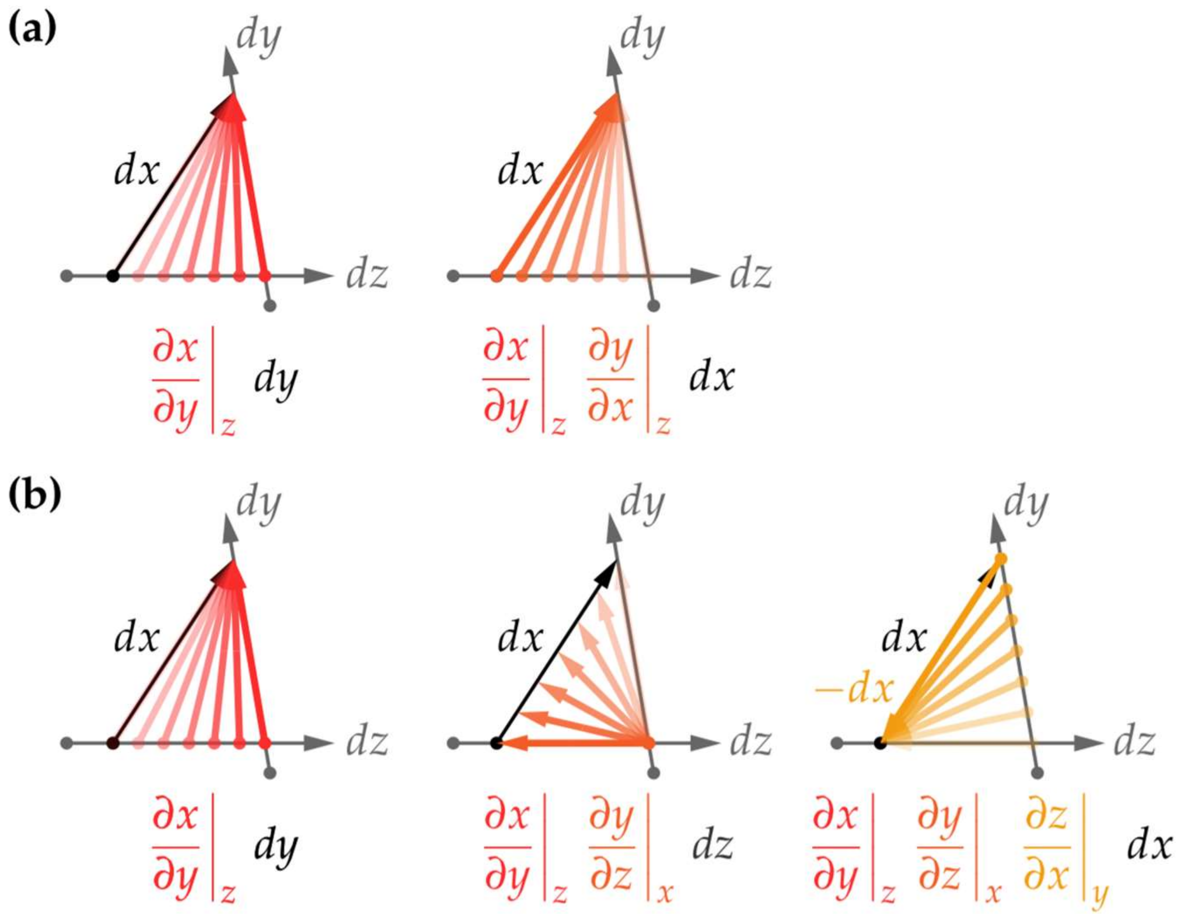
# 2. Sunray Diagrams

❖ Examples

✓ Intuitive explanation of the peculiar $-1$

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial x}\right|_z = 1$$

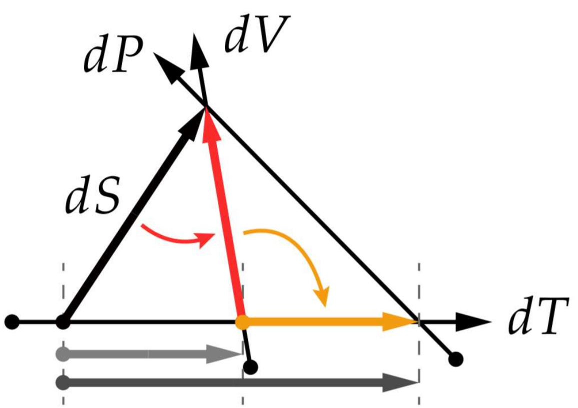
$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1$$



## ❖ Examples

✓ Easy derivation of identities:
comparison with Landau's "Jacobian technique" (p.53)

$$\left.\frac{\partial S}{\partial T}\right|_P - \left.\frac{\partial S}{\partial T}\right|_V = \left.\frac{\partial S}{\partial V}\right|_T \left.\frac{\partial V}{\partial T}\right|_P$$




$$\left.\frac{\partial S}{\partial T}\right|_P - \left.\frac{\partial S}{\partial T}\right|_V = \left.\frac{\partial S}{\partial V}\right|_T \left.\frac{\partial V}{\partial T}\right|_P$$

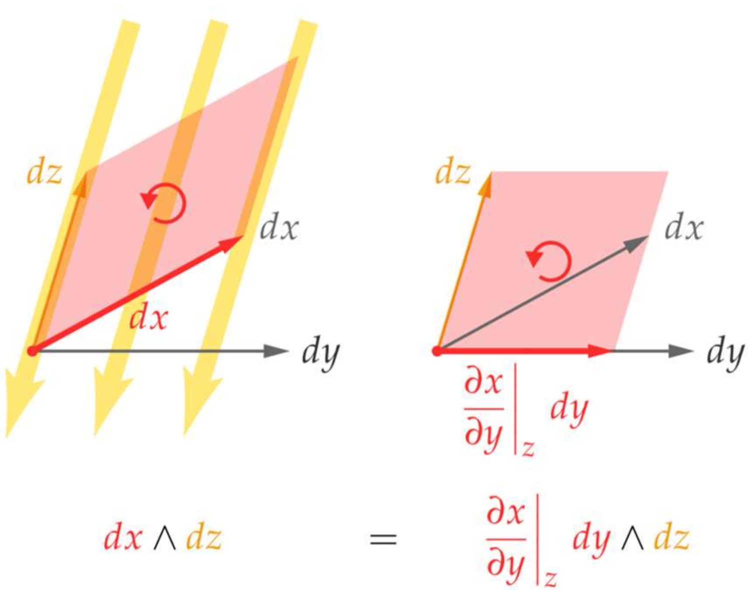
$$\begin{aligned}
&\left.\frac{\partial S}{\partial T}\right|_P - \frac{\partial(S,V)}{\partial(T,V)} \\
&= \left.\frac{\partial S}{\partial T}\right|_P - \frac{\partial(S,V)}{\partial(T,P)}\frac{\partial(T,P)}{\partial(T,V)} \\
&= \left.\frac{\partial S}{\partial T}\right|_P - \left[\left.\frac{\partial S}{\partial T}\right|_P \left.\frac{\partial V}{\partial P}\right|_T - \left.\frac{\partial V}{\partial T}\right|_P \left.\frac{\partial S}{\partial P}\right|_T\right]\frac{\partial(T,P)}{\partial(T,V)} \\
&= \left.\frac{\partial S}{\partial T}\right|_P - \left[\left.\frac{\partial S}{\partial T}\right|_P - \left.\frac{\partial V}{\partial T}\right|_P \left.\frac{\partial S}{\partial V}\right|_w\right] \\
&= \left.\frac{\partial V}{\partial T}\right|_P \left.\frac{\partial S}{\partial V}\right|_T
\end{aligned}$$

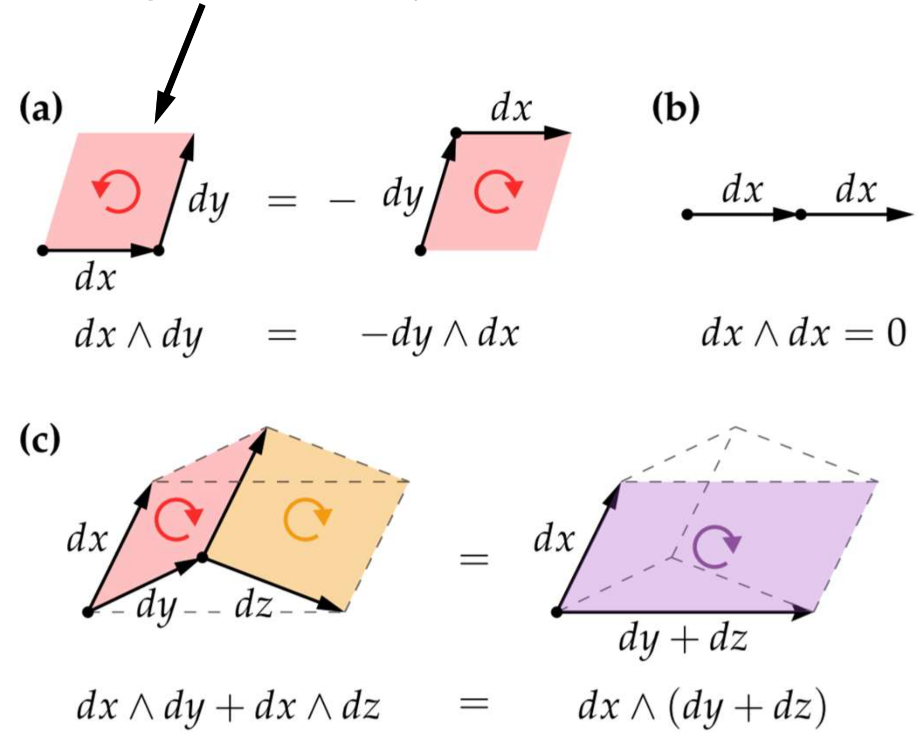
# 2. Sunray Diagrams

## ❖ The Oriented Area and Maxwell's Relations

✓ The Jacobian technique can be imported in the sunray language by introducing the oriented area structure

define $\overrightarrow{dx} \wedge \overrightarrow{dy}$ as the oriented area generated by $\overrightarrow{dx}$ and $\overrightarrow{dy}$



# 2. Sunray Diagrams

## ❖ The Oriented Area and Maxwell's Relations

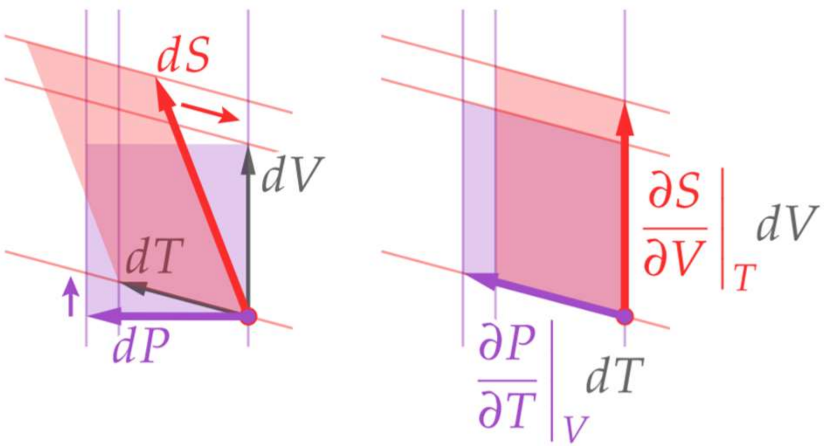
✓ Maxwell's relations can be derived by assuming that

$$\overrightarrow{dP} \wedge \overrightarrow{dV} = \overrightarrow{dT} \wedge \overrightarrow{dS}$$

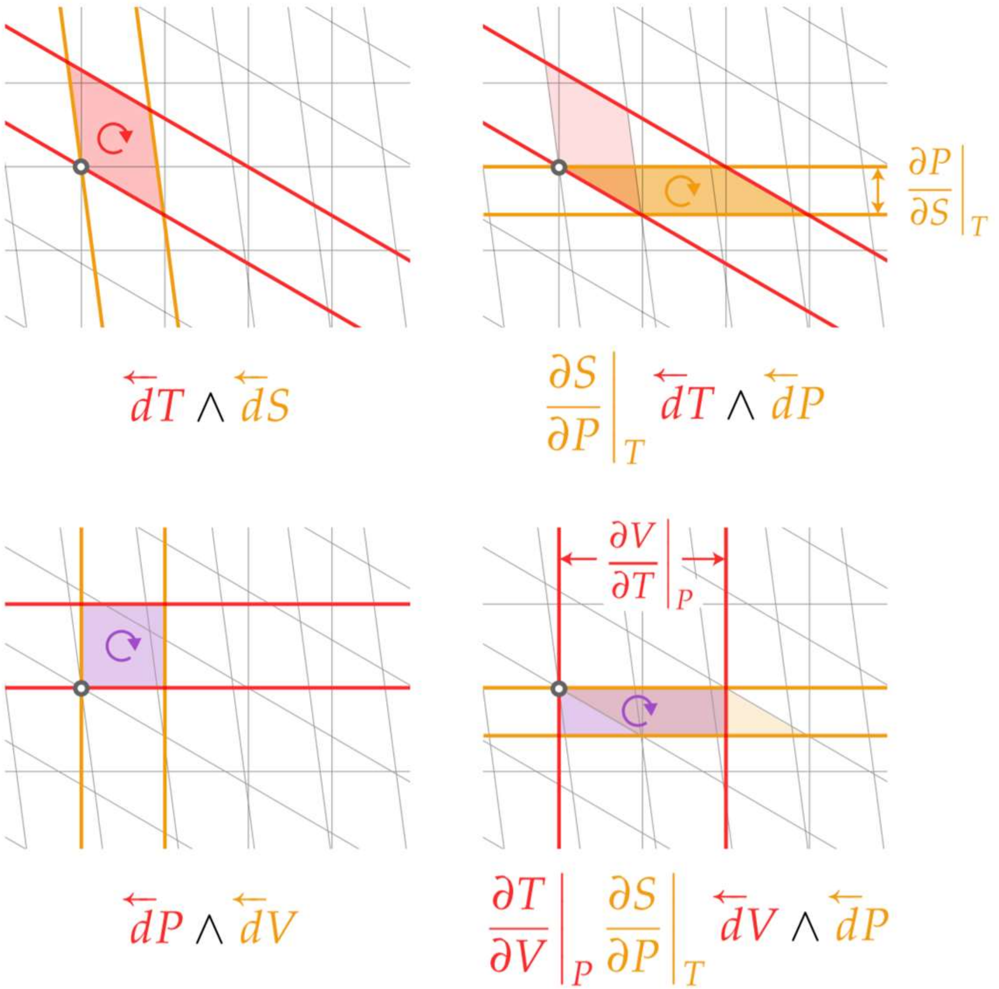
… reminds us of the differential forms derivation…

$$\overleftarrow{d}P \wedge \overleftarrow{d}V = \overleftarrow{d}T \wedge \overleftarrow{d}S$$

## ❖ The Oriented Area and Maxwell's Relations




Sunray diagram derivation

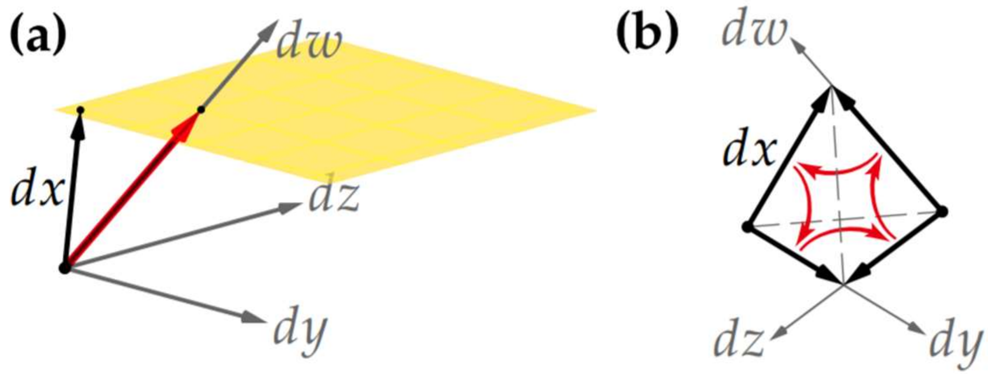



Differential forms derivation
[Maxwell(1871), Nash(1964)]

# 2. Sunray Diagrams

❖ Extensions

✓ $n$-variables generalization possible: "sunplane" diagrams




$$\left.\frac{\partial x}{\partial y}\right|_{z,w}\left.\frac{\partial y}{\partial z}\right|_{x,w}\left.\frac{\partial z}{\partial w}\right|_{x,y}\left.\frac{\partial w}{\partial x}\right|_{y,z}=+1$$

# 2. Sunray Diagrams

## ❖ Semantics: More Than Black Magic

- ✓ What is "$\overrightarrow{dT}$" in sunray diagrams?
  (The arrow representing the differential $dT$)
- ✓ Not only being a mathematical abstraction, its "physical" aspect hints that it should somehow directly interpreted in the equilibrium thermodynamic state space (*P*-*V* or *T*-*S* plane)

$$\overrightarrow{dP} \wedge \overrightarrow{dV} = \overrightarrow{dT} \wedge \overrightarrow{dS}$$

infinitesimal oriented area
in the thermodynamic state space?

- ✓ However, it is not a one-form but a vector.
  (Transforms like an arrow, not contour lines)

## ❖ Semantics: More Than Black Magic

- ✓ What is "$\overrightarrow{dT}$" in sunray diagrams?
  (The arrow representing the differential $dT$ )
- ✓ RG flow of syntax

[I]

$$dx = \left.\frac{\partial x}{\partial y}\right|_z dy + \left.\frac{\partial x}{\partial z}\right|_y dz$$

$$dT \wedge dS = dP \wedge dV$$

Differentials

[II]

$$\overleftarrow{dx} = \left.\frac{\partial x}{\partial y}\right|_z \overleftarrow{dy} + \left.\frac{\partial x}{\partial z}\right|_y \overleftarrow{dz}$$

$$\overleftarrow{dT} \wedge \overleftarrow{dS} = \overleftarrow{dP} \wedge \overleftarrow{dV}$$

Difforms

[III]

$$\overrightarrow{dx} = \left.\frac{\partial x}{\partial y}\right|_z \overrightarrow{dy} + \left.\frac{\partial x}{\partial z}\right|_y \overrightarrow{dz}$$

$$\overrightarrow{dT} \wedge \overrightarrow{dS} = \overrightarrow{dP} \wedge \overrightarrow{dV}$$

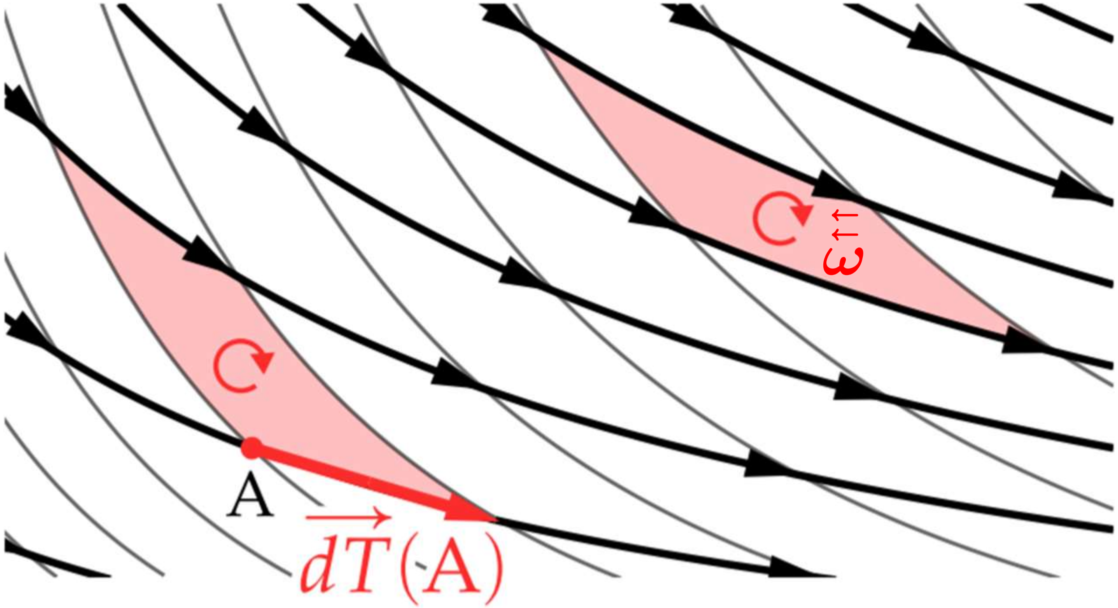
Arrows in sunray diagrams

Figure 14: Two implementations of the partial derivative syntax by differential forms and symplectic gradient vectors. When the arrows ← and → of systems [II] and [III] are "integrated out" (ignored), both of them flows to the system [I].

# 2. Sunray Diagrams

- Semantics: More Than Black Magic
  - ✓ "Vectors as elements of a vector space"
    → sounds nice for mathematicians, but not for physicists
  - ✓ A physicist's answer: Hamiltonian vec. fields $\overrightarrow{dT} \neg \omega = -\overleftarrow{d}T$

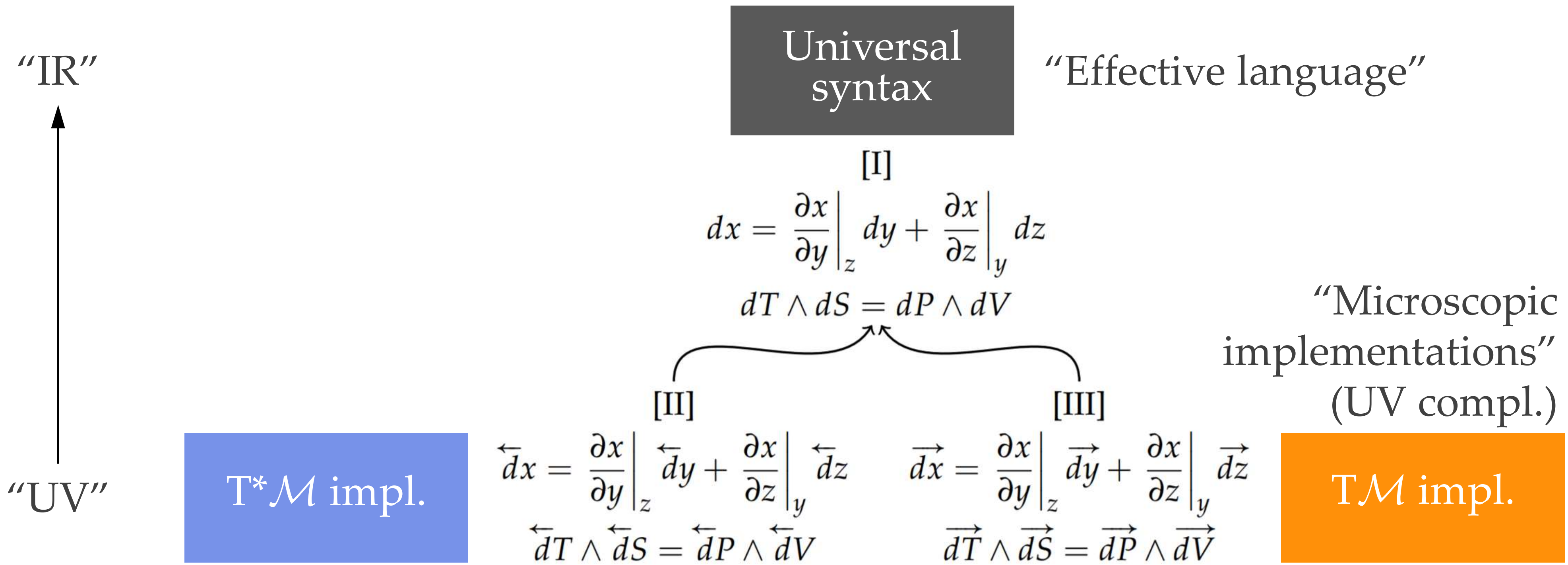
Symplectic form $\overleftarrow{\overleftarrow{\omega}} = \overleftarrow{d}P \wedge \overleftarrow{d}V = \overleftarrow{d}T \wedge \overleftarrow{d}S$




flows through equal-$T$ surface and
increases the conjugate quantity $S$ by one unit

## ❖ Semantics: More Than Black Magic

✓ Microscopic implementations of the universal syntax

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**Notational systems in physics themselves can be investigated through physics**

# 2. Sunray Diagrams

## ❖ Black Magic in General

✓ Microscopic implementation (UV compl.) unknown

A letter to rigour-seeking mathematicians: there is no problem using effective theories (or non-renormalizable theories) in appropriate low-energy regions.

$$\left(1 - \tfrac{d}{dt}\right) x(t) = e^{2t}$$

$$\begin{aligned}
\Rightarrow \quad x(t) &= \frac{1}{1-\frac{d}{dt}} e^{2t} + \frac{1}{1-\frac{d}{dt}} 0 \\
&= \left(1 + \tfrac{d}{dt} + (\tfrac{d}{dt})^2 + \cdots\right) e^{2t} + Ce^t \\
&= \left(1 + 2 + 2^2 + \cdots\right) e^{2t} + Ce^t \\
&= \left(-1\right) e^{2t} + Ce^t
\end{aligned}$$

$$\int \mathcal{D}\phi \, e^{-S[\phi]}$$

Somehow working effective languages

"IR"

"UV"

*Analytic continuation, functional analysis, spectral theory, …*

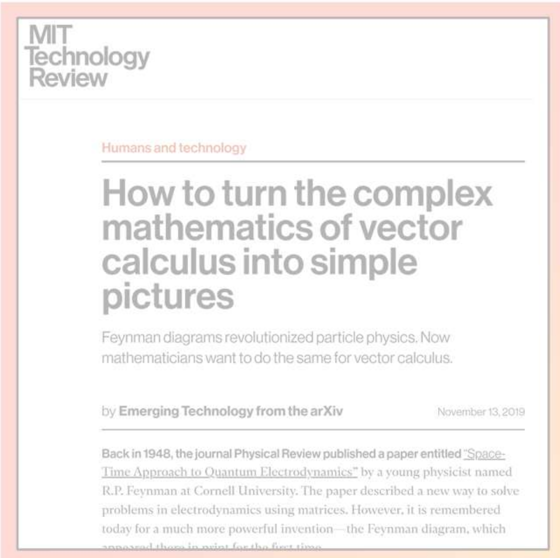
?

Microscopic implementation unknown

Notational systems in physics themselves can be investigated through physics

# Notation Engineering

## ❖ My Contributions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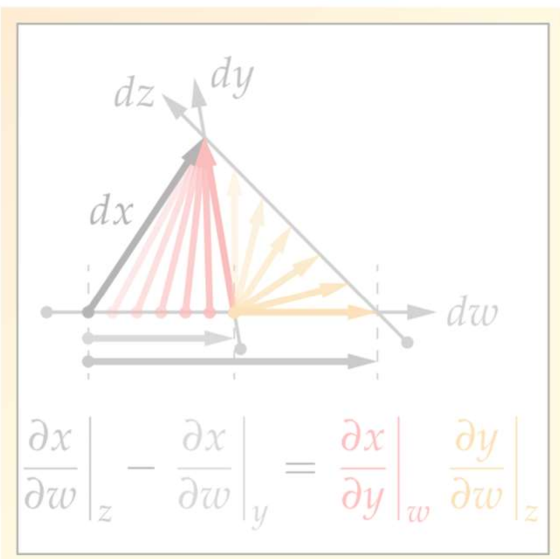


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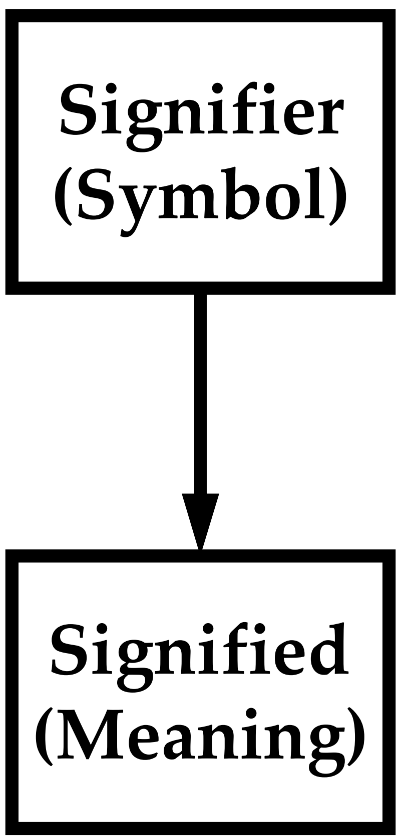
# Suggestion: “Notationology”

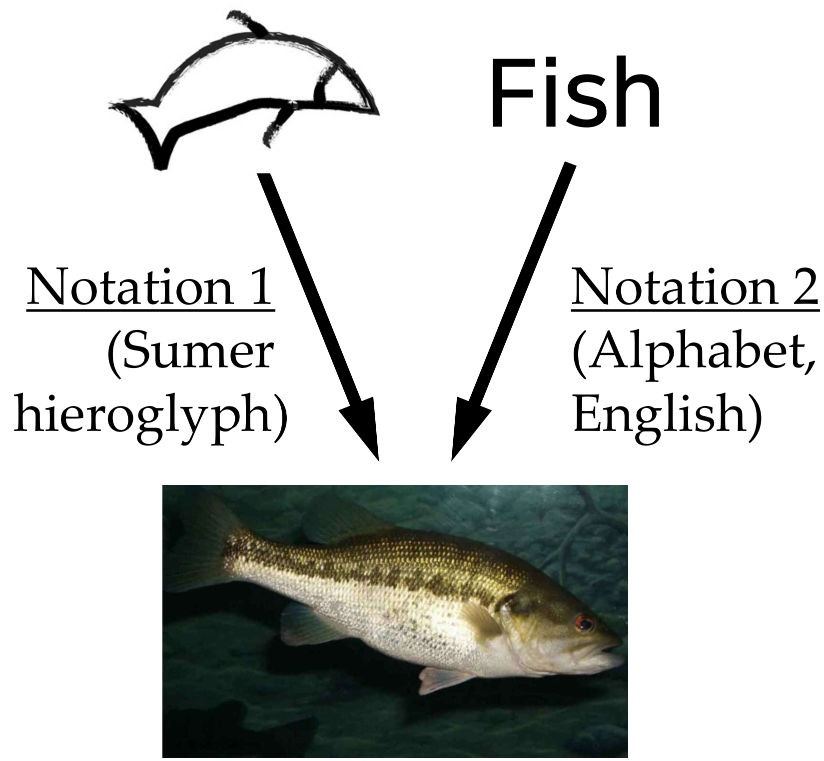
❖ Systematic Understanding of Notational Systems

- ✓ We are left with several questions:
  - Why do particular notations (bra-ket, tensor graphs, …) are better than others? What makes them different?
    **What defines “good notations”?**
  - How can a well-made notation tell us about the physical reality? Isn’t a notation merely a mathematical tool?
- ✓ I found the concept of **signifier** and **signified** in linguistics (semiotic theory of signs) useful to address and understand these issues.

# The Signifier-Signified Separation (SSS)

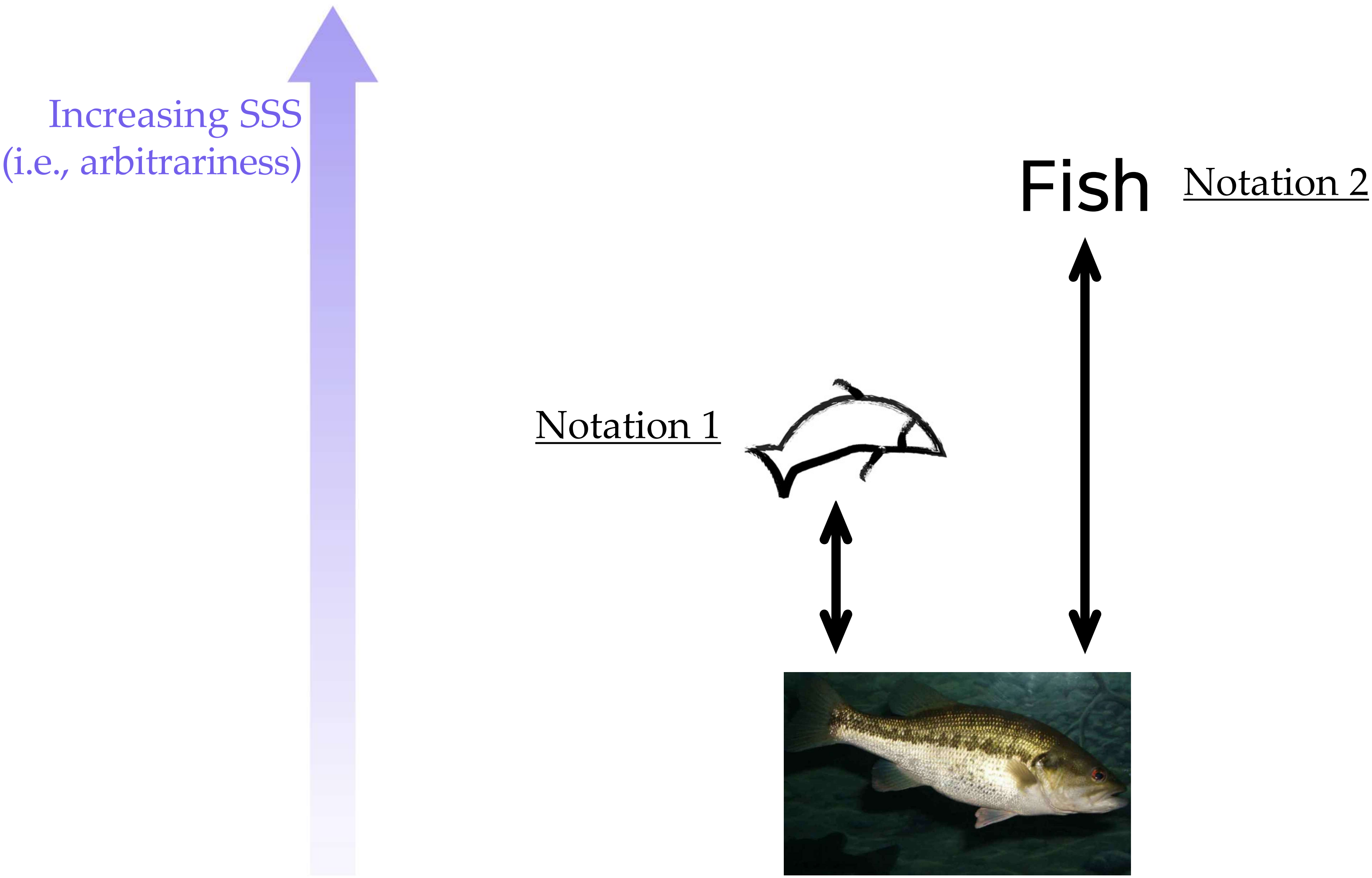
- ❖ Semiotic Theory of Signs (Peirce, Saussure, 1900-1910s)
  - ✓ What is a notational system?
    It is a collection of signifier-to-signified maps!



# The Signifier-Signified Separation (SSS)

❖ *Less Arbitrary* Notations Have *Smaller SSS*



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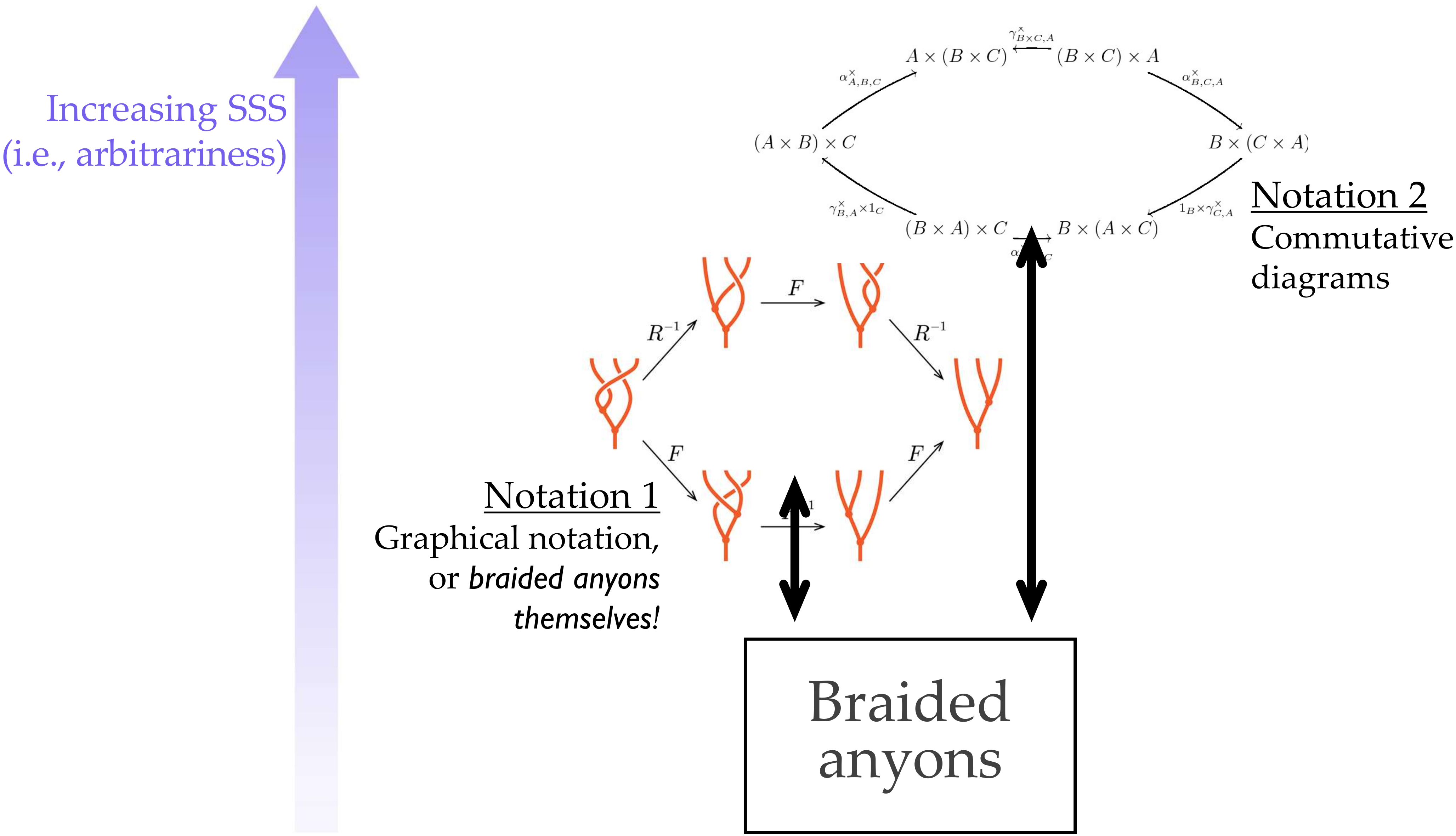
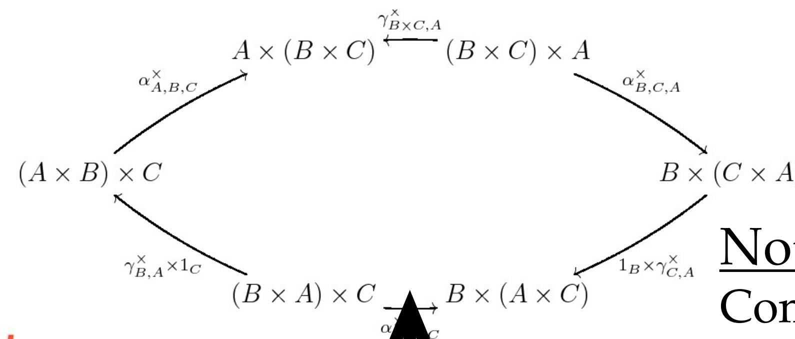


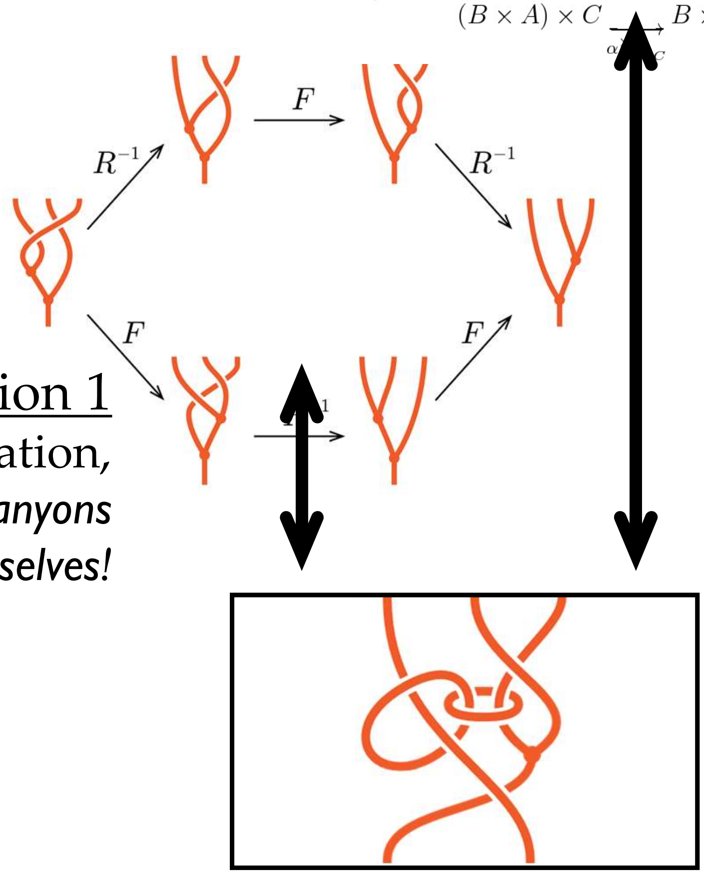
# The Signifier-Signified Separation (SSS)

❖ *Less Arbitrary* Notations Have *Smaller SSS*

Increasing SSS
(i.e., arbitrariness)




Notation 2
Commutative diagrams




Notation 1
Graphical notation,
or *braided anyons themselves!*

# The Signifier-Signified Separation (SSS)

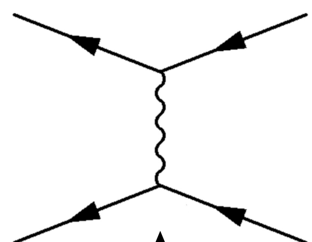
❖ ***Less Arbitrary*** **Notations Have** ***Smaller SSS***

Increasing SSS
(i.e., arbitrariness)

$$(ie)^2 \int d^4x\, d^4x'\, \underleftarrow{\psi}(x)\, \underline{\gamma^\mu}\, \underrightarrow{\psi}(x)\, \underleftarrow{\psi}(x')\, \underline{\gamma^\nu}\, \underrightarrow{\psi}(x')\, \boxed{A_\mu(x)A_\nu(x')}$$

Notation 2
QFT S-matrix

Notation 1
Feynman diagrams,
or *quantum particles themselves!*

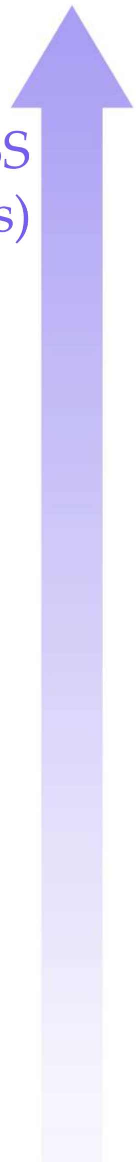


Quantum Particles

# The Signifier-Signified Separation (SSS)

❖ *__Less Arbitrary__* Notations Have *__Smaller SSS__*

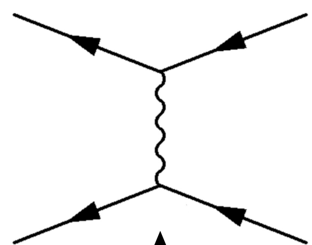
Increasing SSS
(i.e., arbitrariness)

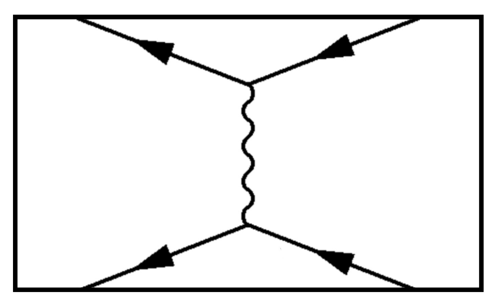


$$(ie)^2 \int d^4x\, d^4x'\, \underset{\leftarrow}{\psi}(x)\,\underline{\;\gamma^\mu\;}\,\underset{\rightarrow}{\psi}(x)\,\underset{\leftarrow}{\psi}(x')\,\underline{\;\gamma^\nu\;}\,\underset{\rightarrow}{\psi}(x')\,\boxed{A_\mu(x)A_\nu(x')}$$

<ins>Notation 2</ins>
QFT S-matrix

<ins>Notation 1</ins>
Feynman diagrams,
or *quantum particles themselves!*





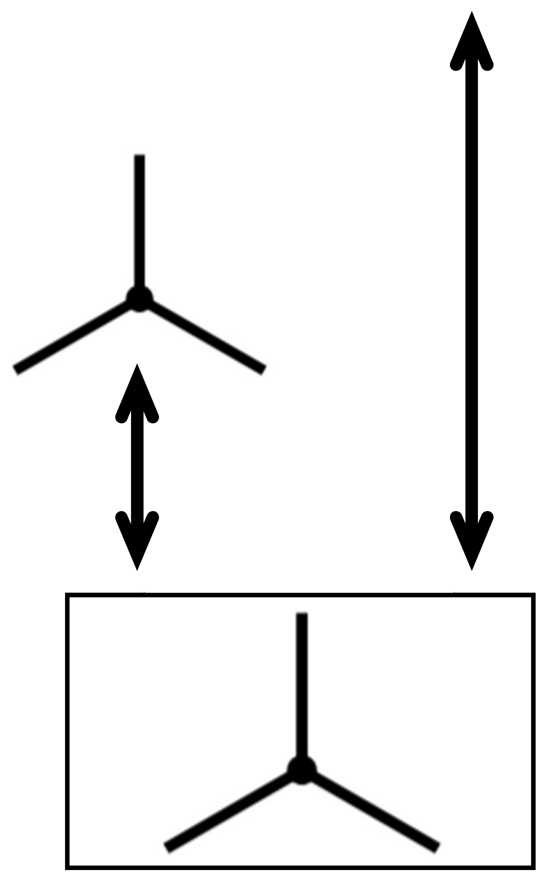
# The Signifier-Signified Separation (SSS)

❖ *Less Arbitrary* Notations Have *Smaller SSS*

Increasing SSS
(i.e., arbitrariness)

$\varepsilon_{ijk}$ Notation 2
Index notation

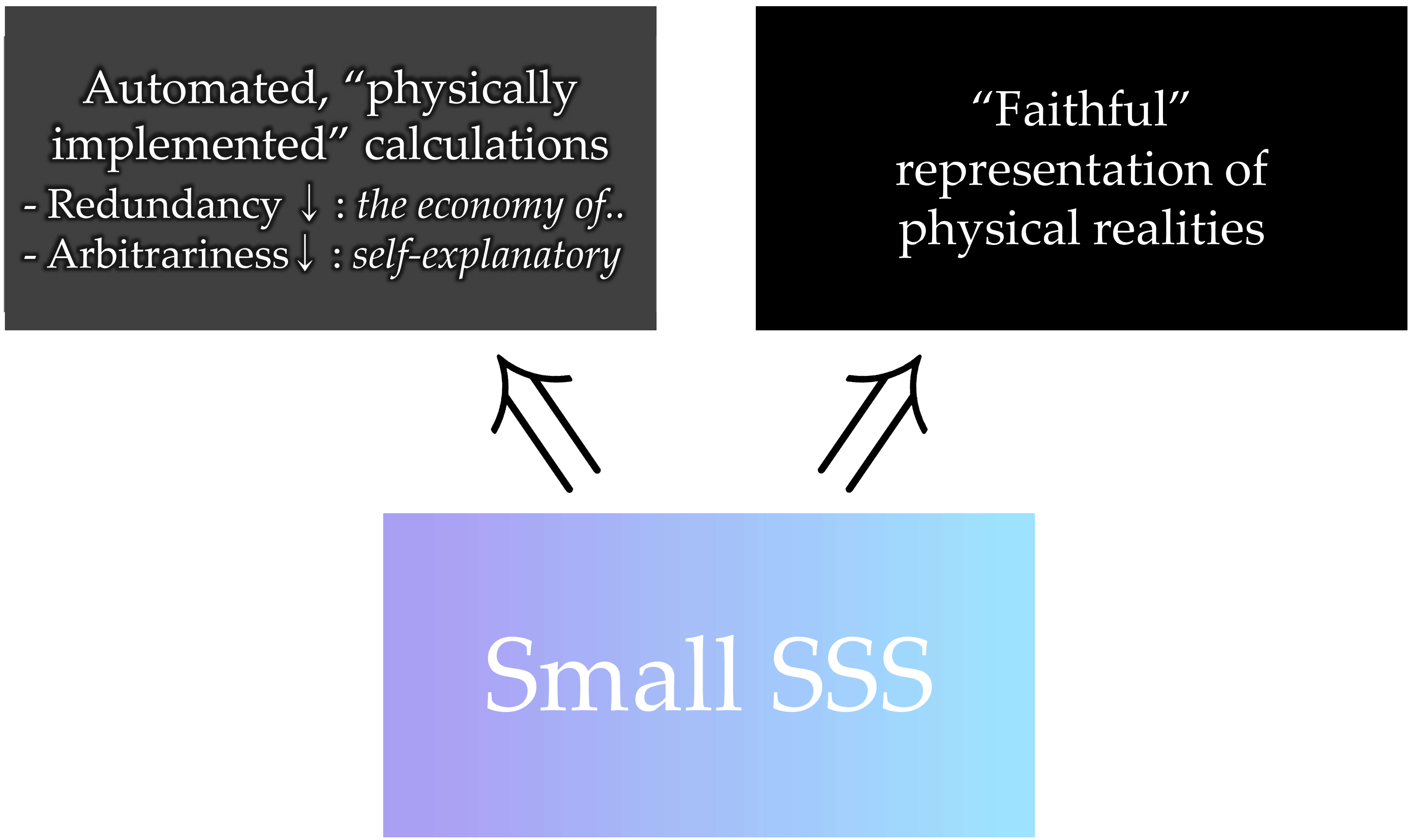
Notation 1
Penrose graphs, or
*"physical implementation"*
*of SO(3)-tensors!*
("SO(3)-*particles*")



If we construct a **TQFT**
(to remove kinematics dof)
with **SO(3) interaction vertices**,
surely it will be the Penrose
notation of SO(3)-tensors!

# Small-SSS Notations

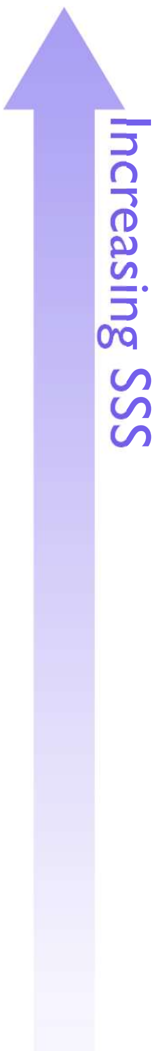
❖ The Working Principle Behind “Good Notations”



# Small-SSS Notations

❖ The Working Principle Behind "Good Notations"

✓ What is the most fundamental "rep" of molecules?

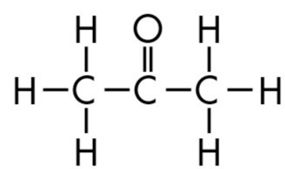


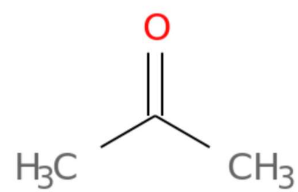

ACE-tone

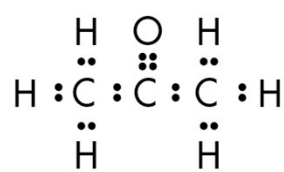
Acetone

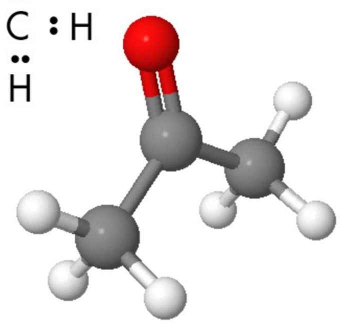
Propan-2-one

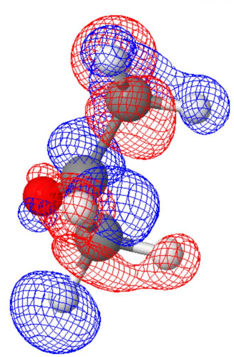
$(CH_3)_2CO$











(The physical molecule itself)

# Small-SSS Notations

❖ The Working Principle Behind "Good Notations"

✓ What is the most fundamental "rep" of molecules?

Increasing SSS

ACE-tone

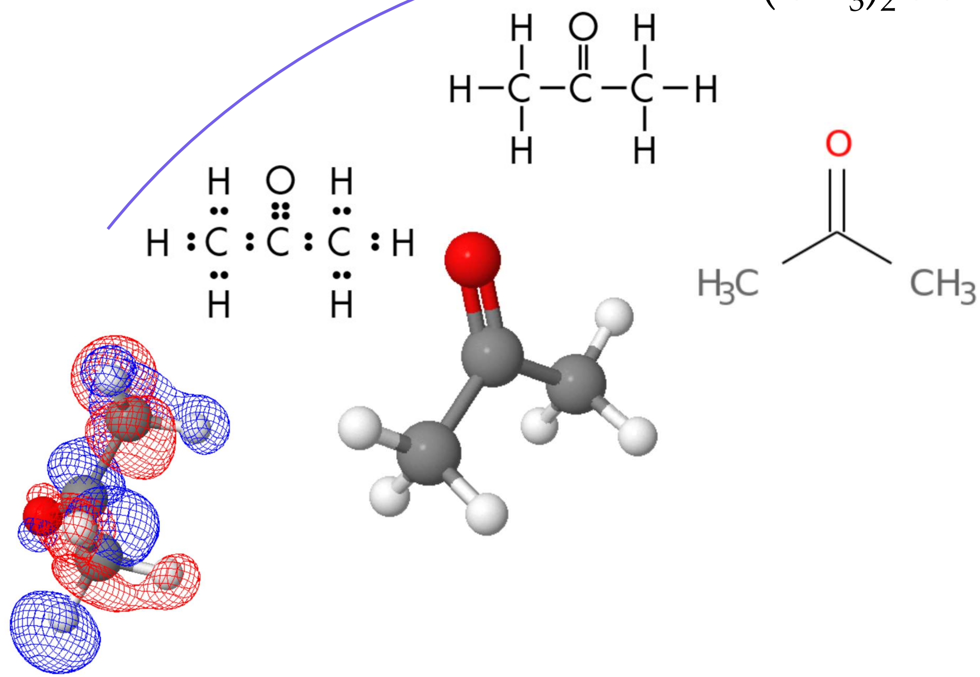
Acetone

Propan-2-one

Early days

Achieved **before** the direct observation of molecular structures!

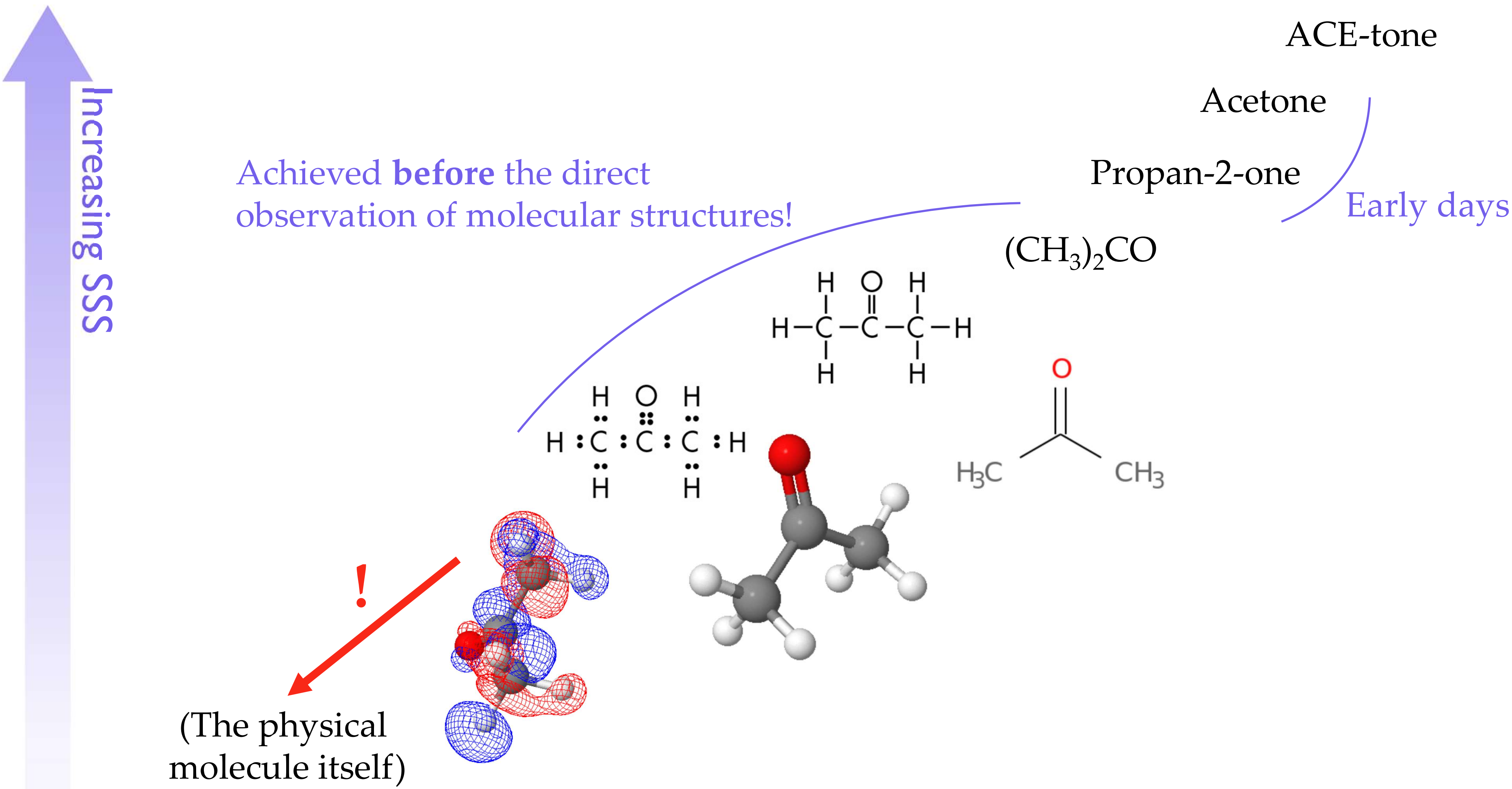
$(CH_3)_2CO$



(The physical molecule itself)

# Small-SSS Notations

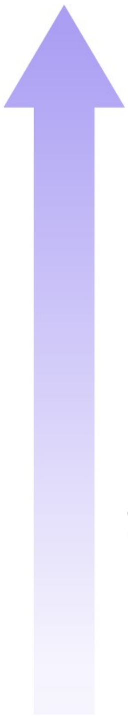
- ❖ The Working Principle Behind "Good Notations"
  - ✓ What is the most fundamental "rep" of molecules?



# Small-SSS Notations

❖ The Working Principle Behind "Good Notations"

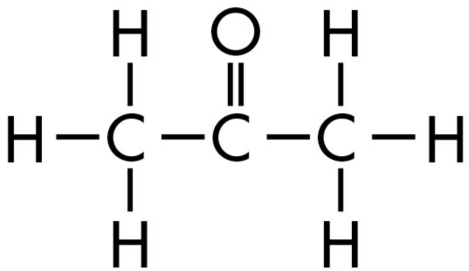
✓ What is the most fundamental "rep" of quantum circuits?



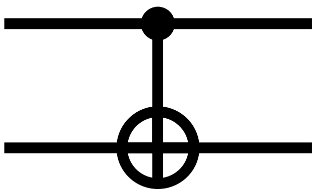
Large SSS

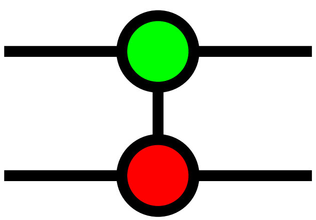
Small SSS

Acetone



CNOT



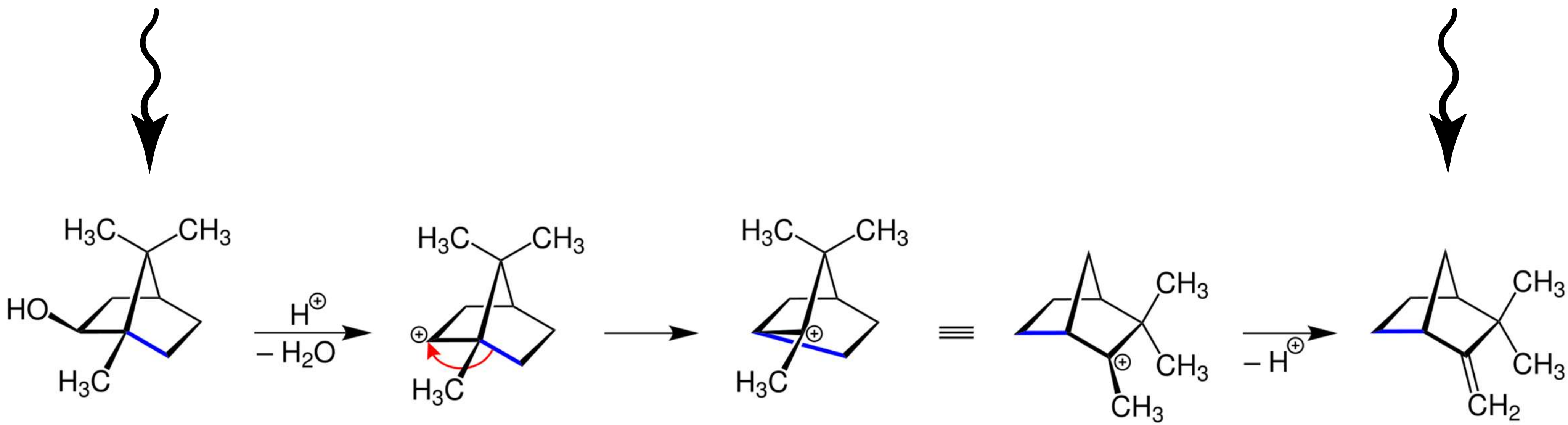


ZX calculus [0906.4725]

# Small-SSS Notations

- The Working Principle Behind "Good Notations"
  - ✓ Less arbitrary notations display "microscopic constituents"
  - ✓ Thus, they enable "derivations" from a smaller grammar (set of axioms)

Isoborneol – Water $\xrightarrow{?}$ Camphene



# Small-SSS Notations

❖ The Working Principle Behind "Good Notations"

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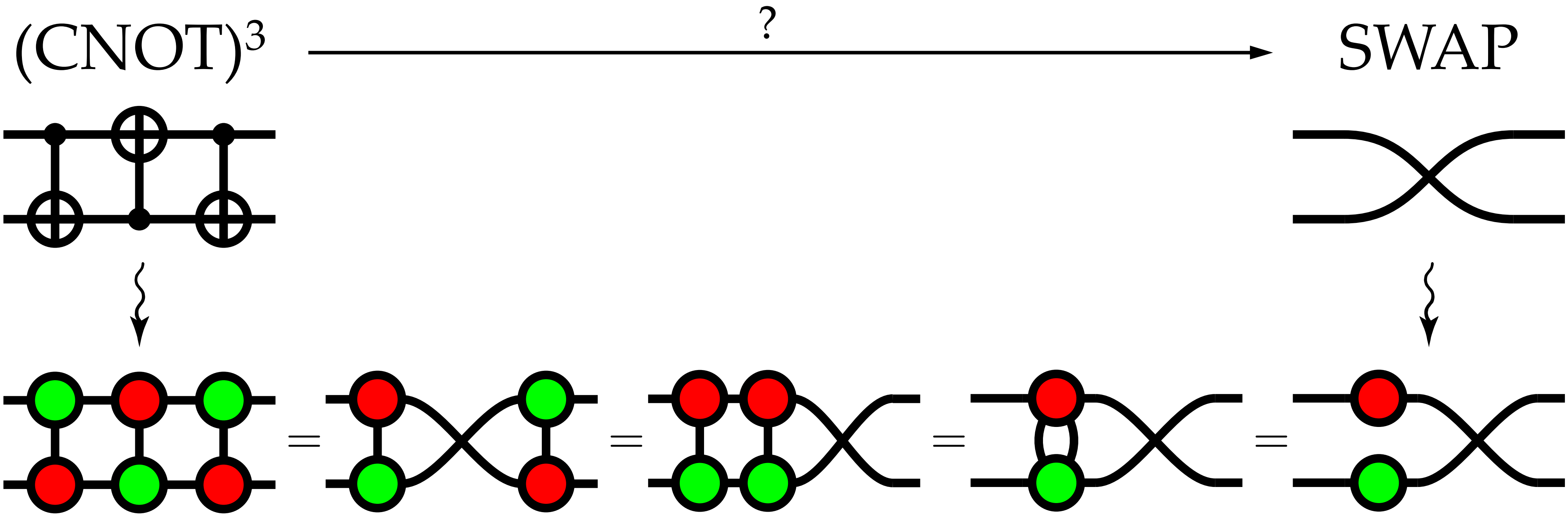
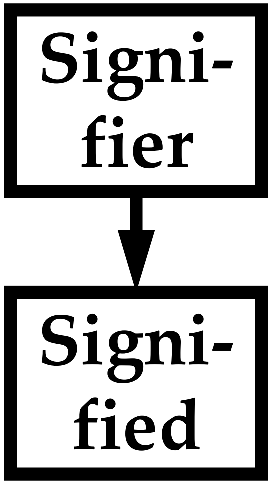


# Suggestion: "Notationology"

- Systematic Understanding of Notational Systems
  - ✓ We are left with several questions:
    - Why do particular notations (bra-ket, tensor graphs, …) are better than others? What makes them different? **What defines "good notations"?** **A) Small SSS**
    - How can a well-made notation tell us about the physical reality? Isn't a notation merely a mathematical tool?

**A) Small-SSS notations (Signifier = Signified) unveil the hidden signified (physical reality) by representing it in the form of signifier (grammar and symbols of the notational system)**

The signifier represents the signified "faithfully"



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The signifier represents the signified "faithfully"

# II. Re:presentation

# SSS Reduction as a Universal Pattern

- Human History of Notations (Math/Physics)

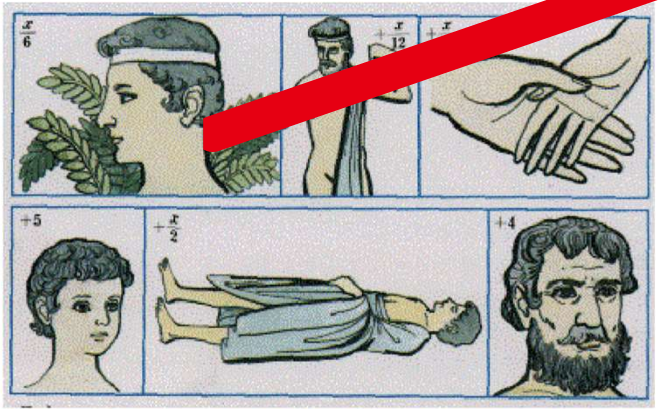
Increasing SSS

Symbolic algebra

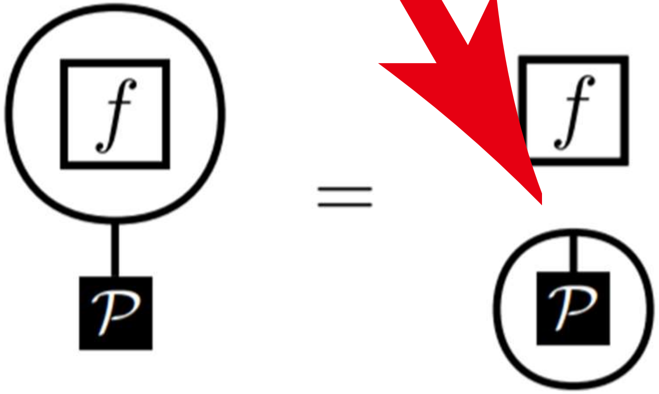
$2x - 2 = 6$

Syncopated algebra

□ is a number that becomes 6 if multiplied by 2 and ...




Rhetorical algebra




Graphical notation

**Ancient** **Medieval** **Early Modern** **Modern**

# SSS Reduction as a Universal Pattern

❖ Human History of Notations (Natural Language)

Increasing SSS

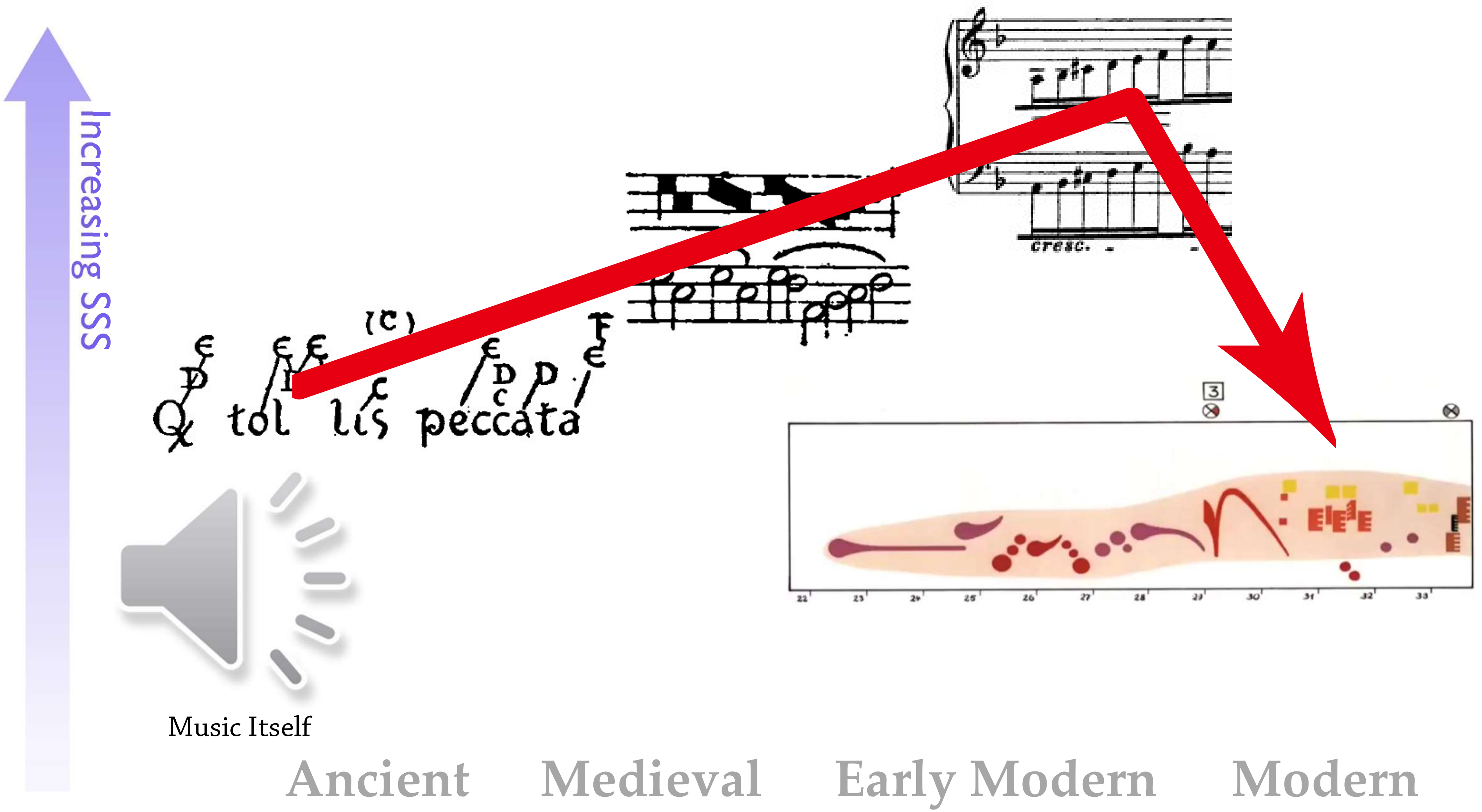

Cafeteria
Baggage
Restroom

Info Center
Parking Lot
Elevator


Ancient | Early Modern | Modern

# SSS Reduction as a Universal Pattern

❖ Human History of Notations (Music)



# SSS Reduction as a Universal Pattern

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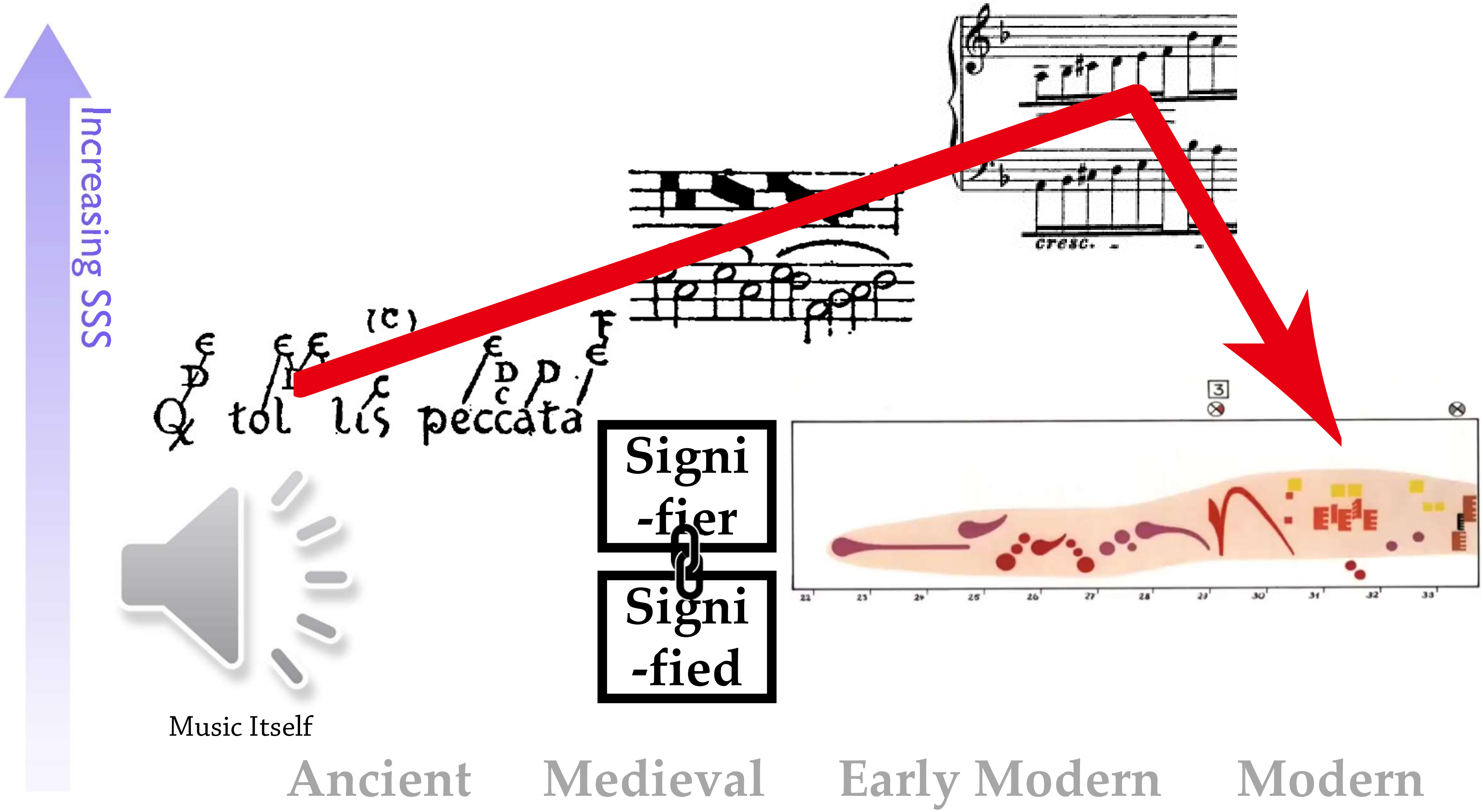
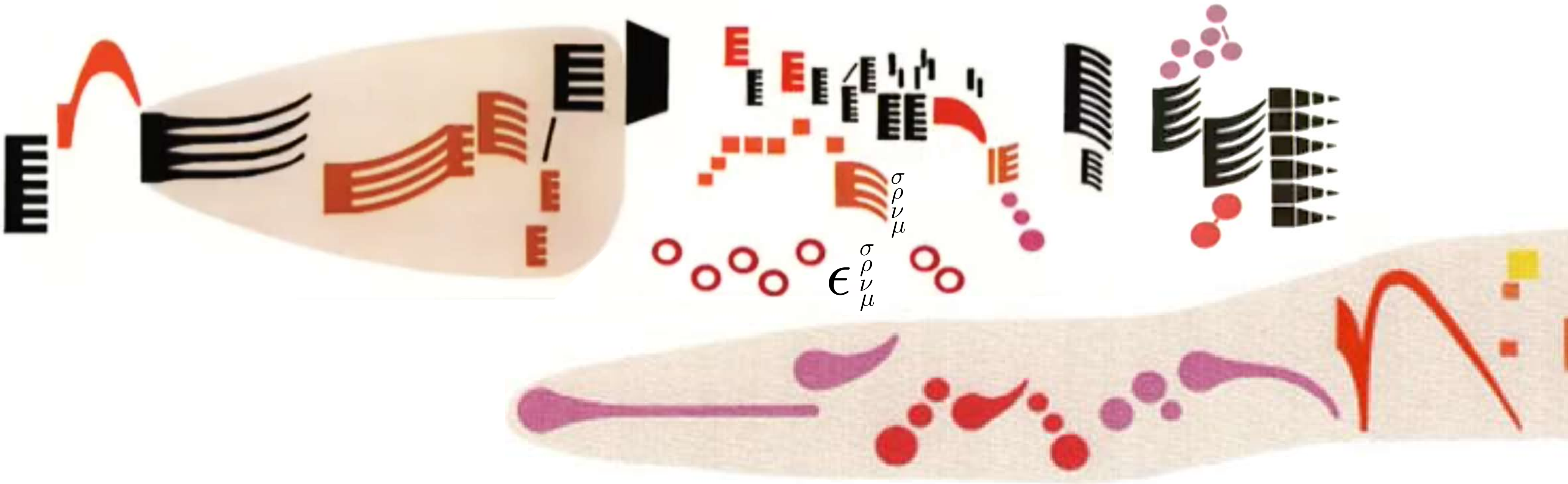


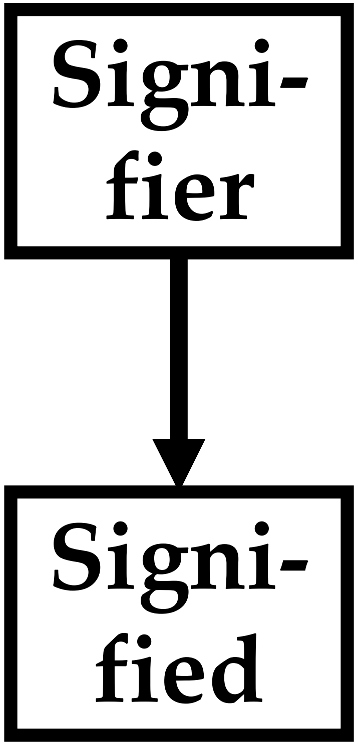
# <Artikulation>: What Do We Feel



- The visual score of Ligeti's <Artikulation> suggests more than just the music it represents: a visual scenery
- SSS reduction in modern art is subtly different from that in linguistics/math/phys
- SSS reduction in <Artikulation> is not to *faithfully* represent the signified (sound) but suggest an undefined semantics

# <Artikulation>: What Do We Feel

- Re-presentation
  - ✓ The visual score of Ligeti's <Artikulation> is not a mere representation of the music but a **re-presentation**
  - ✓ Set the signifier free!




**"Dangling pointer"**

Break the pre-defined signifier-to-signified assignment!

# <Artikulation>: What Do We Feel

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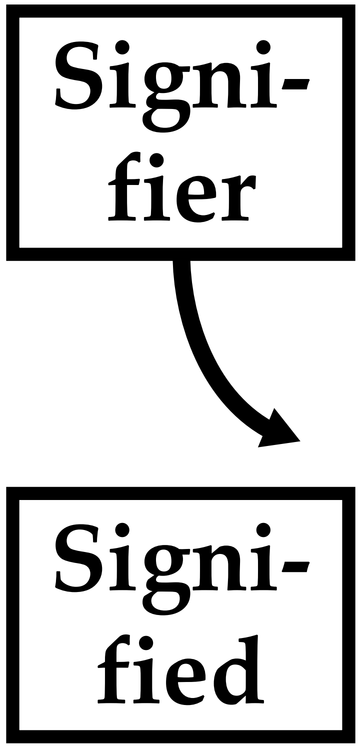
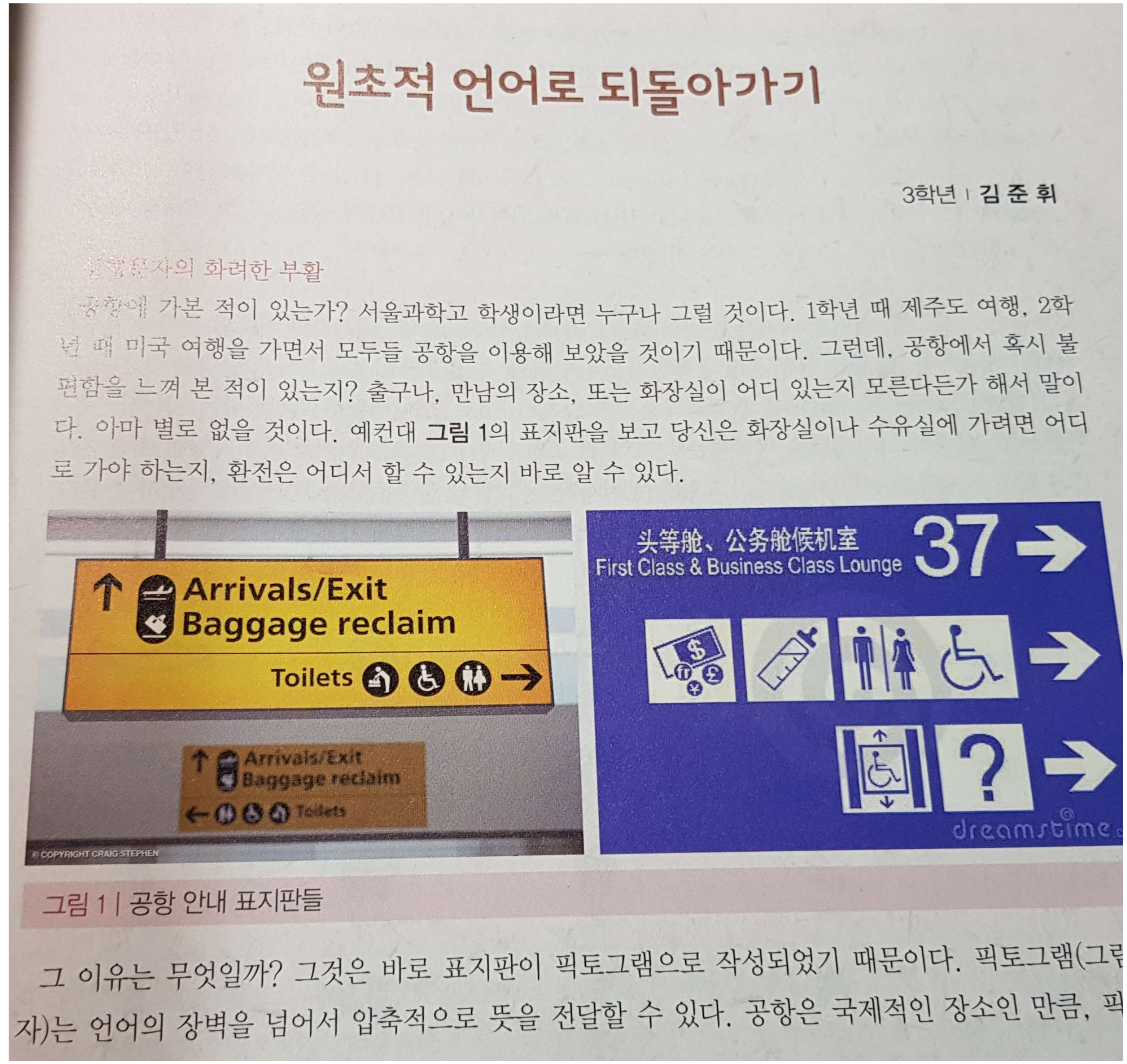



**"Dangling pointer"**

Break the pre-defined signifier-to-signified assignment!

# Experiments



원초적 언어로 되돌아가기

3학년 | 김 준 휘

상형문자의 화려한 부활

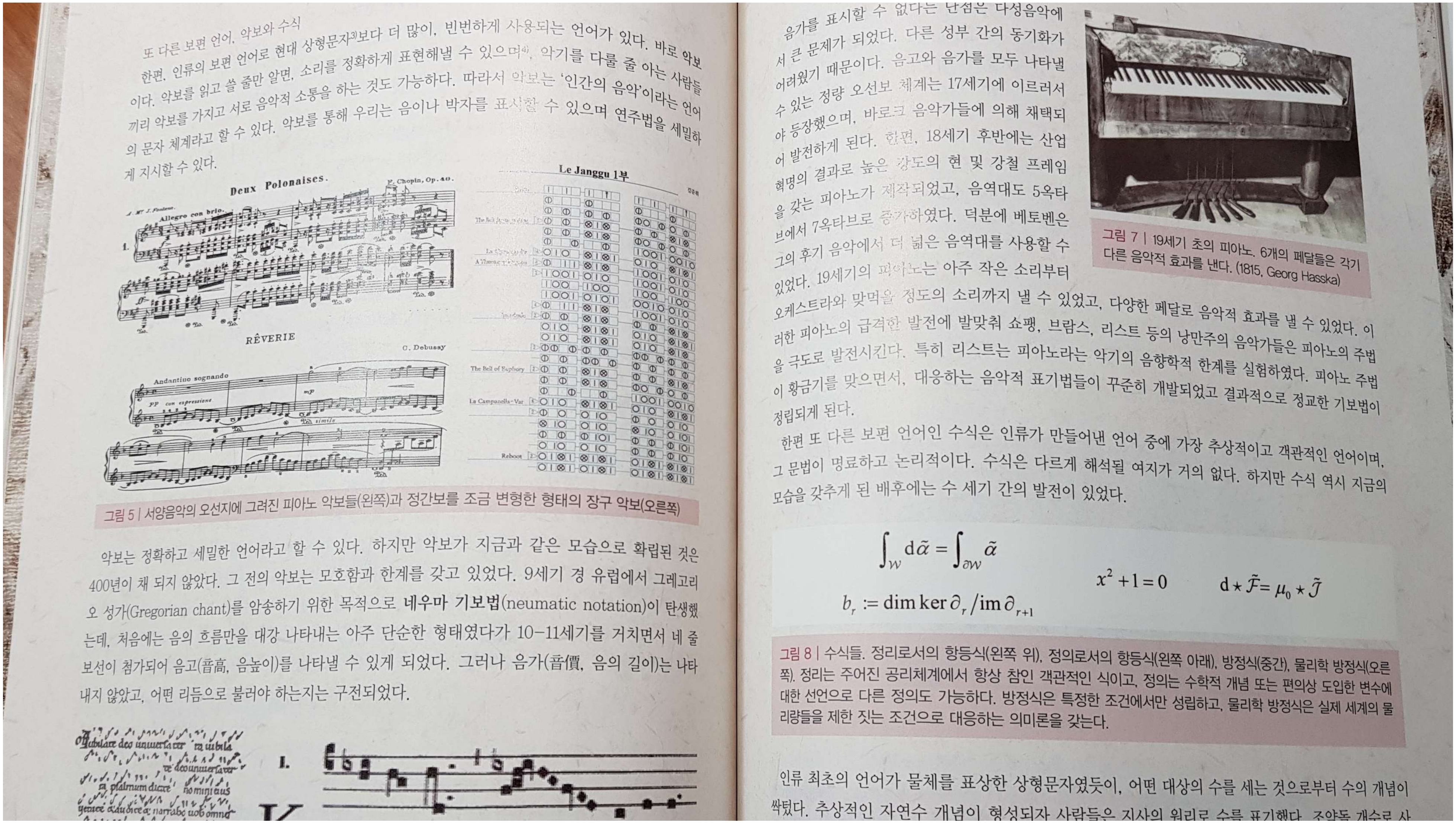
공항에 가본 적이 있는가? 서울과학고 학생이라면 누구나 그럴 것이다. 1학년 때 제주도 여행, 2학년 때 미국 여행을 가면서 모두들 공항을 이용해 보았을 것이기 때문이다. 그런데, 공항에서 혹시 불편함을 느껴 본 적이 있는지? 출구나, 만남의 장소, 또는 화장실이 어디 있는지 모른다든가 해서 말이다. 아마 별로 없을 것이다. 예컨대 **그림 1**의 표지판을 보고 당신은 화장실이나 수유실에 가려면 어디로 가야 하는지, 환전은 어디서 할 수 있는지 바로 알 수 있다.



그림 1 | 공항 안내 표지판들

그 이유는 무엇일까? 그것은 바로 표지판이 픽토그램으로 작성되었기 때문이다. 픽토그램(그림
자)는 언어의 장벽을 넘어서 압축적으로 뜻을 전달할 수 있다. 공항은 국제적인 장소인 만큼, 픽

# Experiments



### 또 다른 보편 언어, 악보와 수식

한편, 인류의 보편 언어로 현대 상형문자[3]보다 더 많이, 빈번하게 사용되는 언어가 있다. 바로 악보이다. 악보를 읽고 쓸 줄만 알면, 소리를 정확하게 표현해낼 수 있으며[4], 악기를 다룰 줄 아는 사람들끼리 악보를 가지고 서로 음악적 소통을 하는 것도 가능하다. 따라서 악보는 '인간의 음악'이라는 언어의 문자 체계라고 할 수 있다. 악보를 통해 우리는 음이나 박자를 표시할 수 있으며 연주법을 세밀하게 지시할 수 있다.

그림 5 | 서양음악의 오선지에 그려진 피아노 악보들(왼쪽)과 정간보를 조금 변형한 형태의 장구 악보(오른쪽)

악보는 정확하고 세밀한 언어라고 할 수 있다. 하지만 악보가 지금과 같은 모습으로 확립된 것은 400년이 채 되지 않았다. 그 전의 악보는 모호함과 한계를 갖고 있었다. 9세기 경 유럽에서 그레고리오 성가(Gregorian chant)를 암송하기 위한 목적으로 **네우마 기보법**(neumatic notation)이 탄생했는데, 처음에는 음의 흐름만을 대강 나타내는 아주 단순한 형태였다가 10–11세기를 거치면서 네 줄 보선이 첨가되어 음고(音高, 음높이)를 나타낼 수 있게 되었다. 그러나 음가(音價, 음의 길이)는 나타내지 않았고, 어떤 리듬으로 불러야 하는지는 구전되었다.

음가를 표시할 수 없다는 단점은 다성음악에서 큰 문제가 되었다. 다른 성부 간의 동기화가 어려웠기 때문이다. 음고와 음가를 모두 나타낼 수 있는 정량 오선보 체계는 17세기에 이르러서야 등장했으며, 바로크 음악가들에 의해 채택되어 발전하게 된다. 한편, 18세기 후반에는 산업혁명의 결과로 높은 강도의 현 및 강철 프레임을 갖는 피아노가 제작되었고, 음역대도 5옥타브에서 7옥타브로 증가하였다. 덕분에 베토벤은 그의 후기 음악에서 더 넓은 음역대를 사용할 수 있었다. 19세기의 피아노는 아주 작은 소리부터 오케스트라와 맞먹을 정도의 소리까지 낼 수 있었고, 다양한 페달로 음악적 효과를 낼 수 있었다. 이러한 피아노의 급격한 발전에 발맞춰 쇼팽, 브람스, 리스트 등의 낭만주의 음악가들은 피아노의 주법을 극도로 발전시킨다. 특히 리스트는 피아노라는 악기의 음향학적 한계를 실험하였다. 피아노 주법이 황금기를 맞으면서, 대응하는 음악적 표기법들이 꾸준히 개발되었고 결과적으로 정교한 기보법이 정립되게 된다.

그림 7 | 19세기 초의 피아노. 6개의 페달들은 각기 다른 음악적 효과를 낸다. (1815, Georg Hasska)

한편 또 다른 보편 언어인 수식은 인류가 만들어낸 언어 중에 가장 추상적이고 객관적인 언어이며, 그 문법이 명료하고 논리적이다. 수식은 다르게 해석될 여지가 거의 없다. 하지만 수식 역시 지금의 모습을 갖추게 된 배후에는 수 세기 간의 발전이 있었다.

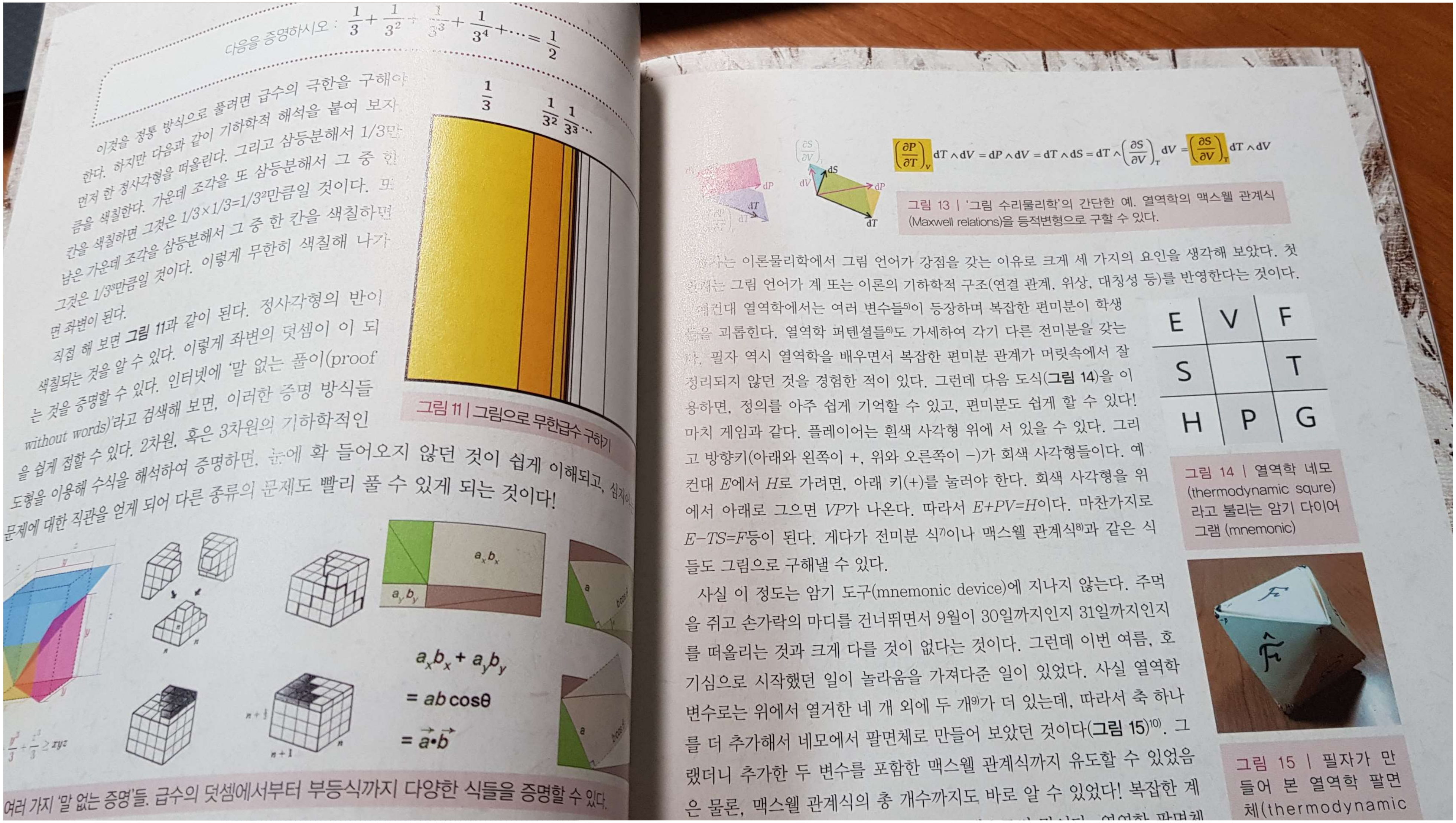
$$\int_W d\tilde{\alpha} = \int_{\partial W} \tilde{\alpha} \qquad x^2 + 1 = 0 \qquad d \star \tilde{\mathcal{F}} = \mu_0 \star \tilde{\mathcal{J}}$$

$$b_r := \dim \ker \partial_r / \operatorname{im} \partial_{r+1}$$

그림 8 | 수식들. 정리로서의 항등식(왼쪽 위), 정의로서의 항등식(왼쪽 아래), 방정식(중간), 물리학 방정식(오른쪽). 정리는 주어진 공리체계에서 항상 참인 객관적인 식이고, 정의는 수학적 개념 또는 편의상 도입한 변수에 대한 선언으로 다른 정의도 가능하다. 방정식은 특정한 조건에서만 성립하고, 물리학 방정식은 실제 세계의 물리량들을 제한 짓는 조건으로 대응하는 의미론을 갖는다.

인류 최초의 언어가 물체를 표상한 상형문자였듯이, 어떤 대상의 수를 세는 것으로부터 수의 개념이 싹텄다. 추상적인 자연수 개념이 형성되자 사람들은 지사의 원리로 수를 표기했다. 조약돌 개수로 사

# Experiments



다음을 증명하시오 : $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots=\frac{1}{2}$

이것을 정통 방식으로 풀려면 급수의 극한을 구해야 한다. 하지만 다음과 같이 기하학적 해석을 붙여 보자. 먼저 한 정사각형을 떠올린다. 그리고 삼등분해서 1/3만큼을 색칠한다. 가운데 조각을 또 삼등분해서 그 중 한 칸을 색칠하면 그것은 $1/3\times1/3=1/3^2$만큼일 것이다. 또 남은 가운데 조각을 삼등분해서 그 중 한 칸을 색칠하면 그것은 $1/3^3$만큼일 것이다. 이렇게 무한히 색칠해 나가면 좌변이 된다.

직접 해 보면 그림 11과 같이 된다. 정사각형의 반이 색칠되는 것을 알 수 있다. 이렇게 좌변의 덧셈이 이 되는 것을 증명할 수 있다. 인터넷에 '말 없는 풀이(proof without words)'라고 검색해 보면, 이러한 증명 방식들을 쉽게 접할 수 있다. 2차원, 혹은 3차원의 기하학적인 도형을 이용해 수식을 해석하여 증명하면, 눈에 확 들어오지 않던 것이 쉽게 이해되고, 심지어 문제에 대한 직관을 얻게 되어 다른 종류의 문제도 빨리 풀 수 있게 되는 것이다!

그림 11 | 그림으로 무한급수 구하기

$a_x b_x + a_y b_y = ab\cos\theta = \vec{a}\cdot\vec{b}$

여러 가지 '말 없는 증명'들. 급수의 덧셈에서부터 부등식까지 다양한 식들을 증명할 수 있다.

$\left(\frac{\partial P}{\partial T}\right)_V dT\wedge dV = dP\wedge dV = dT\wedge dS = dT\wedge\left(\frac{\partial S}{\partial V}\right)_T dV = \left(\frac{\partial S}{\partial V}\right)_T dT\wedge dV$

그림 13 | '그림 수리물리학'의 간단한 예. 열역학의 맥스웰 관계식(Maxwell relations)을 등적변형으로 구할 수 있다.

필자는 이론물리학에서 그림 언어가 강점을 갖는 이유로 크게 세 가지의 요인을 생각해 보았다. 첫 번째는 그림 언어가 계 또는 이론의 기하학적 구조(연결 관계, 위상, 대칭성 등)를 반영한다는 것이다.

예컨대 열역학에서는 여러 변수들[5]이 등장하며 복잡한 편미분이 학생들을 괴롭힌다. 열역학 퍼텐셜들[6]도 가세하여 각기 다른 전미분을 갖는다. 필자 역시 열역학을 배우면서 복잡한 편미분 관계가 머릿속에서 잘 정리되지 않던 것을 경험한 적이 있다. 그런데 다음 도식(**그림 14**)을 이용하면, 정의를 아주 쉽게 기억할 수 있고, 편미분도 쉽게 할 수 있다! 마치 게임과 같다. 플레이어는 흰색 사각형 위에 서 있을 수 있다. 그리고 방향키(아래와 왼쪽이 +, 위와 오른쪽이 −)가 회색 사각형들이다. 예컨대 $E$에서 $H$로 가려면, 아래 키(+)를 눌러야 한다. 회색 사각형을 위에서 아래로 그으면 $VP$가 나온다. 따라서 $E+PV=H$이다. 마찬가지로 $E-TS=F$등이 된다. 게다가 전미분 식[7]이나 맥스웰 관계식[8]과 같은 식들도 그림으로 구해낼 수 있다.

| | | |
|---|---|---|
| E | V | F |
| S | | T |
| H | P | G |

그림 14 | 열역학 네모(thermodynamic squre)라고 불리는 암기 다이어그램 (mnemonic)

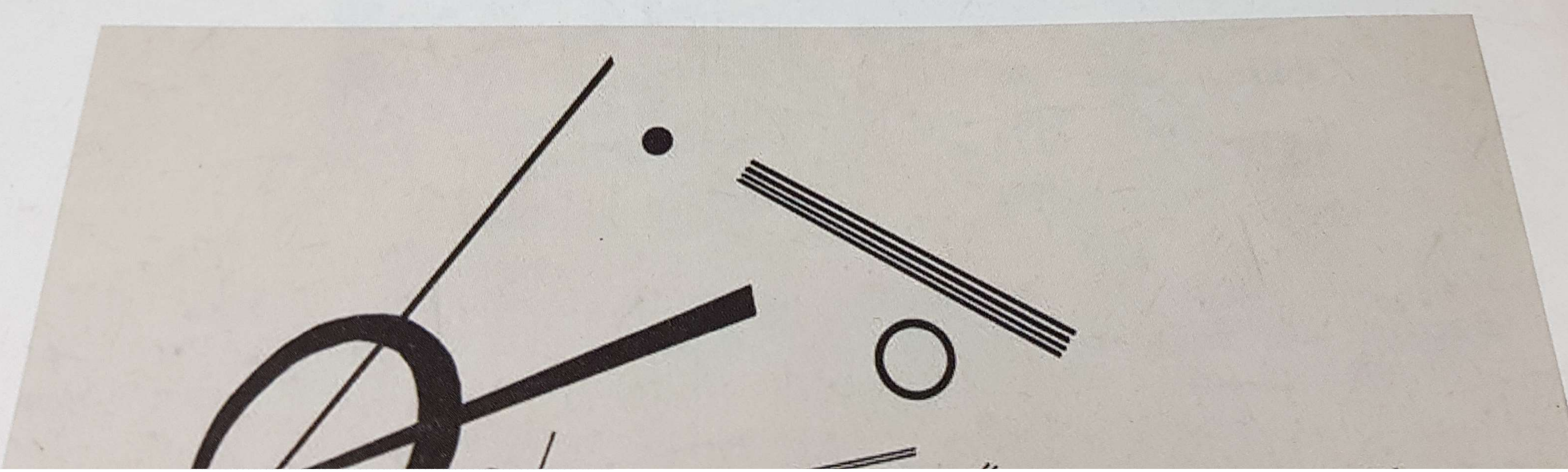
사실 이 정도는 암기 도구(mnemonic device)에 지나지 않는다. 주먹을 쥐고 손가락의 마디를 건너뛰면서 9월이 30일까지인지 31일까지인지를 떠올리는 것과 크게 다를 것이 없다는 것이다. 그런데 이번 여름, 호기심으로 시작했던 일이 놀라움을 가져다준 일이 있었다. 사실 열역학 변수로는 위에서 열거한 네 개 외에 두 개[9]가 더 있는데, 따라서 축 하나를 더 추가해서 네모에서 팔면체로 만들어 보았던 것이다(**그림 15**)[10]. 그랬더니 추가한 두 변수를 포함한 맥스웰 관계식까지 유도할 수 있었음은 물론, 맥스웰 관계식의 총 개수까지도 바로 알 수 있었다! 복잡한 계

그림 15 | 필자가 만들어 본 열역학 팔면체(thermodynamic

# Experiments

그림 언어의 본질은 추상이다. 추상이란 무엇인가? 상징물이자, 치환물이다. 따라서 그 대응의 규약은 자의적이다. 하지만 어떠한 자의적 대응 규칙도 연상의 범주를 벗어날 수 없다. 언어 사용자는 치환물의 조합으로부터 그 뜻을 떠올리거나 유추한다. 그렇지 않으면 사용자는 그 언어를 사용할 수 없기 때문이다. 표현 대상의 외관과 치환물의 모양이 닮았는지의 여부와는 별개로, 그림 문자의 모양은 그 뜻을 제안한다.

따라서 그림 문자의 본질은 결국 어떤 대상을 시각적 심상과 연관 짓는 행위이다. **그림 24**를 보고 혹자는 율동을 느끼겠지만 또 혹자는 음악을 들을 것이고, 혹자는 문장을 읽어 내고, 혹자는 맛을 음미할 것이며, 혹자는 수식을 읽어 내고 또 혹자는 진리를 발견할 것이다. 언젠가 우리는 코와 혀로 물리학을 하고 텐서에서 음악을 듣게 될지도 모르겠다!



# Experiments

❖ Read Tensors as Visual Art

✓ Mathematical physics as art of formal beauty

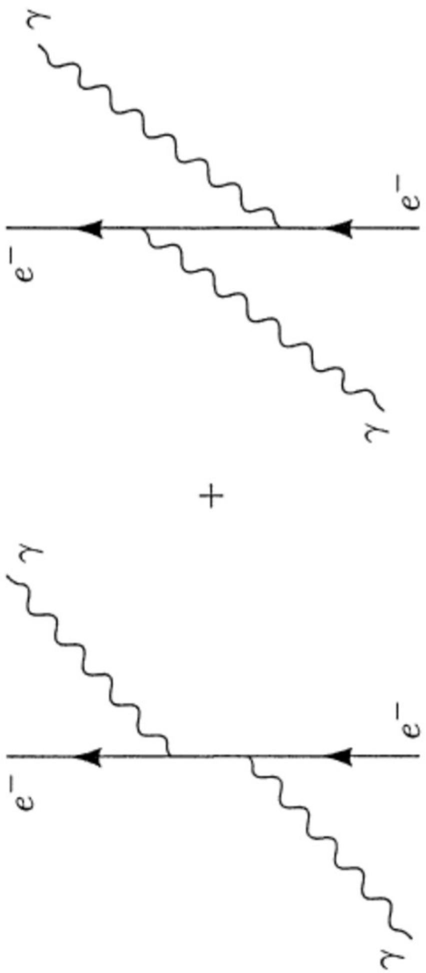



**Figure 1.7.** The two lowest-order diagrams for Compton scattering.

we can write the amplitude as a trace, leaving us with

$$\begin{aligned}
|\mathcal{M}_1|^2 &= \frac{e^4}{4(s-m_e^2)^2}\mathrm{Tr}\left[(\not{p}'+m_e)\gamma^\nu(\not{p}+\not{k}+m_e)\gamma^\mu(\not{p}+m_e)\gamma_\mu(\not{p}+\not{k}+m_e)\gamma_\nu\right] \\
&= \frac{e^4}{4(s-m_e^2)^2}\mathrm{Tr}\left[(-2\not{p}'+4m_e)(\not{p}+\not{k}+m_e)(-2\not{p}+4m_e)(\not{p}+\not{k}+m_e)\right] \\
&= \frac{e^4}{(s-m_e^2)^2}\ \mathrm{Tr}\left[\not{p}'(\not{p}+\not{k})\not{p}(\not{p}+\not{k})+m_e^2\not{p}'\not{p}+4m_e^2(\not{p}+\not{k})(\not{p}+\not{k})\right. \\
&\qquad \left.-4\not{p}'(\not{p}+\not{k})-4(\not{p}+\not{k})\not{p}+4m_e^4\right] \\
&= \frac{e^4}{(s-m_e^2)^2}\ \left[8p'^\mu(p_\mu+k_\mu)p^\nu(p_\nu+k_\nu)-4(p_\mu+k_\mu)^2p'^\nu p_\nu+4m_e^2p'^\nu p_\nu+16m_e^2(p_\mu+k_\mu)^2-\right. \\
&\qquad \left.16p'^\mu(p_\mu+k_\mu)-16p^\nu(p_\nu+k_\nu)+16m_e^4\right] \\
&= \frac{2e^4}{(s-m_e^2)^2}\left[-su+m_e^2(3s+u)+m_e^4\right]. \qquad (2.17)
\end{aligned}$$

In line 4 we used cyclicity of trace and $\gamma^\mu\not{p}\gamma_\mu=-2\not{p}$. As $\mathrm{Tr}\left[\gamma^{\mu_1}\ldots\gamma^{\mu_n}\right]=0$ for $n$ odd, we have only considered terms with even numbers of $\gamma^\mu$ matrices. Next we have

$$\begin{aligned}
|\mathcal{M}_2|^2 &= \frac{e^4}{4(u-m_e^2)^2}\mathrm{Tr}\left[(\not{p}'+m_e)\gamma^\nu\left(\not{p}-\not{k}'+m_e\right)\gamma^\mu(\not{p}+m_e)\gamma_\mu\left(\not{p}-\not{k}'+m_e\right)\gamma_\nu\right] \\
&= \ldots = \frac{2e^4}{(u-m_e^2)^2}\left[-su+(3u+s)m_e^2+m_e^4\right]. \qquad (2.18)
\end{aligned}$$

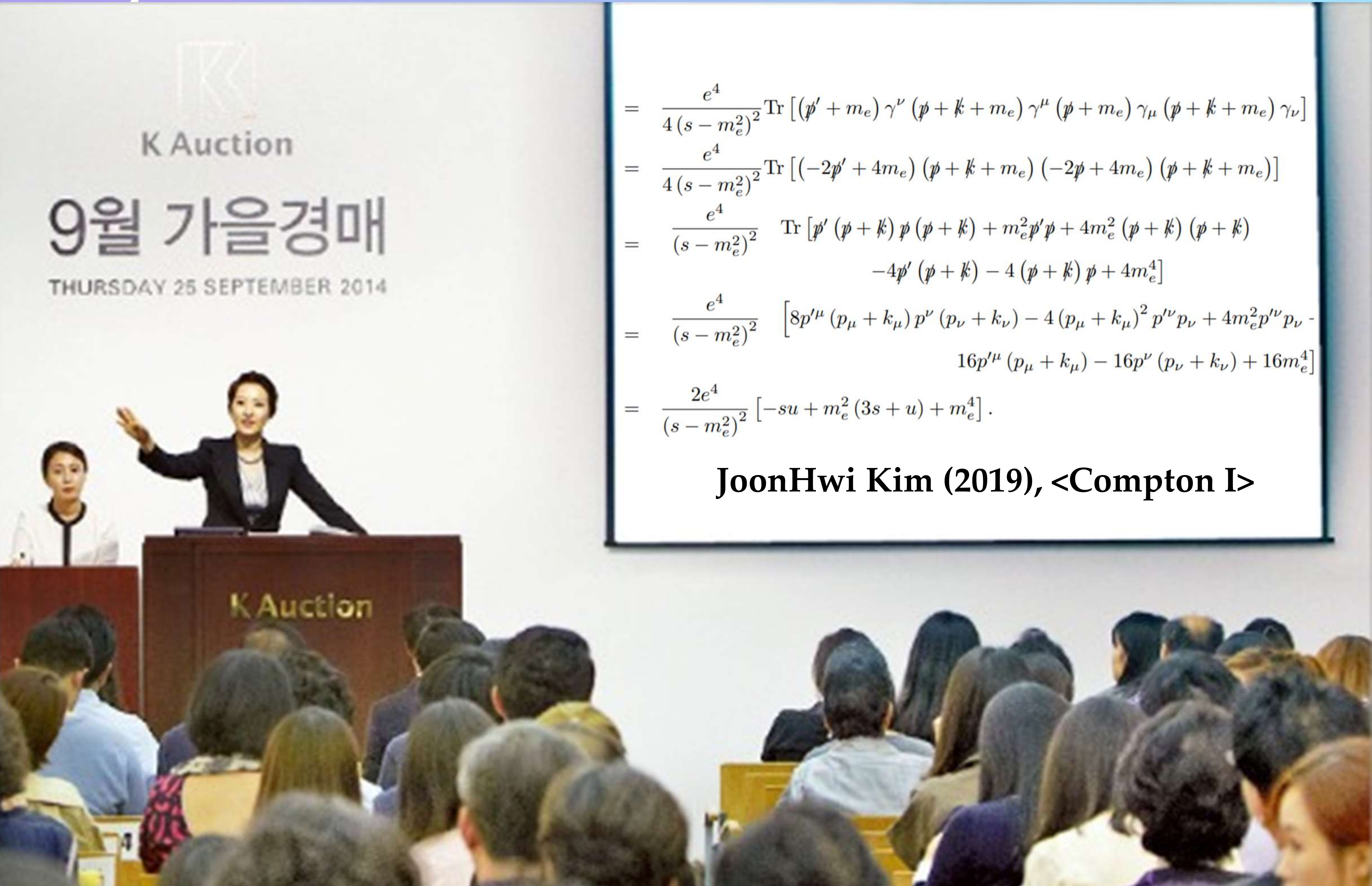
Finally we can calculate

$$\begin{aligned}
\mathcal{M}_1\mathcal{M}_2^*+\mathcal{M}_2\mathcal{M}_1^* = \frac{-e^4}{4(s-m_e^2)(u-m_e^2)}\ &\left\{\mathrm{Tr}\left[(\not{p}'+m_e)\gamma^\nu(\not{p}+\not{k}+m_e)\gamma^\mu(\not{p}+m_e)\gamma_\mu\left(\not{p}-\not{k}'+m_e\right)\gamma_\nu\right]+\right. \\
&\left.\mathrm{Tr}\left[(\not{p}'+m_e)\gamma^\nu\left(\not{p}-\not{k}'+m_e\right)\gamma^\mu(\not{p}+m_e)\gamma_\mu(\not{p}+\not{k}+m_e)\gamma_\nu\right]\right\} \\
= \ldots = \frac{4e^4}{(s-m_e^2)(u-m_e^2)}&\left[2m_e^4+m_e^2(s+u)\right]. \qquad (2.19)
\end{aligned}$$

The full squared matrix element for the Compton scattering process is therefore

$$|\mathcal{M}_1+\mathcal{M}_2|^2=2e^4\left[\frac{-su+m_e^2(3s+u)+m_e^4}{(s-m_e^2)^2}+\frac{-su+m_e^2(3u+s)+m_e^4}{(u-m_e^2)^2}+\frac{8m_e^4+4m_e^2(s+u)}{(s-m_e^2)(u-m_e^2)}\right]. \qquad (2.20)$$

# Experiments

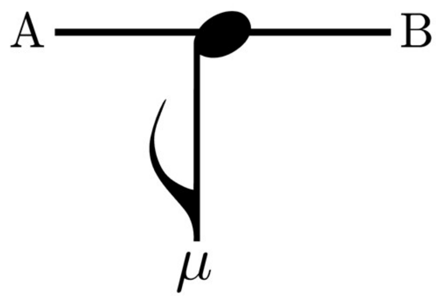


# Experiments

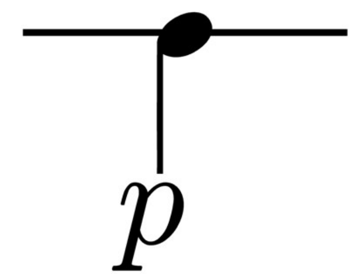
❖ Read Tensors as Visual Art

✓ Employ a graphical notation, to promote the re-reading!

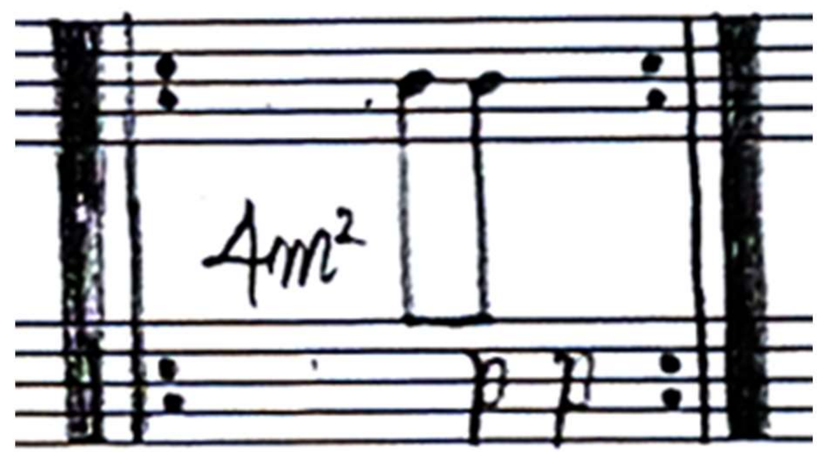
$$(\gamma_\mu)_{\mathrm{A}}{}^{\mathrm{B}} =$$




$$\not{p} = p^\mu \gamma_\mu =$$

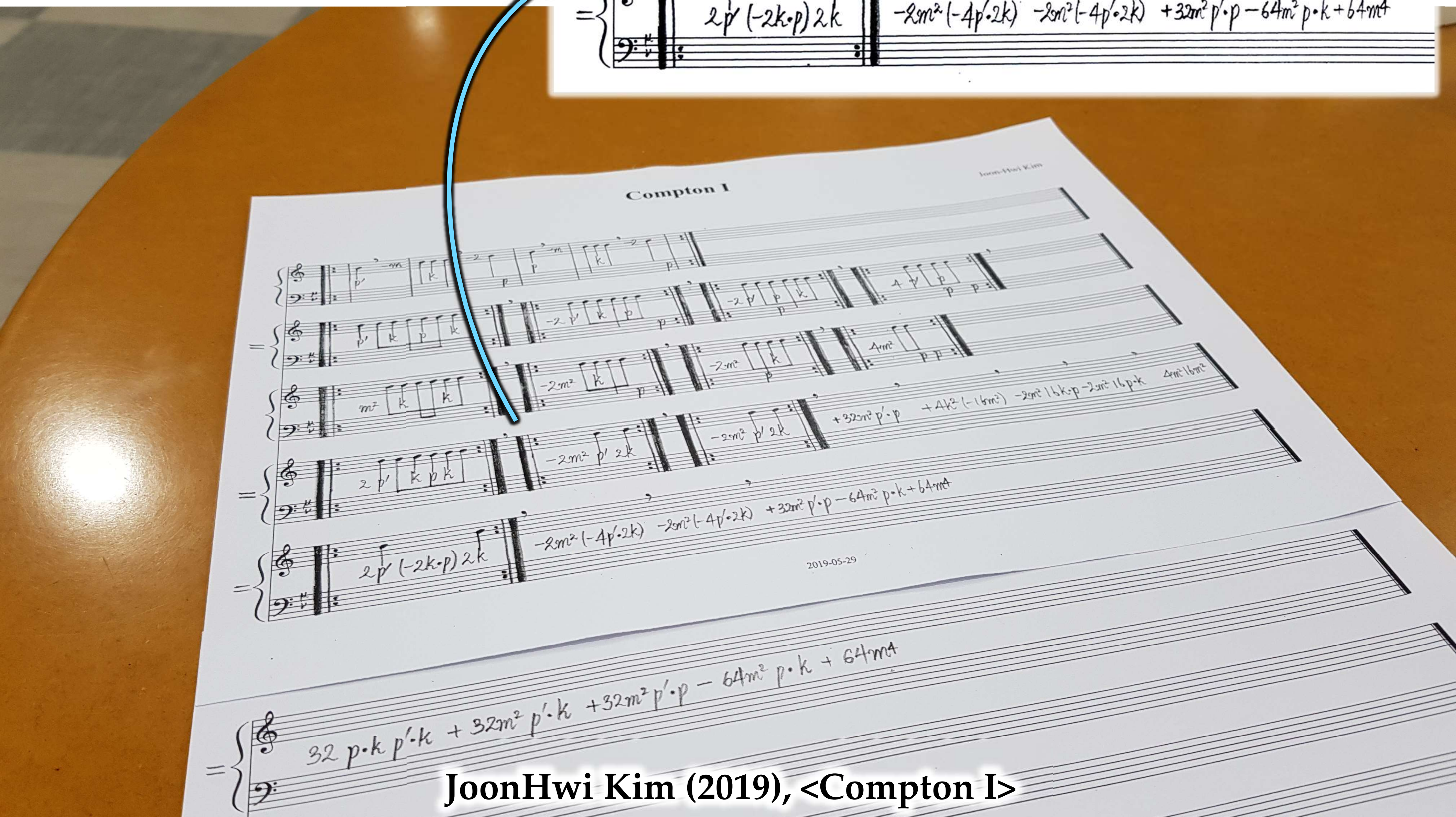


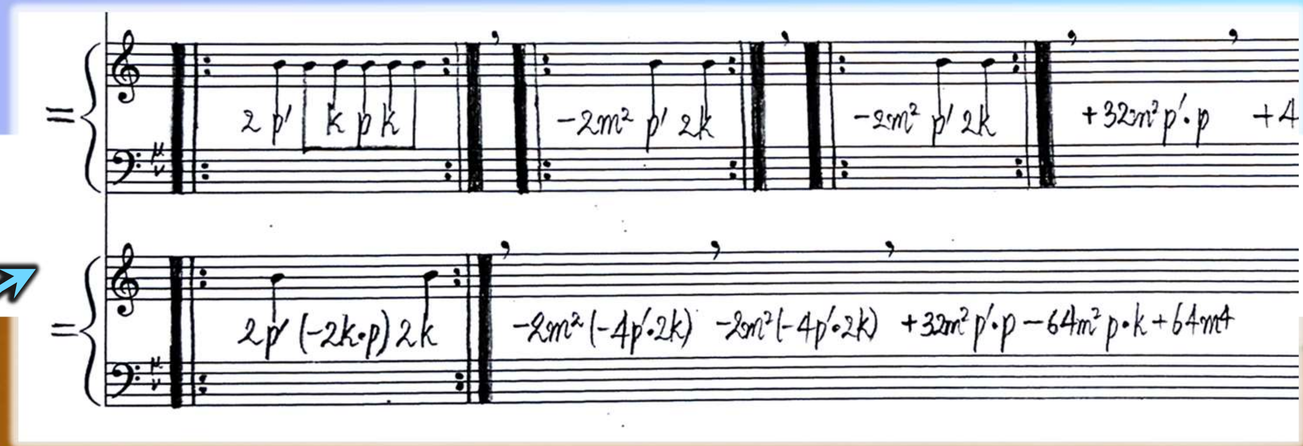

$$4m^2\,\mathrm{Tr}[\not{p}\not{p}] =$$




[JHK(2019), Compton I]

# Experiments



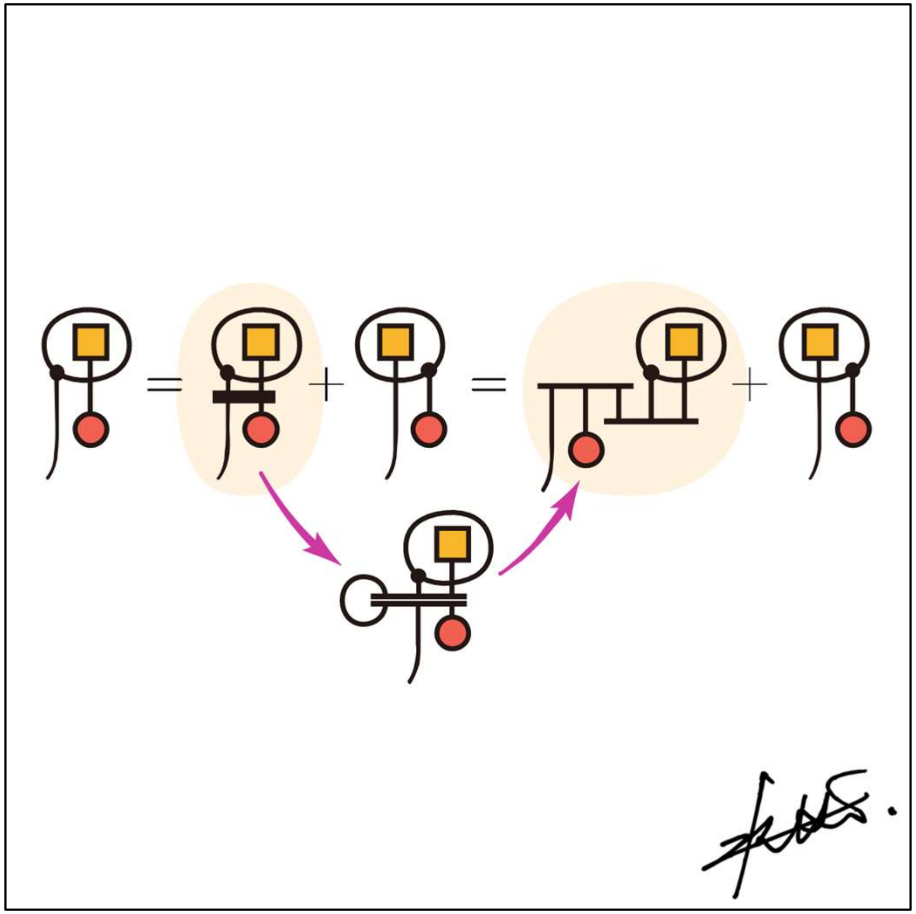



JoonHwi Kim (2019), <Compton I>

# Experiments

❖ Even More, Listen to Tensors as Music!

score (visual rep.)



music

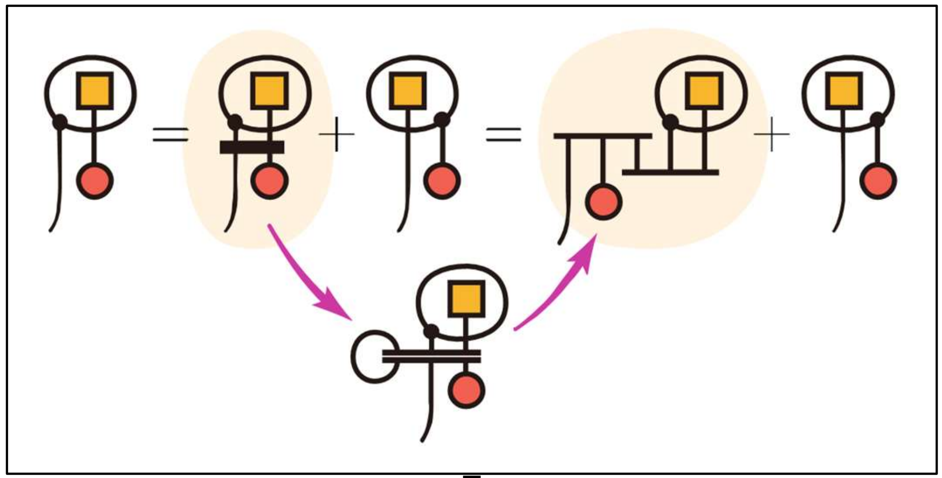
**[JHK(2018), Listen to Tensors]**

# Experiments

Cf. Black hole chirp sound

❖ Even More, Listen to Tensors as Music!

✓ From "auditory notation" to music



Artistic dimension emerges from putting tensor eqns in a different context of interpretation

$$G^j \partial_i F_j = G^j\, 2(\partial_{[i} F_{j]}) + G^j \partial_j F_i = G^j (\partial_l F_m) \varepsilon^{lmk} \varepsilon_{kij} + G^j \partial_j F_i$$
$$= \varepsilon_{ijk} G^j (\varepsilon^{klm} \partial_l F_m) + G^j \partial_j F_i$$

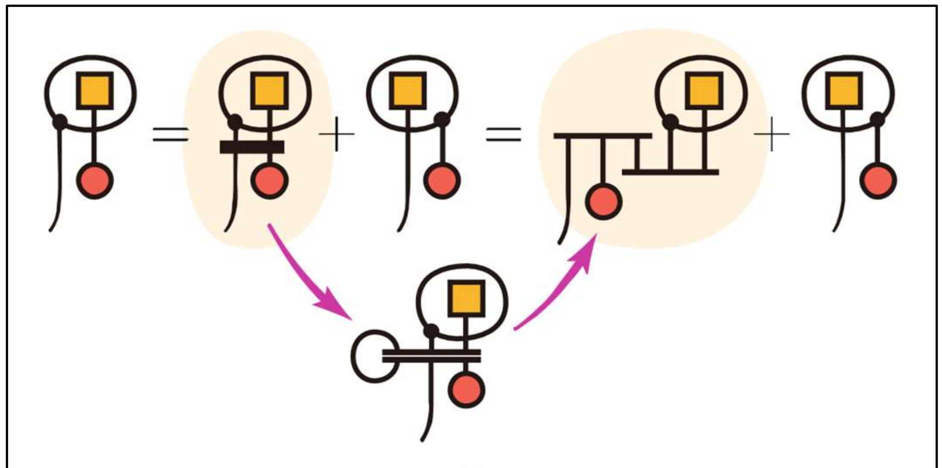
**[JHK(2018), Listen to Tensors]**

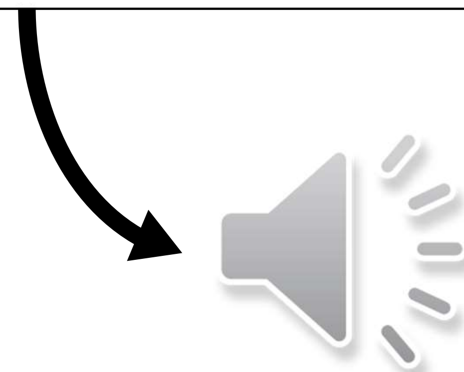
# Experiments

Cf. Black hole chirp sound

❖ Even More, Listen to Tensors as Music!

✓ From "auditory notation" to music





Artistic dimension emerges from putting tensor eqns in a different context of interpretation

$$
\begin{aligned}
G^j\partial_i F_j &= G^j\,2(\partial_{[i}F_{j]}) + G^j\partial_j F_i = G^j(\partial_l F_m)\varepsilon^{lmk}\varepsilon_{kij} + G^j\partial_j F_i \\
&= \varepsilon_{ijk}G^j(\varepsilon^{klm}\partial_l F_m) + G^j\partial_j F_i
\end{aligned}
$$

**[JHK(2018), Listen to Tensors]**

# Re-presentation Art

❖ Tensor-inspired Surrealism

**JoonHwi Kim (2016), <A Pleasant Dream (행복한 꿈) I~IV>, digital printing on canvas**

# Re-presentation Art

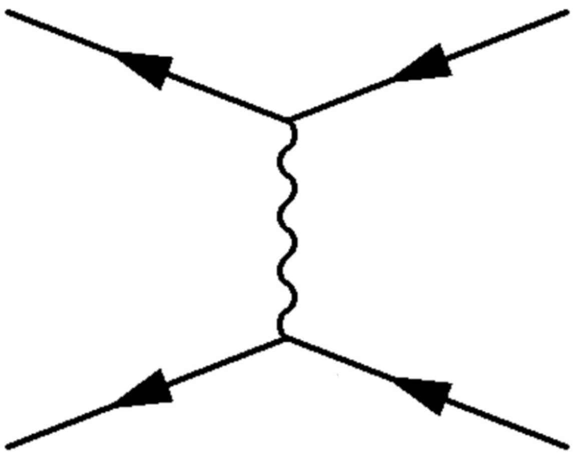
❖ Tensor-inspired Surrealism

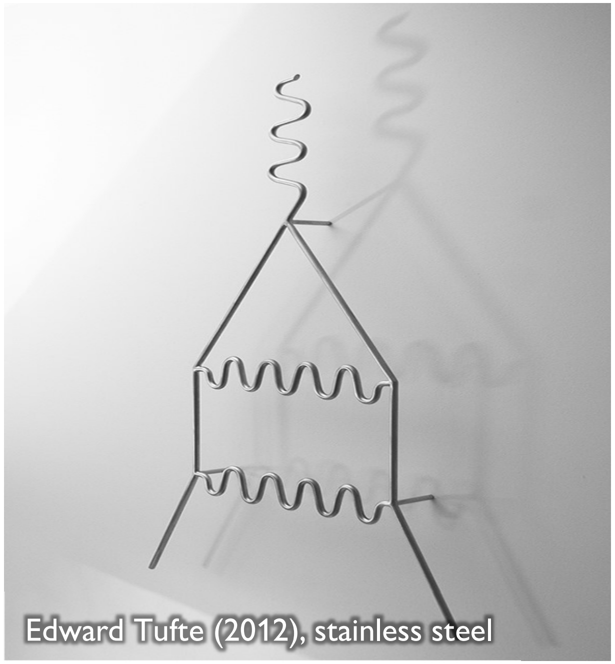
**JoonHwi Kim (2016), <A Pleasant Dream (행복한 꿈) I~IV>, digital printing on canvas**

# III. Closing Remarks

# Summary

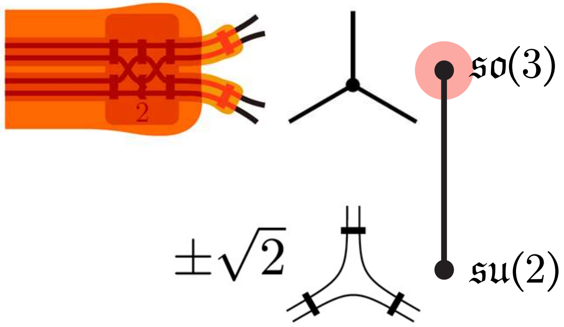
- ✓ **Graphical representation** of tensorial structures have become wide-spread in physics in the 20th C.
- ✓ By **"good notation"** it means that the notation has **small SSS (arbitrariness)**, i.e., the symbol directly represents its meaning.
- ✓ It is this feature that provides **practical benefits** as well as **hints to the physical reality**.
- ✓ By breaking the pre-defined symbol-to-meaning assignment, graphical notations also open up a way towards **"re-presentation art,"** intertwining mathematical physics, modern art, and the semiotic theory of signs.




Edward Tufte (2012), stainless steel

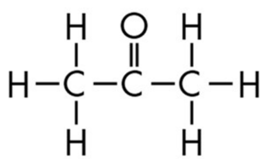
# Summary

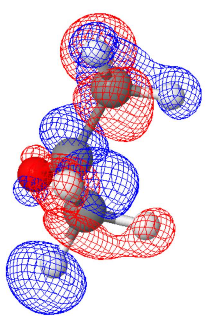
SSS as the RG scale (RG = reducing info!)



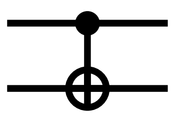

Acetone

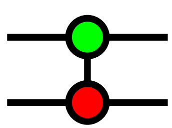
$(CH_3)_2CO$



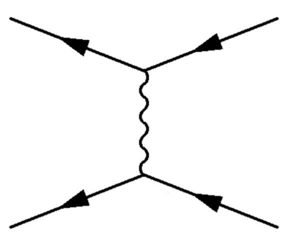
(molecule itself)



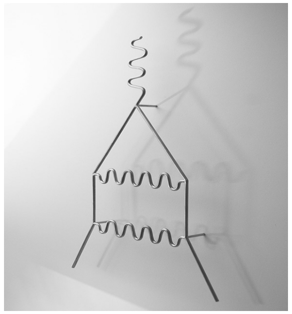


(qubits)

$$(ie)^2 \int d^4x\, d^4x'\, \psi(x)\, \gamma^\mu\, \psi(x)\, \psi(x')\, \gamma^\nu\, \psi(x')\, \boxed{A_\mu(x) A_\nu(x')}$$



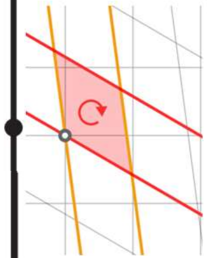
(quantum ptls)



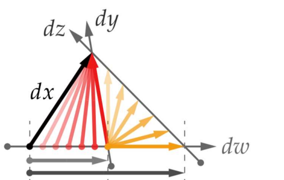
(art)

$$dx = \left.\frac{\partial x}{\partial y}\right|_z dy + \left.\frac{\partial x}{\partial z}\right|_y dz$$

$$dP \wedge dV = dT \wedge dS$$



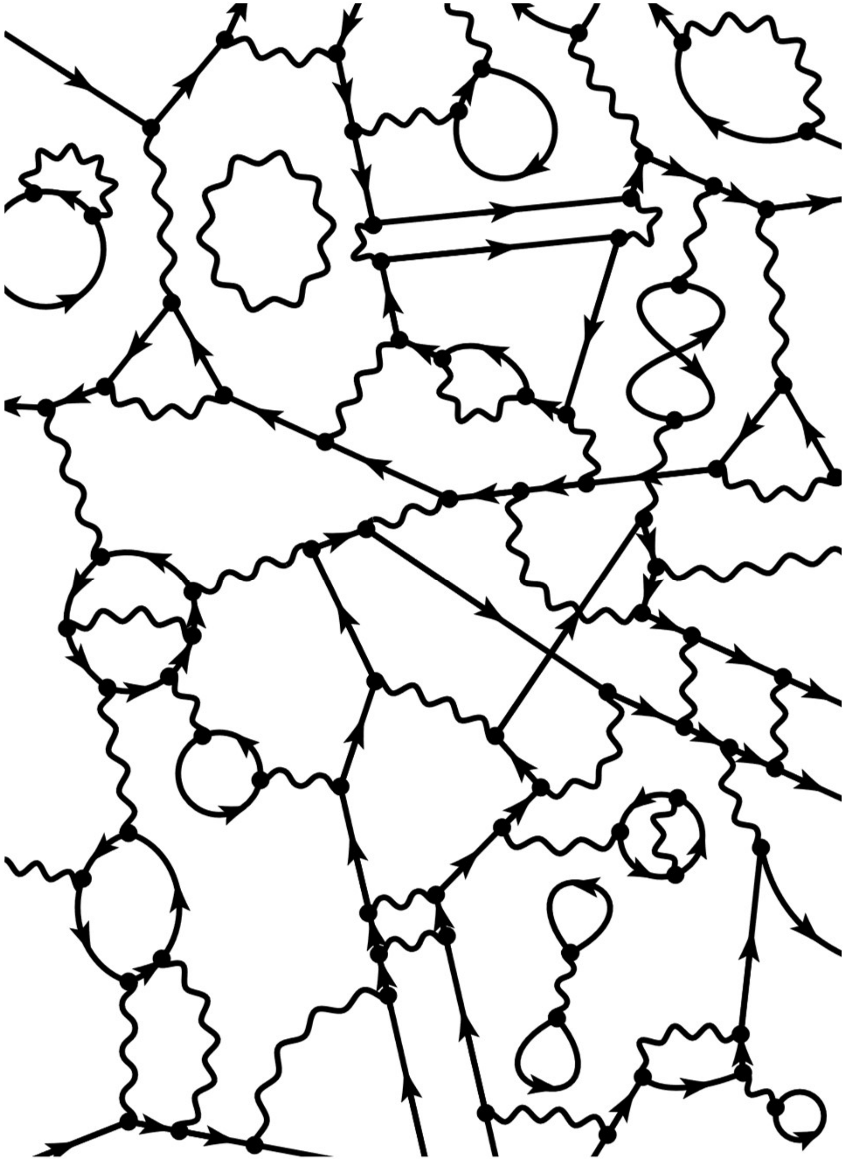
(difforms)

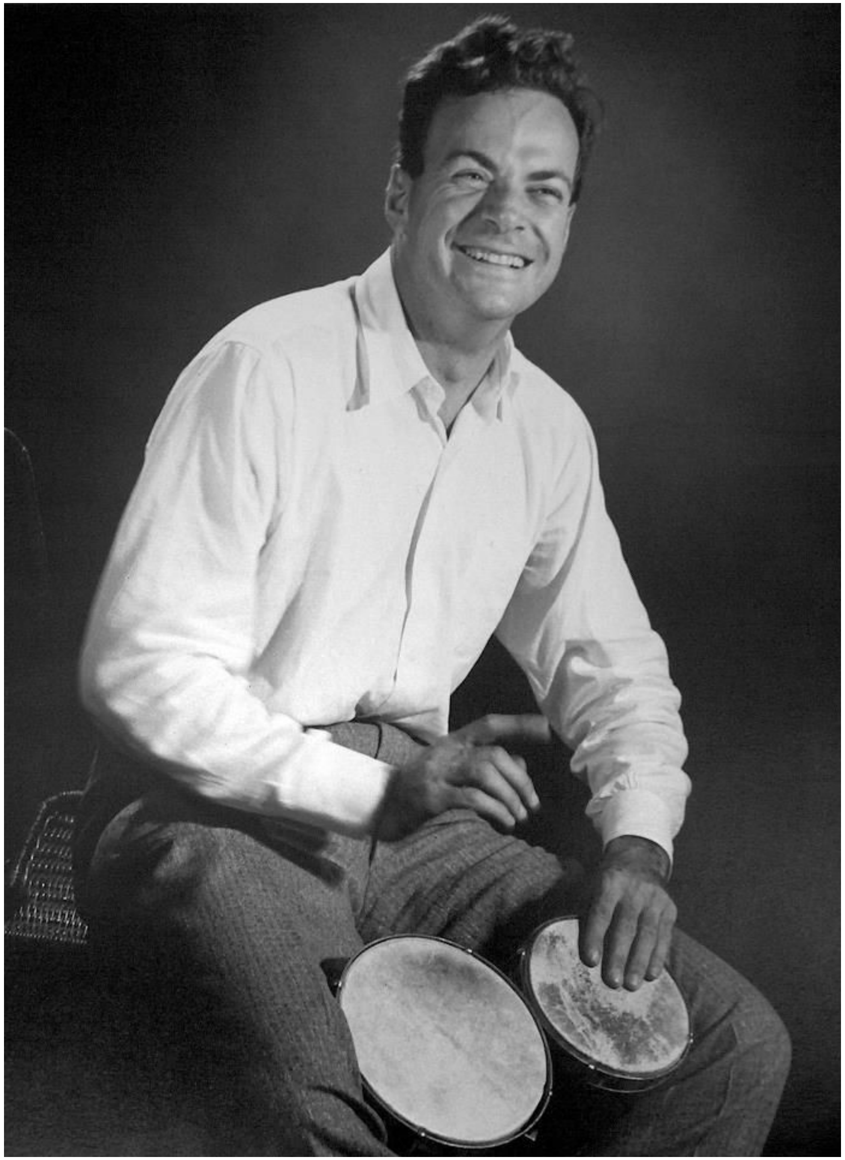



$$\left.\frac{\partial x}{\partial w}\right|_z - \left.\frac{\partial x}{\partial w}\right|_y = \left.\frac{\partial x}{\partial y}\right|_w \left.\frac{\partial y}{\partial w}\right|_z$$

(Ham. vec. fields)

# Closing Remarks





# References

## Scholastic Works


Coecke, B., & Duncan, R. (2011). Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics*, *13*(4), 043016. [arXiv:0906.4725]

Coecke, B. (2006). Kindergarten quantum mechanics: Lecture notes. In *AIP Conference Proceedings 810*(1) 81-98. American Institute of Physics.

Cvitanović, P. (2008). *Group theory: birdtracks, Lie's, and exceptional groups*. Princeton University Press.

Cvitanović, P., & Kennedy, A. D. (1982). Spinors in negative dimensions. *Physica Scripta*, *26*(1), 5.

Herrmann, E., & Trnka, J. (2016). Gravity on-shell diagrams. *Journal of High Energy Physics*, *2016*(11), 1-41.

Hodges, A. (1982). Twistor diagrams. *Physica A: Statistical Mechanics and its Applications*, *114*(1-3), 157-175.

Keppeler, S. (2017). Birdtracks for SU (N). *arXiv preprint arXiv:1707.07280*.

**Kim, J.-H.**, Oh, M. S. H., and Kim, K.-Y. (2021). Boosting Vector Calculus with the Graphical Notation. *American Journal of Physics 89*(2). [arXiv:1911.00892]

**Kim, J.-H.** and Nam, J. (2020). Thermodynamic Identities with Sunray Diagrams. *European Journal of Physics 43*(2). [arXiv:1912.11485]

Kilian, W., Ohl, T., Reuter, J., & Speckner, C. (2012). QCD in the color-flow representation. *Journal of High Energy Physics*, *2012*(10), 1-32. [arXiv:1206.3700]

Maxwell, J. C. (1871). Theory of Heat. London: Longmans, Green and Co.

Nash, L. K. (1964). The Carnot cycle and Maxwell's relations. *Journal of Chemical Education*, *41*(7), 368.

Penrose, R. (1971). Applications of negative dimensional tensors. *Combinatorial mathematics and its applications*, *1*, 221-244.

Penrose, R., & MacCallum, M. A. (1973). Twistor theory: an approach to the quantisation of fields and space-time. *Physics Reports*, *6*(4), 241-315.

Root-Bernstein, R., & Root-Bernstein, M. (2013). Sparks of genius: The 13 thinking tools of the world's most creative people. HMH.

김준휘 (2016). 원초적 언어로 되돌아가기. 叡智 제 27호, 서울과학고등학교. 105-118.


## Artworks


Applebaum, M. (2008). The Metaphysics of Notation. Improvisation with a hand-written graphic score.

Edward Tufte (2012). Space-time Feynman diagrams. Stainless steel.

**Kim, J.-H.** (2019). Compton I. Digital printing and graphite on paper.

**Kim, J.-H.** (2018). Listen to Tensors. Digital composition and illustration.

**Kim, J.-H.** (2016). A Pleasant Dream I~IV. Digital printing on canvas.

Ligeti, G. (1958). Artikulation. Digital composition.

Oh, M. (2016). ABA Video. Single channel video.

# Thank You!

# Back-up Slide

(as coined by Juno Nam)

- ❖ Cadabra-Hwi:

  Calculating Tensors by "Physically Implementing" the Syntax

  - ✓ Tensor calculating software for HEP (better than Mathematica/Cadabra!)
  - ✓ Utilize the "topological computation" property
  - ✓ Graph isomorphism problem? Simulated annealing?
    How to implement it?