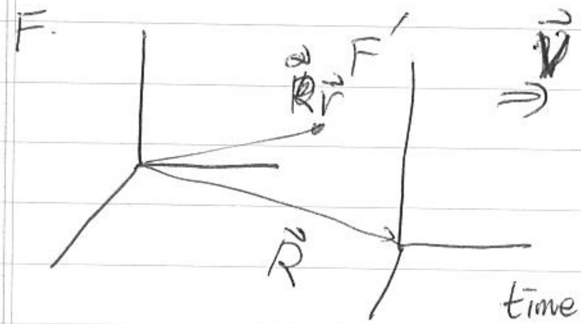
1 Short summary of Relativity.

Unit system. $c = \hbar = 1$. (sometimes) $\pi = 1$.

Dynamics of $v \ll 1$. $\Rightarrow$ Galilean transformation.



$t = t'$
$\vec{r}' = \vec{r} + \vec{R}$
$\vec{v}' = \vec{v} + \vec{V}$

time & position independently transform

For $v \sim 1$. Lorentz transformation
time & position mix with each other

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = L \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

<u>ex</u>. F' moves x-direction with velocity v

$t' = \beta\left(t - \frac{v}{c^2}x\right)$ $x' = \beta(x - vt)$. $y' = y$ $z' = z$

where $\beta = \dfrac{1}{(1 - v^2/c^2)^{1/2}}$

Metric.

1) Galilean

$ds^2 = dx^2 + dy^2 + dz^2$, dt.

2) Special relativity

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$= \underbrace{dt^2 - dx^2 - dy^2 - dz^2}$$

This is the metric of "flat" spacetime.
Minkowski

For a general curved spacetime.

ds^2 = world history length-squared

$$= g_{\mu\nu}(x)\, dx^\mu dx^\nu \quad ; \text{ quadratic invariants.}$$

$g_{\mu\nu}(x)$ measures curvedness of spacetime.

$$g_{\mu\nu} \rightarrow \eta_{\mu\nu} = \underbrace{\mathrm{diag}(+1, -1, -1, -1)}_{\text{flat space-time}}.$$

$$ds^2 = (dt \;\; dx \;\; dy \;\; dz) \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}.$$

Free particle's motion

$$\ddot{x}^a + \Gamma^a_{bc}\, \dot{x}^b \dot{x}^c = 0. \quad \text{where } \dot{x}^a = \frac{\partial x^a}{\partial \tau}$$

τ proper time

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

: Christoffel connection

Einstein's general relativity

Spacetime geometry $\Rightarrow$ particle's motion.

$\Uparrow$

energy-momentum tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} + \Lambda g_{\mu\nu}$$

$\uparrow$

Einstein tensor

Ricci tensor $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$

Riemann tensor $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$

(curvature tensor)

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

G_N : Newton's constant (that shown in Newton's gravity)

$T_{\mu\nu}$: energy-stress tensor

Robertson-Walker metric.

- Cosmological background.

$g_{\mu\nu}(x) = g_{\mu\nu}(t)$: ~~homg~~ homogeneous, isotropic, time-dependent

$\Rightarrow$ non-linear, ordinary 2nd order diff. eq.

$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = dt^2 - g_{ij}(t)\, dx^i dx^j$

At every time t, spatial part is described by

$d\ell_3^2 = g_{ij}(t)\, dx^i dx^j$

- Cosmological principle.

1) At each time, the Universe is homogeneous and isotropic at large scale
2) Matter particles follow the congruence of timelike geodesics diverging from a point in the past.

$$ds^2 = dt^2 - g_{ij}(t)\, dx^i dx^j \leftarrow \text{homogeneity}$$

$$= dt^2 - R^2(t)\, d\ell_3^2 \leftarrow \text{isotropic}$$

- Constructing the metric

consider a 2-dim space embedded in 3-dim space. with 1-constraint.

(x_1, x_2). fictitious x_3.

$$x_1^2 + x_2^2 + x_3^2 = R^2 \Rightarrow x_1 dx_1 + x_2 dx_2 + x_3 dx_3 = 0$$

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2}$$

$$x_1 = r' \cos\theta \qquad x_2 = r' \sin\theta$$

$$\Rightarrow dl^2 = \frac{R^2 dr'^2}{R^2 - r'^2} + r'^2 d\theta^2$$

Reparametrization

$$r = \frac{r'}{R} \quad (0 \leq r \leq 1).$$

$$\Rightarrow dl^2 = R^2 \left\{ \frac{dr^2}{1-r^2} + r^2 d\theta^2 \right\}$$

generalization to 3-sphere embedded in 4-D.

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

$$= dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - x_1^2 - x_2^2 - x_3^2}$$

Coordinate

$x_1 = \cancel{r'\cos\theta}\ r'\sin\theta\cos\phi \quad x_2 = r'\sin\theta\sin\phi \quad x_3 = r'\cos\theta$

& $r = \frac{r'}{R}$

$$\Rightarrow dl^2 = R^2 \left\{ \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right\}$$

Combining it with time direction.

$$ds^2 = dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right\}$$

for $k = +1$: sphere
$k = 0$: flat
$k = -1$: hyperbola

& $R(t)$: evolve with time
"Scale factor"

Sometimes we define

$$d\eta = \frac{1}{R(t)}\, dt \quad ; \text{ conformal time.}$$

$$ds^2 = R^2(\eta)\left\{ d\eta^2 - \frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \right\}$$

* Particle kinematics.

1) a light signal emitted from (r_H, θ_0, ϕ_0) at $t=0$ will reach $r_0 = 0$ in a time t

$$ds^2 = 0 \Rightarrow \int_0^t \frac{dt'}{R(t')} = \int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}}$$

The proper distance to the horizon measured at t

$$d_H(t) = \int_0^{r_H} \sqrt{g_{rr}}\, dr$$

$$= R(t) \int_0^{t} \frac{dt'}{R(t')}$$

2) geodesic motion of a particle

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu_{\nu\alpha}\, u^\nu \frac{dx^\alpha}{d\lambda} = 0.$$

where $u^\mu = \frac{dx^\mu}{ds}$ $\quad \lambda$ is some affine parameter

u^μ can be expressed by ordinary velocity

$$u^\mu = (u^0, u^i) = (\gamma, \gamma v^i)$$

where $\gamma = (1 - v^2)^{-1/2}$

Let's take $\lambda = s$ $\quad (ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu)$

$$\Rightarrow \frac{du^0}{ds} + \Gamma^0_{\nu\alpha}\, u^\nu u^\alpha = 0$$

For RW metric

$$\cancel{R}\ \Gamma^{0}_{\nu\alpha} \rightarrow \Gamma^{0}_{ij} = \left(\frac{\dot{R}}{R}\right) h_{ij} \qquad \left(dl^2 = h_{ij} x^i x^j \right).$$

other ~~ter~~ components are zero.

$$\frac{du^0}{ds} + \frac{\dot{R}}{R} |\vec{u}|^2 = 0 .$$

$$(u^0)^2 - |\vec{u}|^2 = 1$$

$$u^0 du^0 = |\vec{u}|\, d|\vec{u}|$$

$$\Rightarrow \frac{1}{u^0} \frac{d|\vec{u}|}{ds} + \frac{\dot{R}}{R} |\vec{u}| = 0$$

$$\left(u^0 \equiv \frac{dt}{ds} \right)$$

$$\Rightarrow \frac{\dot{|\vec{u}|}}{|\vec{u}|} = -\frac{\dot{R}}{R} \Rightarrow |\vec{u}| \propto \frac{1}{R}$$

$\Rightarrow p^\mu = m u^\mu$ ~~redshifts as~~ $\cancel{\frac{1}{R}}$

3-momentum redshifts as $1/R$

Redshift parameter

$$1 + z = \frac{R(t_0)}{R(t_1)} \begin{matrix} \leftarrow \text{now} \\ \nwarrow \text{past} \end{matrix}$$

, ~~a~~ Taylor expansion.

$$R(t) = R(t_0) + \dot{R}(t_0)(t - t_0) + \frac{1}{2} \ddot{R}(t_0)(t - t_0)^2$$

$$= R(t_0) + \frac{\dot{R}(t_0)}{R(t_0)} \cancel{(t)} R(t_0)(t - t_0)$$

$$- \cancel{4}\ \frac{1}{2} \left(\frac{-\ddot{R}(t_0)}{\dot{R}^2(t_0)} R(t_0) \right) \cdot \frac{\dot{R}^2(t_0)}{R(t_0)} (t - t_0)^2 + \cdots$$

$$\Rightarrow \frac{R(t)}{R(t_0)} = 1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \cdots$$

where $H_0 = \dfrac{\dot{R}(t_0)}{R(t_0)}$ $\quad q_0 = -\dfrac{\ddot{R}(t_0)}{\dot{R}^2(t_0)} \cdot R(t_0)$

$$\Rightarrow z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t)^2 + \cdots$$

Particle kinematics.

$$\int_{t_1}^{t_0} \frac{dt'}{R(t')} = \int_0^{r_1} \frac{dr}{(1 - kr^2)^{1/2}}$$

$$LHS = \frac{1}{R(t_0)} \left((t_0 - t_1) + \frac{1}{2} H_0 (t_0 - t_1)^2 + \cdots \right).$$

$$RHS = \begin{cases} r_1 + \frac{1}{6} r_1^3 + \cdots & k = +1 \\ r_1 & k = 0. \\ r_1 - \frac{1}{6} r_1^3 + \cdots & k = -1 \end{cases}$$

$$\Rightarrow r_1 = \frac{1}{R(t_0)} \left[(t_0 - t_1) + \frac{1}{2} H_0 (t_0 - t_1)^2 + \cdots \right]$$

$$\Rightarrow r_1 = \frac{1}{R(t_0) H_0} \left[z - \frac{1}{2}(1 + q_0) z^2 + \cdots \right]$$

$$\Rightarrow H_0 d_L = z + \frac{1}{2}(1 - q_0) z^2 + \cdots$$: Hubble's law

deceleration parameter

where $d_L = R(t_0) r_1 (1 + z)$. : luminocity distance

Standard Cosmology.

Einstein eq.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

cosmological constant.

$g_{\mu\nu} \Rightarrow$ RW metric

$T_{\mu\nu} \Rightarrow$ that of perfect fluid. $\rho = \rho(t)$. $p = p(t)$.

$$T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p).$$

0 comp. of the conservation of stress energy

$$\nabla_\mu T^{\mu}{}_{\nu} = \partial_\mu T^{\mu}{}_{\nu} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda}{}_{\nu} - \Gamma^{\lambda}_{\mu\nu} T^{\mu}{}_{\lambda} = 0.$$

; continuity eq.

$\nu = 0$ comp.

$$\Rightarrow \dot{\rho} + 3\frac{\dot{R}}{R}\rho + 3\frac{\dot{R}}{R}p = 0.$$

$$\Rightarrow \dot{\rho} R^3 + 3\dot{R}R^2\rho + 3\dot{R}R^2 p = 0.$$

$$\Rightarrow \frac{d}{dt}(\rho R^3) = -p\frac{d}{dt}(R^3)$$

$$\Rightarrow d(\rho R^3) = -p\, d(R^3) \qquad \text{— (1)}$$

$$\Rightarrow d\left[R^3(\rho + p)\right] = R^3 dp$$

energy change

pressure × volume change

Eq. of state.

$$p = \omega \rho \quad \text{where } \omega \text{ is constant.}$$

Sol. $\Rightarrow$ $\rho \propto R^{-3(1+\omega)}$

Rad $\left(p = \frac{1}{3}\rho\right) \Rightarrow \rho \propto R^{-4}$

Mat $\left(p = 0.\right) \Rightarrow \rho \propto R^{-3}$

Vac $\left(p = -\rho\right) \Rightarrow \rho \propto \text{const.}$

Early	mid	late.	current stage.
Rad-dom.	matter	Vac.	; our universe.

From RW metric.

$$R_{00} = -3\frac{\ddot{R}}{R}$$

$$R_{ij} = -\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R} + \frac{2k}{R^2}\right) g_{ij}$$

$$\Rightarrow \mathcal{R} = -6\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R} + \frac{k}{R^2}\right]$$

(0.0) component of the Einstein eq.

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \quad ; \text{ Friedman eq.} \quad -(2)$$

(i.i) -comp.

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R} + \frac{k}{R^2} = -8\pi G p \quad -(3)$$

Only 2 eqs among (1) (2) (3) are independent.
We usually take (1) & (2)

(2) − (3)

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)$$

Current $\dot{R} \geq 0$ & $\rho + 3p > 0 \Rightarrow \ddot{R} < 0$.

$R \to 0$ in the past $\Rightarrow$ Big Bang.

$H \equiv \frac{\dot{R}}{R}$: Hubble parameter $\quad H(t=\text{now}) \equiv H_0$

Hubble constant.

$$(2) \Rightarrow \frac{k}{H^2R^2} = \frac{1}{3H^2}\frac{\rho}{8\pi G} - 1 \equiv \Omega - 1$$

where $\frac{\rho}{\rho_c} \equiv \Omega \qquad \rho_c \equiv \frac{3H^2}{8\pi G}$

$H^2R^2 \geq 0 \Rightarrow \quad k = +1 \Rightarrow \Omega > 1$ closed

$k = 0 \Rightarrow \Omega = 1$ flat

$k = -1 \Rightarrow \Omega < 1$ open

At early times $\dot{R} \gg k$.

$$(2) \Rightarrow H^2 \simeq \frac{8\pi G}{3}\rho \quad \propto R^{-3} \quad \text{MD}$$

$$\propto R^{-4} \quad \text{RD}$$

($\Omega \simeq 1$ today)

$$|\Omega - 1| \simeq \frac{R}{R_0} = (1+z)^{-1} \qquad \text{MD}$$

$$\left(\frac{R_{EQ}}{R_0}\right)\left(\frac{R}{R_{EQ}}\right)^2 \simeq 10^4 (1+z)^{-2} \qquad \text{RD}$$

$\Rightarrow$ At early universe $|\Omega - 1| \lll 1$.

"flatness problem" $\Rightarrow$ inflation.

Expansion age of the Universe

Friedmann eq.

$$\Rightarrow \left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho_0 \frac{R_0}{R} \quad (MD)$$

$$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho_0 \left(\frac{R_0}{R}\right)^2 \quad (RD)$$

$$H = \frac{\dot{R}}{R} = \frac{1}{R}\frac{dR}{dt}$$

$$\Rightarrow \int dt = t = \int_0^{R(t)} \frac{dR'}{\dot{R}'}$$

$$= \begin{cases} H_0^{-1} \displaystyle\int_0^{(1+z)^{-1}} \frac{dx}{\left[1-\Omega+\Omega_0 x^{-1}\right]^{1/2}} & (MD) \\ H_0^{-1} \displaystyle\int_0^{(1+z)^{-1}} \frac{dx}{\left[1-\Omega+\Omega_0 x^{-2}\right]^{1/2}} & (RD) \end{cases}$$

$$\uparrow$$

$$\frac{k}{R_0^2} \equiv H_0^2(\Omega_0 - 1) \quad \& \quad \frac{R_0}{R} = 1+z$$

Equilibrium thermodynamics

- the early universe was to a good approximation in thermal equilibrium, then we can define (or write)

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p \quad \text{; number density}$$

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{p}) f(\vec{p})\, d^3p \quad \text{: energy density}$$

$$p = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p})\, d^3p \quad \text{: pressure}$$

where $E^2 = |\vec{p}|^2 + m^2$

$$f(\vec{p}) = \left[\exp\{(E-\mu)/T\} \pm 1\right]^{-1} \quad \text{; phase space distribution}$$

$+$: Fermi-Dirac $\qquad$ $-$: Bose-Einstein

for particles in kinetic equilibrium.

For particles in chemical equilibrium

$$i + j \longleftrightarrow k + l \quad \Rightarrow \quad \mu_i + \mu_j = \mu_k + \mu_l$$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} E^2\, dE$$

$$n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp[(E-\mu)/T] \pm 1} E\, dE$$

$$p = \frac{g}{6\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{3/2}}{\exp[(E-\mu)/T] \pm 1}\, dE$$

In the relativistic limit $(T \gg m)$, for $T \gg \mu$

$$\rho = \begin{cases} \dfrac{\pi^2}{30} g T^4 & \text{Bose} \\[2ex] \left(\dfrac{7}{8}\right)\left(\dfrac{\pi^2}{30}\right) g T^4 & \text{Fermi} \end{cases}$$

$$n = \begin{cases} \dfrac{\zeta(3)}{\pi^2} g T^3 & \text{Bose} \\[2ex] \dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} g T^3 & \text{Fermi} \end{cases} \qquad \zeta(3) = 1.20206\ldots$$

$$P = \frac{1}{3}\rho$$

while for degenerate fermions $\mu \gg T$

$$\rho = \frac{1}{8\pi^2} g \mu^4$$

$$n = \frac{1}{6\pi^2} g \mu^3$$

$$P = \frac{1}{24\pi^2} g \mu^4$$

For relativistic bosons or fermions with $\mu < 0$ & $|\mu| < T$

$$n = \exp\left(\frac{\mu}{T}\right) \frac{g}{\pi^2} T^3$$

$$\rho = \exp\left(\frac{\mu}{T}\right) \frac{3g}{\pi^2} T^4$$

$$P = \exp\left(\frac{\mu}{T}\right) \frac{g}{\pi^2} T^4$$

In the non-relativistic limit $(m \gg T)$

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[-(m-\mu)/T\right]$$

$$\rho = mn$$

$$p = nT \ll \rho$$

For nondegenerate, relativistic species

$$\langle E \rangle = \frac{\rho}{n} = \left(\frac{\pi^4}{30\zeta(3)}\right)T \simeq 2.701\,T \quad (\text{Bose})$$

$$\left(\frac{7\pi^4}{180\zeta(3)}\right)T \simeq 3.151\,T \quad (\text{Fermi})$$

For a degenerate, relativistic fermion species

$$\langle E \rangle = \frac{\rho}{n} = \frac{3}{4}\mu$$

For a non-relativistic species

$$\langle E \rangle = m + \frac{3}{2}T$$

The total energy density and pressure of all species in equilibrium can be expressed by the photon T.

$$\rho_R = T^4 \sum_{i=\text{all}} \left(\frac{T_i}{T}\right)^4 \left(\frac{g_i}{2\pi^2}\right) \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}\, u^2\, du}{\exp(u - y_i) \pm 1}$$

$$p_R = T^4 \sum_{i=\text{all}} \left(\frac{T_i}{T}\right)^4 \left(\frac{g_i}{6\pi^2}\right) \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{3/2}\, du}{\exp(u - y_i) \pm 1}$$

We can write ρ_R by defining the effective d.o.f

$$\rho_R = \frac{\pi^2}{30} g_* T^4$$

$$p_R = \frac{1}{3}\rho_R = \frac{\pi^2}{90} g_* T^4$$

where g_* counts the total number of ~~d.o.~~ effectively massless d.o.f. ($m_i \ll T$)

$$g_* = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=f} g_* \left(\frac{T_i}{T}\right)^4$$

During the RD. epoch $\rho \simeq \rho_R$, $g_* \simeq$ const. $p_R = \rho_R/3$

$$\Rightarrow R(t) \propto t^{1/2}$$

$$H = \frac{\dot{R}}{R} = \frac{\frac{1}{2}t^{-1/2}}{t^{1/2}} = \frac{1}{2t} = \left(\frac{8\pi G}{3}\rho\right)^{1/2} = 1.66\, g_*^{1/2} \frac{T^2}{m_{pl}}$$

$$t = \frac{1}{2H} = 0.301\, g_*^{-1/2} \frac{m_{pl}}{T^2} \sim \left(\frac{T}{\text{MeV}}\right)^{-2} \text{sec.}$$

* Entropy

$$T dS = \underbrace{d(\rho V)}_{dE} + \underbrace{p dV}_{pdV} = d\left[(\rho + p)V\right] - V dp$$

integrability condition

$$T\frac{dp}{dT} = \rho + p \Rightarrow dp = \left(\frac{\rho + p}{T}\right) dT$$

$$\Rightarrow s = \frac{S}{V} = \frac{\rho + p}{T} = \frac{2\pi^2}{45} g_{*s} T^3$$

where $g_{*s} = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=f} g_i \left(\frac{T_i}{T}\right)^3$

- Dark matter freeze out

the microscopic evolution of phase space distribution

$f(p^\mu, x^\mu) \Rightarrow$ Boltzmann eq.

$$\hat{L}[f] = C[f] \quad \text{— } (*)$$

$$\hat{L} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \quad \leftarrow \text{RW metric.}$$

$$\hat{L}[\underbrace{f(E,t)}_{\uparrow}] = E \frac{\partial f}{\partial t} - \frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial f}{\partial E}$$

spatially homogeneous, isotropic

Note $\quad n(t) = \frac{g}{(2\pi)^3} \int d^3p \, f(E,t)$

$$(*) \Rightarrow \quad \frac{dn}{dt} + 3 \frac{\dot{R}}{R} n = \frac{g}{(2\pi)^3} \underbrace{\int C[f] \frac{d^3p}{E}}$$

$\uparrow$ collision term.
related to DM production/
annihilation
process.

Consider $\quad \underbrace{\psi \bar{\psi}} \longleftrightarrow \underbrace{X \bar{X}}$

DM

all other in Eq.

DM

~~pra particles in~~ TE.

annihil
~~production~~ $\sim f_\psi f_{\bar\psi}$

~~annih~~. prod. $\sim f_\chi f_{\bar\chi}$

collision term $\sim$ ~~$f_\psi f_{\bar\psi} - f_\chi f_{\bar\chi}$~~ $f_\chi f_{\bar\chi} - f_\psi f_{\bar\psi}$

remaining factor is determined by scattering amplitude (~~responsible~~ QFT calculation for a given model)
with $\mu = 0$.

$$f_\chi = \exp[-E_\chi/T] \quad f_{\bar\chi} = \exp[-E_{\bar\chi}/T]$$

equilibrium dist.

Energy conservation $\Rightarrow E_\psi + E_{\bar\psi} = E_\chi + E_{\bar\chi}$

$$f_\chi f_{\bar\chi} = \exp[-(E_\chi + E_{\bar\chi})/T] = \exp[-(E_\psi + E_{\bar\psi})/T]$$

$$= f_\psi^{EQ} f_{\bar\psi}^{EQ}$$

$$\Rightarrow f_\chi f_{\bar\chi} - f_\psi f_{\bar\psi} = f_\psi^{EQ} f_{\bar\psi}^{EQ} - f_\psi f_{\bar\psi}$$

Boltzmann eq.

$$\frac{dn_\psi}{dt} + 3Hn_\psi = -\langle \sigma_{\psi\bar\psi\to\chi\bar\chi} |v| \rangle \left[n_\psi^2 - (n_\psi^{EQ})^2 \right]$$

thermal averaged cross section.