Invertible field transformations with derivatives

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$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

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  How ubiquitous field transformations are ?.

- **Necessary & sufficient conditions for invertible transformations with derivatives**

- **Higher derivative extension of disformal transformation as an example**

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Introduction

Field transformations are ubiquitous in mathematics & physics !!

- **Gauge (global) transformation of fields**
- **Bogliubov transformation**
- **Fourier transformation (series) & Laplace transformation**
- **Galilean, Lorentz, general coordinate transformations**
- **…**

It provides better ways to understand various physical phenomena, and to advance calculations, in particular, to solve more easily differential equations.

Conformal & disformal transformations in gravity

- **Conformal transformation :**

$$\tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu} \quad \xLeftrightarrow{\Omega^2 \neq 0} \quad g_{\mu\nu} = \Omega(x)^{-2} \tilde{g}_{\mu\nu}$$

- **Disformal transformation :** (Bekenstein 1992)

$$\begin{cases} \tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad X = g^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi \\ \tilde{\phi} = \phi \end{cases}$$

$$\Updownarrow \quad \det \left(\frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} \right) = C \left(C - C_X X - D_X X^2 \right) \neq 0.$$

(N.B. No derivatives of the metrics)

$$g_{\mu\nu} = \tilde{C}(\phi, \widetilde{X}) \tilde{g}_{\mu\nu} + \widetilde{D}(\phi, \widetilde{X}) \partial_\mu \phi \partial_\nu \phi, \quad \widetilde{X} = \tilde{g}^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi = \frac{X}{C + DX}$$

$$\left(\tilde{C}(\phi, \widetilde{X}) = \frac{1}{C(\phi, X)}, \quad \widetilde{D}(\phi, \widetilde{X}) = -\frac{D(\phi, X)}{C(\phi, X)}. \right)$$

Inverse function theorem

Field transformations without derivatives :

$$\tilde{\phi}^a(x^\mu) = \tilde{\phi}^a[\phi^b(x^\mu)]$$

$\det|(\partial\tilde{\phi}^a/\partial\phi^b)| \neq 0$ ⟹ **(Local) invertibility**

N.B. this theorem, of course, applies to point particle theories.

$\tilde{q}^a(t) = \tilde{q}^a[q^b(t)]$ ⟹ $\det|(\partial\tilde{q}^a)/\partial q^b| \neq 0$

(Invertibility)

How to judge the invertibility for field transformations with derivatives ??

Lesson

Variable transformation from q(t) to Q(t) with derivatives:

e.g. $Q(t) = q(t) + \dot{q}(t)$

Integration constant !!

Inverse transformation

$$\Longrightarrow \quad q(t) = e^{-t}\left(e^{t_0} q(t_0) + \int_{t_0}^{t} e^{t} Q(t) \right).$$

Given Q(t) completely, q(t) is not uniquely determined !!

⇒ **This is not an invertible transformation !!**

(But, we have such an example of
an invertible transformation with derivatives)

Explicit construction of inverse transformation

One way is to construct the inverse transformation explicitly, which manifestly shows the invertibility.

e.g.

$$\begin{cases} \phi_1 = \psi_1 + \eta^{\mu\nu}\partial_\mu\psi_2\partial_\nu\psi_2, \\ \phi_2 = \psi_2. \end{cases}$$

($\eta^{\mu\nu}$ = diag(-1,1,1,1))

$$\Longleftrightarrow \begin{cases} \psi_1 = \phi_1 - \eta^{\mu\nu}\partial_\mu\phi_2\partial_\nu\phi_2, \\ \psi_2 = \phi_2. \end{cases}$$

$$\phi_1, \phi_2 \Longleftrightarrow \psi_1, \psi_2$$

But, an explicit form of the inverse transformation is not necessarily obtained.

Let's try to extend the inverse function theorem to field transformation with derivatives !!

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu\right)$$

But, how ???

We are going to use
the method of characteristics
by regarding this transformation
as a differential equation w.r.t. ψa.

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu\right)$$

Propagation of waves

Various velocities

Dispersion relation for a propagating wave : $\omega = \omega(k)$

- **phase velocity :** $v_p = \dfrac{\omega}{k}$
- **group velocity :** $v_g = \dfrac{\partial \omega}{\partial k}$
- **front velocity :** $v_s = \left.\dfrac{\omega}{k}\right|_{k \to \infty}$

Which is the signal propagation speed ???

The signal propagation speed is given by front velocity !!

Various velocities with a canonical scalar field

(courtesy of Eugeny)

- **a massive canonical scalar :** $w^2 = k^2 + m^2$

$$\left\{\begin{aligned} v_p &= \frac{\omega}{k} = \sqrt{1+\frac{m^2}{k^2}} \geq 1 \\ v_g &= \frac{\partial\omega}{\partial k} = \frac{k}{\omega} = \frac{1}{\sqrt{1+\omega^2/k^2}} \leq 1 \end{aligned}\right.$$

The phase velocity of EM wave can be larger than unity. This happens in most glasses at X-ray frequencies.

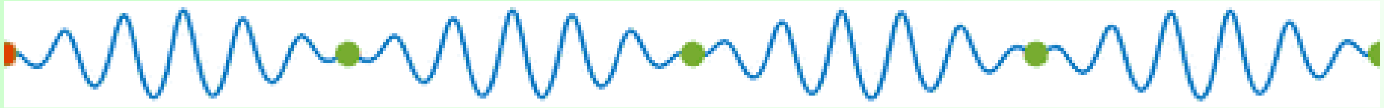
- **a canonical scalar with tachyonic mass :** $w^2 = k^2 - m^2$

$$\left\{\begin{aligned} v_p &= \frac{\omega}{k} = \sqrt{1-\frac{m^2}{k^2}} \leq 1 \\ v_g &= \frac{\partial\omega}{\partial k} = \frac{k}{\omega} = \frac{1}{\sqrt{1-\omega^2/k^2}} \geq 1 \end{aligned}\right.$$

Group velocities faster than light speed were measured in experiments with laser light pulses.

Phase velocity and group velocity

(from wikipedia)



Frequency dispersion in groups of gravity waves on the surface of deep water. The red square moves with the phase velocity, and the green circles propagate with the group velocity. In this deep-water case, the phase velocity is twice the group velocity.
The red square overtakes two green circles when moving from the left to the right of the figure.

Neither of them has something to do with the signal propagation speed in general !!

Front velocities

(courtesy of Eugeny)

$$v_s = \frac{\omega}{k}\bigg|_{k\to\infty}$$

"Physical" definition of signal speed propagation

Mathematically, signals propagate along characteristics of a differential equation.

e.g. $\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \dfrac{dV(\phi)}{d\phi} = 0.$

$\downarrow$

Determine characteristics independent of any potential !!

In case of quadratic potential,

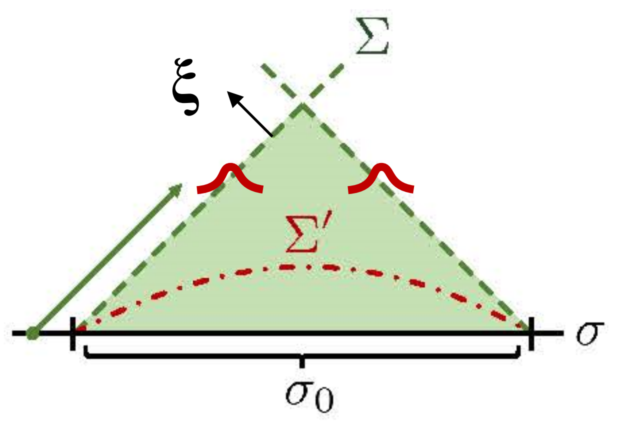
$$v_s = \frac{\omega}{k}\bigg|_{k\to\infty} = \sqrt{1+\frac{m^2}{k^2}}\Bigg|_{k\to\infty} = 1$$

Causal structure and invertibility

EOMs give (determine) causal structure through characteristics.

- **D dim spacetime**
  = (D-1) dim surface + the other direction ξ

- **Characteristic surface**
  = on which, the coefficient of the highest-order derivatives of EOMs to ξ-direction becomes zero.



⇒ **Then, the EOMs cannot be solved uniquely beyond it.**
In fact, this surface coincides with the edge of causality.

⇒ **If one-to-one correspondence (*i.e.* invertibility) would be achieved, the two theories must have the same characteristics.**

Method of characteristics (courtesy of Nori)

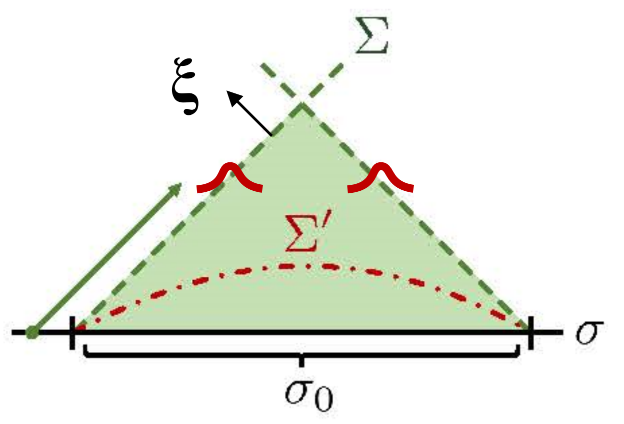
Characteristic surface Σ = wave propagation surface
(Maximum propagation speed is determined by characteristics)

- **For a quasi-linear (hyperbolic) PDE, pick up the highest derivative terms:**

$$E_\phi = \mathcal{G}^{\mu\nu}(\phi, \partial\phi)\partial_\mu\partial_\nu\phi + \cdots = 0.$$

- **Estimate the principal symbol P(ξ):**

$$P(\xi) \equiv \xi_\mu \xi_\nu \frac{\partial E_\phi}{\partial(\partial_\mu\partial_\nu\phi)} = \xi_\mu\xi_\nu\mathcal{G}^{\mu\nu}$$



$(\mathcal{G}^{\mu\nu} = g^{\mu\nu} \Rightarrow \xi^\mu : \text{null})$

- **Solve the characteristic equation (det)P(ξ) = 0:**

$$P(\xi) = \xi_\mu\xi_\nu\mathcal{G}^{\mu\nu} = 0$$

(2 propagation modes = 1 dof)

Time evolution is uniquely determined in the green region (once an initial data is given on σ_0), but not beyond Σ.

Let’s take a lesson

Lesson

e.g. $L = -\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \beta\partial_\mu\phi_1\partial^\mu\phi_2$

$$\Longrightarrow \begin{cases} E_{\phi_1} = \Box\phi_1 - \beta\Box\phi_2 = 0 \\ E_{\phi_2} = \Box\phi_2 - \beta\Box\phi_1 = 0 \end{cases}$$

($\Box = \partial^\mu\partial_\mu$)

(4 propagation modes = 2 dof for β≠ 1)

$$P_\phi = \begin{pmatrix} \xi_\mu\xi_\nu\frac{\partial E_{\phi_1}}{\partial(\partial_\mu\partial_\nu\phi_1)} & \xi_\mu\xi_\nu\frac{\partial E_{\phi_1}}{\partial(\partial_\mu\partial_\nu\phi_2)} \\ \xi_\mu\xi_\nu\frac{\partial E_{\phi_2}}{\partial(\partial_\mu\partial_\nu\phi_1)} & \xi_\mu\xi_\nu\frac{\partial E_{\phi_2}}{\partial(\partial_\mu\partial_\nu\phi_2)} \end{pmatrix} = \begin{pmatrix} \xi^2 & -\beta\xi^2 \\ -\beta\xi^2 & \xi^2 \end{pmatrix} \Longrightarrow \det P_\phi = \left(1-\beta^2\right)(\xi^2)^2$$

Invertible transformation :
$$\begin{cases} \phi_1 = \psi_1 + \frac{\alpha}{2}\partial_\mu\psi_2\partial^\mu\psi_2 \\ \phi_2 = \psi_2 \end{cases}$$

$$\Longrightarrow \begin{cases} E_{\psi_1} = \Box\psi_1 + \alpha\left(\partial_\mu\partial_\nu\psi_2\partial^\mu\partial^\nu\psi_2 + \partial_\mu\psi_2\partial^\mu\Box\psi_2\right) - \beta\Box\psi_2 = 0 \\ E_{\psi_2} = \Box\psi_2 - \beta\left[\Box\psi_1 + \alpha\left(\partial_\mu\partial_\nu\psi_2\partial^\mu\partial^\nu\psi_2 + \partial_\mu\psi_2\partial^\mu\Box\psi_2\right)\right] = 0 \end{cases}$$

$$P_\psi = \begin{pmatrix} \xi_\mu\xi_\nu\frac{\partial E_{\psi_1}}{\partial(\partial_\mu\partial_\nu\psi_1)} & \xi_\mu\xi_\nu\xi_\sigma\frac{\partial E_{\psi_1}}{\partial(\partial_\mu\partial_\nu\partial_\sigma\psi_2)} \\ \xi_\mu\xi_\nu\frac{\partial E_{\psi_2}}{\partial(\partial_\mu\partial_\nu\psi_1)} & \xi_\mu\xi_\nu\xi_\sigma\frac{\partial E_{\psi_2}}{\partial(\partial_\mu\partial_\nu\partial_\sigma\psi_2)} \end{pmatrix} = \begin{pmatrix} \xi^2 & \alpha\xi^2\xi_\mu\partial^\mu\psi_2 \\ -\beta\xi^2 & -\alpha\beta\xi^2\xi_\mu\partial^\mu\psi_2 \end{pmatrix} \Longrightarrow \det P_\psi = 0$$

Identically zero

The third order derivatives are fakes and due not lead to additional d.o.f. (propagation mode). In fact, the characteristic equation trivially vanishes.

Procedures

Field transformations: $\phi_i = \phi_i\left(\psi_a, \partial_\mu\right)$ **(i, a = 1, …, n)**

Necessary for invertibility

necessary conditions

Invertibility ⇨ **No characteristics for ψa associated with derivatives !!**

- **det P for the highest order derivatives must be identically zero, that is, the characteristic matrix must be degenerate (with m degrees).**
- **After taking adequate linear combinations, m EOMs reduce to lower order differential eqs. We repeat this procedure until the derivatives by transformation vanish.**
- **In the last step, the characteristic matrix should not be degenerate, which corresponds to the condition of inverse function theorem.**

Simplest case

$$\begin{cases} \phi_1 = \phi_1\left(\psi_1, \psi_2, \partial_\mu\psi_1, \partial_\mu\psi_2\right) \\ \phi_2 = \phi_2\left(\psi_1, \psi_2, \partial_\mu\psi_1, \partial_\mu\psi_2\right) \end{cases}$$

Two fields with up to first order derivatives

But, the extension to any number of fields with any order derivatives is straightforward.

Necessary conditions

Necessary conditions for invertibility

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu \psi_a\right) \quad (i, a = 1, 2).$$

Two useful matrices: $$A^\mu_{ia} \equiv \frac{\partial \phi_i}{\partial(\partial_\mu \psi_a)}, \quad B_{ia} \equiv \frac{\partial \phi_i}{\partial \psi_a}.$$

- **For all A = 0, the transformation does not involve derivatives.**

det B ≠ 0 ➔ Invertibility (Inverse function theorem).

- **We are interested in the case with at least some A ≠ 0.**

Two (first) derivatives for ψ_1 & ψ_2

➡ **Two degeneracies of the characteristic equations**
are necessary to avoid additional d.o.f.
(They should vanish trivially.)

➡ **But, not three degeneracies to avoid overkill the d.o.f.**

Necessary and sufficient conditions for invertibility

$$\phi_i = \phi_i\left(\psi_a, \partial_\mu \psi_a\right) \quad (i, a = 1, 2).$$

$$\Rightarrow \quad A^\mu_{ia} = \frac{\partial \phi_i}{\partial(\partial_\mu \psi_a)}, \quad B_{ia} = \frac{\partial \phi_i}{\partial \psi_a}$$

$$(1)\ A^\mu_{ia} = a^\mu V_i U_a \quad (V_i V_i = 1, \quad U_a U_a = 1)$$

$$\left(\longleftrightarrow \det\left(A^{(\mu\nu)}_{ia}\right) \equiv \frac{1}{2}\epsilon_{i_1 i_2}\epsilon_{a_1 a_2} A^{(\mu}_{i_1 a_1} A^{\nu)}_{i_2 a_2} = 0\right)$$

$$(2)\ n_i B_{ia} m_a = 0$$

Two degeneracies

$$(3)\ n_i B_{ia} U_a \neq 0, \quad \left(V_i B_{ia} - a^\mu \partial_\mu U_a\right) m_a \neq 0$$

$$\left(n_i := \epsilon_{ij} V_j, \quad m_a := \epsilon_{ab} U_b\right)$$

No third degeneracy
(corresponding to
the standard inverse
function theorem.)

Absence of one-field invertible transformation with derivatives

$$\phi_1 = \phi_1(\psi_1, \partial_\alpha \psi_1, x^\mu), \qquad \phi_2 = \psi_2,$$

$$(1)\ A^\mu_{ia} = a^\mu V_i U_a \quad (V_i V_i = 1, \quad U_a U_a = 1)$$

$$A^\mu_{ia} = \frac{\partial \phi_i}{\partial\left(\partial_\mu \psi_a\right)} = \begin{pmatrix} \frac{\partial \phi_1}{\partial(\partial_\mu \psi_1)} & 0 \\ 0 & 0 \end{pmatrix} = \underset{\neq 0}{\boxed{\frac{\partial \phi_1}{\partial\left(\partial_\mu \psi_1\right)}}} U_a V_i \ , \qquad \begin{pmatrix} U_a = (1, 0) = V_i, \\ m_a = (0, -1) = n_i. \end{pmatrix}$$

$$(2)\ n_i B_{ia} m_a = 0$$

$$B_{ia} = \frac{\partial \phi_i}{\partial \psi_a} = \begin{pmatrix} \frac{\partial \phi_1}{\partial \psi_1} & 0 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad n_i B_{ia} m_a = 1 \neq 0.$$

Contradiction!!

$$(3)\ n_i B_{ia} U_a \neq 0, \quad \left(V_i B_{ia} - a^\mu \partial_\mu U_a\right) m_a \neq 0$$

$$n_i B_{ia} U_a = 0, \qquad \left(V_i B_{ia} - a^\mu \partial_\mu U_a\right) m_a = 0.$$

Contradiction!!

The second and the third conditions are violated.
So, there is no one-field invertible transformation with derivatives.

No-go theorem of higher derivative extension of disformal transformation

Conformal & disformal transformations in gravity

- **Conformal transformation :**

$$\tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu} \quad \xleftrightarrow{\Omega^2 \neq 0} \quad g_{\mu\nu} = \Omega(x)^{-2} \tilde{g}_{\mu\nu}$$

- **Disformal transformation :** **(Bekenstein 1992)**

$$\tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad X = g^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi$$

$$\Big\updownarrow \qquad \det\left(\frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}}\right) = C\left(C - C_X X - D_X X^2\right) \neq 0.$$

(N.B. No derivatives of the metrics)

$$g_{\mu\nu} = \tilde{C}(\phi, \widetilde{X}) \tilde{g}_{\mu\nu} + \widetilde{D}(\phi, \widetilde{X}) \partial_\mu \phi \partial_\nu \phi, \quad \widetilde{X} = \tilde{g}^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi = \frac{X}{C + DX}$$

$$\left(\tilde{C}(\phi, \widetilde{X}) = \frac{1}{C(\phi, X)}, \quad \widetilde{D}(\phi, \widetilde{X}) = -\frac{D(\phi, X)}{C(\phi, X)}.\right)$$

Higher derivative extension of disformal transformation

- **Disformal transformation :** (Bekenstein 1992)

$$\tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad X = g^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi$$

- **Higher derivative extension of disformal transformation :**

$$\tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + E(\phi, X) \nabla_\mu \nabla_\nu \phi, \quad X = g^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi$$

$$\left\{ \begin{array}{l} \nabla_\mu \nabla_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma^\sigma_{\mu\nu} \phi \\ \Gamma^\sigma_{\mu\nu} = \dfrac{1}{2} g^{\sigma\tau} \left(\partial_\mu g_{\tau\nu} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu} \right) \end{array} \right.$$

Derivatives of metric are included as well !!

Is this transformation invertible or not ???

As far as we know, nobody has yet proved it.

We will prove that this transformation is non-invertible for E ≠ 0.

Invertibility condition of higher derivative extension of disformal transformation

- **Higher derivative extension of disformal transformation :**

$$\begin{cases} \tilde{g}_{\mu\nu} = C(\phi, X)g_{\mu\nu} + E(\phi, X)\nabla_\mu\nabla_\nu\phi, \quad X = g^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi \\ \tilde{\phi}(x^\mu) = \phi(x^\mu) \end{cases}$$

Our strategy :

Constrain metric and field to some particular class,

$$\begin{cases} ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\mathfrak{n}(t)dt^2 + \mathfrak{a}(t)d\mathbf{x}^2, \\ \phi = \phi(t), \end{cases}$$

Show the non-invertibility ➔ the whole transformation is non-invertible.

N.B. When one tries to show the invertibility,
this strategy does not work.
But, in the opposite case like ours, it works well.

Invertibility condition of higher derivative extension of disformal transformation II

- **Higer derivative extension of disformal transformation :**

$$\begin{cases} \tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + E(\phi, X) \nabla_\mu \nabla_\nu \phi, \quad X = g^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi \\ \tilde{\phi}(x^\mu) = \phi(x^\mu) \end{cases}$$

Homogeneous and isotropic spacetime and scalar field:

$$\begin{cases} ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\underset{\uparrow}{\mathfrak{n}}(t) dt^2 + \underset{\uparrow}{\mathfrak{a}}(t) d\mathbf{x}^2, \\ \phi = \phi(t), \end{cases}$$

(square of the lapse) **(square of the scale factor)**

$$\Rightarrow \quad d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -\tilde{\mathfrak{n}}(t) dt^2 + \tilde{\mathfrak{a}}(t) d\mathbf{x}^2$$

$$\begin{cases} \tilde{\mathfrak{n}} = C\mathfrak{n} - D\dot{\phi}^2 - E\left(\ddot{\phi} - \dfrac{\dot{\phi}\dot{\mathfrak{n}}}{2\mathfrak{n}}\right), \quad \tilde{\mathfrak{a}} = C\mathfrak{a} - E\dfrac{\dot{\phi}\dot{\mathfrak{a}}}{2\mathfrak{n}}; \\ C = C(\phi, X) = C\left(\phi(t), -\dfrac{\dot{\phi}^2(t)}{\mathfrak{n}(t)}\right), \quad D = D\left(\phi(t), -\dfrac{\dot{\phi}^2(t)}{\mathfrak{n}(t)}\right). \end{cases}$$

Invertibility condition of higher derivative extension of disformal transformation III

By regarding φ(t) as an external variable, the transformation becomes the change of variables, $\{\mathfrak{n}, \mathfrak{a}\} \to \{\tilde{\mathfrak{n}}, \tilde{\mathfrak{a}}\}$

$$g_{\mu\nu}dx^\mu dx^\nu = -\mathfrak{n}(t)dt^2 + \mathfrak{a}(t)d\mathbf{x}^2 \quad \Rightarrow \quad \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -\tilde{\mathfrak{n}}(t)dt^2 + \tilde{\mathfrak{a}}(t)d\mathbf{x}^2$$

$$\begin{cases} \tilde{\mathfrak{n}} = C\mathfrak{n} - D\dot{\phi}^2 - E\left(\ddot{\phi} - \dfrac{\dot{\phi}\dot{\mathfrak{n}}}{2\mathfrak{n}}\right), \quad \tilde{\mathfrak{a}} = C\mathfrak{a} - E\dfrac{\dot{\phi}\dot{\mathfrak{a}}}{2\mathfrak{n}}; \\ C = C\left(\phi, X\right) = C\left(\phi(t), -\dfrac{\dot{\phi}^2(t)}{\mathfrak{n}(t)}\right), \quad D = D\left(\phi(t), -\dfrac{\dot{\phi}^2(t)}{\mathfrak{n}(t)}\right). \end{cases}$$

First derivatives of $\{\mathfrak{n}, \mathfrak{a}\}$

$$\left(A^\mu_{ia} = \frac{\partial \phi_i}{\partial(\partial_\mu \psi_a)}, \quad B_{ia} = \frac{\partial \phi_i}{\partial \psi_a}\right)$$

Our conditions: $A^0_{ia} = \dfrac{E\dot{\phi}}{2\mathfrak{n}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\det A_{ia} = 0$ **only for E = 0.**

Non-invertible !!

Further generalization

$$\begin{cases} \tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi + E(\phi, X) \nabla_\mu \nabla_\nu \phi \\ \tilde{\phi}(x^\mu) = \phi(x^\mu) \qquad\qquad (X = g^{\sigma\tau} \partial_\sigma \phi \partial_\tau \phi) \end{cases}$$

Invertibility conditions :

$$C \left(C - X C_X - X^2 D_X \right) \neq 0, \quad E = 0.$$

Summary

We have derived the necessary and sufficient conditions for the invertibility of field transformations with derivatives, which is the extension of the inverse function theorem.