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Diffractive Gravitational Lensing of Gravitational Wave

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Based on

- "Small-scale shear: peeling off diffuse subhalos with gravitational waves"
  Han Gil Choi, Chanung Park and Sunghoon Jung, arXiv : 2103.08618[astro-ph.CO]

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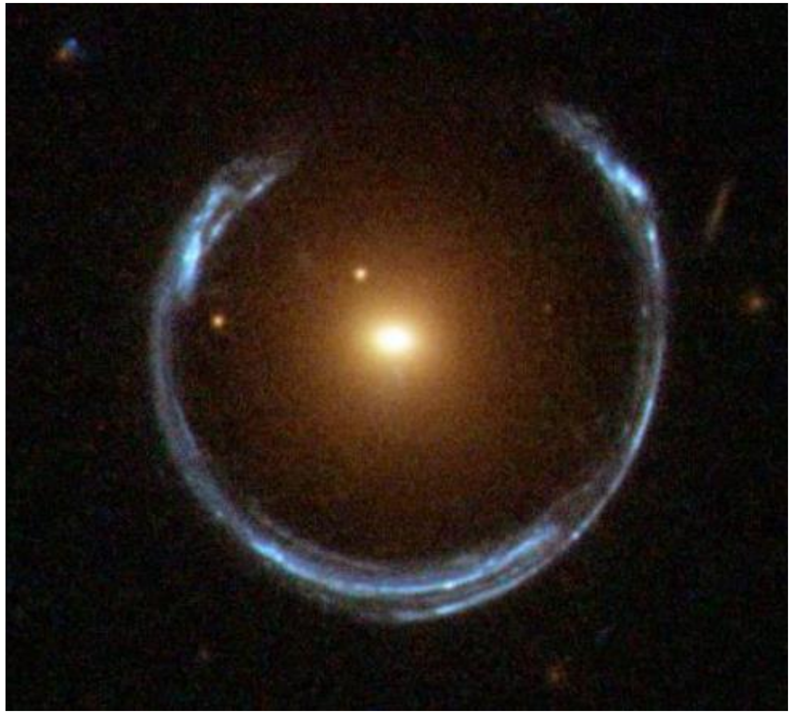
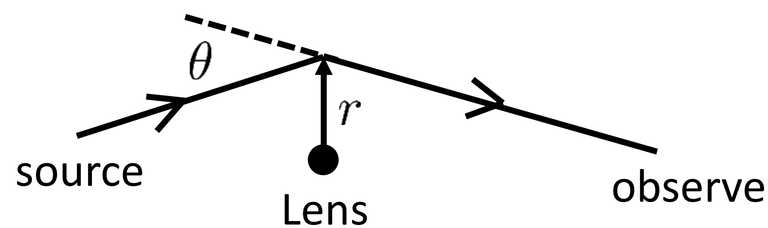




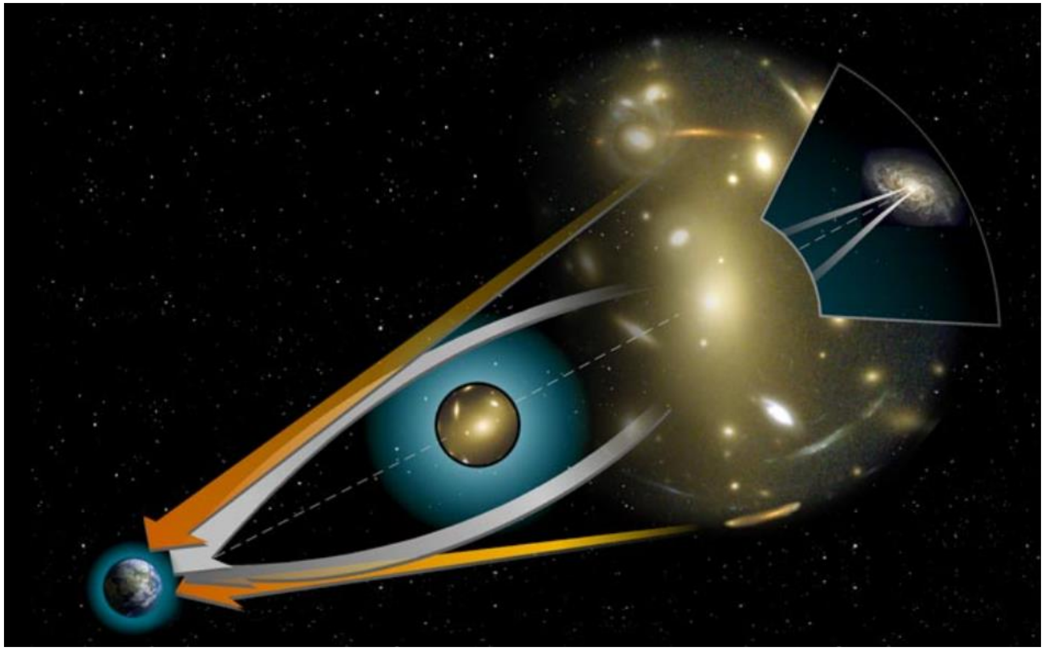
$$\text{Deflection angle} : \theta = \frac{GM}{c^2 r}$$

GR : Mass can deflect light propagation!

I. Basics of Gravitational Lensing

Two types of gravitational lensing(GL)

- Strong GL $\Sigma > \Sigma_{\text{critical}} = c^2/(4\pi G d_{\text{eff}})$
  - Occur when lens surface density **exceeds** critical value
  - Multiple lensed Images (cf. Einstein ring)
  - Arrival time difference between lensed images

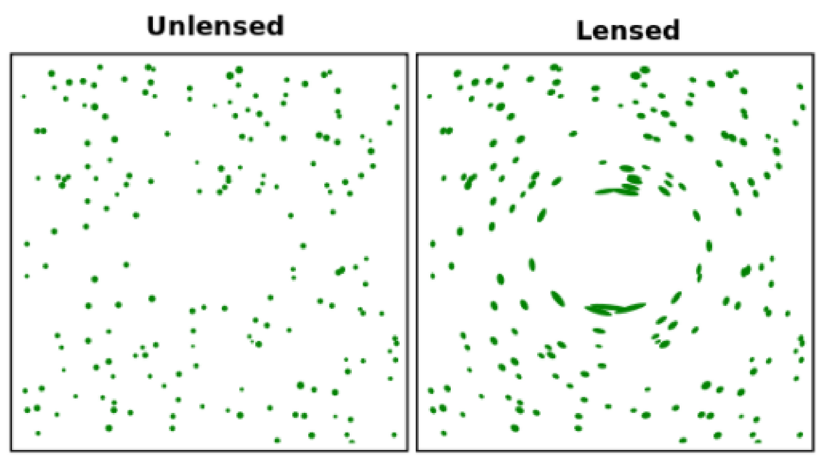


I. Basics of Gravitational Lensing

Two types of gravitational lensing(GL)

- Weak GL $\Sigma < \Sigma_{\rm critical} = c^2/(4\pi G d_{\rm eff})$
  - Occur when lens surface density is **below** critical value
  - No multiple image
  - Image distortion : Convergence(κ), Shear(γ)

$$\kappa = \Sigma/\Sigma_{\rm critical} : \text{Isotropic distortion}$$
$$\gamma \propto \text{tidal field} : \text{Directional distortion}$$



II. Gravitational wave

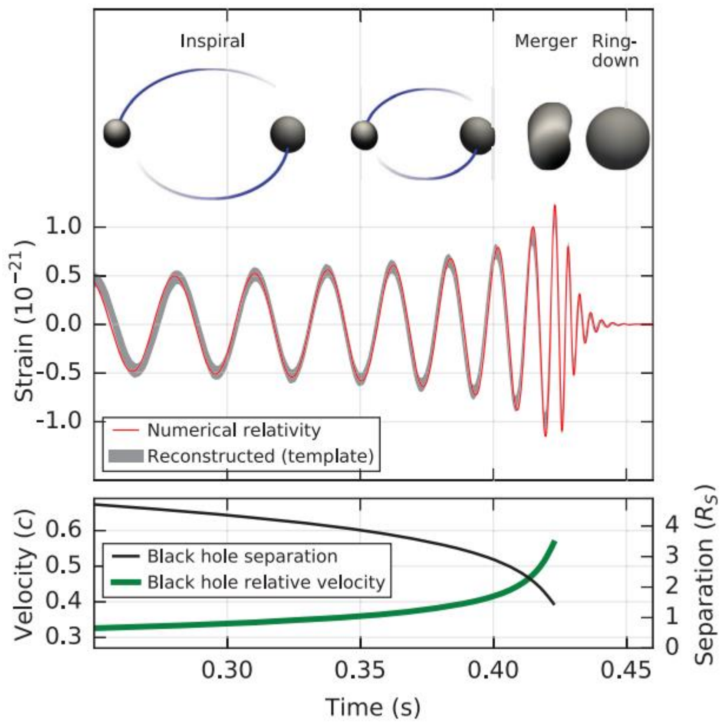
- Gravitational Wave(GW) chirp : GW signal emitted by compact binary inspiral.
- Characteristic frequency & amplitude evolution → great potential to astrophysics

$$h(t) \propto \frac{(G\mathcal{M})^{5/3}}{c^4 d_L}(\pi f(t))^{2/3}\cos(\Phi(t))$$

$$\frac{d}{dt}f(t) = \frac{1}{2\pi}\frac{d^2}{dt^2}\Phi(t) = \frac{96\pi^{\frac{8}{3}}}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f^{11/3}$$

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

Chirp mass



GW150914

II. Gravitational wave

Good/Bad properties of GW chirp as lensing source

Good

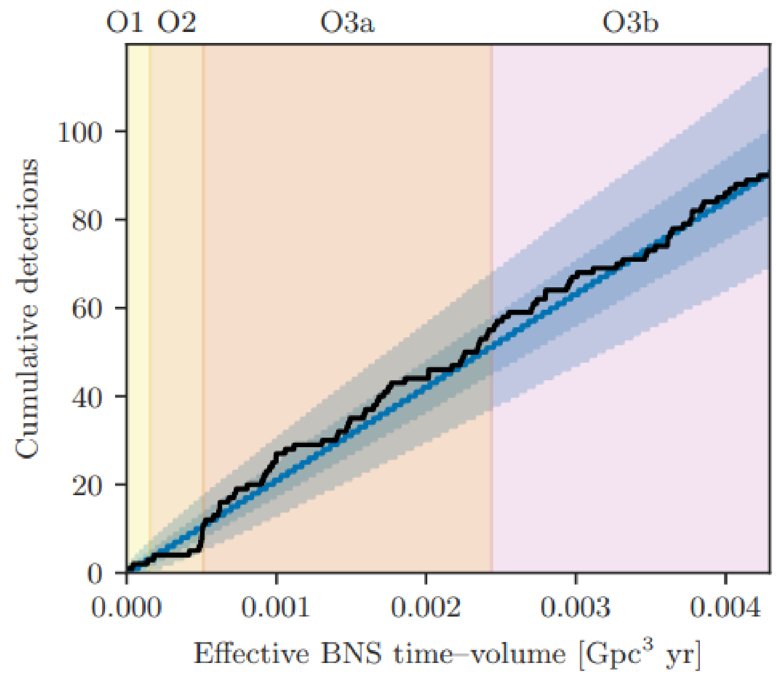
- High accuracy of GW source(binary black hole) modeling
- Small source size $\sim$ O(10~100) Schwarzschild radius = Ideal point source
- GW Interact only with mass $\rightarrow$ small confusion error
- Broad frequency spectrum

Bad

- Relatively small number of sources (compared to galaxy and stars)
- Bad sky localization (> few degrees)
- Large statistical error due to detector noise

II. Gravitational wave

GWTC-3, LSC (2021)



- Total number of GW chirp events is now 90
- Strong lensing probability of GW chirp by galaxy is $\sim 10^{-4}$, no observation until now (LSC, 2021)
- Lensed GW chirp will be detected in near future.
  - GW event rate will be at most $\sim 10^3$/yr at LIGO (LSC, 2010)

III. Wave optics of Gravitational lensing

Long wavelength of GW → Wave optics requirement

Typical observation frequency(wavelength)

GW	Light
LIGO - 100 Hz (3×10^3 km) DECIGO – 0.1 Hz (3×10^6 km) LISA – 0.01 Hz (3×10^7 km)	Visible - 10^6 GHz (3 μm) Radio - 1 GHz (0.3 m)

Propagation of Gravitational wave under gravitational potential (Takahashi, 2003)

$$\nabla^\alpha \nabla_\alpha h_{\mu\nu} = 0$$

Introducing GW polarization $e_{\mu\nu}$ and scalar function h ,

$$h_{\mu\nu} = h e_{\mu\nu}$$

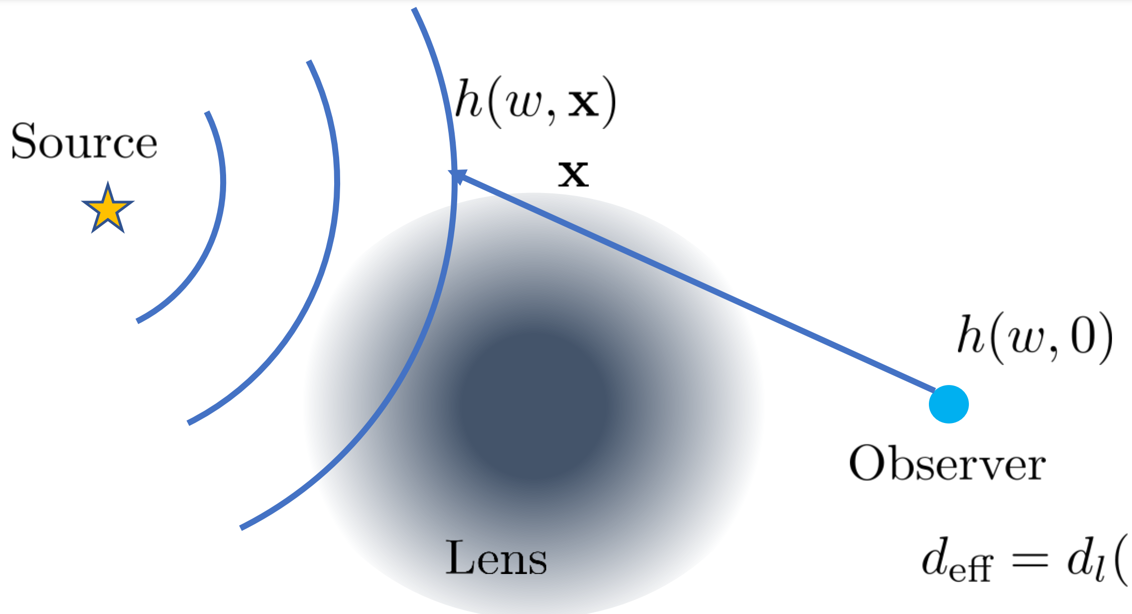
In typical situation, $\delta e_{\mu\nu} \ll 1$, we can write

$$\nabla^\alpha \nabla_\alpha h = 0$$

In Weak gravity & monochromatic wave,

$$(\nabla^2 + w^2) h(w, \mathbf{x}) = 4 w^2 U(\mathbf{x}) h(w, \mathbf{x})$$

III. Wave optics of Gravitational lensing



$$d_{\rm eff} = d_l(1 - \frac{d_l}{d_s})$$

Using the Kirchhoff integral & thin lens approximation , we find the lensed GW at $\vec{x} = \vec{0}$ as

$$\Rightarrow h(w, 0) \simeq \left[\frac{w}{2\pi i d_{\rm eff}} \int d^2x' e^{iwT(x')} \right] h_0(w, 0)$$

$\vec{x}'$ lens plane coordinate

$$F(w) \equiv \frac{w}{2\pi i d_{\rm eff}} \int d^2x' e^{iwT(x')}$$

Lensing amplification factor (F=1 for no lens)

$h_0(w, 0)$: GW without lensing effects

III. Wave optics of Gravitational lensing

$$F(w;\mathbf{y}) = \frac{w}{2\pi i d_{\text{eff}}} \int d^2x' e^{iwT(\mathbf{x}',\mathbf{y})}$$

$\mathbf{y}$: source impact parameter on lens plane

$T(\mathbf{x}',\mathbf{y})$: arrival time of ray passing $\mathbf{x}'$ on lens plane

$$T(\mathbf{x},\mathbf{y}) = \frac{1}{2d_{\text{eff}}}|\mathbf{x}-\mathbf{y}|^2 - \psi(\mathbf{x})$$

$\psi(\mathbf{x}')$: project gravitational potential on lens plane

In geometric optics limit $w \to \infty$, only stationary points of T(x) (classical path) contributes to integral, (Nakamura, 1999)

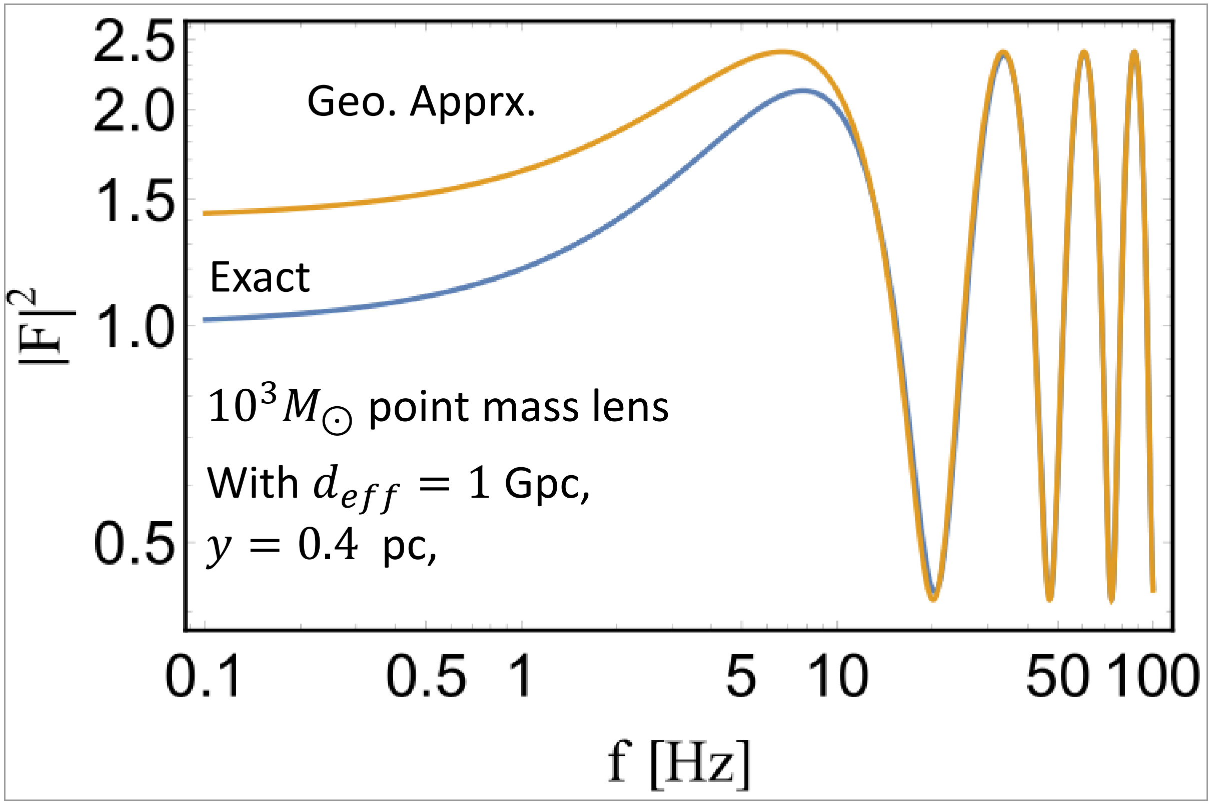
$$F(w;\mathbf{y}) \simeq \sqrt{|\mu_1(\mathbf{y})|} + \sqrt{|\mu_2(\mathbf{y})|} e^{iw\Delta T_{12}(\mathbf{y})}$$

$\mu_{1,2}$: magnification of lensed images

ΔT_{12} : arrival time difference of lensed images

III. Wave optics of Gravitational lensing

example $F(w;\mathbf{y}) \simeq \sqrt{|\mu_1(\mathbf{y})|} - i\sqrt{|\mu_2(\mathbf{y})|}e^{iw\Delta T_{12}(\mathbf{y})}$

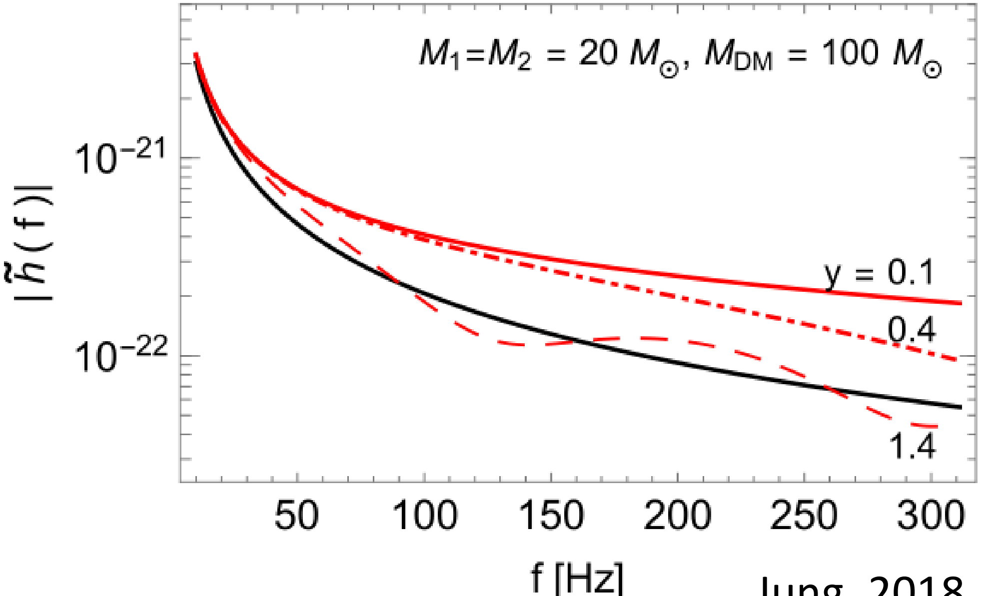


- Frequency dependent lensing due to interference
- Geometric limit approximation breaks down when $w\Delta T \sim Mf < 1$
- The physics of $f \to 0$ limit was not well studied

III. Wave optics of Gravitational lensing

Gravitational lensing of **GW Chirps** is sensitive to

- Intermediate mass black hole (Lai 2018, Jung 2018)
- Dwarf galaxy (Takahashi 2003, Dai 2018)

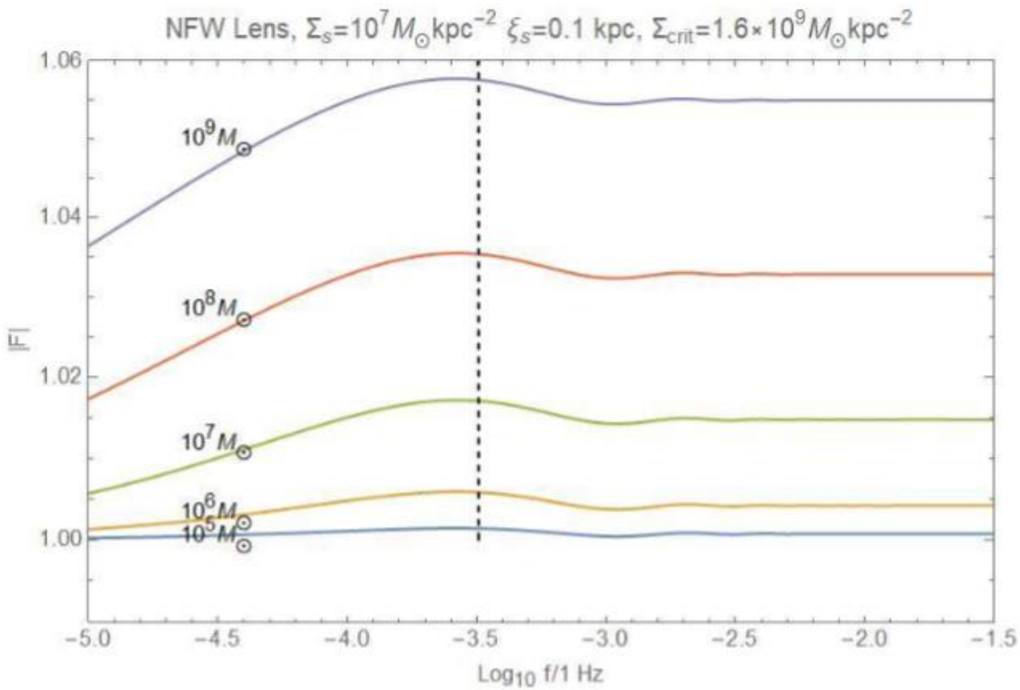


Black : unlensed GW

Jung, 2018

III. Wave optics of Gravitational lensing

- **Lensing of GW chirps might be able to probe subhalo as well.**
  - Subhalos are too diffuse → single image → no ΔT_{12}
  - What cause the frequency dependence at low f ?
  - Can the phenomena be used to detect lensing?
  - → **Diffractive lensing formalism**



$$|F| \simeq \sqrt{\mu(y)}$$

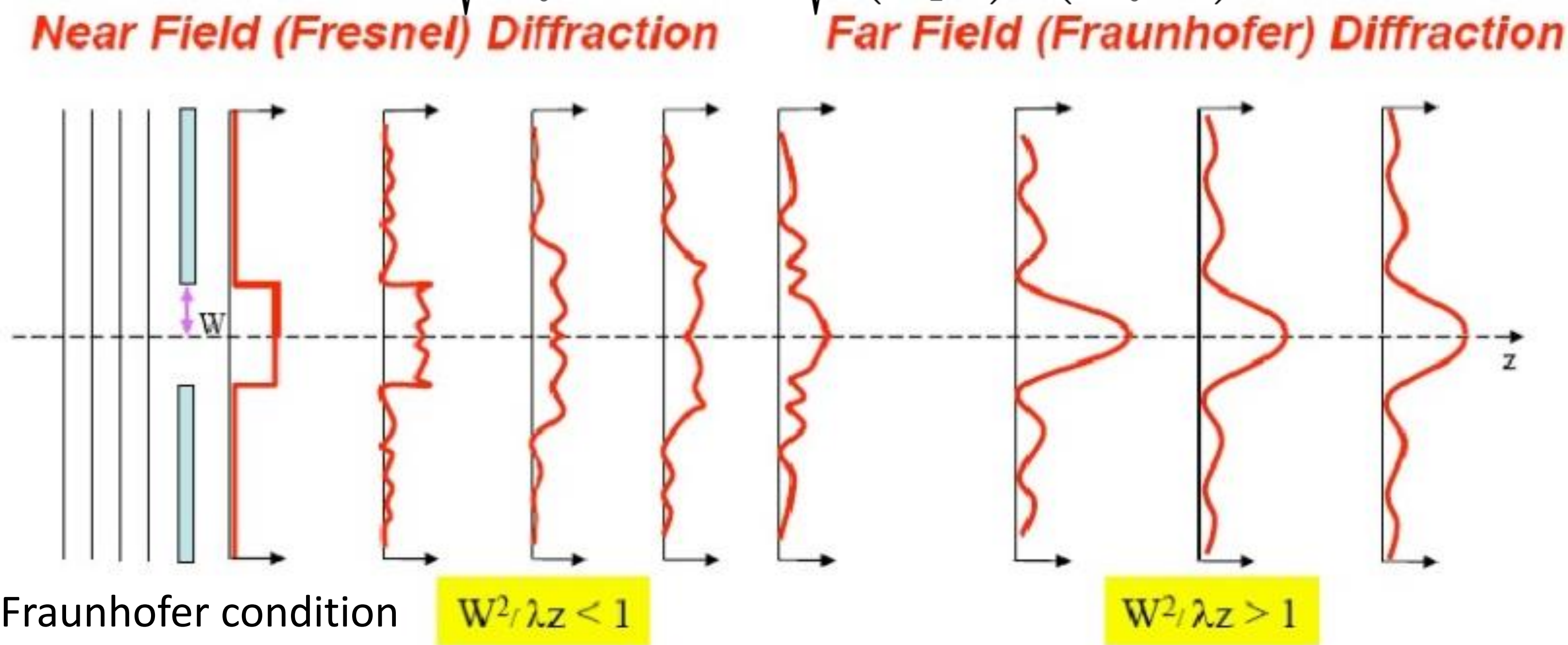
Lensing of Navarro-Frenk-White(NFW) profile

IV. Diffractive lensing

When wave effects are prominent, **Fresnel length** become important.

- **Wave dispersion** length scale on lens plane
- **Resolution** length scale on lens plane
- **Phase variation** length scale on lens plane

$$r_F = \sqrt{\frac{d_{\rm eff}}{\pi f}} \simeq 5.56{\rm pc}\sqrt{\left(\frac{d_{\rm eff}}{\rm Gpc}\right)\left(\frac{0.1{\rm Hz}}{f}\right)}$$



Fraunhofer condition

$r_F < W$ $r_F > W$

IV. Diffractive lensing

Strong and weak diffraction

- Einstein radius $r_E = \left(4\, M_E\, d_{eff}\right)^{1/2}$ $M_E = r_E^2 \pi \Sigma_{\rm critical}$
  - Wave focusing length scale of the lens
- Diffractive lensing is competition between focusing(r_E) and dispersion (r_F)

Weak diffraction, $r_E < r_F$

- Wave dispersion dominant regime
- Gravitational potential effects are small-> Born's approximation

Let y=0, spherical symmetric lens

$$F(w) \simeq 1 - \frac{w^2}{2\pi i d_{\rm eff}} \int d^2x' e^{i\frac{w}{2d_{\rm eff}}|\mathbf{x}'|^2} \psi(\mathbf{x}') \simeq 1 + \overline{\kappa}(\frac{r_F}{\sqrt{2}} e^{i\pi/4})$$

$$\frac{dF(w)}{d\ln w} \simeq \gamma(\frac{r_F}{\sqrt{2}} e^{i\pi/4})$$

$\overline{\kappa}(r)$: mean convergence within the radius r

$\gamma(r)$: shear at the radius r

→ weak lensing quantities!

IV. Diffractive lensing

Strong and weak diffraction

- Einstein radius $r_E = \left(4\, M_E\, d_{eff}\right)^{1/2}$
  - Wave focusing length scale of the lens
- Diffractive lensing is competition between focusing(r_E) and dispersion (r_F)

Strong diffraction, $r_E > r_F$

- Wave focusing dominant regime
- Gravitational potential cannot be ignored

Let y=0, spherical symmetric lens

$$F(w) = \frac{w}{2\pi i d_{\rm eff}} \int d^2x' e^{i\frac{w}{2d_{\rm eff}}|\mathbf{x}'|^2 - w\psi(\mathbf{x}')}$$

$$\simeq \frac{r_E^2}{i r_F}\sqrt{\frac{4\pi}{1-\kappa(r_E)+\gamma(r_E)}}$$

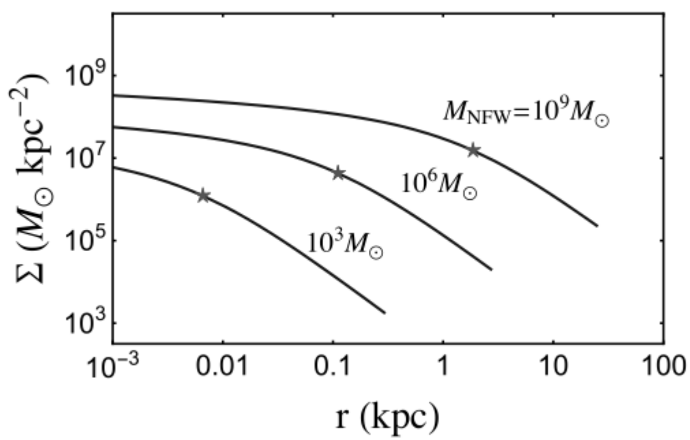
Frequency dependent is universal to all density profile

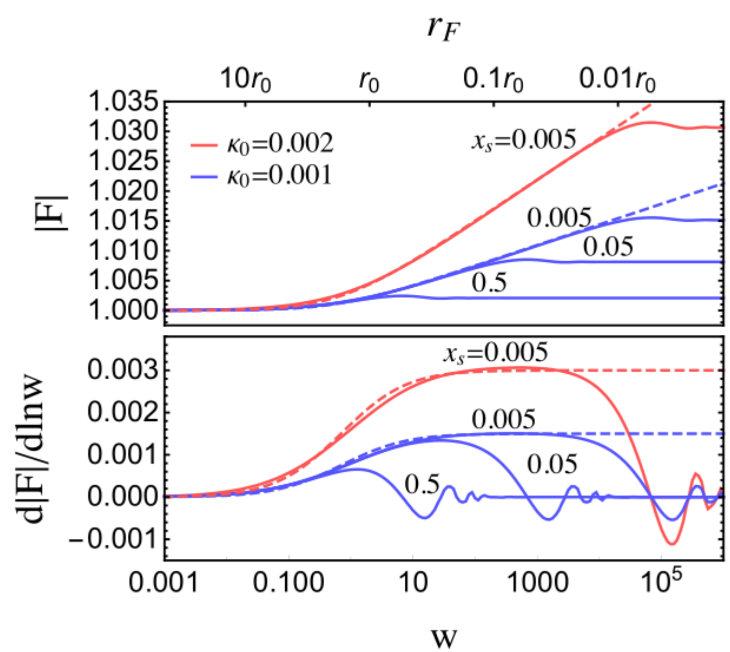
IV. Diffractive lensing

Implication of weak diffraction $F(w) \simeq 1 + \overline{\kappa}(\frac{r_F}{\sqrt{2}} e^{i\pi/4})$

- Analytic treatment of wave optics
- It explains the diffraction pattern of $r_E = 0$ lenses
- Frequency dependence follows density profile of lens

Surface density profile of NFW lens





$$\mathrm{w} = 2\pi f (r_0^2/d_{\rm eff})$$

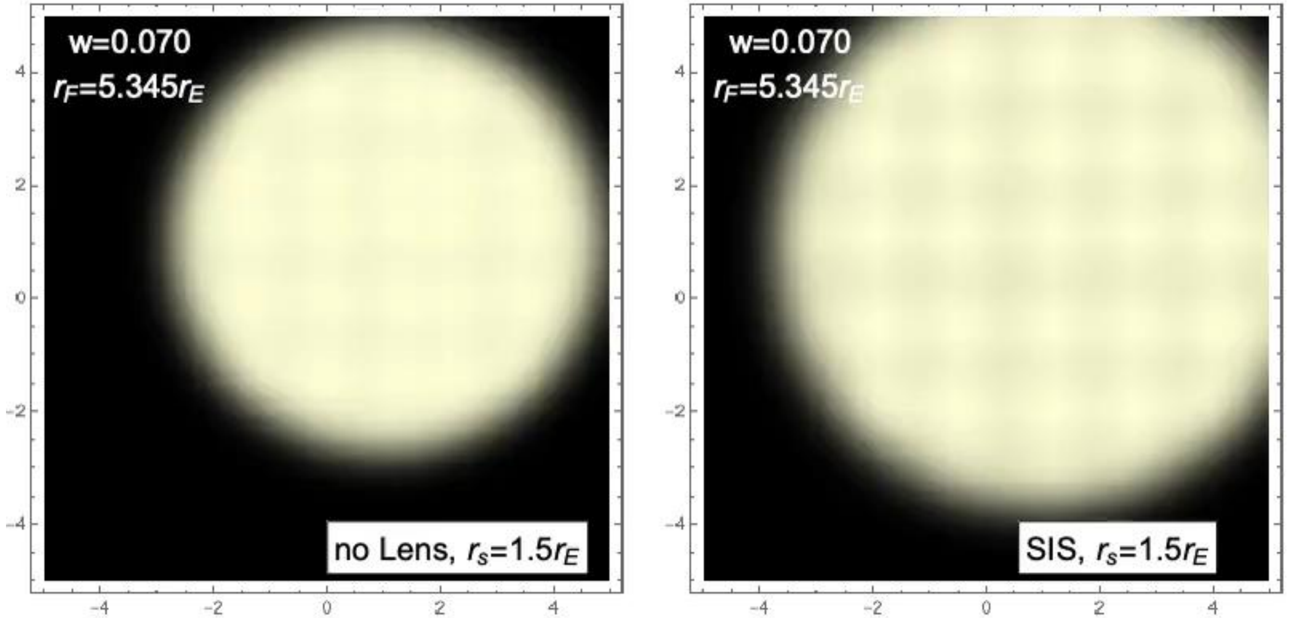
IV. Diffractive lensing

Weak Diffractive lensing – geometric optics lensing transition

Principle : wave only sensitive to **matter within radius r_F** → small phase variation around impact parameter

$$\text{Diffraction condition : } y < r_F$$

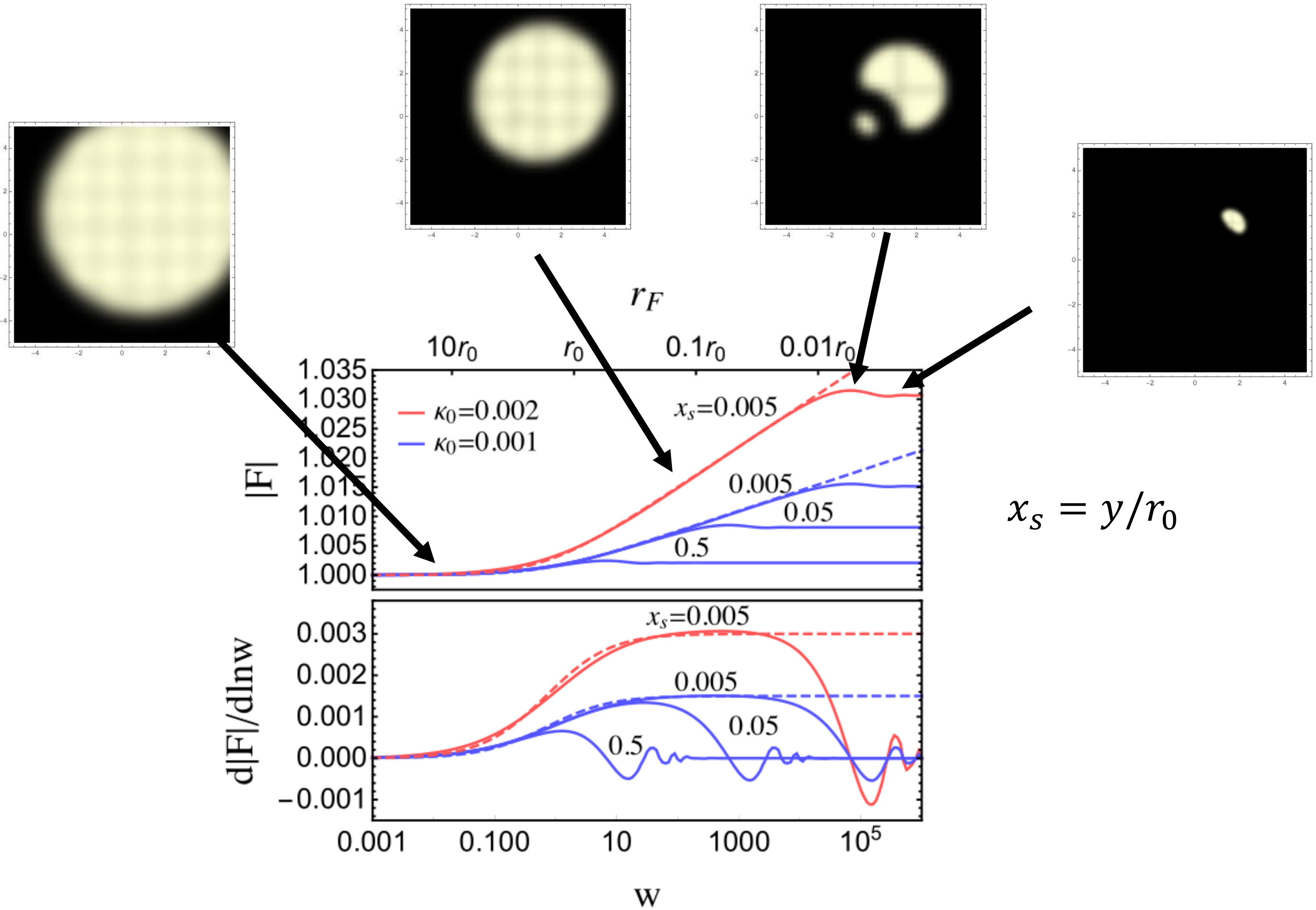
Phase variance < 1 region with respect to increasing frequency



$y = r_s$

IV. Diffractive lensing

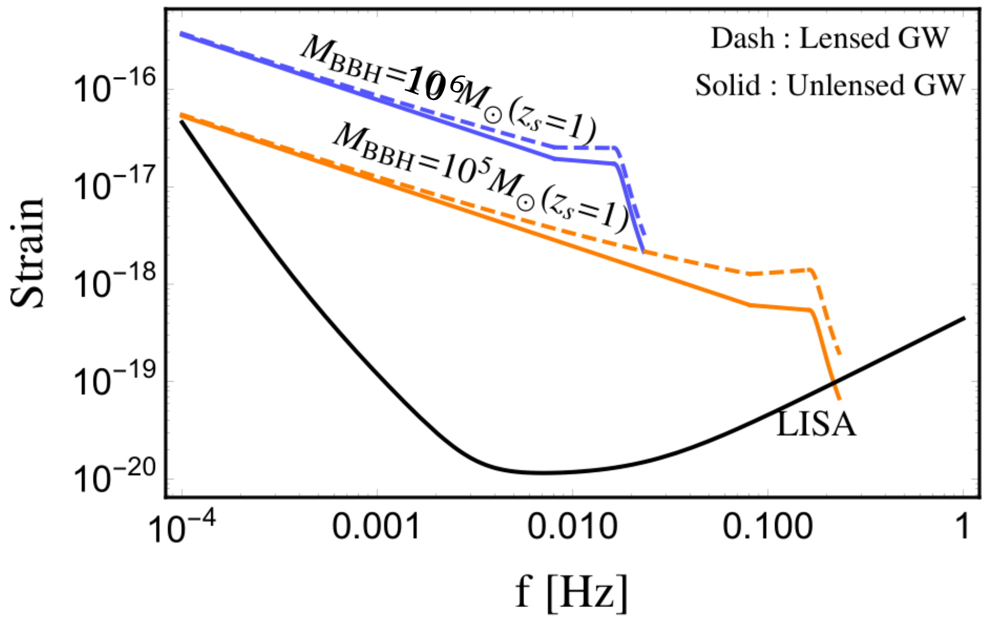
Weak Diffractive lensing – geometric optics limit transition



$x_s = y/r_0$

IV. Diffractive lensing

Weak Diffractive lensing of gravitational wave



Lensing by $\bar{\kappa}(r) \propto r^{-1}$ lens ($M = 10^5\ M_\odot, z_l = 0.35$)

Although we don't know intrinsic luminosity of GW, this Frequency dependent amplification can be detected.

We can measure the difference by log-likelihood of GWs

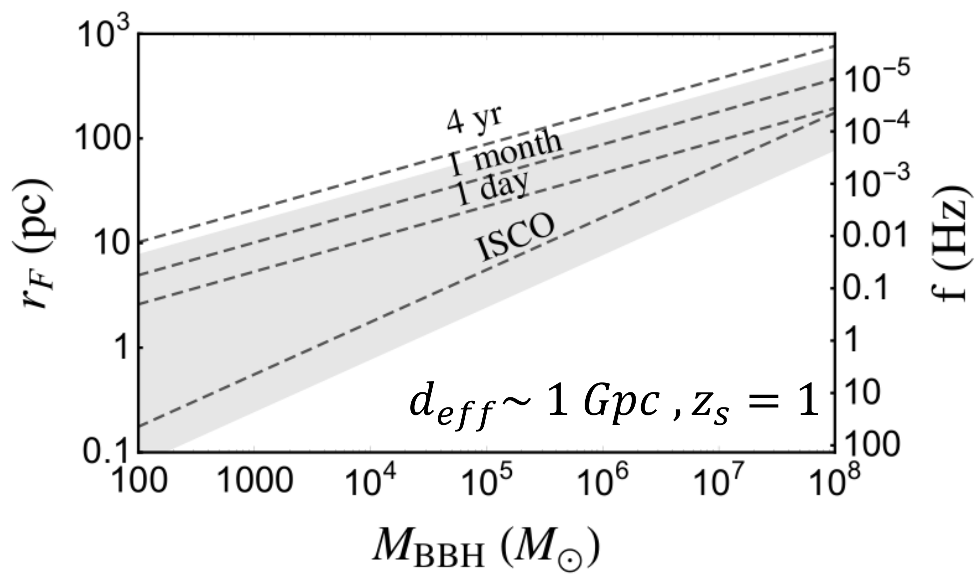
$$\ln p = -\frac{1}{2} \min_{t_c,\phi_c} (h_L - h_0 | h_L - h_0) \qquad (h_1|h_2) = 4\mathrm{Re} \int df \frac{h_1^*(f) h_2(f)}{S_n(f)}$$

IV. Diffractive lensing

Weak Diffractive lensing of gravitational wave

GW chirps from **massive Black hole binaries** is ideal difffractive lensing source : low f, large d_{eff} → large r_F

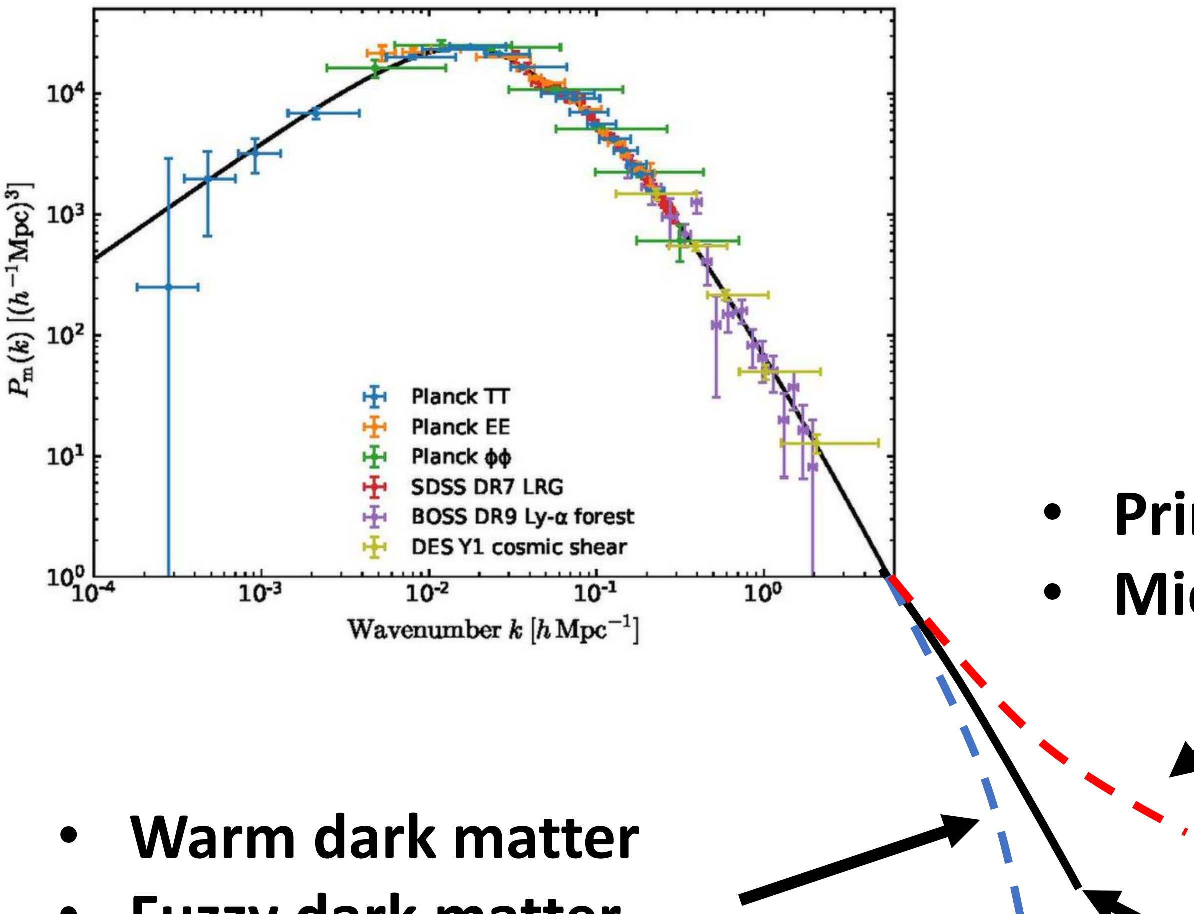
$$r_F \simeq 5.56\text{pc}\sqrt{\left(\frac{d_{\text{eff}}}{\text{Gpc}}\right)\left(\frac{0.1\text{Hz}}{f}\right)} \sim \text{(sub halo length scale)}$$



Ex) $10^5 M_\odot$ BBH spectrum → Scan 1 pc to 50 pc by 1yr observation

V. Weak Diffractive lensing of subhalo

Cosmological structures at small scale depends on Dark matter physics.



- **Primordial Black holes**
- **Micro(or Mini) halos**

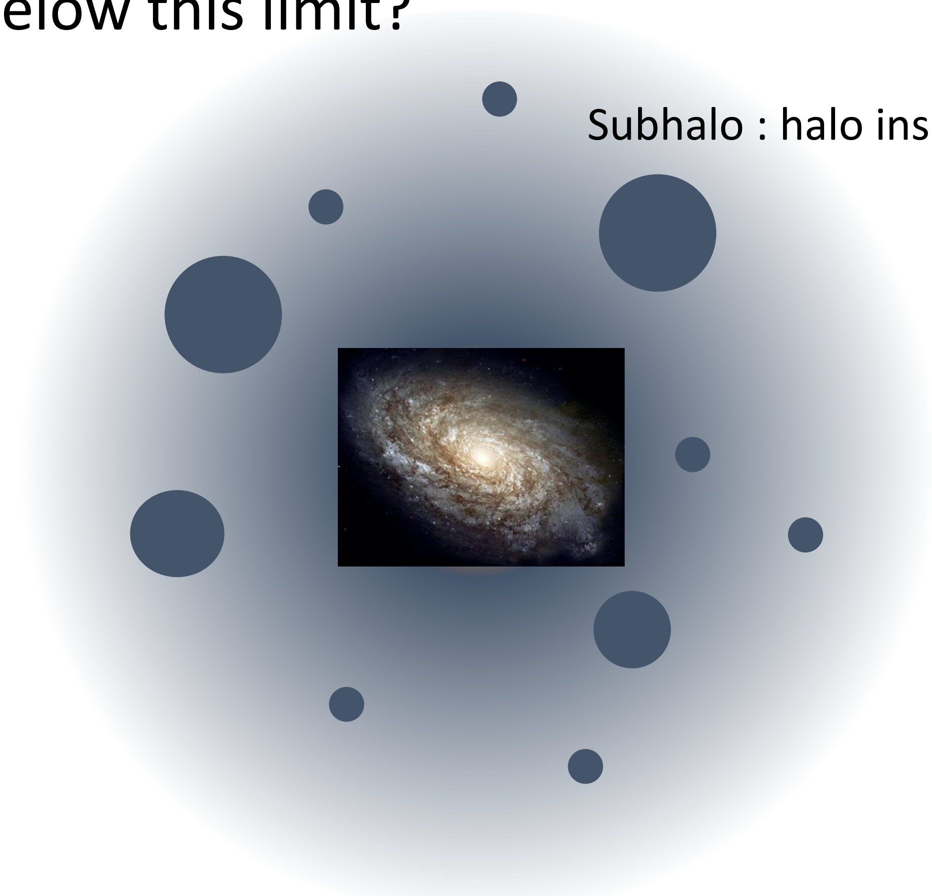
- **Warm dark matter**
- **Fuzzy dark matter**
- **Self-Interacting dark matter**

CDM

V. Weak Diffractive lensing of subhalo

Probing dark matter subhalo is the best option for small scale until now.

- Current limit : $\mathrm{M_{sub}} \sim 10^7 M_\odot$ $(k = 10^{3\sim4} Mpc^{-1})$ (Nadler 2021)
- Can we go below this limit?



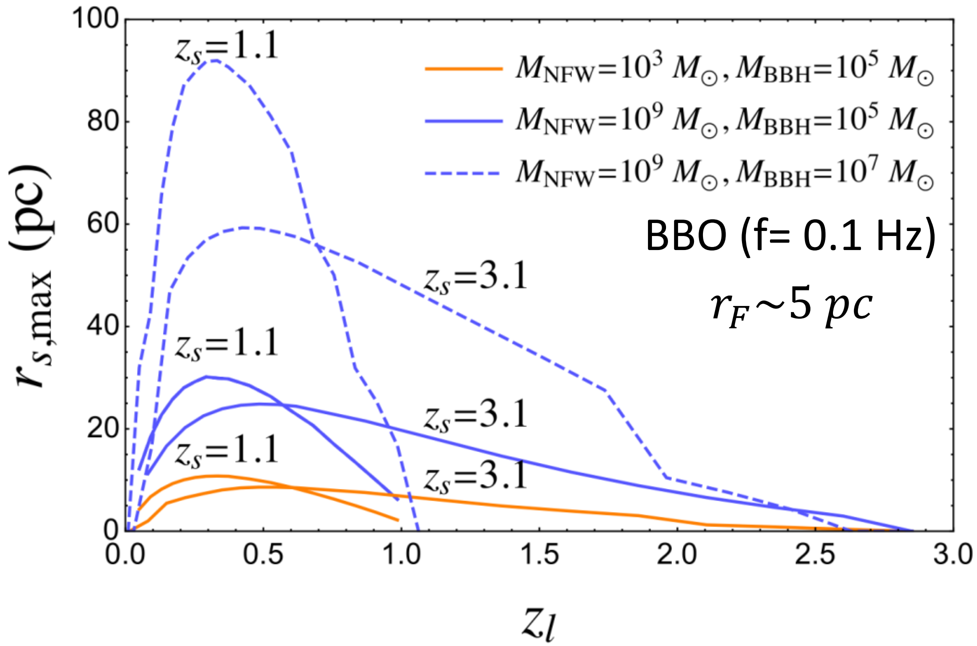
V. Weak Diffractive lensing of subhalo

Set 3-sigma criteria to lnp, we find maximum impact parameter ‘y’-> cross-section

Lensing cross-section : shear at r_F

⇒Insensitive to mass at high SNR limit

$$|\ln p| \simeq \frac{1}{8}\left\{\rho_0 \cdot \left|\gamma\left(\frac{r_F(f_0)e^{i\frac{\pi}{4}}}{\sqrt{2}}\right)\right| \cdot \ln\frac{f_{\max}}{f_{\min}}\right\}^2$$



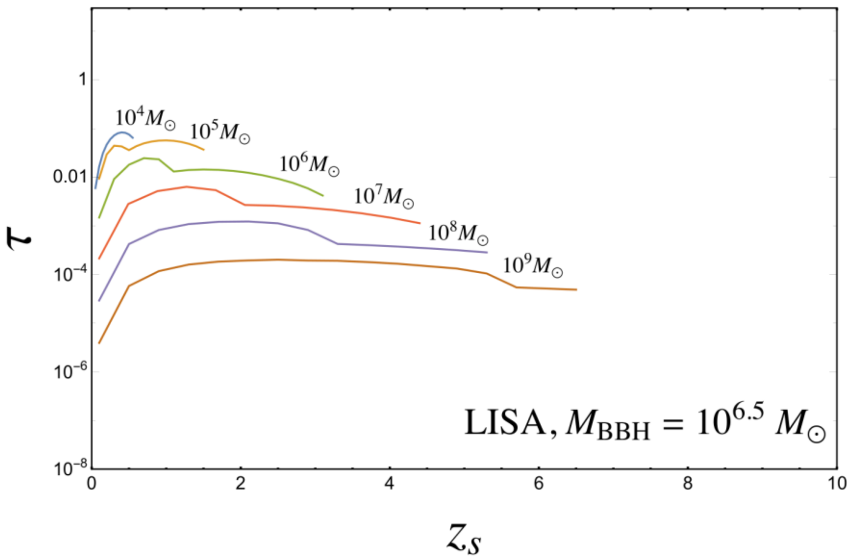
Mass scale difference : 10^6

Cross-section scale difference : $O(1)$

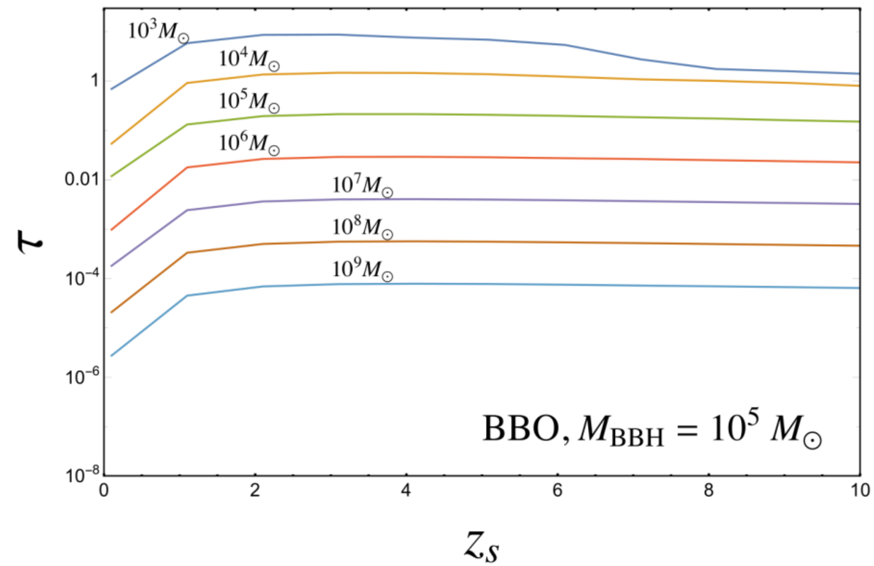
V. Weak Diffractive lensing of subhalo

Optical depth from the lensing cross-section

$\tau \sim 0.01$ for $10^6 M_\odot$ halo



$\tau > 1$ for $10^{3-4} M_\odot$ halo



$$\tau \propto (\text{number of Lens}) \propto 1/(M_{\rm halo})$$

Diffractive lensing is sensitive to low mass halo !

V. Weak Diffractive lensing of subhalo

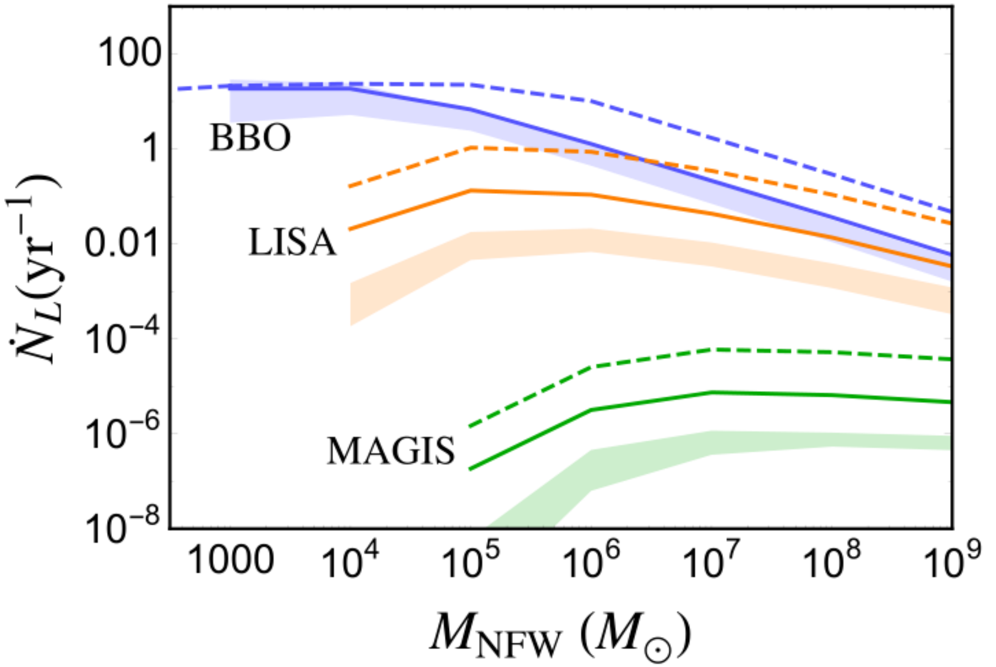
Prospect

We need powerful space-based GW detectors like BBO(0.1Hz), LISA(1 mHz).

- BBO can detect $10^{3-4} M_{\odot}$ halo lensing O(10) events per year.

  In future, BBO will discriminate CDM and the other DM models.

- LISA and the others are less promising.
  - Lack of **High Signal-to-Noise Ratio(>1000)** BBH sources



BBH Merger rate
Solid : 0.01 $Gpc^{-3} yr^{-1}$
Shaded : astrophysical
(Bonetti 2018)

Conclusions

1. Small frequency limit, diffraction dominates wave optics lensing, i.e. diffractive lensing

2. r_F is essential concept to diffractive lensing which characterizes wave dispersion length scale.

3. When r_E which characterize wave focusing length scale is smaller than r_F or zero, Weak diffractive lensing can occur.

4. Weak diffractive lensing is valid when $r_F > y$, otherwise, geometric optics become relevant.

5. Dark matter subhalos are very diffuse objects but might be able to detected by weak diffractive lensing

6. The future GW detector BBO will detect GW chirps of super massive binary black holes lensed by $10^{3\sim4} M_\odot$ DM subhalo with a few tens of events per year.